Generalisations of the Doyen-Wilson Theorem

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Abstract

In 1973, Doyen and Wilson famously solved the problem of when a 3-cycle system can be embedded in another 3-cycle system. There has been much interest in the literature in generalising this result for m-cycle systems when $m > 3$. Although there are several partial results, including complete solutions for some small values of m and strong partial results for even m, this still remains an open problem.

The main results of this thesis concern generalisations of the Doyen-Wilson Theorem for odd m-cycle systems and cycle decompositions of the complete graph with a hole. The complete graph of order v with a hole of size u, $K_v - K_u$, is constructed from the complete graph of order v by removing the edges of a complete subgraph of order u (where $v \geqslant u$).

For each odd $m \geqslant 3$ we completely solve the problem of when an m-cycle system of order u can be embedded in an m-cycle system of order v, barring a finite number of possible exceptions. The problem is completely resolved in cases where u is large compared to m, where m is a prime power, or where $m \leqslant 15$. In other cases, the only possible exceptions occur when $v - u$ is small compared to m. This result is proved as a consequence of a more general result which gives necessary and sufficient conditions for the existence of an m-cycle decomposition of $K_v - K_u$ in the case where $u \geqslant m - 2$ and $v - u \geqslant m + 1$ both hold.

We prove that $K_v - K_u$ can be decomposed into cycles of arbitrary specified lengths provided that the obvious necessary conditions are satisfied, $v - u \geqslant 10$, each cycle has length at most $\min(u, v - u)$, and the longest cycle is at most three times as long as the second longest. This complements existing results for cycle decompositions of graphs such as the complete graph, complete bipartite graph and complete multigraph.

We obtain these cycle decomposition results by applying a cycle switching technique to modify cycle packings of $K_v - K_u$. The tools developed by cycle switching enable us to merge collections of short cycles to obtain longer cycles. The methodology therefore relies on first finding decompositions of various graphs into short cycles, then applying the merging results to obtain the required decomposition. Similar techniques have previously been successfully applied to the complete graph and the complete bipartite graph. These methods also have potential to be further developed for the complete graph with a hole as well as other graphs.

We also give a complete solution to the problem of when there exists a packing of the complete multigraph with cycles of arbitrary specified lengths. The proof of this result relies on applying cycle switching to modify cycle decompositions of the complete multigraph obtained from known results.

The results in this thesis make substantial progress toward generalising the Doyen-Wilson Theorem for arbitrary odd cycle systems and toward constructing cycle decompositions of the complete graph with a hole. However there still remain unsolved cases. Moreover, the cycle switching and base decomposition methods used to obtain these results give rise to several interesting open problems.

Declaration

This thesis contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

RHoyte

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19 December 2016

Publications

This thesis contains work from the following publications.

D. Horsley and R. A. Hoyte, Doyen-Wilson results for odd length cycle systems. *J. Combin. Des.*, 24 (7), 308–335, 2016.
D. Horsley and R. A. Hoyte, Decomposing $K_{u+w} - K_u$ into cycles of prescribed lengths. *Discrete Math.*, http://dx.doi.org/10.1016/j.disc.2017.03.011, 2017.
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Soli Deo Gloria.

Contents

Chapter 1

Introduction

A *decomposition* of a graph G is a collection of subgraphs of G, $\{G_1, \ldots, G_\tau\}$ whose edge sets partition the edge set of G. Many classical graph theoretic problems, such as finding colourings and factorisations, can also be posed as graph decomposition problems. The study of graph decompositions is closely related to topics in combinatorial design theory and finite geometry. Techniques arising from the study of graph decompositions have been effective in obtaining solutions to problems such as designing effective experiments, optimising the flow of data through fibre optic networks, and data sampling via compressed sensing [16, 17, 36, 47]. This thesis makes significant progress on two well-studied graph decomposition problems, namely, generalising the Doyen-Wilson Theorem and packing the complete multigraph with cycles.

We now provide some definitions, and any graph theoretic terminology that is not defined here can be found in [95]. An *m-cycle decomposition* of a graph G is a decomposition $\{G_1, \ldots, G_\tau\}$ such that G_i is a cycle of length m for $i \in \{1, \ldots, \tau\}$, where a cycle of length m is a 2-regular connected graph on m vertices. An *m-cycle system of order v* is an m-cycle decomposition of the complete graph of order v. An m-cycle system $\mathcal{A}$ is said to be *embedded* in another m-cycle system $\mathcal{B}$ when $\mathcal{A} \subseteq \mathcal{B}$. In 1973, Doyen and Wilson [51] found a complete solution to the question of when one 3-cycle system can be embedded in another.

Theorem 1.1 (Doyen-Wilson Theorem [51])**.** *There exists an embedding of a 3-cycle system of order u in a 3-cycle system of order v if and only if $u, v \equiv 1$ or $3 \pmod 6$ and $v \geqslant 2u + 1$.*

The requirement that u and v are 1 or $3 \pmod 6$ is both necessary and sufficient for the existence of 3-cycle systems of these orders. In fact, for each $m \geqslant 3$ it is known [4, 83] exactly which values of v are the order of an m-cycle system. Thus it is a natural question to ask whether one m-cycle system of order u can be embedded in another of order v for $m > 3$. A complete solution to this question is known for all $m \leqslant 14$ [37, 38, 40], and for even values of m when $m \leqslant v - u$ [64]. It has also been solved when m is odd and $u, v \equiv 1$ or $m \pmod{2m}$, with the exception of some cases when v is small [38]. Overall, weaker results had been obtained for odd m than for even m.

A further generalisation of the Doyen-Wilson Theorem is to consider cycle decompositions of the complete graph with a hole. The complete graph of order v with a hole of size u, $K_v - K_u$, is constructed from the complete graph of order v by removing the edges of a complete subgraph of order u (where $v \geqslant u$). For even m, the problem of when $K_v - K_u$ can be decomposed into m-cycles is solved completely when $m \leqslant 14$ [40] and for $4 \leqslant m \leqslant \min(u, v-u)$ [64]. Again, less is known for odd m and this problem had only been solved completely when $m \in \{3, 5, 7\}$ [37, 40, 75].

The main focus of this thesis is on generalising the Doyen-Wilson Theorem. In particular, we find embeddings of odd cycle systems and decompositions of the complete graph with a hole into uniform length odd cycles. We also extend these results to decompositions of the complete graph with a hole into cycles of arbitrary lengths. The literature review in Section 1.1 outlines known results on cycle decompositions of various graphs and progress that has been made to date on generalising the Doyen-Wilson Theorem. The specific research questions and methodology of this thesis are outlined in Section 1.2.

1.1 Literature review

Cycle decomposition of various graphs has been an interesting graph theoretic problem since at least the 1800s when Walecki proposed a construction for a Hamilton cycle decomposition of the complete graph (as reported by Lucas [71]). Another early result is due to Kirkman in 1847 [68] who constructed a 3-cycle decomposition of the complete graph of order v whenever $v \equiv 1$ or $3 \pmod 6$. These results have since been generalised in various ways and this review outlines some of the major results on cycle decompositions of various graphs.

1.1.1 Cycle decompositions of the complete graph

Each vertex in a cycle is incident with exactly two edges of that cycle, so every vertex in a cycle decomposition has even degree. It follows that if a complete graph has a decomposition into cycles then it has an odd number of vertices. The intuitive variation of this problem for the complete graph on an even number of vertices is to decompose the complete graph minus a 1-factor. Building on the early results by Walecki and Kirkman, research in the literature first focused on decomposing the complete graph into cycles of uniform length and in particular on proving Theorem 1.2 (stated below). Later results, inspired by the 1981 conjecture by Alspach [3], addressed cases with various cycle lengths.

Uniform length cycles

Interest in the problem of decomposing the complete graph into uniform length cycles was renewed in the 1960s with various results for specific cycle lengths [69, 81, 82]. Many of these earlier results relied on cyclic constructions and had limited potential to be extended to further results. However, following on from numerous partial results, as outlined in the surveys [23, 39], research on this problem culminated in 2002 with a complete solution due to Alspach, Gavlas and Šajna [4, 83]. Here we briefly outline the main elements of this proof.

Theorem 1.2 ([4, 83]). *Let v and m be integers such that $3 \leqslant m \leqslant v$. If v is odd then there exists an m-cycle decomposition of K_v whenever $v(v-1) \equiv 0 \pmod{2m}$. If v is even then there exists an m-cycle decomposition of $K_v - I$ whenever $v(v-2) \equiv 0 \pmod{2m}$, where I is a 1-factor of K_v.*

In the proof of Theorem 1.2, different cases are considered depending on the parity of m and v, and each case has two main steps. The first step is to show that there exists an m-cycle decomposition of K_v (or $K_v - I$) for all v that satisfy the necessary conditions, provided that this holds for all $m \leqslant v \leqslant km$ (where $k \in \{2, 3\}$ depending on the parity of m and v) [4, 18, 61, 79, 84]. The second step is to find the decompositions required by this reduction.

Proving the reduction of this problem involves first decomposing K_v into a collection of smaller graphs which in turn have m-cycle decompositions. These results rely on the existence of cycle decompositions of graphs such as the complete bipartite and multipartite graph, and the complete graph with a hole (see results from Sections 1.1.2 and 1.1.3). As we will see in the remainder of this chapter, this general approach is often a useful method for obtaining cycle decomposition results (see also surveys on cycle decompositions [39, 23, 79]).

The proof of Theorem 1.2 is completed by obtaining m-cycle decompositions of K_v when $m \leqslant v \leqslant 3m$ (or $v \leqslant 2m$ when m and v are even). These decompositions are found using various cycle decomposition techniques involving decomposing circulant graphs, results due to Haggkvist [57] and Tarsi [93], and other constructions (see [4, 83]). For definitions and descriptions of these and other techniques see the survey [23].

Cycles of various lengths

Decomposing the complete graph into cycles of arbitrary specified lengths is a natural generalisation of Theorem 1.2. The problem was articulated by Alspach in Conjecture 1.3. For some list of positive integers $m_1, \dots, m_\tau$, an $(m_1, \dots, m_\tau)$*-decomposition of a graph* G is a decomposition of G into τ cycles of lengths $m_1, \dots, m_\tau$. Throughout the following, for a list of integers $M = m_1, \dots, m_\tau$, an $(M)^*$-decomposition of K_v denotes an (M)-decomposition of K_v if v is odd and an (M)-decomposition of $K_v - I$ if v is even, where I is a 1-factor of K_v.

Conjecture 1.3 ([3]). *Let v be a positive integer, and $m_1, \dots, m_\tau$ a list of integers. There exists an $(m_1, \dots, m_\tau)^*$-decomposition of K_v if and only if $3 \leqslant m_i \leqslant v$ for $i \in \{1, \dots, \tau\}$ and $m_1 + \cdots + m_\tau = v\lfloor (v-1)/2 \rfloor$.*

The necessity of the conditions given in Conjecture 1.3 follows from first observing that the total number of edges in the decomposition is $\binom{v}{2}$ when v is odd, and $\binom{v}{2} - \frac{v}{2}$ when v is even. Thus it follows that $m_1 + \cdots + m_\tau = v\lfloor (v-1)/2 \rfloor$. Furthermore, the length of the longest cycle cannot exceed the size of the graph, so $3 \leqslant m_i \leqslant v$ for $i \in \{1, \dots, \tau\}$. Hence it remains to prove the sufficiency of the conditions in Conjecture 1.3.

Results for decomposing the complete graph into cycles of different lengths first appeared in the 1980s [57, 58]. These results considered existence of a $(m_1, \dots, m_\tau)$-decompositions when $m_i \in S$ for each $i \in \{1, \dots, \tau\}$, where S is a set of allowed cycle lengths. Several other results of this type were obtained up until 2011 [59, 66] as listed in the surveys [39, 23].

In 2001, Balister [14] verified Conjecture 1.3 for $v \leqslant 14$, building on the solution by Rosa [80] for $v \leqslant 10$. Other partial results in the literature include verifying Conjecture 1.3 for all $v \geqslant N$ for some large value of N. In earlier results of this form, N is a rapidly increasing function of the longest cycle in the decomposition [14, 42]. These asymptotic results were later improved by Bryant and Horsley when v is odd [29].

A complete solution to Conjecture 1.3 was obtained in 2014 by Bryant, Horsley and Pettersson. This result is given as Theorem 1.4 and we outline the proof below.

Theorem 1.4 ([33]). *Let v be an integer and let $m_1, \dots, m_\tau$ be a list of integers. There exists an $(m_1, \dots, m_\tau)^*$-decomposition of K_v if and only if $3 \leqslant m_i \leqslant v$ for $i \in \{1, \dots, \tau\}$ and $m_1 + \cdots + m_\tau = v\lfloor (v-1)/2 \rfloor$.*

We first require the following definition, which is given in full in [29, 33]. A *v-ancestor list* is defined as a list that contains at most one k-cycle for $6 \leqslant k \leqslant v-1$ where the number of cycles of lengths 3, 4, 5 and v are also subject to some constraints, including ensuring that the list satisfies the necessary conditions given in Conjecture 1.3. So a v-ancestor list is of the form

$$(3, 3, \dots, 3, 4, 4, \dots, 4, 5, 5, \dots, 5, k, v, v, \dots, v).$$

The proof of Theorem 1.4 has two main steps. The first is given by Theorem 1.5 which reduces the proof of Theorem 1.4 to the problem of finding decompositions corresponding to ancestor lists. The second step is to find these decompositions.

Theorem 1.5 ([29]). *For each positive integer v, if there exists an $(M')^*$-decomposition of K_v for each v-ancestor list M', then there exists an $(M)^*$-decomposition of K_v for each list $M = m_1, \dots, m_\tau$ that satisfies $3 \leqslant m_i \leqslant v$ for $i \in \{1, \dots, \tau\}$ and $m_1 + \cdots + m_\tau = v\lfloor (v-1)/2 \rfloor$.*

The key ingredients in the proof of this result are Lemmas 1.6 and 1.7, which rely on the cycle switching technique introduced below in Section 1.1.4. Ancestor lists are defined so that, using results such as these, a decomposition corresponding to any list $m_1, \dots, m_\tau$ that satisfies $3 \leqslant m_i \leqslant v$ for $i \in \{1, \dots, \tau\}$ and $m_1 + \cdots + m_\tau = v\lfloor (v-1)/2 \rfloor$ can be obtained from a decomposition corresponding to some v-ancestor list.

Lemma 1.6 ([28]). *Let v, m_1, m_2, m_1' and m_2' be positive integers such that $m_1 \leqslant m_1' \leqslant m_2' \leqslant m_2$ and $m_1 + m_2 = m_1' + m_2'$, and let M be a list of integers. If there exists an $(M, m_1, m_2)^*$-decomposition of K_v in which an m_1-cycle and an m_2-cycle share at least two vertices, then there exists an $(M, m_1', m_2')^*$-decomposition of K_v.*

Lemma 1.6 'equalises' the lengths of two cycles in a decomposition. A result that complements Lemma 1.6 is the following lemma which, under certain conditions, 'merges' two cycles in a decomposition to form one longer cycle.

Lemma 1.7 ([29]). *Let M be a list of integers and let v, m, m' and h be positive integers such that $m + m' \leqslant 2h$ and $m + m' + h \leqslant v + 1$. If there exists an $(M, h, m, m')^*$-decomposition of K_v, then there exists an $(M, h, m + m')^*$-decomposition of K_v.*

The proof of Theorem 1.4 was obtained by using both the 'equalising' and 'merging' methods (Lemmas 1.6 and 1.7 respectively) as well as numerous constructions for cycle decompositions corresponding to v-ancestor lists. The constructions concerning ancestor list decompositions consist of two main cases, ancestor list decompositions with at least two v-cycles and those with at most one v-cycle. The case with at least two v-cycles relies on various constructions involving circulant graphs and difference methods. These constructions often assume that the decomposition has at least two v-cycles, so a different approach is required for the second case. The second case is proved by induction on v, relying on the known result for $v \leqslant 14$ [14]. There are separate sub-cases for whether the decomposition contains one or no v-cycles and whether it has many cycles of length 3, 4 or 5.

Finding the required decompositions completes the proof of Theorem 1.4, solving the problem of when there exists a decomposition of the complete graph into cycles of arbitrary specified lengths. Theorem 1.4 is applied numerous times in this thesis (see Sections 2.2, 2.3, 3.2 and 3.3). Some of the other results in this thesis include ideas similar to Lemma 1.7 (see Sections 2.1 and 3.1), thereby reducing the problem to finding decompositions in a smaller subset of the problem (see Sections 2.2 and 3.2).

Lemma 1.7 has been extended to decompositions of the complete bipartite graph in order to prove a recent result by Horsley [64] on decomposing the complete bipartite graph into cycles of arbitrary lengths (see Theorem 1.21). Using a merging result alone (as is done in the case of the complete bipartite graph in [64]) tends to produce decompositions only when the cycle lengths are at most half of the obvious necessary upper bound. In particular, while Theorem 1.21 did significantly improve on previously known results for decomposing the complete bipartite graph, the lack of a bipartite counterpart to Lemma 1.6 limits the strength of results that can be obtained. Similarly, when the equalising result alone was applied to cycle decompositions of K_v, Conjecture 1.3 was only verified for cases when all of the cycle lengths are at least about $v/2$ [28].

Related decomposition problems for K_v

There are several results worth mentioning here concerning problems that are closely related to cycle decomposition of the complete graph.

A *packing* of a graph G is a decomposition of some subgraph H of G and the *leave* of the packing is the graph obtained by removing the edges of H from G. We define the *reduced leave* of a packing of a graph G as the graph obtained from its leave by deleting any isolated vertices. For a list of positive integers $M = m_1, \dots, m_\tau$ an (M)*-packing of* G is a packing of G with τ cycles of lengths $m_1, \dots, m_\tau$.

The following result extends Theorem 1.2 to packings of the complete graph with uniform length cycles.

Theorem 1.8 ([63]). *Let v and m be positive integers such that $3 \leqslant m \leqslant v$. If v is odd, let k be the largest integer such that $km \leqslant \binom{v}{2}$ and $\binom{v}{2} - km \notin \{1, 2\}$, and if v is even, let k be the largest integer such that $km \leqslant \binom{v}{2} - \frac{v}{2}$. Then there exists a cycle packing of K_v with k cycles of length m.*

Theorem 1.9 is the analogous version of Theorem 1.4 for λK_v, where λK_v is the complete multigraph with λ edges between each pair of v distinct points. Theorem 1.9 and other results in [31, 32] also include 2-cycles (pairs of parallel edges) as these are present in λK_v for $\lambda \geqslant 2$.

Theorem 1.9 ([32]). *There is an $(m_1, \ldots, m_\tau)$-decomposition of λK_v if and only if*

- $\lambda(v-1)$ *is even;*
- $2 \leqslant m_1, m_2, \ldots, m_\tau \leqslant v$*;*
- $m_1 + \cdots + m_\tau = \lambda\binom{v}{2}$*;*
- $\max(m_1, \ldots, m_\tau) + \tau - 2 \leqslant \frac{\lambda}{2}\binom{v}{2}$ *when λ is even; and*
- $\sum_{m_i=2} m_i \leqslant (\lambda - 1)\binom{v}{2}$ *when λ is odd.*

There is an $(m_1, \ldots, m_\tau)$-decomposition of $\lambda K_v - I$, where I is a 1-factor in λK_v, if and only if

- $\lambda(v-1)$ *is odd;*
- $2 \leqslant m_1, m_2, \ldots, m_\tau \leqslant v$*;*
- $m_1 + \cdots + m_\tau = \lambda\binom{v}{2} - \frac{v}{2}$*; and*
- $\sum_{m_i=2} m_i \leqslant (\lambda - 1)\binom{v}{2}$.

Resolvable decompositions of the complete graph into cycles have also been of interest in the literature. Particular problems include the Oberwolfach problem [24, 41, 49] and Hamilton-Waterloo problem [1, 25]. Other problems of interest are when complete graph can be decomposed into circuits [13, 30], paths [93] or stars [70, 92] (note that a circuit is a connected graph in which each vertex has even degree, and a star of size k is the complete bipartite graph $K_{1,k}$). For results on these and other problems see the survey [39].

1.1.2 Doyen-Wilson type results: $K_v - K_u$

In 1973, Doyen and Wilson proved Theorem 1.1 concerning embedding 3-cycle systems. A natural generalisation of the Doyen-Wilson Theorem is to find necessary and sufficient conditions for embedding m-cycle systems when $m > 3$. A further generalisation is to consider m-cycle decompositions of the complete graph with a hole, $K_v - K_u$. Here we outline the progress that has been made to date on these problems, but we first mention some other generalisations that have featured in the literature.

One alternative generalisation of the Doyen-Wilson Theorem is given by pairwise balanced designs of order v with a mandatory block of size u. This gives a decomposition of $K_v - K_u$ into blocks (or cliques) of given sizes. For a precise definition and known results see [52, 76]. Another related problem is enclosings of m-cycle decompositions of the complete multigraph λK_u inside decompositions of $(\lambda + \mu)K_v$ or, more generally, decompositions of complete multigraphs with holes $(\lambda+\mu)K_v - \lambda K_u$ [9, 10, 77]. Other generalisations concern embedding maximum packings and minimum coverings of triple systems, and embedding partial triple systems [46, 48].

Any embedding of an m-cycle system of order u in another of order v yields an m-cycle decomposition of $K_v - K_u$ (via removing the cycles in the original system). The problem of finding m-cycle decompositions of complete graphs with holes is more general because the orders of the graph and hole need not be feasible orders for m-cycle systems. Cycle decompositions of complete graphs with holes are useful for constructing decompositions of other graphs. For example, the reduction of Theorem 1.2 which is outlined above relies on results concerning cycle decompositions of $K_v - K_u$ (see Theorem 1.15 below).

The following result of Mendelsohn and Rosa in 1983 gives the conditions for when the complete graph with a hole can be decomposed into 3-cycles, expanding on Theorem 1.1 which concerns the more specific embedding problem.

Theorem 1.10 ([75]). *Let u and v be integers such that $u < v$. There exists a 3-cycle decomposition of $K_v - K_u$ if and only if u and v are odd, $v \geqslant 2u + 1$ and $u, v \equiv 1$ or $3 \pmod 6$, or $u \equiv v \equiv 5 \pmod 6$.*

Other known results similar to Theorem 1.10 are included below. We first present Lemma 1.11 which gives necessary conditions for the existence of an m-cycle decomposition of the complete graph with a hole.

Lemma 1.11 ([40]). *For an integer $m \geqslant 3$ and for positive integers u and v, if there exists an m-cycle decomposition of $K_v - K_u$ then the following conditions hold.*

(N1) u *and* v *are odd;*

(N2) $\binom{v}{2} - \binom{u}{2} \equiv 0 \pmod m$*;*

(N3) $(v - m)(v - 1) \geqslant u(u - 1)$*; and*

(N4) $v \geqslant \frac{u(m+1)}{m-1} + 1$ *if* m *is odd.*

The following proof of this result expands on some details of the proof given in [40].

Proof. Suppose there exists a decomposition of $K_v - K_u$ into m-cycles. Since the degree of each vertex must be even, we have $v - u \equiv 0 \pmod 2$ and $v - 1 \equiv 0 \pmod 2$ so (N1) follows. Condition (N2) holds since m divides the number of edges in $K_v - K_u$. Note that there are $\frac{v(v-1)-u(u-1)}{2m}$ cycles in the decomposition and a fixed vertex outside the hole must be in at least $\frac{v-1}{2}$ cycles. So $\frac{v(v-1)-u(u-1)}{2m} \geqslant \frac{v-1}{2}$ and (N3) follows. Finally, any odd cycle in $K_v - K_u$ must contain at least one edge that is not incident with a vertex in the hole, so if m is odd then $\frac{(v-u)(v-u-1)}{2} \geqslant \frac{v(v-1)-u(u-1)}{2m}$ and (N4) follows. $\square$

So, as a consequence of Lemma 1.11, to generalise the Doyen-Wilson Theorem to m-cycle decompositions of $K_v - K_u$ it suffices to show that (N1)–(N4) are sufficient conditions. In particular, an implication of (N4) (noted by Rodger [78]) is that approaches to this problem are quite different depending on the parity of m. This is reflected in the following results and more generally in the fact that more is known for even cycle decompositions of $K_v - K_u$ than for odd cycles.

Even cycle decompositions of $K_v - K_u$

When m is even, the following theorem proved by Bryant, Rodger and Spicer in 1997 reduces the problem of decomposing $K_v - K_u$ into m-cycles to finding decompositions of smaller graphs when u and v are not too large. This idea is similar to the first step in proving Theorem 1.2 as was discussed in Section 1.1.1.

Theorem 1.12 ([40])**.** *Let m be an even integer and let u and v be integers such that $u = 1$, $u > m/2$ or $v > u$. If there exists an m-cycle decomposition of $K_v - K_u$ then there exists an m-cycle decomposition of $K_{v+xm} - K_{u+ym}$ for $x \geqslant y \geqslant 0$ and $x \equiv y \pmod 2$.*

The proof of Theorem 1.12 relies on the 1981 result by Sotteau for decomposing the complete bipartite graph into uniform length cycles (see Theorem 1.19), as well as the then-known partial results on the existence of m-cycle systems.

Theorem 1.13 fills in the cases required by Theorem 1.12 to solve the problem for $m \leqslant 14$.

Theorem 1.13 ([40])**.** *Let u, v and m be integers such that $m \in \{4, 6, \ldots, 14\}$ and $u < v$. There exists an m-cycle decomposition of $K_v - K_u$ if and only if (N1)–(N3) hold.*

The following more recent result of Horsley shows the existence of some m-cycle decompositions of $K_v - K_u$ when m is even.

Theorem 1.14 ([64]). *Let u and v be odd positive integers. If $m \geqslant 4$ is an even integer such that $\binom{v}{2} - \binom{u}{2} \equiv 0 \pmod{m}$, $u \geqslant m+1$ and $v-u \geqslant m$, then there is an m-cycle decomposition of $K_v - K_u$.*

The proof of this result uses Sotteau's result on decompositions of the complete bipartite graph, and the result mentioned above on cycle packings of the complete graph (see Theorem 1.8). The main idea behind the proof is to consider $K_v - K_u$ as the edge-disjoint union of $K_{u,v-u}$ and K_{v-u} and apply the relevant results to decompose each of these separately. However, the proof also allows for the decomposition to include an m-cycle that is not contained entirely within either of these graphs. Note that if we are considering the embedding problem and u is the order of an m-cycle system, then $u \geqslant m+1$. However, in general $u \geqslant m+1$ and $v-u \geqslant m$ are much stronger conditions than (N3) so there are still unsolved cases when $m \geqslant 16$.

Odd cycle decompositions of $K_v - K_u$

The following theorem is one of the earlier results concerning m-cycle decompositions of the complete graph with a hole when m is odd. It is used in the proof of Theorem 1.2 when m and v are both odd, but it only solves a small fraction of the cases for each value of m.

Theorem 1.15 ([61]). *Let u and m be odd integers and let ℓ, q and r be integers such that $m = 2\ell+1$, $u = q\ell + r$ and $1 \leqslant r \leqslant \ell$. If $q \leqslant m + 2r - 1$ then there exists an m-cycle decomposition of $K_{u+2m} - K_u$.*

The next result due to Bryant and Rodger considers more cases than Theorem 1.15, however it only applies to the more specific embedding problem.

Theorem 1.16 ([38]). *Let m be odd and let u and v be 1 or $m \pmod{2m}$. Any m-cycle system of order u can be embedded in an m-cycle system of order v if and only if $v \geqslant (m+1)u/(m-1)+1$, except sometimes when $u \equiv v \equiv m \pmod{2m}$ and $(m+1)u/(m-1)+1 \leqslant v \leqslant (m+1)u/(m-1)+2m$.*

Theorem 1.16 is simplified here but the full version in [38] gives more details on which of the cases are missing. The same paper also develops some methods for reducing the number of examples that need to be found in order to solve the exceptions to Theorem 1.16. These methods are then applied to $m \in \{7, 9\}$ [38] and $m \in \{11, 13\}$ [40], building on the result for $m = 5$ [37] to obtain the following theorem.

Theorem 1.17 ([37, 38, 40, 51]). *Let u, v and m be integers such that $u < v$ and $m \in \{3, 5, \ldots, 13\}$. There exists an m-cycle system of order v containing an m-cycle system or order u if and only if u and v are odd, $v \geqslant \frac{m+1}{m-1}u + 1$, and $u, v \equiv 1$ or $m \pmod{2m}$.*

The previous theorem fills in some of the gaps of Theorem 1.16, however for odd values of $m \leqslant 13$ there are still numerous cases that remain unsolved for the general problem of m-cycle decompositions of $K_v - K_u$. These are solved for $m = 5$ [37] and $m = 7$ [40], which extends Theorem 1.10.

Theorem 1.18 ([37, 40, 75])**.** *Let u, v and m be integers such that $u < v$ and $m \in \{3, 5, 7\}$. There exists an m-cycle decomposition of $K_v - K_u$ if and only if (N1)–(N4) hold.*

For odd values of $m \geqslant 9$, the problem of decomposing $K_v - K_u$ into m-cycles remains largely unsolved. The difficulty is, in part, due to (N4) and the fact that each cycle must contain an edge that is not incident with a vertex in the hole which makes any constructions more cumbersome. In contrast, as we have seen, more is known about the equivalent problem for even cycles, including a complete solution for even $m \leqslant 14$ and the general result given by Theorem 1.14.

1.1.3 Cycle decompositions of other graphs

The previous sections have outlined some results for the complete graph and complete graph with a hole, but other graphs with a high degree of structure such as complete bipartite and complete multipartite graphs have also been studied in depth. Here we present some of the central results for cycle decompositions of these graphs, including those which have been referred to in the previous two sections. The aim of many of these investigations is to prove results analogous to Theorems 1.2 and 1.4.

The first result we present here is due to Sotteau [89] and gives a complete solution for when the complete bipartite graph can be decomposed into uniform length cycles.

Theorem 1.19 ([89])**.** *Let m, p and q be positive even integers. There exists an m-cycle decomposition of $K_{p,q}$ if and only if $pq \equiv 0 \pmod{m}$ and $m \leqslant 2\min(p, q)$.*

Theorem 1.19 has been used to obtain various other cycle decomposition results. For example, it formed the basis of the reduction of Theorem 1.2 in the case when v is odd and m is even [79]. Often, in applications of Theorem 1.19, the graph that is being decomposed is first partitioned into smaller graphs, one of which is a complete bipartite graph to which this result can be applied.

Extensions of Theorem 1.19 include results such as the following theorem of Chou, Fu and Huang for decomposing the complete bipartite graph into short cycles.

Theorem 1.20 ([45]). *Let b, d, f, p and q and be nonnegative integers such that p and q are even. There exists a decomposition of $K_{p,q}$ into b 4-cycles, d 6-cycles and f 8-cycles if and only if $4b+6d+8f=pq$, $p,q \geqslant 4$ if $d+f \geqslant 1$, and $(b,f) \neq (2,1)$ if $p=q=4$.*

Results analogous to Theorems 1.19 and 1.20 for the complete bipartite graph minus a 1-factor, $K_{n,n}-I$ have been proved [7, 45, 72]. Because the existence of a 1-factor is required for these results, these extensions are restricted to complete bipartite graphs with both parts the same size. Other results include a characterisation of the existence of decompositions of the complete bipartite graph (and the complete bipartite graph minus a 1-factor) into cycles of lengths 4 and $2t$ [44].

A more general result than Theorem 1.20 is the following theorem of Horsley which gives a strong partial result for when the complete bipartite graph can be decomposed into cycles of arbitrary specified lengths. It is used in the proof of Theorem 1.14 on decomposing the complete graph with a hole into uniform even length cycles.

Theorem 1.21 ([64]). *Let p and q be positive integers such that either p and q are even or $p=q$, and let $K^*_{p,q}$ be the graph $K_{p,q}$ if p and q are even and the graph $K_{p,q}-I$ if $p=q$ and p is odd, where I is a 1-factor of $K_{p,q}$. If $m_1,\ldots,m_\tau$ are even integers such that $4 \leqslant m_1 \leqslant \cdots \leqslant m_\tau \leqslant \min(p,q,3m_{\tau-1})$ and $m_1+\cdots+m_\tau = |E(K^*_{p,q})|$, then there is an $(m_1,\ldots,m_\tau)$-decomposition of $K^*_{p,q}$.*

Theorem 1.21 is proved by applying a bipartite graph version of Lemma 1.7 to decompositions into short cycles obtained via Theorem 1.20. While this result is in some ways more general than Theorems 1.19 and 1.20 it does not supersede them. For example, consider the possible decompositions of $K_{6,10}$. By Theorem 1.19 there exists a 10-cycle decomposition of $K_{6,10}$ since $10 \mid 60$ and $10 \leqslant 2\min(6,10)=12$. However, the hypotheses of Theorem 1.21 are not satisfied. Similarly, by Theorem 1.20, if $4b+6d+8f=60$ for nonnegative integers b, d and f then there also exists a decomposition of $K_{6,10}$ into b 4-cycles, d 6-cycles and f 8-cycles, but if $f \geqslant 1$ then Theorem 1.21 cannot be applied here.

In general it is still an open problem to determine when a complete bipartite graph can be decomposed into cycles of arbitrary specified lengths. It is clear that necessary conditions for such a decomposition to exist are that the longest cycle in the decomposition is at most twice the size of the smallest part, the degree of each vertex is even, and the sum of the lengths of the cycles in the decomposition is equal to the number of edges in the graph. The only known example where these conditions are not sufficient is that there is no $(4,4,8)$-decomposition of $K_{4,4}$, which is straightforward to verify.

Other graphs that have been of interest in the cycle decomposition problem include tripartite graphs [21], multipartite graphs [20, 64, 65] and circulant graphs [35].

1.1.4 Cycle switching technique

Cycle switching is a technique for altering some of the cycles in a cycle decomposition of a graph. It has been a powerful tool for obtaining graph decomposition results for the complete graph and has also been extended to apply to the complete bipartite graph and the complete multigraph. Results that have been obtained through applications of the cycle switching technique include Theorem 1.4 and Theorem 1.21.

This technique is related to early ideas for edge-colourings [67, 94] and switching in triple systems [5]. It also has similarities to the amalgamation technique [6, 12], and a method included in [58] for decomposing the complete graph into cycles of lengths 2^k and 2^{k+1}. The results included here originate in the more recent version of cycle switching which was first introduced by Bryant, Horsley and Maenhaut in 2005 [30] to obtain results for decompositions of the complete graph into 2-regular subgraphs. It has since been further developed to obtain cycle decomposition results. See [28, 64] and the recent survey [62].

We now introduce some notation that will be necessary for this subsection. The neighbourhood $\mathrm{N}_G(x)$ of a vertex x in a graph G is the set of vertices in G that are adjacent to x (not including x itself). We say vertices x and y of a graph G are *twin in G* if $\mathrm{N}_G(x) \setminus \{y\} = \mathrm{N}_G(y) \setminus \{x\}$. For positive integers u and v, *the hole of $K_v - K_u$* is the set of vertices of degree $v - u$. We say that the remaining vertices (those of degree $v - 1$) are outside the hole. Note that if $u = 1$ then the hole is trivial and any vertex can be specified as the hole. We say an edge xy of $K_v - K_u$ is a *pure edge* if both x and y are outside the hole, and we say that it is a *cross edge* if either x or y is in the hole. Given a permutation π of a set V, a subset S of V and a graph G with $V(G) \subseteq V$, $\pi(S)$ is defined to be the set $\{\pi(x) : x \in S\}$ and $\pi(G)$ is defined to be the graph with vertex set $\pi(V(G))$ and edge set $\{\pi(x)\pi(y) : xy \in E(G)\}$.

The following result presents the cycle switching idea for a general graph. It is almost identical to Lemma 2.1 of [64] and the proof given in that paper suffices to prove this result as well. A similar proof is also included in the recent survey [62].

Lemma 1.22 ([64])**.** *Let G be a graph, and let M be a list of integers. Let $\mathcal{P}$ be an (M)-packing of G with leave L, let α and β be twin vertices in G, and let π be the transposition $(\alpha\beta)$. Then there exists a partition of the set $Z(\mathcal{P}, \alpha, \beta) = (\mathrm{N}_L(\alpha) \cup \mathrm{N}_L(\beta)) \setminus ((\mathrm{N}_L(\alpha) \cap \mathrm{N}_L(\beta)) \cup \{\alpha, \beta\})$ into pairs such that for each pair $\{x, y\}$ of the partition, there exists an (M)-packing $\mathcal{P}'$ of G*

whose leave L' *differs from* L *only in that each of* αx, αy, βx *and* βy *is an edge in* L' *if and only if it is not an edge in* L.

Furthermore, if $\mathcal{P} = \{C_1, C_2, \ldots, C_\tau\}$, *then* $\mathcal{P}' = \{C'_1, C'_2, \ldots, C'_\tau\}$, *where, for each* $i \in \{1, \ldots, \tau\}$, C'_i *is a cycle of the same length as* C_i *such that*

(i) if neither α *nor* β *is in* $V(C_i)$, *then* $C'_i = C_i$;

(ii) if exactly one of α *and* β *is in* $V(C_i)$, *then either* $C'_i = C_i$ *or* $C'_i = \pi(C_i)$; *and*

(iii) if both α *and* β *are in* $V(C_i)$, *then* $C'_i \in \{C_i, \pi(C_i), \pi(P_i) \cup P_i^\dagger, P_i \cup \pi(P_i^\dagger)\}$, *where* P_i *and* $P_i^\dagger$ *are the two paths in* C_i *which have end vertices* α *and* β.

We include a proof of this result since it will be central to the results in Chapters 2 and 3 of this thesis. The following is based on the proofs by Horsley [62, 64].

Proof. Let $Z = Z(\mathcal{P}, \alpha, \beta)$, and $\mathcal{P} = \{C_1, \ldots, C_\tau\}$. We will construct a (multi)graph G^* from $\mathcal{P}$ such that $V(G^*) = V(G) \cup J$ and $E(G^*) = E(G) \cup F$, where J and F are defined below, and F is a set of coloured red and green edges. Furthermore, the set of vertices in G^* that are the endpoints of maximal alternating red-green paths are the set Z.

G^* is constructed as follows. For each $i \in \{1, \ldots, \tau\}$

- If $\alpha \in V(C_i)$ and $\beta \notin V(C_i)$, add a red edge xy where $\mathrm{N}_{C_i}(\alpha) = \{x, y\}$;
- If $\beta \in V(C_i)$ and $\alpha \notin V(C_i)$, add a green edge xy where $\mathrm{N}_{C_i}(\beta) = \{x, y\}$;
- If $\{\alpha, \beta\} \subseteq V(C_i)$, then $C_i = P_i \cup P_i^\dagger$, where P_i and $P_i^\dagger$ are the two paths from α to β in C_i. For each of these paths P, if $P = [\alpha, x, \ldots, y, \beta]$ has length at least three then add a red edge xz and a green edge yz, where z is a new vertex added to J.

Note that G^* may not be a simple graph, but that $G^* - G$ is simple. For example, if $\mathcal{P}$ contains a 3-cycle (α, x, y), then a red edge xy is added to G^* parallel to the original edge $xy \in E(G)$.

The set $Z \cap \mathrm{N}_L(\alpha)$ is equal to the set of vertices that are incident with a green edge and no red edges. Likewise, $Z \cap \mathrm{N}_L(\beta)$ is equal to the set of vertices that are incident with a red edge and no green edges. This follows from the assumption that α and β are twin, and by the definition of G^*. Consider a vertex $x \in Z \cap \mathrm{N}_L(\alpha)$. Since x is in $\mathrm{N}_L(\alpha)$, then x is incident with β in G because α and β are twin. Since x is also in Z then it follows that $x\beta$ is an edge of some cycle in $\mathcal{P}$ and a green edge incident with x was added to G^*. Furthermore, since $x\alpha \in E(L)$ then x is not incident with a red edge in G^*.

Similarly, for a vertex x in $\mathrm{N}_G(\alpha) \setminus Z$, x is incident with both a red and a green edge in G^*. This holds since αx must be an edge of C_i for some $i \in \{1, \dots, \tau\}$ so there is a red edge added in the above construction. There is also a green edge added since $\mathrm{N}_G(\alpha) \setminus \{\beta\} = \mathrm{N}_G(\beta) \setminus \{\alpha\}$.

Therefore, the set of maximal alternating red-green paths in G^* forms a partition Π of Z by pairing the end vertices of these paths. Let $\{x, y\}$ be a pair in Π, and let Q be the red-green path from x to y in G^*. Then $\mathcal{P}'$ is constructed as follows.

For each edge pq of Q such that $\{p, q\} \subseteq V(G)$, there is some cycle $C \in \mathcal{P}$ such that $[p, \gamma, q]$ is a subpath of C, where $\gamma \in \{\alpha, \beta\}$. Then in $\mathcal{P}'$ replace C with $\pi(C)$ (recall that π is the transposition $(\alpha\beta)$).

For each subpath $[p, z, q]$ of Q such that $\{p, q\} \subseteq V(G)$ and $z \in J$, there is some cycle $C \in \mathcal{P}$ such that $C = P \cup P^*$ where P and $P^\dagger$ are the two paths from α to β and $\{p, q\} \subseteq V(P)$. Then in $\mathcal{P}'$, replace P with $\pi(P)$. Note that there may also be a subpath $[p^\dagger, z^\dagger, q^\dagger]$ of Q where $\{p^\dagger, q^\dagger\} \subseteq V(P^\dagger)$.

The remaining cycles in $\mathcal{P}$ remain unchanged since the only cycles that are affected are those along the red-green path Q. We can see that $\mathcal{P}'$ is the required (M)-packing of G since we have examined the changes that occur in each of the cycles along Q. □

As in the statement of Lemma 1.22, for a packing $\mathcal{P}$ with leave L and vertices α and β, let $Z(\mathcal{P}, \alpha, \beta)$ denote the set $(\mathrm{N}_L(\alpha) \cup \mathrm{N}_L(\beta)) \setminus ((\mathrm{N}_L(\alpha) \cap \mathrm{N}_L(\beta)) \cup \{\alpha, \beta\})$. In the proof above, the partition of $Z(\mathcal{P}, \alpha, \beta)$ is constructed by considering the cycles that contain the edges αx and βx for each $x \in V(G)$. Figures 1.1–1.2 depict some of the cases where all of the cycles in $\mathcal{P}$ that are involved in the switch contain exactly one of α and β. That is, all of the edges of the maximal red-green path Q correspond to cycles which contain exactly one of α and β.

If we are applying Lemma 1.22 we say that we are performing the (α, β)-switch with origin x and terminus y (equivalently, with origin y and terminus x). Figure 1.1 shows one case for the (α, β)-switch, where the edges αx and βy in the leave are replaced by edges βx and αy, and the cycles in the packing are modified as shown. The case shown by Figure 1.2 proceeds in a similar manner, as do cases where the switch involves cycles containing both α and β. However the latter is slightly more complicated than the examples shown by Figures 1.1–1.2.

Since the partition of $Z(\mathcal{P}, \alpha, \beta)$ is not given explicitly by Lemma 1.22, in performing an (α, β)-switch we can specify the origin of the switch but must examine all possible cases for the terminus. This will be seen in the proofs given in Chapters 2 and 3.

When Lemma 1.22 is applied to a packing of the complete graph, the condition that α and β are twin is trivial for any pair of vertices (α, β). If

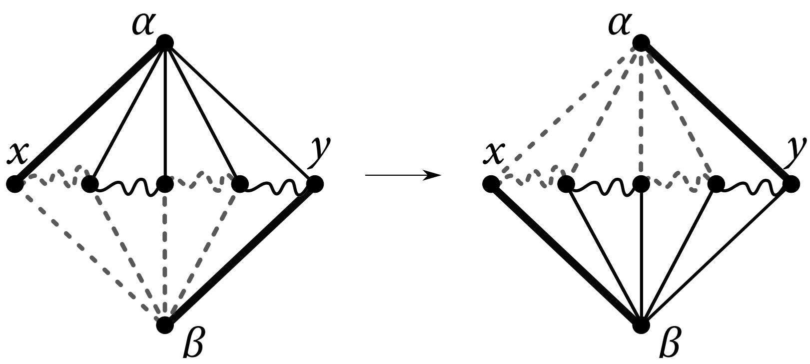

Figure 1.1

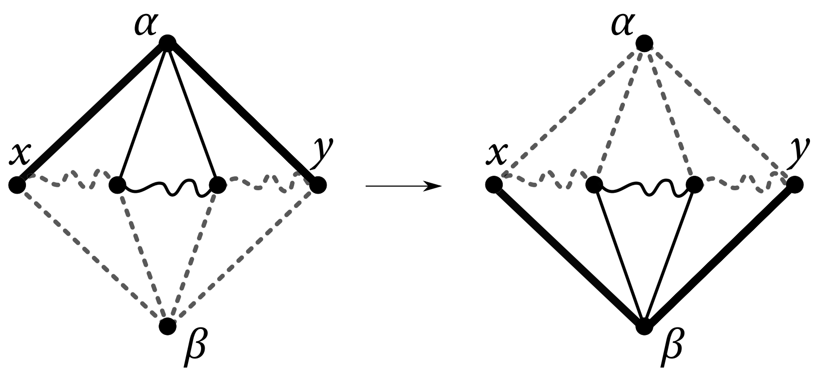

Figure 1.2

G is a complete bipartite graph then this condition is equivalent to specifying that α and β are in the same part. Similarly, if G is a complete graph with a non-trivial hole then this is equivalent to specifying that α and β are both inside or both outside the hole. If we consider packings of graphs other than the complete graph, omitting this condition could result in some of the edges αx, αy, βx and βy being non-existent in G.

Applying a switch to a packing preserves certain properties of its cycles and the following definition enables us to formalise this idea.

Definition. Let G be a graph, and let $\mathcal{P} = \{G_1, \ldots, G_\tau\}$ be a packing of G. We say that another packing $\mathcal{P}'$ of G is a *repacking* of $\mathcal{P}$ if $\mathcal{P}' = \{G'_1, \ldots, G'_\tau\}$ where for each $i \in \{1, \ldots, \tau\}$ there is a permutation π_i of $V(G)$ such that $\pi_i(G_i) = G'_i$ and x and $\pi_i(x)$ are twin in G for each $x \in V(G)$.

Obviously, for any list of integers M, a repacking of an (M)-packing of a graph G is also an (M)-packing of G. If G is a complete graph with a hole, then the above definition implies that G_i and G'_i have the same number of pure and cross edges for each $i \in \{1, \ldots, \tau\}$ and hence also that the leaves of

$\mathcal{P}$ and $\mathcal{P}'$ have the same number of pure and cross edges. If $\mathcal{P}$ is a packing of a graph G, $\mathcal{P}'$ is a repacking of $\mathcal{P}$ and $\mathcal{P}''$ is a repacking of $\mathcal{P}'$, then $\mathcal{P}''$ is also a repacking of $\mathcal{P}$. If $\mathcal{P}$ is a packing of a graph G and $\mathcal{P}'$ is another packing of G obtained from $\mathcal{P}$ by applying Lemma 1.22, then $\mathcal{P}'$ is necessarily a repacking of $\mathcal{P}$.

Since its application to decompositions of the complete graph, cycle switching has also been developed for use on decompositions of the complete bipartite graph [64]. A result analogous to Lemma 1.7 was proved for merging cycles in the complete bipartite graph [64, Lemma 3.6]. Along with Theorem 1.20, this is the main ingredient in proving Theorem 1.21. Results in [64] for the complete graph with a hole and complete multipartite graphs are then obtained as consequences of Theorem 1.21 rather than by applying cycle switching techniques directly to these graphs.

Lemma 1.22 can be applied to any graph with twin vertices, but it is most useful for highly structured graphs. Using cycle switching techniques often relies on repeatedly applying Lemma 1.22 to different pairs of vertices in the leave of a packing. Hence it is most effective when the graph contains large sets of pairwise twin vertices as is the case with the complete graph, the complete bipartite graph, and the complete graph with a hole.

1.2 Research questions

As a generalisation of the Doyen-Wilson Theorem we investigate the following problem for embedding m-cycle systems. We focus on odd values of m since less is known in this case than in the case of even m.

Problem 1.23. *Given an odd integer $m \geqslant 3$, for which values of u and v can an m-cycle system of order u be embedded in an m-cycle system of order v?*

Problem 1.23 is a specific case of the following problem for decompositions of the complete graph with a hole. Solutions to Problem 1.24 also give solutions to Problem 1.23 by removing the edges of the cycle system of order u. However the converse does not always hold.

Problem 1.24. *Given an odd integer $m \geqslant 3$, for which values of u and v does there exist an m-cycle decomposition of $K_v - K_u$?*

Strong partial results to Problems 1.23 and 1.24 are given in Chapter 2.

We saw above that Theorem 1.2 for the existence of m-cycle systems was later generalised to Theorem 1.4 for decompositions of the complete graph into arbitrary length cycles. Chapter 3 contains results related to the following question.

Problem 1.25. *For which lists of integers $m_1, \ldots, m_\tau$, and values of u and v does there exist an $(m_1, \ldots, m_\tau)$-decomposition of $K_v - K_u$?*

The partial solutions to Problems 1.23–1.25 that are given in Chapters 2 and 3 have a similar format. In both chapters, the proof of the main results relies on starting with a 'base' decomposition of the complete graph with a hole and then merging short cycles into cycles of the required length. The cycles in the base decomposition consist of two types, short cycles that each contain at most one pure edge and cycles of the desired lengths that contain no cross edges (m-cycles in the case of Problems 1.23 and 1.24). These decompositions are given in Sections 2.2 and 3.2. Sections 2.1 and 3.1 give results for merging cycles. The merging result in Section 3.1 is similar to Lemma 1.7 and applies to cycles with at most one pure edge. Section 2.1 contains a result adapted to obtaining a collection of m-cycles rather than arbitrary length cycles. Both of these merging results are obtained by applications of Lemma 1.22. The remaining lemmas in Sections 2.1 and 3.1 are required to prove these merging results.

The following question is a natural extension of the cycle packing result Theorem 1.8 in light of results for cycle decompositions, namely Theorems 1.4 and 1.9.

Problem 1.26. *For which lists of integers $m_1, \ldots, m_\tau$, and values of v and λ does there exist an $(m_1, \ldots, m_\tau)$-packing of λK_v?*

A complete solution to this question is given in Chapter 4. The proof of this result uses Theorems 1.4 and 1.9 as well as cycle switching on the complete graph and the complete multigraph (see [31]).

Chapter 2

Uniform Length Cycles

The first main result of this chapter is a generalisation of the Doyen-Wilson Theorem to embeddings of odd cycle systems. Theorem 2.1 is a complete solution to this embedding problem for odd m-cycle systems in the case when $u > \frac{(m-1)(m-2)}{2}$ and, for other values of m and u, a solution apart from cases where $u < v \leqslant u + m - 1$. This means that, for each odd m, the Doyen-Wilson Theorem is generalised with the exception of finitely many possible cases. These possible exceptions are resolved when m is an odd prime power by applying Theorem 1.16 (see [38]).

The embedding results in this chapter are obtained as a consequence of results for decomposing the complete graph with a hole into uniform length odd cycles. Theorem 2.2 gives necessary and sufficient conditions for the existence of an m-cycle decomposition of $K_v - K_u$ in the case where m is odd, $u \geqslant m-2$ and $v - u \geqslant m + 1$ all hold. As in the solution to the embedding problem, for each odd m, the possible exceptions are reduced to a finite number of cases. These cases are then fully resolved for $m \in \{9, 11, 13, 15\}$, supplementing known results for $m \in \{3, 5, 7\}$ (see Theorem 1.18).

Theorem 2.1. *Let $m \geqslant 3$ be an odd integer and let u and v be positive integers with $u < v$.*

(i) If $u > \frac{(m-1)(m-2)}{2}$ or if m is a prime power, then an m-cycle system of order u can be embedded in an m-cycle system of order v if and only if u and v are odd, $\binom{u}{2}, \binom{v}{2} \equiv 0 \pmod{m}$ and $v \geqslant \frac{u(m+1)}{m-1} + 1$.

(ii) If $u \leqslant \frac{(m-1)(m-2)}{2}$ and m is not a prime power, then an m-cycle system of order u can be embedded in an m-cycle system of order v if and only if u and v are odd, $\binom{u}{2}, \binom{v}{2} \equiv 0 \pmod{m}$ and $v \geqslant \frac{u(m+1)}{m-1} + 1$, except that the embedding may not exist when $\frac{u(m+1)}{m-1} + 1 \leqslant v \leqslant u + m - 1$.

Theorem 2.2. *Let $m \geqslant 3$ be an odd integer and let u and v be integers such that $u \geqslant m-2$ and $v-u \geqslant m+1$. There exists an m-cycle decomposition of $K_v - K_u$ if and only if*

(i) *u and v are odd;*

(ii) $\binom{v}{2} - \binom{u}{2} \equiv 0 \pmod{m}$*; and*

(iii) $v \geqslant \frac{u(m+1)}{m-1} + 1$.

Theorem 2.2 complements a similar result for cycles of fixed even length; see Theorem 1.14 (from [64]). It is proved by beginning with a cycle decomposition of $K_v - K_u$ that involves many short cycles and iteratively altering our decomposition of $K_v - K_u$ so as to 'merge' a number of short cycle lengths until we eventually obtain an m-cycle decomposition of $K_v - K_u$. For an odd integer $m \geqslant 3$, we say that a pair (u, v) of positive integers is *m-admissible* if u and v satisfy the conditions (N1)–(N4) of Lemma 1.11. As a consequence of Theorem 2.2 we find the following.

Corollary 2.3. *Let m and u be odd integers and let $\omega_m(u)$ be the smallest integer $x > u$ such that (u, x) is m-admissible.*

(i) *If $3 < u < m-2$ and there exists an m-cycle decomposition of $K_{v'} - K_u$ for each integer v' such that (u, v') is m-admissible and $\omega_m(u) \leqslant v' \leqslant \omega_m(u) + m - 1$, then there exists an m-cycle decomposition of $K_v - K_u$ if and only if (u, v) is m-admissible.*

(ii) *If $m-2 \leqslant u \leqslant \frac{(m-1)(m-2)}{2}$, then there exists an m-cycle decomposition of $K_v - K_u$ if and only if (u, v) is m-admissible, except that this decomposition may not exist when $\omega_m(u) \leqslant v \leqslant u + m - 1$.*

(iii) *If $u > \frac{(m-1)(m-2)}{2}$ or $u \in \{1, 3\}$, then there exists an m-cycle decomposition of $K_v - K_u$ if and only if (u, v) is m-admissible.*

Note that $\omega_m(u)$ in the above corollary is at most the smallest integer $y \equiv u \pmod{2m}$ such that $y \geqslant \frac{u(m+1)}{m-1} + 1$, because (u, y) is m-admissible for any such integer. Corollary 2.3 makes it clear that for a given odd m, we can establish the existence of an m-cycle decomposition of $K_v - K_u$ for all m-admissible (u, v) provided we can construct a number of "small" decompositions. We have been able to do this for $m \in \{9, 11, 13, 15\}$ and thus have resolved the problem for each $m \leqslant 15$, building on Theorems 1.13 and 1.18 for $m \in \{3, 4, 5, 6, 7, 8, 10, 12, 14\}$.

Theorem 2.4. *Let m, u and v be positive integers such that $3 \leqslant m \leqslant 15$ and $v > u$. Then there exists an m-cycle decomposition of $K_v - K_u$ if and only if (u, v) is m-admissible.*

We now give some notation and the definitions for two types of graphs, rings and chains, which will be useful in both this chapter and Chapter 3. For a set V, let K_V denote the complete graph with vertex set V. For disjoint sets U and W, let $K_{U,W}$ denote the complete bipartite graph with parts U and W. For graphs G and H, we denote by $G \cup H$ the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, we denote by $G - H$ the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(H)$, and, if $V(G)$ and $V(H)$ are disjoint, we denote by $G \vee H$ the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup E(K_{V(G),V(H)})$ (our use of this last notation will imply that $V(G)$ and $V(H)$ are disjoint). The m-cycle with vertices $x_0, x_1, \ldots, x_{m-1}$ and edges $x_i x_{i+1}$ for $i \in \{0, \ldots, m-1\}$ (with subscripts modulo m) is denoted by $(x_0, x_1, \ldots, x_{m-1})$ and the n-path with vertices $y_0, y_1, \ldots, y_n$ and edges $y_j y_{j+1}$ for $j \in \{0, 1, \ldots, n-1\}$ is denoted by $[y_0, y_1, \ldots, y_n]$. We will say that y_0 and y_n are the *end vertices* of this path.

Definition. An $(a_1, a_2, \ldots, a_s)$-chain (or s-chain if we do not wish to specify the lengths of the cycles) is the edge-disjoint union of $s \geqslant 2$ cycles $A_1, A_2, \ldots, A_s$ such that

- A_i is a cycle of length a_i for $1 \leqslant i \leqslant s$; and
- for $1 \leqslant i < j \leqslant s$, $|V(A_i) \cap V(A_j)| = 1$ if $j = i+1$ and $|V(A_i) \cap V(A_j)| = 0$ otherwise.

We call A_1 and A_s the *end cycles* of the chain, and for $1 < i < s$ we call A_i an *internal cycle* of the chain. A vertex which is in two cycles of the chain is said to be the *link vertex* of those cycles. We denote a 2-chain with cycles P and Q by $P \cdot Q$.

Definition. An $(a_1, a_2, \ldots, a_s)$-ring (or s-ring if we do not wish to specify the lengths of the cycles) is the edge-disjoint union of $s \geqslant 2$ cycles $A_1, A_2, \ldots, A_s$ such that

- A_i is a cycle of length a_i for $1 \leqslant i \leqslant s$;
- for $s \geqslant 3$ and $1 \leqslant i < j \leqslant s$, $|V(A_i) \cap V(A_j)| = 1$ if $j = i + 1$ or if $(i, j) = (1, s)$, and $|V(A_i) \cap V(A_j)| = 0$ otherwise; and
- if $s = 2$ then $|V(A_1) \cap V(A_2)| = 2$.

We refer to the cycles $A_1, A_2, \ldots, A_s$ as the *ring cycles* of the ring in order to distinguish them from the other cycles that can be found within the graph. A vertex which is in two ring cycles of the ring is said to be a *link vertex* of those cycles.

Definition. For disjoint sets U and W, an s-chain that is a subgraph of $K_{U\cup W} - K_U$ is *good* if $s = 2$ or if $s \geqslant 3$ and

- one end cycle of the chain contains at least one pure edge and has its link vertex in W; and
- each internal cycle of the chain has one link vertex in W and one link vertex in U.

Definition. For disjoint sets U and W, an s-ring that is a subgraph of $K_{U\cup W} - K_U$ is *good* if either

- s is even, and each of the ring cycles has one link vertex in U and one link vertex in W; or
- s is odd, one ring cycle has both link vertices in W and contains at least one pure edge, and each other ring cycle has one link vertex in U and one link vertex in W.

2.1 Merging cycle lengths

Our aim in this section is to prove Lemma 2.5. This lemma allows us to begin with a packing of $K_v - K_u$ satisfying various conditions and find a repacking whose leave can be decomposed into two m-cycles, each with exactly one pure edge. Finding m-cycles of this form is important because, in an m-cycle decomposition of $K_v - K_u$ with m odd and $v = \frac{u(m+1)}{m-1} + 1$ (that is, with equality in necessary condition Lemma 1.11(N4)), every cycle must contain exactly one pure edge. Thus Lemmas 2.5 and 2.11–2.19 concern packings of $K_v - K_u$ whose leaves have exactly two pure edges (recall that repacking preserves the number of pure and cross edges in the leave).

Lemma 2.5. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even, and let M be a list of integers. Let m be a positive odd integer such that $7 \leqslant m \leqslant \min(|U|+2, |W|-1)$. Let $a_1, \ldots, a_s$ and $b_1, \ldots, b_t$ be lists of integers such that $a_1 + \cdots + a_s = m$ and $b_1 + \cdots + b_t = m$. Suppose there exists an (M)-packing $\mathcal{P}$ of $K_{U\cup W} - K_U$ with a reduced leave that contains exactly two pure edges and is the edge-disjoint union of cycles of lengths $a_1, \ldots, a_s, b_1, \ldots, b_t$. Then there exists an (M, m, m)-decomposition $\mathcal{D}$ of $K_{U\cup W} - K_U$ containing two m-cycles C' and C'' such that $\mathcal{D} \setminus \{C', C''\}$ is a repacking of $\mathcal{P}$.*

The results in Subsection 2.1.1 show when the required m-cycle can be obtained from a packing whose reduced leave is a 2-chain of size at least $m + 3$. In Subsection 2.1.2, this is extended to the case when the reduced leave is an

s-chain or s-ring with specified properties. The main result of Subsection 2.1.2 is proved by induction on s, relying on results in Subsection 2.1.1 for 2-chains. Finally, Lemma 2.5 is proved by using the results in Subsection 2.1.3 to obtain a packing whose reduced leave is an s-chain with the required properties. Lemma 2.6 is a technical lemma which will be used in Subsections 2.1.2 and 2.1.3.

Lemma 2.6. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even, and suppose that L is a subgraph of $K_{U\cup W} - K_U$ such that L contains exactly two pure edges and each vertex of L has positive even degree.*

(i) If $|E(L)| \leqslant 2(|U|+1)$ and U contains a vertex of degree at least 4 in L, then there is a vertex y in U such that $y \notin V(L)$.

(ii) If $|E(L)| \leqslant 2\min(|U|+2, |W|)$ and S is an element of $\{U, W\}$ such that S contains either at least two vertices of degree 4 in L or at least one vertex of degree at least 6 in L, then there is a vertex y in S such that $y \notin V(L)$.

(iii) If $|E(L)| \leqslant 2\min(|U|+2, |W|)$ and L contains either at least two vertices of degree 4 or at least one vertex of degree at least 6, then there are twin vertices x and y in $K_{U\cup W} - K_U$ such that $\deg_L(x) \geqslant 4$ and $y \notin V(L)$.

Proof. Let $l = |E(L)|$. Because L contains exactly two pure edges, we have

$$\sum_{x\in V(L)\cap U} \deg_L(x) = l-2 \qquad \text{and} \qquad \sum_{x\in V(L)\setminus U} \deg_L(x) = l+2.$$

Proof of (i). Suppose that $l \leqslant 2(|U|+1)$ and U contains a vertex of degree at least 4 in L. Suppose for a contradiction that $U \subseteq V(L)$. Then we have $l-2 = \sum_{x\in V(L)\cap U} \deg_L(x) \geqslant 2|U|+2$ since every vertex of L in U has degree at least 2. This contradicts $l \leqslant 2(|U|+1)$.
Proof of (ii). Suppose that $l \leqslant 2\min(|U|+2, |W|)$ and S is an element of $\{U, W\}$ such that S contains either at least two vertices of degree 4 in L or at least one vertex of degree at least 6 in L. Suppose for a contradiction that $S \subseteq V(L)$. Then we have $\sum_{x\in V(L)\cap S} \deg_L(x) \geqslant 2|S|+4$ since every vertex of L in S has degree at least 2. So, if $S = U$, then $l-2 \geqslant 2|U|+4$, contradicting $l \leqslant 2(|U|+2)$. If $S = W$, then $l+2 \geqslant 2|W|+4$, contradicting $l \leqslant 2|W|$.
Proof of (iii). Because we have proved (ii), it only remains to show that if L contains two vertices of degree 4, one in U and one in W, and every other vertex of L has degree 2, then there are twin vertices x and y in $K_{U\cup W} - K_U$ such that $\deg_L(x) \geqslant 4$ and $y \notin V(L)$. Suppose otherwise. Then it must be the case that $V(L) = U \cup W$, $l-2 = 2|U|+2$ and $l+2 = 2|W|+2$. But then

$l = 2|U| + 4$ and $l = 2|W|$, so $|W| = |U| + 2$ which contradicts the fact that $|U|$ is odd and $|W|$ is even. □

2.1.1 Packings whose leaves are 2-chains

In this subsection we focus on starting with a packing whose reduced leave is a 2-chain, and finding a repacking whose reduced leave is the edge-disjoint union of two cycles of specified lengths. Our main goal here is to prove Lemma 2.13. The other lemmas in this subsection are used only in order to prove it. Lemmas 2.7–2.10 apply to packings of arbitrary graphs, while in Lemmas 2.11–2.13 we concentrate on packings of complete graphs with holes whose leaves have exactly two pure edges. Lemma 2.9 will also be used in Chapter 3.

Lemma 2.7. *Let G be a graph and let M be a list of integers. Let m, p and q be positive integers with $m \geqslant p$ and $p + q - m \geqslant 3$. Suppose there exists an (M)-packing $\mathcal{P}$ of G whose reduced leave is a (p, q)-chain $(x_1, x_2, \ldots, x_{p-1}, c) \cdot (c, y_1, y_2, \ldots, y_{q-1})$ such that x_1 and y_{m-p+1} are twin in G. Then there exists a repacking of $\mathcal{P}$ whose reduced leave is either*

- *the edge-disjoint union of an m-cycle and a $(p + q - m)$-cycle; or*
- *the $(m-p+2, 2p+q-m-2)$-chain given by $(x_1, y_{m-p}, y_{m-p-1}, \ldots, y_1, c) \cdot (c, x_{p-1}, x_{p-2}, \ldots, x_2, y_{m-p+1}, y_{m-p+2}, \ldots, y_{q-1})$.*

Proof. Note that $p + q - m \geqslant 3$ implies that $m - p + 1 \leqslant q - 2$. If $p = m$ then we are finished, so assume $p < m$. Since x_1 and y_{m-p+1} are twin in G, we can perform the (x_1, y_{m-p+1})-switch with origin x_2. If the switch has terminus y_{m-p}, then we obtain a repacking of $\mathcal{P}$ whose reduced leave is the $(m - p + 2, 2p + q - m - 2)$-chain $(x_1, y_{m-p}, y_{m-p-1}, \ldots, y_1, c) \cdot (c, x_{p-1}, x_{p-2}, \ldots, x_2, y_{m-p+1}, y_{m-p+2}, \ldots, y_{q-1})$. Otherwise the switch has terminus y_{m-p+2} or c and in either case we obtain a repacking of $\mathcal{P}$ whose reduced leave is the edge-disjoint union of a $(p + q - m)$-cycle and the m-cycle $(y_1, y_2, \ldots, y_{m-p+1}, x_2, x_3, \ldots, x_{p-1}, c)$. □

Figure 2.1 provides an illustration of the different cases in Lemma 2.7. The grey edges in the original leave indicate the m-path from x_1 to y_{m-p+1}. We can see that this results in an m-cycle when the terminus of the switch is y_{m-p+2} or c.

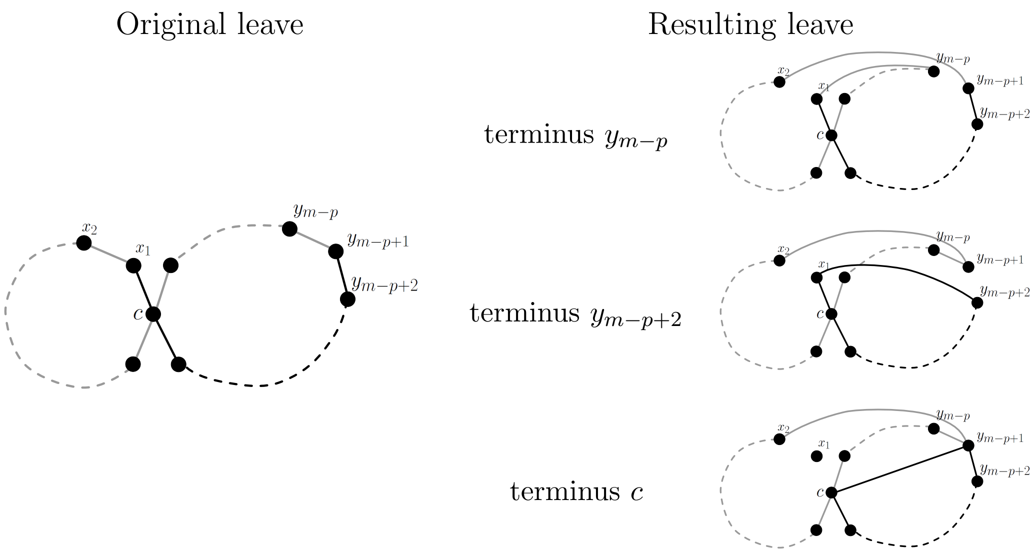

Figure 2.1

Lemma 2.8. *Let G be a graph and let M be a list of integers. Let m, p and q be positive integers with $m \geqslant p$ and $p+q-m \geqslant 3$. Suppose there exists an (M)-packing $\mathcal{P}$ of G whose reduced leave is a (p,q)-chain $(x_1, x_2, \ldots, x_{p-1}, c) \cdot (c, y_1, y_2, \ldots, y_{q-1})$ such that x_2 and y_{m-p+2} are twin in G. Then there exists a repacking of $\mathcal{P}$ whose reduced leave is either*

- *the edge-disjoint union of an m-cycle and a $(p+q-m)$-cycle; or*
- *the $(m-p+4, 2p+q-m-4)$-chain given by $(x_1, x_2, y_{m-p+1}, y_{m-p}, \ldots, y_1, c) \cdot (c, x_{p-1}, x_{p-2}, \ldots, x_3, y_{m-p+2}, y_{m-p+3}, \ldots, y_{q-1})$.*

Proof. Note that $p+q-m \geqslant 3$ implies that $m-p+2 \leqslant q-1$. If $p = m$ then we are finished, so assume $p < m$. Since x_2 and y_{m-p+2} are twin in G, we can perform the (x_2, y_{m-p+2})-switch with origin x_3. If the switch has terminus y_{m-p+1}, then we obtain a repacking of $\mathcal{P}$ whose reduced leave is the $(m-p+4, 2p+q-m-4)$-chain $(x_1, x_2, y_{m-p+1}, y_{m-p}, \ldots, y_1, c) \cdot (c, x_{p-1}, x_{p-2}, \ldots, x_3, y_{m-p+2}, y_{m-p+3}, \ldots, y_{q-1})$. Otherwise the switch has terminus y_{m-p+3} or x_1 and in either case we obtain a repacking of $\mathcal{P}$ whose reduced leave is the edge-disjoint union of a $(p+q-m)$-cycle and the m-cycle $(y_1, y_2, \ldots, y_{m-p+2}, x_3, x_4, \ldots, x_{p-1}, c)$. $\square$

Lemma 2.9. *Let G be a graph and let M be a list of integers. Let m, p and q be positive integers with m odd, $m \geqslant p$ and $p+q-m \geqslant 3$. Suppose there exists an (M)-packing $\mathcal{P}$ of G whose reduced leave is a (p,q)-chain $(x_1, x_2, \ldots, x_{p-1}, c) \cdot (c, y_1, y_2, \ldots, y_{q-1})$ such that either* *(i) p is odd, $x_1, y_3, y_5, \ldots, y_{m-p+1}$ are pairwise twin in G and $y_2, y_4, \ldots, y_{m-p+2}$ are pairwise twin in G; or*

(ii) p is even, $x_1, x_3, \ldots, x_{p-3}$ are pairwise twin in G and $y_{m-p+2}, x_2, x_4, \ldots, x_{p-2}$ are pairwise twin in G.

Then there exists a repacking of $\mathcal{P}$ whose reduced leave is the edge-disjoint union of an m-cycle and a $(p+q-m)$-cycle.

Proof. If $p = m$, then we are finished. If $p = 4$, then x_2 and y_{m-2} are twin in G and we can apply Lemma 2.8 to obtain the required packing. So we may assume $p \notin \{4, m\}$. Let $p_0, p_1, \ldots, p_\ell$ be the sequence $m, 4, m-2, 6, \ldots, 7, m-3, 5, m-1, 3$. For some $k \in \{2, \ldots, \ell\}$ assume that the lemma holds for $p = p_{k-1}$. We will now show that it holds for $p = p_k$.
Case 1. Suppose $p = p_k$ is odd. Since x_1 and y_{m-p+1} are twin in G, Lemma 2.7 can be applied to obtain a repacking $\mathcal{P}'$ of $\mathcal{P}$. Either we are finished, or the reduced leave of $\mathcal{P}'$ is a (p', q')-chain where $p' = m-p+2$ and $q' = 2p+q-m-2$. We give this chain and, below it, a relabelling of its vertices.

$$\begin{array}{l}
(x_1, y_{m-p}, y_{m-p-1}, \ldots, y_1, c) \cdot (c, x_{p-1}, x_{p-2}, \ldots, x_2, y_{m-p+1}, y_{m-p+2}, \ldots, y_{q-1}) \\
(x'_1, x'_2, x'_3, \ldots, x'_{p'-1}, c) \cdot (c, y'_1, y'_2, \ldots, y'_{p-2}, y'_{p-1}, y'_p, \ldots, y'_{q'-1})
\end{array}$$

Note that $p' = p_{k-1}$ and p' is even. Since $x'_1 = x_1$ and $\{x'_3, x'_5, \ldots, x'_{p'-3}\} = \{y_3, y_5, \ldots, y_{m-p-1}\}$, the vertices $x'_1, x'_3, \ldots, x'_{p'-3}$ are pairwise twin in G. Similarly, since $y'_{m-p'+2} = y'_p = y_{m-p+2}$ and $\{x'_2, x'_4, \ldots, x'_{p'-2}\} = \{y_2, y_4, \ldots, y_{m-p}\}$, the vertices $y'_{m-p'+2}, x'_2, x'_4, \ldots, x'_{p'-2}$ are pairwise twin in G. Thus $\mathcal{P}'$ satisfies (ii) and we are finished by our inductive hypothesis.
Case 2. Suppose $p = p_k$ is even. Then, since x_2 and y_{m-p+2} are twin in G, Lemma 2.8 can be applied to obtain a repacking $\mathcal{P}'$ of $\mathcal{P}$. Either we are finished, or the reduced leave of $\mathcal{P}'$ is a (p', q')-chain where $p' = m-p+4$ and $q' = 2p+q-m-4$. We give this chain and, below it, a relabelling of its vertices.

$$\begin{array}{l}
(x_1, x_2, y_{m-p+1}, y_{m-p}, \ldots, y_1, c) \cdot (c, x_{p-1}, x_{p-2}, \ldots, x_3, y_{m-p+2}, y_{m-p+3}, \ldots, y_{q-1}) \\
(x'_1, x'_2, x'_3, x'_4, \ldots, x'_{p'-1}, c) \cdot (c, y'_1, y'_2, \ldots, y'_{p-3}, y'_{p-2}, y'_{p-1}, \ldots, y'_{q'-1})
\end{array}$$

Note that $p' = p_{k-1}$ and p' is odd. Since $x'_1 = x_1$ and $\{y'_3, y'_5, \ldots, y'_{m-p'+1}\} = \{x_3, x_5, \ldots, x_{p-3}\}$, the vertices $x'_1, y'_3, y'_5, \ldots, y'_{m-p'+1}$ are pairwise twin in G. Similarly, since $\{y'_2, y'_4, \ldots, y'_{m-p'+2}\} = \{x_4, x_6, \ldots, x_{p-2}\} \cup \{y_{m-p+2}\}$, the vertices $y'_2, y'_4, \ldots, y'_{m-p'+2}$ are pairwise twin in G. Thus $\mathcal{P}'$ satisfies (i) and we are finished by our inductive hypothesis. $\square$

Lemma 2.10. *Let G be a graph and let M be a list of integers. Let m, p and q be positive integers with m odd, $m \geqslant p$, $p+q-m \geqslant 3$ and $q \geqslant 5$. Suppose there exists an (M)-packing $\mathcal{P}$ of G whose reduced leave is a (p,q)-chain $(x_1, x_2, \ldots, x_{p-1}, y_0) \cdot (y_0, y_1, \ldots, y_{q-1})$ such that y_0 and y_{q-2} are twin in G. Then there exists a repacking of $\mathcal{P}$ whose reduced leave is either*

- *a $(p+2, q-2)$-chain containing the $(q-2)$-cycle $(y_0, y_1, \ldots, y_{q-3})$; or*
- *the (p,q)-chain $(x_1, x_2, \ldots, x_{p-1}, y_0) \cdot (y_0, y_{q-1}, y_{q-2}, y_1, y_2, \ldots, y_{q-3})$.*

Proof. Perform the (y_0, y_{q-2})-switch with origin y_{q-3} (note that y_0 and y_{q-2} are twin in G and that $q \geqslant 5$). If the terminus of the switch is y_1, then the reduced leave of the resulting packing is the (p,q)-chain $(x_1, x_2, \ldots, x_{p-1}, y_0) \cdot (y_0, y_{q-1}, y_{q-2}, y_1, y_2, \ldots, y_{q-3})$. Otherwise the terminus of the switch is x_1 or x_{p-1} and in either case the leave of the resulting packing is a $(p+2, q-2)$-chain containing the $(q-2)$-cycle $(y_0, y_1, \ldots, y_{q-3})$. □

Lemma 2.11. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even, and let M be a list of integers. Let m, p and q be positive integers with m odd and $m, p+q-m \geqslant 3$. Suppose there exists an (M)-packing $\mathcal{P}$ of $K_{U \cup W} - K_U$ whose reduced leave L is a (p,q)-chain such that each cycle of L contains exactly one pure edge and the link vertex of L is in W if $3 \in \{m, p+q-m\}$. Then there exists a repacking of $\mathcal{P}$ whose reduced leave is the edge-disjoint union of an m-cycle and a $(p+q-m)$-cycle.*

Proof. We can assume without loss of generality that $m \geqslant p+q-m$ and that $p \leqslant q$. Note that this implies $p \leqslant m$. Since each cycle of L must contain an even number of cross edges, p and q are odd. If $p = m$, then we are finished immediately, so we can assume that $p \leqslant m-2$. We will show that we can obtain a repacking of $\mathcal{P}$ whose reduced leave is either a $(p+2, q-2)$-chain or the edge-disjoint union of an m-cycle and a $(p+q-m)$-cycle. This will suffice to complete the proof, because by iteratively applying this procedure we will eventually obtain a repacking of $\mathcal{P}$ whose reduced leave is the edge-disjoint union of an m-cycle and a $(p+q-m)$-cycle.
Case 1. Suppose that L can be labelled $(x_1, x_2, \ldots, x_{p-1}, y_0) \cdot (y_0, y_1, \ldots, y_{q-1})$ so that $y_0 x_1$ is not a pure edge and $y_r y_{r+1}$ is a pure edge (subscripts modulo q) for an integer r such that $m-p+2 \leqslant r \leqslant q-1$. Then the hypotheses of Lemma 2.9(i) are satisfied and we can apply it to obtain a repacking of $\mathcal{P}$ whose reduced leave is the edge-disjoint union of an m-cycle and a $(p+q-m)$-cycle.
Case 2. Suppose that L cannot be labelled as in Case 1. Without loss of generality we can label L as $(x_1, x_2, \ldots, x_{p-1}, y_0) \cdot (y_0, y_1, \ldots, y_{q-1})$ so that $y_0 x_1$ is not a pure edge and $y_r y_{r+1}$ is a pure edge (subscripts modulo q) for an integer r such that $\frac{q-1}{2} \leqslant r \leqslant q-1$, r is even if $y_0 \in W$, and r is odd if $y_0 \in U$. It must be that $r \leqslant m-p+1$, for otherwise we would be in Case 1. Then we can iteratively apply Lemma 2.10 to obtain a repacking of $\mathcal{P}$ whose reduced leave L' is either a $(p+2, q-2)$-chain or a (p,q)-chain which can be labelled $(x'_1, x'_2, \ldots, x'_{p-1}, y'_0) \cdot (y'_0, y'_1, \ldots, y'_{q-1})$ so that $y'_0 x'_1$ is not a pure edge, and $y'_{r'} y'_{r'+1}$ is a pure edge (subscripts modulo q), where r' is the element of $\{m-p+2, m-p+3\}$ such that $r' \equiv r \pmod 2$. Note that $r' \leqslant q-1$ because

if $p+q-m \geqslant 4$ then $m-p+3 \leqslant q-1$, and if $p+q-m=3$ then $y_0 \in W$, r is even and $r' = m-p+2 = q-1$. If L' is a $(p+2, q-2)$-chain then we are finished, and if L' is a (p,q)-chain then we can proceed as we did in Case 1. □

Lemma 2.12. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even, and let M be a list of integers. Let m, p and q be positive integers with m odd and $m, p+q-m \geqslant 3$. Suppose there exists an (M)-packing $\mathcal{P}$ of $K_{U\cup W} - K_U$ whose reduced leave L is a (p,q)-chain such that one cycle in L contains no pure edges, the other contains exactly two pure edges, and the link vertex of L is in W if $3 \in \{m, p+q-m\}$. Then there exists a repacking of $\mathcal{P}$ whose reduced leave is the edge-disjoint union of an m-cycle and a $(p+q-m)$-cycle.*

Proof. We can assume without loss of generality that $m \geqslant p+q-m$ and that a p-cycle in L contains no pure edges. Since each cycle of L must contain an even number of cross edges, p and q are even.
Case 1. Suppose that L can be labelled $(x_1, x_2, \ldots, x_{p-1}, y_0) \cdot (y_0, y_1, \ldots, y_{q-1})$ so that $y_r y_{r+1}$ and $y_s y_{s+1}$ are pure edges (subscripts modulo q) for integers r and s such that $0 \leqslant r < s \leqslant q-1$, $r \leqslant m-2$ and $s \geqslant m-p+1$. Observe that, in particular, such a labelling is always possible when $q = 4$ (any labelling with $r < s$ and $s \in \{2,3\}$ will suffice, because then $r \leqslant 2 < m-2$ since $m \geqslant p+4-m$ and $m-p+1 \leqslant 2 \leqslant s$ since $p+4-m \geqslant 3$). Let $x_0 = y_0$ and $t = \max(r+1, m-p+1)$. Consider the vertices x_{m-t} and y_t. Note that $1 \leqslant m-t \leqslant p-1$ because $r \leqslant m-2$, $p \geqslant 3$ and $t \geqslant m-p+1$, and that $r < t \leqslant s$ because $t \geqslant r+1$, $r < s$ and $s \geqslant m-p+1$. Since $r+1 \leqslant t \leqslant s$, there is exactly one pure edge in the m-path $[x_{m-t}, x_{m-t-1}, \ldots, x_1, y_0, y_1, \ldots, y_t]$ and hence x_{m-t} and y_t are twin in $K_{U\cup W} - K_U$. Let L' be the reduced leave of the repacking of $\mathcal{P}$ obtained by performing the (x_{m-t}, y_t)-switch with origin x_{m-t-1}. If the terminus of the switch is not y_{t-1}, L' is the edge-disjoint union of an m-cycle and a $(p+q-m)$-cycle and we are finished. If the terminus of the switch is y_{t-1}, then L' is a $(p+2t-m, q+m-2t)$-chain with one pure edge in each cycle and whose link vertex is in W if $3 \in \{m, p+q-m\}$, and we can apply Lemma 2.11 to complete the proof.
Case 2. Suppose that L cannot be labelled as in Case 1. From our comments in Case 1 we may assume $q \geqslant 6$. We will show that we can obtain a repacking of $\mathcal{P}$ whose reduced leave either satisfies the conditions of Case 1 or is a $(p+2, q-2)$-chain in which a $(p+2)$-cycle contains no pure edges. Since any reduced leave which is a $(p+q-4, 4)$-chain with exactly two pure edges in which a $(p+q-4)$-cycle contains no pure edges must fall into Case 1, repeating this procedure will eventually result in a repacking of $\mathcal{P}$ whose reduced leave satisfies the conditions of Case 1. We can then proceed as we did in Case 1 to complete the proof.

Without loss of generality we can label L as $(x_1, x_2, \ldots, x_{p-1}, y_0) \cdot (y_0, y_1, \ldots, y_{q-1})$ so that $y_r y_{r+1}$ and $y_s y_{s+1}$ are pure edges (subscripts modulo q) for integers such that $0 \leqslant r < s \leqslant q-1$ and $r \leqslant \frac{q}{2}$. Because $r \leqslant \frac{q}{2}$ and $\frac{q}{2} \leqslant m-2$ (note that $m \geqslant \frac{p+q}{2} \geqslant \frac{q+4}{2}$), it must be that $s < m-p+1$, for otherwise we would be in Case 1. So we can repeatedly apply Lemma 2.10 to obtain a repacking of $\mathcal{P}$ whose reduced leave L' is either a $(p+2, q-2)$-chain in which a $(p+2)$-cycle contains no pure edges or a (p, q)-chain which can be labelled $(x'_1, x'_2, \ldots, x'_{p-1}, y'_0) \cdot (y'_0, y'_1, \ldots, y'_{q-1})$ so that $y'_r y'_{r+1}$ and $y'_{s'} y'_{s'+1}$ are pure edges for integers r' and s' such that $0 \leqslant r' < s' \leqslant q-1$ and $s' \in \{m-p+1, m-p+2\}$ (note that $m-p+2 \leqslant q-1$ since $p+q-m \geqslant 3$). Observe that in the latter case L' satisfies the conditions of Case 1. □

Lemma 2.13. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even, and let M be a list of integers. Let m, p and q be positive integers with m odd, and $m, p+q-m \geqslant 3$. Suppose there exists an (M)-packing of $K_{U \cup W} - K_U$ whose reduced leave L is a (p, q)-chain such that L contains exactly two pure edges and the link vertex of L is in W if $3 \in \{m, p+q-m\}$. Then there exists a repacking of $\mathcal{P}$ whose reduced leave is the edge-disjoint union of an m-cycle and a $(p+q-m)$-cycle.*

Proof. If each cycle of L contains exactly one pure edge, then we can apply Lemma 2.11 to complete the proof. If one cycle in L contains no pure edges and the other contains exactly two pure edges, then we can apply Lemma 2.12 to complete the proof. □

2.1.2 Packings whose leaves are s-chains

In this subsection we use Lemma 2.13 to prove an analogous result for chains with more than two cycles, namely Lemma 2.16. Given a packing whose reduced leave is an s-chain that contains two pure edges and satisfies certain other properties, Lemma 2.16 allows us to find a repacking whose reduced leave is the edge-disjoint union of two cycles of specified lengths.

Lemma 2.14. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even, and let M be a list of integers. Let p and s be positive integers such that $p \geqslant 5$ is odd and $s \geqslant 2$. Suppose there exists an (M)-packing $\mathcal{P}$ of $K_{U \cup W} - K_U$ whose reduced leave L is a good s-chain that has a decomposition $\{P, L-P\}$ into two paths such that P has length p and each path contains exactly one pure edge and has both end vertices in W. Suppose further that P has a subpath $P_0 = [x_0, \ldots, x_r]$ such that $2 \leqslant r \leqslant p-1$, x_0 is an end vertex of P, P_0 contains no pure edge, and $\deg_L(x_{r-1}) = \deg_L(x_r) = 2$. Then there is a repacking of $\mathcal{P}$ whose reduced leave L' is a good s-chain that has a decomposition $\{P', L'-P'\}$ into two paths such that P' has length $p-2$, each path contains exactly one* *pure edge and has both end vertices in W, and P' contains a pure edge in an end cycle of L' with link vertex in W if P contains a pure edge in an end cycle of L with link vertex in W.*

Proof. We prove the result by induction on the length of P_0. If $|E(P_0)| = 2$, then $\{P', L - P'\}$ where $P' = [x_2, \ldots, x_p]$ is a decomposition of L with the required properties. So we can assume that $|E(P_0)| \geqslant 3$. By induction we can assume that P_0 is the shortest subpath of P satisfying the required conditions. Because $r \geqslant 3$ this implies $\deg_L(x_{r-2}) = 4$. Label the vertices in $V(P) \setminus V(P_0)$ so that $P = [x_0, \ldots, x_p]$.

The vertices x_r and x_{r-2} are twin in $K_{U \cup W} - K_U$ because they are joined by a path of length 2 containing no pure edge. Let L' be the reduced leave of the repacking of $\mathcal{P}$ obtained by performing the (x_r, x_{r-2})-switch with origin x_{r-3}. Note that L' is a good s-chain irrespective of the terminus of the switch. If the terminus of the switch is not x_{r+1}, then $\{P', L' - P'\}$ where $P' = [x_0, x_1, \ldots, x_{r-3}, x_r, x_{r+1}, \ldots, x_p]$ is a decomposition of L' with the required properties. If the terminus of the switch is x_{r+1}, then $\{P', L' - P'\}$ where $P' = [x_0, x_1, \ldots, x_{r-3}, x_r, x_{r-1}, x_{r-2}, x_{r+1}, x_{r+2}, \ldots, x_p]$ is a decomposition of L' into two paths such that P' has length p and each path contains exactly one pure edge and has both end vertices in W. Further P' has the subpath $P'_0 = [x_0, \ldots, x_{r-3}, x_r, x_{r-1}]$ and we know that x_0 is an end vertex of P', P'_0 contains no pure edge, and $\deg_{L'}(x_r) = \deg_{L'}(x_{r-1}) = 2$. Thus, because $|E(P'_0)| = r-1$, we are finished by our inductive hypothesis. □

Lemma 2.15. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even, and let M be a list of integers. Let m_1, m_2 and s be positive integers such that m_1 and m_2 are odd, $m_1, m_2 \geqslant s$ and $s \geqslant 3$. Suppose there exists an (M)-packing $\mathcal{P}$ of $K_{U \cup W} - K_U$ whose reduced leave is a good s-chain of size $m_1 + m_2$ that contains exactly two pure edges. Then there exists a repacking of $\mathcal{P}$ whose reduced leave is a good s-chain that has a decomposition into an m_1-path and an m_2-path such that each path contains exactly one pure edge.*

Proof. Suppose without loss of generality that $m_1 \leqslant m_2$, and let L be the reduced leave of $\mathcal{P}$. Note that $|E(L)| = m_1 + m_2$. Because L is good and contains exactly two pure edges, we can find some decomposition $\{P, L - P\}$ of L into two odd length paths each of which has both end vertices in W and contains exactly one pure edge. Without loss of generality we can assume that P is at least as long as $L - P$ if $m_1 \geqslant s + 1$ and that P contains a pure edge in an end cycle of L with link vertex in W if $m_1 = s$. Let p be the length of P and note that in each case $p \geqslant m_1$ because $p \geqslant \frac{m_1+m_2}{2} \geqslant m_1$ if $m_1 \geqslant s + 1$ and $p \geqslant s = m_1$ if $m_1 = s$. We are finished if $p = m_1$, so we may assume $p \geqslant m_1 + 2$.

Case 1. Suppose each cycle of L contains at most two edges of P. Then exactly $p-s$ cycles of L contain two edges of P and the rest contain one edge of P. Because L is good and both end vertices of P are in W, if C is a cycle of L that contains two edges of P, then either

- C is an internal cycle of L and C contains the pure edge of P; or
- C is an end cycle of L with link vertex in U and C contains the pure edge of P; or
- C is an end cycle of L with link vertex in W and C does not contain the pure edge of P.

From this it follows that $p-s \leqslant 3$. Note that $p \geqslant m_1+2 \geqslant s+2$ and hence that $p \in \{s+2, s+3\}$. If $p = s+2$, then $m_1 = s$. But then P contains a pure edge in an end cycle of L with link vertex in W by its definition and it can be seen that no cycle of L contains two edges of P, contradicting $p = s+2$. So it must be that $p = s+3$ and thus $m_1 = s+1 = p-2$ because m_1 and p are odd. Because $p = s+3$, P contains two edges of each end cycle of L and two edges, including a pure edge, of some internal cycle of L. Let P' be the path obtained from P by deleting both end vertices of P and their incident edges. Then $\{P', L-P'\}$ is a decomposition of L into an m_1-path and an m_2-path such that each path contains exactly one pure edge.
Case 2. Suppose there is a cycle C in L such that $C \cap P$ is a path of length at least 3. Let $P_0 = [x_0, \ldots, x_r]$ be a subpath of P such that x_0 is an end vertex of P, P_0 contains no pure edge, and P_0 contains exactly two edges in $C \cap P$. If $C \cap P$ contains no pure edge or if $C \cap P$ has length at least 4, then it is easy to see such a subpath exists. If $C \cap P$ has length 3 and contains a pure edge, then the facts that L is good and that the end vertices of P are in W imply that C is an end cycle of L with link vertex in W and hence that such a subpath exists. So we can apply Lemma 2.14 to obtain a repacking of $\mathcal{P}$ whose reduced leave L' is a good s-chain that has a decomposition $\{P', L'-P'\}$ into two paths such that P' has length $p-2$, each path contains exactly one pure edge and has both end vertices in W, and P' contains a pure edge in an end cycle of L' with link vertex in W if $m_1 = s$. It is clear that by repeating this procedure we will eventually obtain a repacking of $\mathcal{P}$ whose reduced leave either has a decomposition into an m_1-path and an m_2-path such that each path contains exactly one pure edge or has a decomposition into odd length paths which satisfies the hypotheses for Case 1. In the former case we are finished and in the latter we can proceed as we did in Case 1. □

Lemma 2.16. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even, and let M be a list of integers. Let m_1, m_2 and s be positive integers such that*

$s \geqslant 2$, m_1 and m_2 are odd, $m_1, m_2 \geqslant s$, $m_1 + m_2 \leqslant 2\min(|U|+2, |W|)$, and $m_1 + m_2 \leqslant 2(|U|+1)$ if $3 \in \{m_1, m_2\}$. Suppose there exists an (M)-packing $\mathcal{P}$ of $K_{U\cup W} - K_U$ whose reduced leave has size $m_1 + m_2$, contains exactly two pure edges, is either a good s-ring or a good s-chain that, if $3 \in \{m_1, m_2\}$, is not a 2-chain with link vertex in U. Then there exists a repacking of $\mathcal{P}$ whose reduced leave is the edge-disjoint union of an m_1-cycle and an m_2-cycle.

Proof. Let L be the reduced leave of $\mathcal{P}$. We first show that the result holds for $s = 2$. If L is a 2-chain, then the result follows by Lemma 2.13. If L is a 2-ring, then it follows from our hypotheses and Lemma 2.6 that there are twin vertices x and y in $K_{U\cup W} - K_U$ such that $\deg_L(x) \geqslant 4$ and $y \notin V(L)$, and such that if $3 \in \{m_1, m_2\}$ then $x \in U$ (if $3 \in \{m_1, m_2\}$, then apply Lemma 2.6(i) and otherwise apply Lemma 2.6(iii)). Performing an (x, y)-switch results in a repacking of $\mathcal{P}$ whose reduced leave is a 2-chain whose link vertex is in W if $3 \in \{m_1, m_2\}$ and the result follows by Lemma 2.13. So it is sufficient to show, for each integer $s' \geqslant 3$, that if the result holds for $s = s' - 1$ then it holds for $s = s'$.

Case 1. Suppose that L is a good s'-chain. By Lemma 2.15 we can obtain a repacking of $\mathcal{P}$ whose reduced leave is a good s'-chain with a decomposition into paths of length m_1 and m_2 each containing exactly one pure edge. Let $[x_0, x_1, \ldots, x_{m_1}]$ be the path of length m_1. Observe that x_0 and x_{m_1} are twin in $K_{U\cup W} - K_U$ because they are joined by an odd length path containing exactly one pure edge, and perform the (x_0, x_{m_1})-switch with origin x_1.

If the terminus of the switch is not x_{m_1-1}, then we obtain a repacking of $\mathcal{P}$ whose reduced leave is the edge-disjoint union of an m_1-cycle and an m_2-cycle and we are finished. If the terminus of the switch is x_{m_1-1}, then we obtain a repacking of $\mathcal{P}$ whose reduced leave is a good $(s'-1)$-ring that contains exactly two pure edges and the result follows by our inductive hypothesis.

Case 2. Suppose that L is a good s'-ring. Let A be a ring cycle of L such that A contains a pure edge and if s' is odd then A has both link vertices in W. Let x and y be twin vertices in $K_{U\cup W} - K_U$ such that x is a link vertex in A, $x \in U$ if s' is even, and $y \notin V(L)$. Such a vertex y exists by Lemma 2.6(ii) because $|E(L)| \leqslant 2\min(|U|+2, |W|)$, W contains two vertices of degree 4 in L if s' is odd, and U contains two vertices of degree 4 in L if s' is even (for then $s' \geqslant 4$). By performing an (x, y)-switch with origin in $V(A)$ we obtain a repacking of $\mathcal{P}$ whose reduced leave contains exactly two pure edges, is a good s'-chain if the terminus of the switch is also in $V(A)$, and is a good $(s'-1)$-ring otherwise. In the former case we can proceed as in Case 1 and in the latter case the result follows by our inductive hypothesis. □

2.1.3 Proof of Lemma 2.5

In this subsection we use Lemma 2.16 to prove Lemma 2.5, which is our primary goal in this section. Given a cycle decomposition of $K_v - K_u$ that satisfies certain conditions, Lemma 2.5 allows us to find a new cycle decomposition of $K_v - K_u$ in which some of the shorter cycle lengths have been merged into cycles of length m.

Lemma 2.17. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even, and let M be a list of integers. Let m_1, m_2, t and k be positive integers such that m_1 and m_2 are odd, $m_1, m_2 \geqslant k + t - 1$, $m_1 + m_2 \leqslant 2\min(|U| + 2, |W|)$, and $m_1 + m_2 \leqslant 2(|U| + 1)$ if $3 \in \{m_1, m_2\}$. Suppose there exists an (M)-packing $\mathcal{P}$ of $K_{U\cup W} - K_U$ with a reduced leave L of size $m_1 + m_2$ such that L contains exactly two pure edges and L has exactly k components, $k - 1$ of which are cycles and one of which is a good t-chain that, if $3 \in \{m_1, m_2\}$, is not a 2-chain with link vertex in U. Then there exists a repacking of $\mathcal{P}$ whose reduced leave is the edge-disjoint union of an m_1-cycle and an m_2-cycle.*

Proof. By Lemma 2.16 it is sufficient to show that we can construct a repacking of $\mathcal{P}$ whose reduced leave is a good s-chain, for some $s \in \{2, \ldots, k+t-1\}$, that is not a 2-chain with link vertex in U if $3 \in \{m_1, m_2\}$. If $k = 1$, then we are finished, so we can assume $k \geqslant 2$. By induction on k, it suffices to show that there is a repacking of $\mathcal{P}$ with a reduced leave L' such that L' has exactly $k - 1$ components, one component of L' is a good t'-chain for $t' \in \{t, t + 1\}$, each other component of L' is a cycle, and a degree 4 vertex of L' is in W if $3 \in \{m_1, m_2\}$.

Let H be the component of L which is a good t-chain, and let C be a component of L such that C is a cycle and C contains at least one pure edge if H contains at most one pure edge. Let H_1 and H_t be the end cycles of H where H_1 contains a pure edge if H does and the link vertex of H_1 is in W if $t \geqslant 3$.

Case 1. Suppose that either $t \geqslant 3$ or it is the case that $t = 2$, H_1 contains a pure edge, and the link vertex of H is in W. Let x and y be vertices such that $x \in V(H_t)$, x is not a link vertex of H, $y \in V(C)$, $x, y \in W$ if t is odd, and $x, y \in U$ if t is even. Let $\mathcal{P}'$ be a repacking of $\mathcal{P}$ obtained by performing an (x, y)-switch with origin in $V(H_t)$. The reduced leave L' of $\mathcal{P}'$ has exactly $k - 1$ components, $k - 2$ of which are cycles and one of which is a good t'-chain, where $t' = t + 1$ if the terminus of the switch is also in $V(H_t)$ and $t' = t$ otherwise. Further, a degree 4 vertex of L' is in W if $3 \in \{m_1, m_2\}$. So we are finished by induction.

Case 2. Suppose that $t = 2$ and either H contains exactly one pure edge and has its link vertex in U or H contains no pure edges. Then C contains a pure edge. Let w and x be vertices such that $w \in V(C) \cap W$, $x \in V(H_1) \cap U$, and x

is not the link vertex of H. Let $\mathcal{P}'$ be a repacking of $\mathcal{P}$ obtained by performing a (w,x)-switch with origin in $V(H_1)$ and let L' be the reduced leave of $\mathcal{P}'$. If the terminus of this switch is in C, then L' has exactly $k-1$ components, $k-2$ of which are cycles and one of which is a 2-chain, and the link vertex of this chain is in W if $3 \in \{m_1, m_2\}$. In this case we are finished by induction. Otherwise the terminus of this switch is in $V(H_1)$ and L' has exactly $k-1$ components, $k-2$ of which are cycles and one of which is a 3-chain H' one of whose end cycles contains a pure edge and has its link vertex in W. If H' is good, then we are done. Otherwise, it must be that both link vertices of H' are in W. In this latter case we proceed as follows.

Let H'_1 and H'_3 be the end cycles of H' such that H'_1 has a pure edge. Let $y, z \in W$ be vertices such that y is the link vertex in $V(H'_3)$ and $z \notin V(L')$ (note that z exists by Lemma 2.6(ii) because $m_1 + m_2 \leqslant 2\min(|U|+2, |W|)$ and both link vertices of H' are in W). Let $\mathcal{P}''$ be a repacking of $\mathcal{P}$ obtained from $\mathcal{P}'$ by performing a (y,z)-switch with origin in $V(H'_3)$ and let L'' be the reduced leave of $\mathcal{P}''$. If the terminus of this switch is not in $V(H'_3)$, then L'' has exactly $k-1$ components, $k-2$ of which are cycles and one of which is a 2-chain whose link vertex is in W. In this case we are finished by induction. Otherwise, the terminus of this switch is in $V(H'_3)$ and L'' has exactly k components, $k-1$ of which are cycles and one of which is a 2-chain that contains a pure edge and has its link vertex in W. In this case we can proceed as we did in Case 1.

Case 3. Suppose that $t = 2$, H contains two pure edges and the link vertex of H is in U. Note that, from our hypotheses, $m_1, m_2 \geqslant 4$. Let x be the link vertex of H and let y be a vertex in $V(C) \cap U$. Let $\mathcal{P}'$ be a repacking of $\mathcal{P}$ obtained by performing an (x,y)-switch with origin in $V(H_2)$ and let L' be the reduced leave of $\mathcal{P}'$. If the terminus of this switch is in $V(C)$, then L' has exactly $k-1$ components, $k-2$ of which are cycles and one of which is a 2-chain. In this case we are finished by induction. Otherwise the terminus of this switch is in $V(H_2)$ and L' has exactly k components, $k-1$ of which are cycles and one of which is a 2-chain that contains at most one pure edge and has its link vertex in U. In this case we can proceed as we did in Case 2. □

Lemma 2.18. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even. If L is a subgraph of $K_{U\cup W} - K_U$ such that L contains at most two pure edges, L has one vertex of degree 4, and each other vertex of L has degree 2, then L has at most $\left\lfloor \frac{|E(L)|-6}{4} \right\rfloor + 1$ components.*

Proof. Because each vertex of L has even degree, L has a decomposition $\mathcal{D}$ into cycles. Since there are at most two pure edges in L, at most two cycles in $\mathcal{D}$ have length 3 and each other cycle in $\mathcal{D}$ has length at least 4. Thus $|E(L)| \geqslant 4(|\mathcal{D}|-2)+6$ which implies $|\mathcal{D}| \leqslant \left\lfloor \frac{|E(L)|-6}{4} \right\rfloor + 2$. At least one component of L contains a vertex of degree 4 and hence contains at least two cycles and each other component of L contains at least one cycle. The result follows. $\square$

Lemma 2.19. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even, and let M be a list of integers. Suppose there exists an (M)-packing $\mathcal{P}_0$ of $K_{U\cup W}-K_U$ with a reduced leave L_0 such that $|E(L_0)| \leqslant 2\min(|U|+2,|W|)$, L_0 has exactly two pure edges, and L_0 has at least one vertex of degree at least 4. Then there exists a repacking $\mathcal{P}^\star$ of $\mathcal{P}_0$ with a reduced leave $L^\star$ such that exactly one vertex of $L^\star$ has degree 4 and every other vertex of $L^\star$ has degree 2.*

Proof. Let $d=\frac{1}{2}\sum_{x\in V(L_0)}(\deg_{L_0}(x)-2)$, and construct a sequence of packings $\mathcal{P}_0,\mathcal{P}_1,\dots,\mathcal{P}_{d-1}$, where for $i\in\{0,\dots,d-2\}$ $\mathcal{P}_{i+1}$ is a repacking of $\mathcal{P}_i$ obtained from $\mathcal{P}_i$ by performing an (x_i,y_i)-switch where x_i and y_i are twin vertices in $K_{U\cup W}-K_U$ such that the degree of x_i in the reduced leave of $\mathcal{P}_i$ is at least 4 and y_i is not in the reduced leave of $\mathcal{P}_i$. Such vertices exist by Lemma 2.6(iii) since $|E(L_0)|\leqslant 2\min(|U|+2,|W|)$ and $i\leqslant d-2$. Exactly one vertex of the reduced leave of $\mathcal{P}_{d-1}$ has degree 4 and all its other vertices have degree 2. $\square$

We now prove the main result of this section.

Proof of Lemma 2.5. Let L be the reduced leave of $\mathcal{P}$. It obviously suffices to find a repacking of $\mathcal{P}$ whose reduced leave is the edge-disjoint union of two m-cycles.

We prove the result by induction on $s+t$. If $s=1$ and $t=1$, then the result is trivial. So suppose that $s+t\geqslant 3$. Assume without loss of generality that $s\geqslant t$ and note that $s\geqslant 2$.

Case 1. Suppose that some vertex of L has degree at least 4. Then by Lemma 2.19, there is a repacking $\mathcal{P}'$ of $\mathcal{P}$ with a reduced leave L' such that exactly one vertex of L' has degree 4 and every other vertex of L' has degree 2. So one component of L' is a 2-chain, and any other component of L' is a cycle. Furthermore L' contains at most $\lfloor\frac{2m-6}{4}\rfloor+1$ components by Lemma 2.18 and obviously $m\geqslant\lfloor\frac{2m-6}{4}\rfloor+2$. Thus, applying Lemma 2.17 with $m_1=m_2=m$ to $\mathcal{P}'$, there is a repacking of $\mathcal{P}$ whose reduced leave is the edge-disjoint union of two m-cycles.

Case 2. Suppose that every vertex of L has degree 2. Then the components of L are cycles of lengths $a_1,\dots,a_s,b_1,\dots,b_t$. Let x and y be vertices in W such that x and y are in two distinct cycles of L which have lengths a_1 and a_2 respectively. Let $\mathcal{P}'$ be a repacking of $\mathcal{P}$ obtained by performing an (x,y)-switch and let L' be the reduced leave of $\mathcal{P}'$. If the origin and terminus of this switch are in the same cycle, then one vertex of L' has degree 4 and every other vertex of L' has degree 2, and we can proceed as we did in Case 1. If the origin and terminus of this switch are in different cycles, then L' is the edge-disjoint union of cycles of lengths $a_1+a_2,a_3,\dots,a_s,b_1,\dots,b_t$ (lengths

$a_1 + a_2, b_1, \dots, b_t$ if $s = 2$) and we can complete the proof by applying our inductive hypothesis. □

2.2 Base decompositions

Our goal in this section is to prove Lemmas 2.23 and 2.24 which provide the 'base' decompositions of $K_v - K_u$ into short cycles and m-cycles to which we apply Lemma 2.5 in order to prove Theorem 2.2. Lemma 2.23 is used in the case where $m \geqslant 11$ and Lemma 2.24 is used when $m = 9$. We first require two preliminary results. Lemma 2.20 is a method for decomposing certain graphs into 3-cycles and 5-cycles, and Theorem 2.22 is an existing result on decomposing the complete bipartite graph into cycles.

We require some additional notation. For a positive integer v, let K_v^c denote a graph of order v with no edges and, for a set V, let K_V^c denote the graph with vertex set V and no edges. For technical reasons we shall consider a 0-cycle to be a trivial graph with no vertices or edges. Because we can add any number of 0-cycles to a packing without altering its leave, we shall not distinguish between packings that differ only in their number of 0-cycles nor between lists that differ only in their number of 0s. For a nonnegative integer i, let x^i denote a list containing i entries all equal to x.

For each even integer $\ell \geqslant 4$ we define a list $\mathcal{R}_\ell$ as follows

$$\mathcal{R}_\ell = \begin{cases} 4^{\ell/4} & \text{if } \ell \equiv 0 \pmod 4; \\ 4^{(\ell-6)/4}, 6 & \text{if } \ell \equiv 2 \pmod 4). \end{cases}$$

We also define $\mathcal{R}_0$ to be the empty list. Given a list $\mathcal{R}_\ell$ and a positive integer i we define $\mathcal{R}_\ell^i$ to be the list obtained by concatenating i copies of $\mathcal{R}_\ell$.

Lemma 2.20. *Let a and k be nonnegative integers such that $k \geqslant 3$, $a \leqslant k$ and a is even. Let C be a cycle of length k, and let N be a vertex set of size $k-a$ such that $V(C) \cap N = \emptyset$. Then there exists a $(3^a, 5^{k-a})$-decomposition of $K_2^c \vee (C \cup K_N^c)$ such that each cycle in the decomposition contains exactly one edge of C.*

Proof. Let $U' = \{y, z\}$, $C = (c_1, c_2, \dots, c_k)$, and $N = \{x_1, x_2, \dots, x_{k-a}\}$. Let

$$\mathcal{D}_1 = \{(y, c_k, c_1), (z, c_1, c_2), (y, c_2, c_3), (z, c_3, c_4), \dots, (y, c_{a-2}, c_{a-1}), (z, c_{a-1}, c_a)\};$$

and

$$\mathcal{D}_2 = \{(y, c_a, c_{a+1}, z, x_1), (y, c_{a+1}, c_{a+2}, z, x_2), \dots, (y, c_{k-2}, c_{k-1}, z, x_{k-a-1}),$$
$$(y, c_{k-1}, c_k, z, x_{k-a})\};$$

where $\mathcal{D}_1$ is understood to be empty and $c_0 = c_k$ if $a = 0$, and $\mathcal{D}_2$ is understood to be empty if $a = k$. Then $\mathcal{D}_1 \cup \mathcal{D}_2$ is a decomposition with the required properties. □

Theorem 2.22 below is slightly stronger than Theorem 1.21 (see [64]) but is easily proved using the following lemma which is Lemma 3.6 of the same paper.

Lemma 2.21 ([64]). *Let M be a list of integers and let a, b, h, n and n' be positive integers such that $a \leqslant b$, $n + n' \leqslant 3h$, $n + n' + h \leqslant 2a + 2$ if $a < b$, and $n + n' + h \leqslant 2a$ if $a = b$. If there exists an (M, h, n, n')-decomposition of $K_{a,b}$, then there exists an $(M, h, n + n')$-decomposition of $K_{a,b}$.*

Theorem 2.22. *Let a and b be positive integers such that a and b are even and $a \leqslant b$, and let $m_1, m_2, \ldots, m_\tau$ be even integers such that $4 \leqslant m_1 \leqslant m_2 \leqslant \ldots \leqslant m_\tau$. If*

(B1) *$m_\tau \leqslant 3m_{\tau-1}$;*

(B2) *$m_{\tau-1} + m_\tau \leqslant 2a + 2$ if $a < b$ and $m_{\tau-1} + m_\tau \leqslant 2a$ if $a = b$; and*

(B3) *$m_1 + m_2 + \cdots + m_\tau = ab$;*

then there exists an $(m_1, m_2, \ldots, m_\tau)$-decomposition of $K_{a,b}$.

Proof. Suppose for a contradiction that there exists a nondecreasing list of integers that satisfies the hypotheses of the theorem but for which there is no corresponding decomposition of $K_{a,b}$, and amongst all such lists let $Z = z_1, \ldots, z_\tau$ be one with a maximum number of entries.

It follows from Theorem 1.20 that $z_\tau > 8$. Let $Z^\star$ be the list $z_1, z_2, \ldots, z_{\tau-1}$, $4, z_\tau - 4$ reordered so as to be nondecreasing. Since Z satisfies the conditions of the claim, so must $Z^\star$, and since $Z^\star$ has more entries than Z, there exists a $(Z^\star)$-decomposition of $K_{a,b}$. However, by applying Lemma 2.21 with $n = 4$, $n' = z_\tau - 4$ and $h = z_{\tau-1}$ we obtain a (Z)-decomposition of $K_{a,b}$ which is a contradiction. □

We now prove the existence of the required base decompositions for odd $m \geqslant 11$.

Lemma 2.23. *Let u, v and m be odd integers such that $m \geqslant 11$, (u, v) is m-admissible, $v - u \geqslant m+1$, $u \geqslant m$ if $m \in \{11, 13, 15\}$, and $u \geqslant m-2$ if $m \geqslant 17$. Let k, t and x be the nonnegative integers such that $u(v - u) = (m - 1)k + t$, $t < m - 1$ and $m(k + x) = \binom{v}{2} - \binom{u}{2}$. Then, for some $h \in \{4, 6, \ldots, m - 7\} \cup \{m - 3\}$, there exists an $(m^x, 3^k, h, \mathcal{R}^{k-1}_{m-3}, \mathcal{R}^1_{m-h-3})$-decomposition of $K_v - K_u$ in which each cycle of length less than m contains at most one pure edge.*

Proof. Observe that k is the maximum number of pairwise edge-disjoint m-cycles in $K_v - K_u$ that each contain exactly one pure edge. Also note that t is even so $t \leqslant m-3$. Let $w = v-u$, note that w is even, and let p and q be the nonnegative integers such that $k = w(p+\frac{1}{2})+q$ and $q < w$. We will make use of the following facts throughout this proof.

$$uw = (w(p+\tfrac{1}{2})+q)(m-1)+t \tag{2.2.1}$$

$$2(u-2p) \geqslant \begin{cases} \frac{4}{3}m+\frac{22}{3} & \text{if } p=0 \\ \frac{4}{3}m+\frac{34}{3} & \text{if } p \geqslant 1 \end{cases} \tag{2.2.2}$$

Note that (2.2.1) follows directly from the definitions of k, t, p and q. To see that (2.2.2) holds, observe that when $p=0$ and $m \in \{11,13,15\}$ we have $2(u-2p) = 2u \geqslant 2m$, which implies $2(u-2p) \geqslant \frac{4}{3}m+\frac{22}{3}$ since $m \geqslant 11$. When $p=0$ and $m \geqslant 17$ we have $2(u-2p) = 2u \geqslant 2m-4$, which implies $2(u-2p) \geqslant \frac{4}{3}m+\frac{22}{3}$ since $m \geqslant 17$. Also, (2.2.1) implies $uw \geqslant w(p+\frac{1}{2})(m-1)$ and so $u \geqslant (p+\frac{1}{2})(m-1)$. Thus $2(u-2p) \geqslant 2p(m-3)+m-1$. So when $p \geqslant 1$ we have $2(u-2p) \geqslant 3m-7$, which implies $2(u-2p) \geqslant \frac{4}{3}m+\frac{34}{3}$ since $m \geqslant 11$.

Let $U = \{y_1, y_2, \ldots, y_u\}$ and $W = \{z_1, z_2, \ldots, z_w\}$ be disjoint sets of vertices. We will construct a decomposition of $K_{U\cup W} - K_U$ with the desired properties. Let I be a 1-factor with vertex set W. The proof divides into two cases depending on whether $t=0$ or $t>0$.

Case 1. Suppose that $t > 0$. Then $q > 0$, for otherwise (2.2.1) implies $w(u-(p+\frac{1}{2})(m-1)) = t > 0$ which contradicts the facts that $w > m > t$ and $u-(p+\frac{1}{2})(m-1)$ is an integer.

Depending on the value of q, we define integers p', q' and q'' so that $w(p'+\frac{1}{2})+q'+q'' = k$ according to the following table.

case	p'	q'	q''
$q \in \{1,2,3,4\}$	$p-1$	w	q
$q \in \{5,7,\ldots,w-1\}$	p	$q-1$	1
$q \in \{6,8,\ldots,w-2\}$	p	$q-2$	2

We show that $p' \geqslant 0$ by establishing that it cannot be the case that both $p=0$ and $q \in \{1,2,3,4\}$. If $p=0$ and $q \in \{1,2,3,4\}$, then (2.2.1) implies $uw \leqslant (m-1)(\frac{w}{2}+4)+(m-3)$ and hence $w(u-\frac{m-1}{2}) \leqslant 5m-7$. Because $u \geqslant m$ if $m \in \{11,13,15\}$ and $u \geqslant m-2$ if $m \geqslant 17$, we have that $u-\frac{m-1}{2} \geqslant 6$ and we obtain a contradiction by noting that $w > m$.

We define h to be the smallest integer in $\{4,6,\ldots,m-7\} \cup \{m-3\}$ such that $h \geqslant \frac{2q''+t}{3}$. Using the facts that $q'' \in \{1,2,3,4\}$ and $t \leqslant m-3$ it is routine to check that if $\frac{2q''+t}{3} > m-7$, then $m=11$, $q'' \in \{3,4\}$ and $h=8$. Thus h is well-defined and, if it is not the case that $\frac{2q''+t}{3} \leqslant h \leqslant \frac{2q''+t+5}{3}$, then either

$2q'' + t \leqslant 6$ and $h = 4$ or $m = 11$, $q'' \in \{3, 4\}$ and $h = 8$. We claim that

$$2q'' + t + h \leqslant \begin{cases} \frac{4}{3}m + \frac{28}{3} & \text{if } p' = p - 1; \\ \frac{4}{3}m + 3 & \text{if } p' = p. \end{cases} \tag{2.2.3}$$

To see that this is the case note that if $2q'' + t \leqslant 6$ and $h = 4$ or if $m = 11$, $q'' \in \{3, 4\}$ and $h = 8$, then (2.2.3) holds (recall that $t \leqslant m - 3$ and $m \geqslant 11$). Otherwise, $2q''+t+h \leqslant \frac{4}{3}(2q''+t)+\frac{5}{3}$ and hence $2q''+t+h \leqslant \frac{4}{3}m+\frac{1}{3}(8q''-7)$ using $t \leqslant m - 3$. Because $q'' \leqslant 2$ if $p' = p$ and $q'' \leqslant 4$ if $p' = p - 1$, (2.2.3) holds.

We will complete the proof by constructing an $(m^x, 3^k, h, \mathcal{R}^{k-1}_{m-3}, \mathcal{R}^1_{m-h-3})$-decomposition of $K_{U \cup W} - K_U$ in which each cycle of length less than m contains at most one pure edge. We will construct this decomposition in such a way that the pure edges in the 3-cycles of the decomposition form p' w-cycles, a 1-factor with w vertices, a q'-cycle, and a q''-path (recall that $p'w + w/2 + q' + q'' = k$). Our required decomposition can be obtained as

$$(\mathcal{D}_1 \setminus \{H_1, \ldots, H_p, C', C''\}) \cup (\mathcal{D}_2 \setminus \{C^\dagger\}) \cup \mathcal{D}_3 \cup \mathcal{D}_4 \cup \mathcal{D}_5$$

where $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_5$ are given as follows.

- $\mathcal{D}_1$ is a $(w^{p'}, m^{x-1}, q', m + q'' - t)$-decomposition of $K_W - I$, that includes p' w-cycles $H_1, \ldots, H_{p'}$, a q'-cycle C', and an $(m+q''-t)$-cycle C'' containing the path $[z_1, z_2, \ldots, z_{q''+1}]$ and not containing the $\frac{t}{2} - 1$ vertices $z_{q''+2}, z_{q''+3}, \ldots, z_{q''+\frac{t}{2}}$. A $(w^{p'}, m^{x-1}, q', m + q'' - t)$-decomposition of $K_W - I$ exists by Theorem 1.4 because $mx + p'w + q' + q'' - t = mx+k-t-\frac{w}{2} = \binom{w}{2}-\frac{w}{2}$ (note that the definitions of k, t and x imply that $\binom{w}{2} = mx + k - t$). We can relabel the vertices of this decomposition to ensure that C'' has the specified properties because $(m+q''-t)+(\frac{t}{2}-1) = m - \frac{t}{2} + q'' - 1 \leqslant w$. (If $m - \frac{t}{2} + q'' - 1 \geqslant w + 1$, then $t \geqslant 2$, $q'' \leqslant 4$ and $w \geqslant m + 1$ imply $(t, q'', w) = (2, 4, m + 1)$ and hence $q = 4$. However, in this case (2.2.1) implies that $u(m + 1) = (m + 1)(m - 1)(p + \frac{1}{2}) + 4m - 2$ and hence that $(m + 1)$ divides $4m - 2$. This contradicts $m \geqslant 11$.)

- $\mathcal{D}_2$ is a $(2q'' + t, h, \mathcal{R}^{k-1}_{m-3}, \mathcal{R}^1_{m-h-3})$-decomposition of $K_{\{y_1,\ldots,y_{u-2p'-1}\},W} - K_{U',V(C')}$ including the $(2q''+t)$-cycle $C^\dagger = (y_1, z_1, y_2, z_2, \ldots, y_{q''+\frac{t}{2}}, z_{q''+\frac{t}{2}})$, where $U' = \{y_{u-2p'-2}, y_{u-2p'-1}\}$.

  We form $\mathcal{D}_2$ by first decomposing the complete bipartite graph $K_{U',W\setminus V(C')}$ into $\frac{w-q'}{2}$ 4-cycles (if $q' < w$) and then decomposing $K_{\{y_1,y_2,\ldots,y_{u-2p'-3}\},W}$ into cycles of the remaining lengths. Note that $\mathcal{R}^{k-1}_{m-3}$ contains at least $k-1$ 4s since $m \geqslant 11$, and also that $k-1 = w(p+\frac{1}{2})+q-1 \geqslant \frac{w}{2}+q-1 \geqslant \frac{w-q'}{2}$.

A decomposition of $K_{\{y_1,y_2,\ldots,y_{u-2p'-3}\},W}$ into cycles of the remaining lengths exists by Theorem 2.22. Let m_τ and $m_{\tau-1}$ be respectively the greatest and second greatest of the remaining cycle lengths. To see that (B1), (B2) and (B3) hold, we first suppose that $(m_\tau, m_{\tau-1}) = (2q''+t, h)$. It follows from the definition of h that $2q''+t \leqslant 3h$ and hence that (B1) holds. If $u-2p'-3 \geqslant w$, then (B2) holds because, using $m \geqslant 11$ and (2.2.3), we have

$$2w \geqslant 2(m+1) \geqslant \tfrac{4}{3}m + \tfrac{28}{3} \geqslant 2q'' + t + h.$$

If $u-2p'-3 < w$ and $p' = p$, then (B2) holds because, using (2.2.2) and (2.2.3), we have

$$2(u-2p'-3)+2 = 2(u-2p)-4 \geqslant (\tfrac{4}{3}m+\tfrac{22}{3})-4 > \tfrac{4}{3}m+3 \geqslant 2q''+t+h.$$

If $u-2p'-3 < w$ and $p' = p-1$, then $p \geqslant 1$ and (B2) holds because, using (2.2.2) and (2.2.3), we have

$$2(u-2p'-3)+2 = 2(u-2p) \geqslant \tfrac{4}{3}m + \tfrac{34}{3} > \tfrac{4}{3}m + \tfrac{28}{3} \geqslant 2q''+t+h.$$

Finally, (B3) holds because, using the definitions of t, k, p and q, and the fact that $q'+q''-q = w(p-p')$, we have

$$\begin{aligned}
&(k-1)(m-3)+(m-h-3)+h+(2q''+t)-4\tfrac{w-q'}{2}\\
&= k(m-3)+t-2w+2(q'+q'')\\
&= (uw-2k)-2w+2(q'+q'')\\
&= (u-2p-3)w+2(q'+q''-q)\\
&= (u-2p'-3)w.
\end{aligned}$$

Now suppose that $(m_\tau, m_{\tau-1}) \neq (2q''+t, h)$. Then it must be that $m_\tau \leqslant 12$ and $m_{\tau-1} \in \{4,6\}$. Clearly (B1) holds. Also, $m_\tau + m_{\tau-1} \leqslant 18$. So if $w \leqslant u-2p'-3$ then (B2) holds because $2w \geqslant 2(m+1) \geqslant 24$, and if $w > u-2p'-3$ then (B2) holds by (2.2.2) since $2(u-2p'-3)+2 \geqslant 18$ (note that $p' \leqslant p$ and $m \geqslant 11$). Finally, (B3) holds by the argument above. We can relabel the vertices of this decomposition to ensure that $C^\dagger = (y_1, z_1, y_2, z_2, \ldots, y_{q''+\frac{t}{2}}, z_{q''+\frac{t}{2}})$.

- $\mathcal{D}_3$ is a $(3^{q'})$-decomposition of $K^c_{\{y_{u-2p'-2}, y_{u-2p'-1}\}} \vee C'$ which exists by Lemma 2.20.
- $\mathcal{D}_4$ is a $(3^{p'w+w/2})$-decomposition of $K^c_{\{y_{u-2p'},\ldots,y_u\}} \vee (I \cup H_1 \cup \cdots \cup H_{p'})$ which exists by applying Lemma 2.20 to $K^c_{\{y_{u-2p'-1+i}, y_{u-p'-1+i}\}} \vee H_i$ for

$i \in \{1, 2, \dots, p'\}$ and taking the obvious decomposition of $K_{\{y_u\}} \vee I$.

- $\mathcal{D}_5$ is the $(3^{q''}, m)$-decomposition of $C'' \cup C^\dagger$ given by

$$\{(z_1, y_2, z_2), (z_2, y_3, z_3), \dots, (z_{q''}, y_{q''+1}, z_{q''+1})\} \cup$$
$$\{(C''-[z_1, z_2, \dots, z_{q''+1}]) \cup [z_{q''+1}, y_{q''+2}, z_{q''+2}, y_{q''+3}, \dots, y_{q''+\frac{t}{2}}, z_{q''+\frac{t}{2}}, y_1, z_1]\}.$$

Case 2. Suppose that $t = 0$. Then (2.2.1) reduces to $uw = (w(p+\frac{1}{2})+q)(m-1)$. Note that $q \neq 1$, since if $q = 1$ then $uw = w(p+\frac{1}{2})(m-1)+m-1$ and so w divides $m-1$ which contradicts $w \geqslant m+1$. Depending on the value of q, we define integers p', q' and q'' so that $w(p'+\frac{1}{2})+q'+q'' = k$ according to the following table.

case	p'	q'	q''
$q \in \{0, 3, 5\}$	$p-1$	w	q
$q = 2$	$p-1$	$w-2$	4
$q \in \{4, 6, \dots, w-2\}$	p	q	0
$q \in \{7, 9, \dots, w-1\}$	p	$q-3$	3

We show that $p' \geqslant 0$ by establishing that it cannot be the case that both $p = 0$ and $q \in \{0, 2, 3, 5\}$. If $p = 0$ and $q \in \{0, 2, 3, 5\}$, then (2.2.1) implies $uw \leqslant (m-1)(\frac{w}{2}+5)$ and hence $w(u-\frac{m-1}{2}) \leqslant 5m-5$. Because $u \geqslant m$ if $m \in \{11, 13, 15\}$ and $u \geqslant m-2$ if $m \geqslant 17$, we have that $u - \frac{m-1}{2} \geqslant 6$ and we obtain a contradiction by noting that $w > m$.

We will complete the proof by constructing an $(m^x, 3^k, \mathcal{R}^k_{m-3})$-decomposition of $K_{U \cup W} - K_U$ in which each cycle of length less than m contains at most one pure edge. We will construct this decomposition in such a way that the pure edges in the 3-cycles of the decomposition form p' w-cycles, a 1-factor with w vertices, a q'-cycle, and a q''-cycle if $q'' \neq 0$ (recall that $p'w+w/2+q'+q'' = k$). The desired decomposition can be obtained as

$$(\mathcal{D}_1 \setminus S_1) \cup (\mathcal{D}_2 \setminus S_2) \cup \mathcal{D}_3 \cup \mathcal{D}_4 \cup \mathcal{D}_5$$

where $(S_1, S_2) = (\{H_1, H_2, \dots, H_{p'}, C', C''\}, \{C^\dagger\})$ if $q'' \neq 0$, $(S_1, S_2) = (\{H_1, H_2, \dots, H_{p'}, C'\}, \emptyset)$ if $q'' = 0$, and $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_5$ are given as follows.

- $\mathcal{D}_1$ is an $(m^x, w^{p'}, q', q'')$-decomposition of $K_W - I$ that includes p' w-cycles $H_1, \dots, H_{p'}$, a q'-cycle C' and, if $q'' \neq 0$, the q''-cycle $C'' = (z_1, z_2, \dots, z_{q''})$. An $(m^x, w^{p'}, q', q'')$-decomposition of $K_W - I$ exists by Theorem 1.4 because $mx+p'w+q'+q'' = mx+k-\frac{w}{2} = \binom{w}{2}-\frac{w}{2}$ (note that the definitions of k, t and x imply that $\binom{w}{2} = mx+k-t$). We can relabel the vertices of this decomposition to ensure that $C'' = (z_1, z_2, \dots, z_{q''})$.

- $\mathcal{D}_2$ is a $(2q'', 3^k, \mathcal{R}^k_{m-3})$-decomposition of $K_{\{y_1,y_2,\ldots,y_{u-2p'-1}\},W} - K_{U',V(C')}$ that includes the $(2q'')$-cycle $C^\dagger = (y_1, z_1, y_2, z_2, \ldots, y_{q''}, z_{q''})$ if $q'' \neq 0$, where $U' = \{y_{u-2p'-2}, y_{u-2p'-1}\}$.

  We form $\mathcal{D}_2$ by first decomposing $K_{U',W\setminus V(C')}$ into $\frac{w-q'}{2}$ 4-cycles (if $q' < w$) and then decomposing $K_{\{y_1,y_2,\ldots,y_{u-2p'-3}\},W}$ into cycles of the remaining lengths. Note that $\mathcal{R}^k_{m-3}$ contains at least k 4s since $m \geqslant 11$, and also that $k = w(p + \frac{1}{2}) + q \geqslant \frac{w}{2} + q \geqslant \frac{w-q'}{2}$.

  A decomposition of $K_{\{y_1,y_2,\ldots,y_{u-2p'-3}\},W}$ into cycles of the remaining lengths exists by Theorem 2.22. Let m_τ and $m_{\tau-1}$ be respectively the greatest and second greatest of the remaining cycle lengths. Note that $m_\tau \leqslant \max(2q'', 6) \leqslant 10$ and $m_{\tau-1} \in \{4, 6\}$. Clearly (B1) holds. If $w \leqslant u - 2p' - 3$, then (B2) holds because $m_{\tau-1} + m_\tau \leqslant 16$ and $2w \geqslant 2(m + 1) \geqslant 24$. If $w > u - 2p' - 3$, then (B2) holds because $m_{\tau-1} + m_\tau \leqslant 16$ and (2.2.2) implies that $2(u - 2p' - 3) + 2 \geqslant 18$ (note that $p' \leqslant p$ and $m \geqslant 11$). Finally, (B3) holds by a similar argument to that used in Case 1. We can relabel the vertices of this decomposition to ensure that $C^\dagger = (y_1, z_1, y_2, z_2, \ldots, y_{q''}, z_{q''})$ if $q'' \neq 0$.

- $\mathcal{D}_3$ is a $(3^{q'})$-decomposition of $K^c_{\{y_{u-2p'-2},y_{u-2p'-1}\}} \vee C'$ which exists by Lemma 2.20.

- $\mathcal{D}_4$ is a $(3^{p'w+w/2})$-decomposition of $K^c_{\{y_{u-2p'},\ldots,y_u\}} \vee (I \cup H_1 \cup \cdots \cup H_{p'})$ which exists by applying Lemma 2.20 to $K^c_{\{y_{u-2p'-1+i},y_{u-p'-1+i}\}} \vee H_i$ for $i \in \{1, 2, \ldots, p'\}$ and taking the obvious decomposition of $K_{\{y_u\}} \vee I$.

- $\mathcal{D}_5$ is the $(3^{q''})$-decomposition of $C'' \cup C^\dagger$ given by

$$\{(z_1, y_2, z_2), (z_2, y_3, z_3), \ldots, (z_{q''-1}, y_{q''}, z_{q''}), (z_{q''}, y_1, z_1)\}$$

  if $q'' \neq 0$, and $\mathcal{D}_5 = \emptyset$ if $q'' = 0$. □

Lemma 2.24. *Let u and v be positive integers such that (u, v) is 9-admissible, $v - u \geqslant 10$ and $u \geqslant 9$. Let k, t and x be the nonnegative integers such that $u(v - u) = 8k + t$, $t < 8$ and $9(x + k) = \binom{v}{2} - \binom{u}{2}$. Then, for some nonnegative integer $k' \leqslant k$, there exists a $(3^{k-k'}, 4^{k'}, 5^{k'}, 6^{k-k'}, 9^x)$-decomposition of $K_v - K_u$ in which each cycle of length less than 9 contains at most one pure edge.*

Proof. Observe that k is the maximum number of pairwise edge-disjoint 9-cycles in $K_v - K_u$ that each contain exactly one pure edge. Also note that t is even and so $t \leqslant 6$. Let $w = v - u$, note that w is even, and let p and q be the nonnegative integers such that $k = (p + 1)\frac{w}{2} + q$ and $q < \frac{w}{2}$. We will make use of the following fact, which follows directly from the definitions of k, t, p and q, throughout this proof.

$$uw = 8((p+1)\tfrac{w}{2} + q) + t \tag{2.2.4}$$

Note that $p \geqslant 1$, for otherwise $uw \leqslant 4w + 8q + t \leqslant 4w + 8(\frac{w}{2} - 1) + 6 < 8w$ which contradicts $u \geqslant 9$. Also note that $u \geqslant 4p + 5$, because u is odd and (2.2.4) implies $uw \geqslant 4(p+1)w$. From this, we can see that

$$u - 2p - 3 \geqslant 2p + 2 \geqslant 4. \tag{2.2.5}$$

Let $U = \{y_1, y_2, \ldots, y_u\}$ and $W = \{z_1, z_2, \ldots, z_w\}$ be disjoint sets of vertices. We will construct a decomposition of $K_{U \cup W} - K_U$ with the desired properties. Let I be a 1-factor with vertex set W. The proof divides into two cases depending on whether $t = 0$ or $t > 0$.

Case 1. Suppose that $t > 0$. Then $q > 0$, for otherwise (2.2.4) implies $w(u - 4p - 4) = t > 0$ which contradicts the facts that $w \geqslant 10 > t$ and $u - 4p - 4$ is an integer. Depending on the value of q, we define integers p', q_3', q_5' and q'' so that $(p'+1)\frac{w}{2} + q_3' + q_5' + q'' = k$ according to the following table.

case	p'	q_3'	q_5'	q''
$q \in \{1, 2, 3\}$	$p - 1$	$2q - 2$	$\frac{w}{2} + 1 - q$	1
$q \in \{4, \ldots, \frac{w}{2} - 1\}$	p	0	$q - 1$	1

We will complete the proof by constructing a $(3^{w/2+q_3'+q''}, 4^{p'w/2+q_5'}, 5^{p'w/2+q_5'}, 6^{w/2+q_3'+q''}, 9^x)$-decomposition of $K_{U\cup W} - K_U$ in which each cycle of length less than 9 contains at most one pure edge. We will construct this decomposition in such a way that the pure edges in the 3-cycles and 5-cycles of the decomposition form p' $\frac{w}{2}$-cycles, a 1-factor with w vertices, a $(q_3' + q_5')$-cycle, and a q''-path (recall that $(p'+1)\frac{w}{2} + q_3' + q_5' + q'' = k$). The desired decomposition can be obtained as

$$(\mathcal{D}_1 \setminus \{H_1, H_2, \ldots, H_{p'}, C', C''\}) \cup (\mathcal{D}_2 \setminus \{C^\dagger\}) \cup \mathcal{D}_3 \cup \mathcal{D}_4 \cup \mathcal{D}_5$$

where $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, $\mathcal{D}_4$ and $\mathcal{D}_5$ are given as follows.

- $\mathcal{D}_1$ is a $(9^{x-1}, (\frac{w}{2})^{p'}, q_3' + q_5', 9 + q'' - t)$-decomposition of $K_W - I$ that includes p' $(\frac{w}{2})$-cycles $H_1, H_2, \ldots, H_{p'}$, a $(q_3'+q_5')$-cycle C', and a $(9+q''-t)$-cycle C'' containing the edge z_1z_2 and not containing the $\frac{t}{2} - 1$ vertices $z_3, z_4, \ldots, z_{\frac{t}{2}+1}$ if $t > 2$. A $(9^{x-1}, (\frac{w}{2})^{p'}, q_3' + q_5', 9 + q'' - t)$-decomposition of $K_W - I$ exists by Theorem 1.4 because $9(x-1) + p'\frac{w}{2} + q_3' + q_5' + 9 + q'' - t = 9x + k - t - \frac{w}{2} = \binom{w}{2} - \frac{w}{2}$ (note that the definitions of k, t and x imply that $\binom{w}{2} = 9x + k - t$). We can relabel the vertices of this decomposition to ensure that C'' has the specified properties because $(9+q''-t)+(\frac{t}{2}-1)=9-\frac{t}{2}<10\leqslant w$.

- $\mathcal{D}_2$ is a $(4^{p'w/2+q'_5}, 6^{w/2+q'_3+q''}, t+2q'')$-decomposition of $K_{\{y_1,\ldots,y_{u-2p'-1}\},W} - K_{\{y_{u-2p'-2},y_{u-2p'-1}\},V(C')\cup Q}$, where Q is a subset of $W\setminus V(C')$ of size q'_5, that includes the $(t+2q'')$-cycle $C^\dagger=(y_1,z_1,y_2,z_2,\ldots,y_{\frac{t}{2}+1},z_{\frac{t}{2}+1})$. Note that $|W\setminus V(C')|=w-q'_3-q'_5\geqslant q'_5$ follows from our choice of q'_3 and q'_5.
  We form $\mathcal{D}_2$ by first decomposing $K_{\{y_{u-2p'-2},y_{u-2p'-1}\},W\setminus(V(C')\cup Q)}$ into $\frac{w-q'_3-2q'_5}{2}$ 4-cycles and then decomposing $K_{\{y_1,y_2,\ldots,y_{u-2p'-3}\},W}$ into cycles of the remaining lengths. Note that we desire $p'\frac{w}{2}+q'_5$ 4-cycles in $\mathcal{D}_2$ and that $p'\frac{w}{2}+q'_5\geqslant\frac{w-q'_3-2q'_5}{2}$ follows from our choice of p', q'_3 and q'_5 (recall that $p\geqslant 1$).
  A decomposition of $K_{\{y_1,y_2,\ldots,y_{u-2p'-3}\},W}$ into cycles of the remaining lengths exists by Theorem 1.20 because $t+2q''\in\{4,6,8\}$. To see that the conditions of Theorem 1.20 hold, note that $u-2p'-3\geqslant u-2p-3\geqslant 4$ using (2.2.5), that $w\geqslant 10$, and that
  $$\begin{aligned}&4(p'\tfrac{w}{2}+q'_5-\tfrac{w-q'_3-2q'_5}{2})+6(\tfrac{w}{2}+q'_3+q'')+(t+2q'')\\&=8((p'+1)\tfrac{w}{2}+q'_3+q'_5+q'')+t-(2p'+3)w\\&=uw-(2p'+3)w\\&=(u-2p'-3)w.\end{aligned}$$
  We can relabel the vertices of this decomposition to ensure that $C^\dagger=(y_1,z_1,y_2,z_2,\ldots,y_{\frac{t}{2}+1},z_{\frac{t}{2}+1})$.
- $\mathcal{D}_3$ is a $(3^{q'_3},5^{q'_5})$-decomposition of $K^c_{\{y_{u-2p'-2},y_{u-2p'-1}\}}\vee(C'\cup K^c_Q)$ which exists by Lemma 2.20.
- $\mathcal{D}_4$ is a $(3^{w/2},5^{p'w/2})$-decomposition of $K_{\{y_{u-2p'},\ldots,y_u\},W}\cup I\cup H_1\cup\cdots\cup H_{p'}$ which exists by applying Lemma 2.20 (with $a=0$ and $n=\frac{w}{2}$) to $K^c_{\{y_{u-2p'-1+i},y_{u-p'-1+i}\}}\vee(H_i\cup K^c_{W\setminus V(H_i)})$ for $i\in\{1,2,\ldots,p'\}$ and taking the obvious decomposition of $K_{\{y_u\}}\vee I$.
- $\mathcal{D}_5$ is the following $(3^{q''},9^1)$-decomposition of $C''\cup C^\dagger$ (recall that $q''=1$).
  $$\{(z_1,y_2,z_2)\}\cup\{(C''-[z_1,z_2])\cup[z_2,y_3,z_3,y_4,\ldots,y_{\frac{t}{2}+1},z_{\frac{t}{2}+1},y_1,z_1]\}$$

Case 2. Suppose that $t=0$. Then (2.2.4) reduces to $uw=8((p+1)\frac{w}{2}+q)$. Depending on the value of q, we define integers p', q' and q'' so that $(p'+1)\frac{w}{2}+q'+q''=k$ according to the following table.

case	p'	q'	q''
$q = 0$	$p-1$	$\frac{w}{2}$	0
$q \in \{1,2\}$	$p-1$	$\frac{w}{2} - 3 + q$	3
$q \in \{3,4,\ldots,\frac{w}{2}-1\}$	p	q	0

We will complete the proof by constructing a $(3^{w/2+q''}, 4^{p'w/2+q'}, 5^{p'w/2+q'}, 6^{w/2+q''}, 9^x)$-decomposition of $K_{U \cup W} - K_U$ in which each cycle of length less than 9 contains at most one pure edge. We will construct this decomposition in such a way that the pure edges in the 3-cycles and 5-cycles of the decomposition form p' $\frac{w}{2}$-cycles, a 1-factor with w vertices, a q'-cycle, and a q''-cycle if $q'' \neq 0$ (recall that $(p'+1)\frac{w}{2}+q'+q'' = k$). The desired decomposition can be obtained as

$$(\mathcal{D}_1 \setminus S_1) \cup (\mathcal{D}_2 \setminus S_2) \cup \mathcal{D}_3 \cup \mathcal{D}_4 \cup \mathcal{D}_5$$

where $(S_1, S_2) = (\{H_1, H_2, \ldots, H_{p'}, C', C''\}, \{C^\dagger\})$ if $q'' \neq 0$, $(S_1, S_2) = (\{H_1, H_2, \ldots, H_{p'}, C'\}, \emptyset)$ if $q'' = 0$, and $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, $\mathcal{D}_4$ and $\mathcal{D}_5$ are given as follows.

- $\mathcal{D}_1$ is a $(9^x, (\frac{w}{2})^{p'}, q', q'')$-decomposition of $K_W - I$ that includes p' $(\frac{w}{2})$-cycles $H_1, H_2, \ldots, H_{p'}$, a q'-cycle C' and a q''-cycle $C'' = (z_1, z_2, \ldots, z_{q''})$ if $q'' \neq 0$. A $(9^x, (\frac{w}{2})^{p'}, q', q'')$-decomposition of $K_W - I$ exists by Theorem 1.4 because $9x + p'\frac{w}{2} + q' + q'' = 9x + k - \frac{w}{2} = \binom{w}{2} - \frac{w}{2}$ (note that the definitions of k, t and x imply that $\binom{w}{2} = 9x + k - t$). We can relabel the vertices of this decomposition to ensure that $C'' = (z_1, z_2, \ldots, z_{q''})$ if $q'' \neq 0$.

- $\mathcal{D}_2$ is a $(4^{p'w/2+q'}, 6^{w/2+q''}, 2q'')$-decomposition of $K_{\{y_1,y_2,\ldots,y_{u-2p'-1}\},W} - K_{\{y_{u-2p'-2},y_{u-2p'-1}\},V(C')\cup Q}$, where Q is a subset of $W \setminus V(C')$ of size q', that includes the $(2q'')$-cycle $C^\dagger = (y_1, z_1, y_2, z_2, \ldots, y_{q''}, z_{q''})$ if $q'' \neq 0$. Note that $|W \setminus V(C')| = w - q' \geqslant q'$ follows from our choice of q'.

  We form $\mathcal{D}_2$ by first decomposing $K_{\{y_{u-2p'-2},y_{u-2p'-1}\},W\setminus(V(C')\cup Q)}$ into $\frac{w}{2}-q'$ 4-cycles (if $q' < \frac{w}{2}$) and then decomposing $K_{\{y_1,y_2,\ldots,y_{u-2p'-3}\},W}$ into cycles of the remaining lengths. Note that we desire $p'\frac{w}{2}+q'$ 4-cycles in $\mathcal{D}_2$ and that $p'\frac{w}{2}+q' \geqslant \frac{w}{2}-q'$ follows from our choice of p', q' and q'' (recall that $p \geqslant 1$ and $w \geqslant 10$).

  A decomposition of $K_{\{y_1,y_2,\ldots,y_{u-2p'-3}\},W}$ into cycles of the remaining lengths exists by Theorem 1.20 because $2q'' \in \{0,6\}$. The conditions of Theorem 1.20 can be shown to hold by a similar argument to that used in Case 1. We can relabel the vertices of this decomposition to ensure that $C^\dagger = (y_1, z_1, y_2, z_2, \ldots, y_{q''}, z_{q''})$ if $q'' \neq 0$.

- $\mathcal{D}_3$ is a $(5^{q'})$-decomposition of $K^c_{\{y_{u-2p'-2},y_{u-2p'-1}\}} \vee (C' \cup K^c_Q)$ which exists by Lemma 2.20.

- $\mathcal{D}_4$ is a $(3^{w/2}, 5^{p'w/2})$-decomposition of $K_{\{y_{u-2p'},\ldots,y_u\},W} \cup I \cup H_1 \cup \cdots \cup H_{p'}$ which exists by applying Lemma 2.20 (with $a = 0$ and $n = \frac{w}{2}$) to $K^c_{\{y_{u-2p'-1+i},y_{u-p'-1+i}\}} \vee (H_i \cup K^c_{W\setminus V(H_i)})$ for $i \in \{1, 2, \ldots, p'\}$ and taking the obvious decomposition of $K_{\{y_u\}} \vee I$.

- $\mathcal{D}_5$ is the $(3^{q''})$-decomposition of $C'' \cup C^\dagger$ given by

$$\{(z_1, y_2, z_2), (z_2, y_3, z_3), \ldots, (z_{q''-1}, y_{q''}, z_{q''}), (z_{q''}, y_1, z_1)\}$$

if $q'' \neq 0$, and $\mathcal{D}_5 = \emptyset$ if $q'' = 0$. □

2.3 Proof of main results

We first prove the following lemma, which does most of the work toward proving Theorem 2.2.

Lemma 2.25. *Let m, u and v be positive odd integers such that $m \geqslant 9$, $v - u \geqslant m + 1$ and (u, v) is m-admissible. If either $m \geqslant 17$ and $u \geqslant m - 2$ or $m \in \{9, 11, 13, 15\}$ and $u \geqslant m$, then there exists an m-cycle decomposition of $K_v - K_u$.*

Proof. Let k and x be the integers such that $k = \lfloor (u(v - u))/(m - 1) \rfloor$ and $m(x + k) = \binom{v}{2} - \binom{u}{2}$. From Lemmas 2.23 and 2.24, it follows that there is an $(m^x, M_1, \ldots, M_k)$-decomposition $\mathcal{D}_0$ of $K_v - K_u$ in which each cycle of length less than m contains at most one pure edge, where for $j \in \{1, \ldots, k\}$ M_j is a list of at least two integers with sum m that contains exactly one odd integer.

We will now construct a sequence of decompositions $\mathcal{D}_1, \ldots, \mathcal{D}_{k-1}$ such that, for each $i \in \{1, \ldots, k-1\}$, $\mathcal{D}_i$ is an $(m^{x+i+1}, M_1, \ldots, M_{k-i-1})$-decomposition of $K_v - K_u$ such that, with the exception of x m-cycles, each cycle in $\mathcal{D}_i$ contains at most one pure edge. This will suffice to complete the proof because $\mathcal{D}_{k-1}$ will be an m-cycle decomposition of $K_v - K_u$.

Let $\mathcal{D}_1$ be the decomposition obtained by applying Lemma 2.5 to the packing $\mathcal{P}_0 = \mathcal{D}_0 \setminus \mathcal{C}_0$, where $\mathcal{C}_0$ is a set of cycles in $\mathcal{D}_0$ with lengths given by the list M_{k-1}, M_k. For each $i \in \{1, \ldots, k-2\}$, let $\mathcal{D}_{i+1}$ be the decomposition obtained by applying Lemma 2.5 to the packing $\mathcal{P}_i = \mathcal{D}_i \setminus (\{C_i\} \cup \mathcal{C}_i)$, where C_i is an m-cycle in $\mathcal{D}_i$ that contains exactly one pure edge and $\mathcal{C}_i$ is a set of cycles in $\mathcal{D}_i$ with lengths given by the list M_{k-i-1}. For each $i \in \{0, \ldots, k - 2\}$, it is easy to verify that $\mathcal{D}_{i+1}$ has the required properties because $\mathcal{D}_i$ does and, by Lemma 2.5, $\mathcal{D}_{i+1} \setminus \{C', C''\}$ is a repacking of $\mathcal{P}_i$ for some distinct m-cycles $C', C'' \in \mathcal{D}_{i+1}$. □

The following lemma exploits the fact that two m-cycle decompositions of complete graphs with holes can be "nested" to create another. It will be used to help deal with the remaining cases of Theorem 2.2 and in the proof of Corollary 2.3.

Lemma 2.26. *Let $m \geqslant 9$ and u be odd integers. If there exists an m-cycle decomposition of $K_{u^\star} - K_u$ for some positive integer $u^\star \geqslant m$, then there exists an m-cycle decomposition of $K_v - K_u$ for each integer v such that $v \geqslant u^\star + m + 1$ and (u, v) is m-admissible.*

Proof. Let v be an integer such that $v \geqslant u^\star + m + 1$ and (u, v) is m-admissible. Let U, $U^\star$ and V be sets such that $|U| = u$, $|U^\star| = u^\star$, $|V| = v$, and $U \subseteq U^\star \subseteq V$. By our hypotheses, there exists an m-cycle decomposition $\mathcal{D}_1$ of $K_{U^\star} - K_U$. Now note that $v - u^\star \geqslant m + 1$ and $u^\star \geqslant m$ from our hypotheses, and that $(u^\star, v)$ is m-admissible because $(u, u^\star)$ and (u, v) are m-admissible. Thus, by Lemma 2.25, there is an m-cycle decomposition $\mathcal{D}_2$ of $K_V - K_{U^\star}$. Then $\mathcal{D}_1 \cup \mathcal{D}_2$ is an m-cycle decomposition of $K_V - K_U$. □

Proof of Theorem 2.2. We first note that Theorem 2.2 is equivalent to showing that for $u \geqslant m - 2$ and $v - u \geqslant m + 1$, there exists an m-cycle decomposition of $K_v - K_u$ if and only if (u, v) is m-admissible (note that $v - u \geqslant m + 1$ guarantees that $(v - m)(v - 1) \geqslant u(u - 1)$). By Lemma 1.11, if there exists an m-cycle decomposition of $K_v - K_u$, then (u, v) is m-admissible. So it is sufficient to prove that for any m-admissible pair (u, v) of integers such that $u \geqslant m - 2$ and $v - u \geqslant m + 1$, there exists an m-cycle decomposition of $K_v - K_u$. By Theorem 1.18, this is established for $m \leqslant 7$ (see [26, 40, 75]), so we can suppose that $m \geqslant 9$. By Lemma 2.25 there exists an m-cycle decomposition of $K_v - K_u$ if either $m \geqslant 17$ and $u \geqslant m - 2$ or $m \in \{9, 11, 13, 15\}$ and $u \geqslant m$, so we can further suppose that $m \in \{9, 11, 13, 15\}$ and $u = m - 2$.

Because $u = m - 2$ and $m \in \{9, 11, 13, 15\}$, it follows from $v - u \geqslant m + 1$ that $v \geqslant 2m - 1$. Furthermore, it is routine to check that (N1) and (N2) of Lemma 1.11 imply that $v \equiv 3 \pmod{2m}$ or $v \equiv m - 2 \pmod{2m}$. Provided that there exists an m-cycle decomposition of $K_{2m+3} - K_{m-2}$, by Lemma 2.26 there exists an m-cycle decomposition of $K_v - K_{m-2}$ if $v \geqslant 4m + 3$. Thus, because $v \geqslant 2m - 1$, it suffices to show that there is an m-cycle decomposition of $K_{3m-2} - K_{m-2}$ and of $K_{2m+3} - K_{m-2}$. By Theorem 1.15 (with $q = 1$), there exists an m-cycle decomposition of $K_{3m-2} - K_{m-2}$ for each $m \geqslant 3$. For each $m \in \{9, 11, 13, 15\}$, we have an m-cycle decomposition of $K_{2m+3} - K_{m-2}$ using a computer program that implements basic cycle switching techniques to augment decompositions. These decompositions are included in Appendix A. □

Proof of Corollary 2.3. Part (i) follows from Lemma 2.26 (with $u^\star = \omega_m(u)$). Part (ii) follows from Theorem 2.2. For $u = 1$ part (iii) follows from Theorem 1.4 and for $u = 3$ it follows by removing a 3-cycle from a decomposition of a complete graph into m-cycles and a single 3-cycle which also exists by Theorem 1.4. If $u > 3$, then part (iii) follows from Theorem 2.2, noting that $u > \frac{(m-1)(m-2)}{2}$ implies that $\frac{u(m+1)}{m-1} + 1 > u + m - 1$. □

Now we shall prove Theorem 2.1. Theorem 2.2 can be shown to cover the exceptions to Theorem 1.16, and as a consequence we can completely solve the embedding problem for m-cycle systems in the case where m is an odd prime power.

Lemma 2.27. *Let m be an odd prime power. For positive integers u and v with $u < v$, an m-cycle system of order u can be embedded in an m-cycle system of order v if and only if $u, v \equiv 1$ or $m \pmod{2m}$ and $v \geqslant \frac{u(m+1)}{m-1} + 1$.*

Proof. Let $m = p^n$ for some odd prime p and some integer $n \geqslant 1$. If there exists an m-cycle system of order v containing a subsystem of order u then p^n divides $\binom{u}{2} = \frac{u(u-1)}{2}$ and $\binom{v}{2} = \frac{v(v-1)}{2}$. Since p cannot divide both u and $u-1$, then $u \equiv 1$ or $m \pmod{2m}$ and by a similar argument $v \equiv 1$ or $m \pmod{2m}$. Also note that $v \geqslant \frac{(m+1)u}{m-1} + 1$ by Lemma 1.11.

Conversely, suppose that $v \geqslant \frac{(m+1)u}{m-1} + 1$ and $u, v \equiv 1$ or $m \pmod{2m}$. If $u \equiv v \equiv m \pmod{2m}$, then $u \geqslant m$, $v - u \geqslant 2m$ and the result follows by Theorem 2.2. Otherwise either $u \equiv 1 \pmod{2m}$ or $v \equiv 1 \pmod{2m}$ and the result follows directly from Theorem 1.16. □

Proof of Theorem 2.1. Part (i) follows directly from Theorem 1.4, Corollary 2.3(iii) and Lemma 2.27. Part (ii) follows from Theorem 1.4 and Corollary 2.3(ii) (note that an m-cycle system of order one is trivially embedded in any m-cycle system and that any nontrivial m-cycle system has order at least m). □

Finally we shall prove Theorem 2.4.

Proof of Theorem 2.4. By Theorem 1.13 and Corollary 2.3 it is sufficient to find m-cycle decompositions of $K_v - K_u$ when $m \leqslant 15$ is odd, (u, v) is m-admissible and either

- $u < m - 2$ and $v \leqslant \omega_m(u) + m - 1$; or
- $m - 2 \leqslant u \leqslant \frac{(m-1)(m-2)}{2}$ and $v \leqslant u + m - 1$.

By Theorem 1.16 we also know that, for pairs (u, v) such that $u \equiv 1 \pmod{2m}$ and $v \equiv m \pmod{2m}$, there exists an m-cycle decomposition of $K_v - K_u$ when $v \geqslant \frac{u(m+1)}{m-1} + 1$. Finally, by Theorem 1.15 with $q = 1$ we know that there exists an m-cycle decomposition of $K_v - K_u$ when $v - u = 2m$ and $u \leqslant m - 1$. So simple calculation reveals that it suffices to find an m-cycle decomposition of $K_v - K_u$ for the values of m, u and v given in the following table.

m	(u, v)
9	$(5, 11)$, $(5, 17)$, $(11, 17)$, $(17, 23)$
11	$(5, 29)$, $(7, 27)$, $(13, 21)$, $(25, 31)$, $(35, 43)$
13	$(5, 35)$, $(9, 31)$, $(15, 25)$, $(29, 37)$, $(41, 51)$
15	$(5, 41)$, $(7, 19)$, $(7, 27)$, $(9, 19)$, $(9, 27)$, $(11, 35)$, $(15, 25)$, $(17, 29)$, $(21, 31)$, $(27, 37)$, $(27, 39)$, $(33, 43)$, $(37, 49)$, $(39, 49)$, $(45, 55)$, $(47, 59)$, $(49, 57)$, $(51, 61)$, $(57, 67)$, $(57, 69)$, $(63, 73)$, $(67, 79)$, $(77, 89)$

We have found the desired decompositions in each of these cases using the same computer program that was used to prove the cases in Theorem 2.2. These decompositions are included in Appendix A. $\square$

Chapter 3

Cycles of Arbitrary Lengths

The focus of this chapter is on decomposing the complete graph with a hole into cycles of arbitrary specified lengths. The main result here is that the complete graph of order v with a hole of size u can be decomposed into cycles of lengths $m_1, m_2, \ldots, m_\tau$ provided that the obvious necessary conditions are satisfied, $v - u \geqslant 10$, and $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_\tau \leqslant \min(u, v-u, 3m_{\tau-1})$. This generalises results from Chapter 2 for uniform length cycles. Specifically, if $m_i = m$ for $i \in \{1, \ldots, \tau\}$ then Theorem 3.1 specialises to Theorem 2.2 when m is odd and Theorem 1.14 when m is even. Theorem 3.1 also complements known results for decomposing various graphs into cycles of arbitrary specified lengths, namely Theorems 1.4, 1.9 and 1.21.

Theorem 3.1. *Let u and v be integers with $v - u \geqslant 10$, and let $m_1, \ldots, m_\tau$ be a nondecreasing list of integers such that $m_\tau \leqslant \min(u, v-u, 3m_{\tau-1})$. There exists a decomposition of $K_v - K_u$ into cycles of lengths $m_1, \ldots, m_\tau$ if and only if*

(i) u and v are odd;

(ii) $m_1 \geqslant 3$;

(iii) $m_1 + \cdots + m_\tau = \binom{v}{2} - \binom{u}{2}$; and

(iv) there are at most $\binom{v-u}{2}$ odd entries in $m_1, \ldots, m_\tau$.

It is not difficult to see that conditions (i)-(iv) in Theorem 3.1 are necessary for the existence of a decomposition of $K_v - K_u$ into cycles of lengths $m_1, \ldots, m_\tau$. We establish this in Lemma 3.2. Note that if $m_1 = m_2 = \cdots = m_\tau$, then Lemma 3.2 specialises to Lemma 1.11, which is the analogous result for cycles of uniform length.

Lemma 3.2. *Let $m_1, \dots, m_\tau$ be a nondecreasing list of integers and let u and v be positive integers. If there exists a decomposition of $K_v - K_u$ into cycles of lengths $m_1, \dots, m_\tau$ then*

(i) *u and v are odd;*

(ii) *$m_1 \geqslant 3$ and $m_\tau \leqslant \min(v, 2(v-u))$;*

(iii) *$m_1 + \cdots + m_\tau = \binom{v}{2} - \binom{u}{2}$;*

(iv) *there are at most $\binom{v-u}{2}$ odd entries in $m_1, \dots, m_\tau$; and*

(v) *$\tau \geqslant \frac{v-1}{2}$.*

Proof. Suppose there exists a decomposition of $K_v - K_u$ into cycles of lengths $m_1, \dots, m_\tau$. Since the degree of each vertex must be even, we have $v-u$ and $v-1$ are even so (i) follows. Clearly $m_1 \geqslant 3$ and $m_\tau \leqslant v$. Also, every cycle has at least half of its vertices outside the hole, so $m_\tau \leqslant 2(v-u)$ and thus (ii) follows. Condition (iii) clearly holds. Any odd cycle in $K_v - K_u$ must contain at least one edge that is not incident with a vertex in the hole, thus (iv) follows. Finally, a fixed vertex outside the hole must be in at least $\frac{v-1}{2}$ cycles, so (v) follows. □

The remainder of this chapter is concerned with proving the existence of cycle decompositions of $K_v - K_u$. Our general approach is to first construct decompositions of $K_v - K_u$ that contain collections of short cycles and then merge these cycles together to construct longer cycles. This method is similar to that used in Chapter 2.

3.1 Merging two cycle lengths

The aim of this section is to prove Lemma 3.3 which, given a cycle packing of $K_v - K_u$ with certain properties, allows us to obtain a repacking where two cycle lengths have been merged into a single cycle. This lemma plays a similar role here to Lemma 2.5 in Chapter 2. In Chapter 2 we applied Lemma 2.5 to produce two odd cycles each containing one pure edge and so we only considered leaves containing two pure edges. Here, however, we might also wish to produce two even cycles containing no pure edges, or one odd cycle containing one pure edge and one even cycle containing no pure edges. Thus we must consider leaves with 0, 1 and 2 pure edges.

Lemma 3.3. *Let $u \geqslant 5$ and v be odd integers such that $v > u$, and let M be a list of integers. Suppose there exists an (M)-packing $\mathcal{P}$ of $K_v - K_u$ with a reduced leave that has exactly μ pure edges, where $\mu \in \{0, 1, 2\}$, and has a* *decomposition into an h-cycle, an m_1-cycle and an m_2-cycle where h is odd if $\mu = 2$. If $m_1 + m_2 \leqslant 3h$ and $m_1 + m_2 + h \leqslant \min(2u + 3, 2(v - u) + 1, v)$, then there exists a repacking of $\mathcal{P}$ whose reduced leave has a decomposition into an h-cycle and an $(m_1 + m_2)$-cycle each containing at most one pure edge.*

The work required to prove the $\mu = 0$ case of this lemma is done in the proof of Lemma 2.21 (see [64]).

Lemma 3.4. *Let u and v be positive odd integers, and let M be a list of integers. Suppose there exists an (M)-packing $\mathcal{P}$ of $K_v - K_u$ with a reduced leave that has no pure edges, and has a decomposition into an h-cycle, an m-cycle and an m'-cycle. If $m+m' \leqslant 3h$ and $m+m'+h \leqslant 2\min(u+1, v-u+1)$, then there exists a repacking of $\mathcal{P}$ whose reduced leave has a decomposition into an h-cycle and an $(m + m')$-cycle.*

Proof. This lemma is very similar to Lemma 2.21, but it applies to packings of $K_v - K_u$ rather than to packings of $K_{u,v-u}$ (the leaves of the packings are still subgraphs of $K_{u,v-u}$). It can be proved exactly as per the proof of Lemma 2.21, except that Lemma 1.22 is used in place of [64, Lemma 2.1]. Note that two vertices of $K_v - K_u$ are twin if and only if they are both in the hole or both outside it. Also note that $u \neq v - u$ because u and v are odd. □

It remains to prove Lemma 3.3 in the cases $\mu = 1$ and $\mu = 2$. To help with this task we first prove two useful lemmas. Lemma 3.6 below extends Lemma 2.6 to include leaves with one pure edge.

Lemma 3.5. *Let G be an even graph, let y and z be twin vertices in G, and let $\mathcal{P}$ be an (M)-packing of G with (unreduced) leave L. If $\deg_L(y) > \deg_L(z)$, then there is an (M)-packing $\mathcal{P}'$ of G with an (unreduced) leave L' such that $\deg_{L'}(y) = \deg_L(y) - 2$, $\deg_{L'}(z) = \deg_L(z) + 2$, and $\deg_{L'}(x) = \deg_L(x)$ for each $x \in V(G) \setminus \{y, z\}$. Furthermore the number of nontrivial components in L' is at most one greater than the number of nontrivial components in L.*

Proof. Let $\mathcal{P}'$ be the repacking of $\mathcal{P}$ obtained by applying a (y, z)-switch whose origin and terminus are both adjacent to y in L. Such a switch exists because $\deg_L(y) > \deg_L(z)$. The result can be seen to follow by examining the differences between L' and L. □

Lemma 3.6. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even, and suppose that L is a subgraph of $K_{U\cup W} - K_U$ such that L contains exactly μ pure edges, where $\mu \in \{1, 2\}$, and each vertex of L has positive even degree.*

- *(i) If $|E(L)| \leqslant 2(|U| + 1)$ and U contains a vertex of degree at least 4 in L, then there is a vertex x in U such that $x \notin V(L)$.*

(ii) *If* $|E(L)| \leqslant \min(2(|U| + 2), 2|W| + 1)$ *and* S *is an element of* $\{U, W\}$ *such that* S *contains either at least two vertices of degree* 4 *in* L *or at least one vertex of degree at least* 6 *in* L*, then there is a vertex* x *in* S *such that* $x \notin V(L)$*.*

(iii) *If* $|E(L)| \leqslant \min(2(|U|+2), 2|W|+1, |U|+|W|)$ *and* L *contains either at least two vertices of degree* 4 *or at least one vertex of degree at least* 6*, then there are twin vertices* x *and* y *in* $K_{U\cup W} - K_U$ *such that* $\deg_L(x) \geqslant 4$ *and* $y \notin V(L)$*.*

Proof. Let $\ell = |E(L)|$ and note that $\ell \equiv \mu \pmod 2$. If $\mu = 2$ then the result follows by Lemma 2.6. So suppose that $\mu = 1$. Then we have

$$\sum_{x\in V(L)\cap U} \deg_L(x) = \ell - 1, \text{ and}$$
$$\sum_{x\in V(L)\cap W} \deg_L(x) = \ell + 1.$$

Proof of (i). Suppose that $\ell \leqslant 2(|U| + 1)$ and U contains a vertex of degree at least 4 in L. Then $\ell \leqslant 2|U| + 1$ since ℓ is odd. If $U \subseteq V(L)$ then $\ell - 1 = \sum_{x\in V(L)\cap U} \deg_L(x) \geqslant 2|U| + 2$ which contradicts $\ell \leqslant 2|U| + 1$.
Proof of (ii). Suppose that $\ell \leqslant \min(2(|U| + 2), 2|W| + 1, |U| + |W|)$ and S is an element of $\{U, W\}$ such that S contains either at least two vertices of degree 4 in L or at least one vertex of degree at least 6 in L. Suppose for a contradiction that $S \subseteq V(L)$. Then we have $\sum_{x\in V(L)\cap S} \deg_L(x) \geqslant 2|S| + 4$. So, if $S = U$, then $\ell - 1 \geqslant 2|U| + 4$, contradicting $\ell \leqslant 2|U| + 3$. If $S = W$, then $\ell + 1 \geqslant 2|W| + 4$, contradicting $\ell \leqslant 2|W| + 1$.
Proof of (iii). Because we have proved (ii), it only remains to show that if L contains two vertices of degree 4, one in U and one in W, and every other vertex of L has degree 2, then there are twin vertices x and y in $K_{U\cup W} - K_U$ such that $\deg_L(x) \geqslant 4$ and $y \notin V(L)$. Suppose otherwise. Then it must be the case that $V(L) = U \cup W$, $\ell - 1 = 2|U| + 2$ and $\ell + 1 = 2|W| + 2$. But then $\ell = 2|U| + 3$ and $\ell = 2|W| + 1$, so $|U \cup W| = 2|U| + 1$ and $\ell = |U| + |W| + 2$, contradicting $\ell \leqslant |U| + |W|$. □

3.1.1 Proof of Lemma 3.12

In order to prove Lemma 3.3 we will require Lemma 3.12 which is an analogue of Lemma 2.17 for packings whose leaves have one pure edge. The proof of Lemma 3.12 proceeds as follows. Lemmas 2.9 and 3.7 are used in proving Lemma 3.8, which gives conditions under which we can repack to transform a 2-chain leave into a union of two cycles of specified lengths. Lemma 3.8

then acts as a base case and is used, along with Lemmas 3.9 and 3.10, in an induction proof of Lemma 3.11. Lemma 3.11 gives conditions under which we can repack to transform a good s-chain or s-ring leave into a union of two cycles of specified lengths. Finally Lemma 3.12 is proved from Lemma 3.11.

Lemma 3.7. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even, and let M be a list of integers. Let m, p and q be positive integers with m odd and $m, p+q-m \geqslant 3$. Suppose there exists an (M)-packing $\mathcal{P}$ of $K_{U \cup W} - K_U$ whose reduced leave L is a (p,q)-chain $(x_1, x_2, \ldots, x_{p-1}, x_0) \cdot (x_0, y_1, y_2, \ldots, y_{q-1})$ such that L contains exactly one pure edge, namely $x_r x_{r+1}$ for some $r \in \{0, \ldots, p-1\}$ (subscripts modulo p). If $p \leqslant m$, or if $p \geqslant m+2$ and $r \leqslant m-3$, then there exists a repacking of $\mathcal{P}$ whose reduced leave is the edge-disjoint union of an m-cycle and a $(p+q-m)$-cycle.*

Proof. The proof relies on several applications of Lemma 1.22. We consider the case when $p \leqslant m$ and the case $p \geqslant m+2$ and $r \leqslant m-3$ separately. Note that since the p-cycle in L contains exactly one pure edge and the q-cycle contains no pure edges, then p is odd and q is even.

Case 1. First suppose that $p \leqslant m$. If $p = m$ then we are done so assume $p < m$. Without loss of generality, assume $x_0 x_1$ is not a pure edge (otherwise relabel vertices in L). Then the result follows by Lemma 2.9(i) because $[x_1, x_0, y_1, \ldots, y_{m-p+2}]$ is a path with no pure edges and hence the vertices $x_1, y_3, y_5, \ldots, y_{m-p+1}$ are pairwise twin in $K_{U \cup W} - K_U$ and $y_2, y_4, \ldots, y_{m-p+2}$ are pairwise twin in $K_{U \cup W} - K_U$.

Case 2. Now assume that $p \geqslant m+2$ and $r \leqslant m-3$. Then by a simple induction it is sufficient to obtain either the required decomposition, or a $(p-2, q+2)$-chain $(x'_1, x'_2, \ldots, x'_{p-3}, x'_0) \cdot (x'_0, y'_1, y'_2, \ldots, y'_{q+1})$ that contains exactly one pure edge $x'_r x'_{r+1}$ for some $r \in \{0, \ldots, m-3\}$.

Let $\mathcal{P}'$ be the repacking of $\mathcal{P}$ obtained by performing the (y_1, x_{m-1})-switch with origin x_0. Note that $\{y_1, x_{m-1}\}$ and $\{x_m, x_{m-2}\}$ are twin pairs in $K_{U \cup W} - K_U$ because $r \leqslant m-3$ and hence $[x_{m-2}, x_{m-1}, \ldots, x_{p-1}, x_0, y_1]$ is a path with no pure edges. If the terminus of the switch is not x_{m-2} then the reduced leave of $\mathcal{P}'$ has a decomposition into an m-cycle and a $(p+q-m)$-cycle and we are done. So assume that the terminus is x_{m-2}. Then the reduced leave of $\mathcal{P}'$ is the $(q+m-2, p-m+2)$-chain $(x_1, x_2, \ldots, x_{m-2}, y_1, y_2, \ldots, y_{q-1}, x_0) \cdot (x_0, x_{m-1}, x_m, \ldots, x_{p-1})$.

Let $\mathcal{P}''$ be the repacking of $\mathcal{P}'$ obtained by performing the (x_m, x_{m-2})-switch with origin x_{m-1}. If the terminus of the switch is not x_{m-3}, then the reduced leave of $\mathcal{P}''$ has a decomposition into an m-cycle and a $(p+q-m)$-cycle and we are done. Otherwise, the terminus is x_{m-3} and the reduced leave of $\mathcal{P}''$ is the $(p-2, q+2)$-chain $(x_1, \ldots, x_{m-3}, x_m, x_{m+1}, \ldots, x_{p-1}, x_0) \cdot (x_0, x_{m-1}, x_{m-2}, y_1, y_2, \ldots, y_{q-1})$ where $x_r x_{r+1}$ is a pure edge. $\square$

Lemma 3.8. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even, and let M be a list of integers. Let m, p and q be positive integers with m odd and $m, p+q-m \geqslant 3$. Suppose there exists an (M)-packing $\mathcal{P}$ of $K_{U\cup W} - K_U$ whose reduced leave L is a (p,q)-chain such that L contains exactly one pure edge and the link vertex of L is in W if $m = 3$. Then there exists a repacking of $\mathcal{P}$ whose reduced leave is the edge-disjoint union of an m-cycle and a $(p+q-m)$-cycle.*

Proof. Since L contains exactly one pure edge, $p+q$ must be odd. Without loss of generality suppose p is odd. Then the p-cycle in L contains the pure edge and the q-cycle in L contains no pure edges.

Case 1. Suppose either that $p \leqslant m$ or that $p \geqslant m+2$ and L can be labelled $(x_1, x_2, \ldots, x_{p-1}, x_0) \cdot (x_0, y_1, y_2, \ldots, y_{q-1})$ such that $x_r x_{r+1}$ is the pure edge for some $r \in \{0, \ldots, m-3\}$. Then the result follows by Lemma 3.7.

Case 2. Suppose that $p \geqslant m+2$ and there is no such labelling. Let L be labelled $(x_1, x_2, \ldots, x_{p-1}, x_0) \cdot (x_0, y_1, y_2, \ldots, y_{q-1})$ such that $x_r x_{r+1}$ is the pure edge for some $r \in \{m-2, \ldots, p-1\}$ (subscripts modulo p). Then $r \geqslant 2$, using the fact that $x_0 \in W$ if $m = 3$. It is sufficient to show that there exists a repacking $\mathcal{P}'$ of $\mathcal{P}$ whose reduced leave is either a (p,q)-chain that can be labelled as $(x'_1, x'_2, \ldots, x'_{p-1}, x'_0) \cdot (x'_0, y'_1, y'_2, \ldots, y'_{q-1})$ where the pure edge is $x'_{r-2} x'_{r-1}$, or a $(p-2, q+2)$-chain. By repeating this process we eventually obtain a repacking of $\mathcal{P}$ which satisfies the criteria of Case 1.

Let $\mathcal{P}'$ be the repacking of $\mathcal{P}$ obtained by performing the (x_0, x_2)-switch with origin x_3. Note that x_0 and x_2 are twin in $K_{U\cup W} - K_U$ because $r \geqslant 2$ and hence $[x_0, x_1, x_2]$ is a path with no pure edges. If the terminus of the switch is x_{p-1}, then the reduced leave of $\mathcal{P}'$ is the (p,q)-chain $(x_3, x_4, \ldots, x_{p-1}, x_2, x_1, x_0) \cdot (x_0, y_1, y_2, \ldots, y_{q-1})$ and we are done. If the terminus of the switch is not x_{p-1} then the reduced leave of $\mathcal{P}'$ is a $(p-2, q+2)$-chain. □

Lemma 3.9. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even, and let M be a list of integers. Let p and s be integers such that $p \geqslant 4$ and $s \geqslant 2$. Suppose there exists an (M)-packing $\mathcal{P}$ of $K_{U\cup W} - K_U$ whose reduced leave L is a good s-chain that contains exactly one pure edge and has a decomposition $\{P, L-P\}$ into two paths such that P has length p and each path has both end vertices in W. Suppose further that P has a subpath $P_0 = [x_0, \ldots, x_r]$ such that $2 \leqslant r \leqslant p-1$, x_0 is an end vertex of P, P_0 contains no pure edge, and $\deg_L(x_{r-1}) = \deg_L(x_r) = 2$. Then there is a repacking of $\mathcal{P}$ whose reduced leave L' is a good s-chain that has a decomposition $\{P', L'-P'\}$ into two paths such that P' has length $p-2$, each path has both end vertices in W, and P' contains a pure edge if and only if P does.*

Proof. Label the vertices in $V(P) \setminus V(P_0)$ so that $P = [x_0, \ldots, x_p]$. We prove the result by induction on the length of P_0. If $|E(P_0)| = 2$, then $\{P', L-P'\}$ where $P' = [x_2, \ldots, x_p]$ is a decomposition of L with the required properties.

So we can assume that $|E(P_0)| \geqslant 3$. By induction we can assume that P_0 is the shortest subpath of P satisfying the required conditions. Because $r \geqslant 3$, this implies that $\deg_L(x_{r-2}) = 4$ and x_{r-2} is a link vertex of L.

The vertices x_r and x_{r-2} are twin in $K_{U\cup W} - K_U$ because $[x_{r-2}, x_{r-1}, x_r]$ is a path with no pure edges. Let L' be the reduced leave of the repacking of $\mathcal{P}$ obtained by performing the (x_r, x_{r-2})-switch with origin x_{r-3}. Note that L' is a good s-chain irrespective of the terminus of the switch. If the terminus of the switch is not x_{r+1}, then $\{P', L' - P'\}$ where $P' = [x_0, x_1, \ldots, x_{r-3}, x_r, x_{r+1}, \ldots, x_p]$ is a decomposition of L' such that P' has length $p-2$, each path has both end vertices in W, and P' contains a pure edge if and only if P does. If the terminus of the switch is x_{r+1}, then $\{P', L' - P'\}$ where $P' = [x_0, x_1, \ldots, x_{r-3}, x_r, x_{r-1}, x_{r-2}, x_{r+1}, x_{r+2}, \ldots, x_p]$ is a decomposition of L' into two paths such that P' has length p and contains a pure edge if and only if P does, and each path has both end vertices in W. Further P' has the subpath $P'_0 = [x_0, \ldots, x_{r-3}, x_r, x_{r-1}]$ and we know that x_0 is an end vertex of P', P'_0 contains no pure edge, and $\deg_{L'}(x_r) = \deg_{L'}(x_{r-1}) = 2$. Thus, because $|E(P'_0)| = r - 1$, we are finished by our inductive hypothesis. □

Lemma 3.10. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even, and let M be a list of integers. Let m, m' and s be integers such that $m + m'$ is odd, $m, m' \geqslant \max(s, 3)$ and $s \geqslant 2$. Suppose there exists an (M)-packing $\mathcal{P}$ of $K_{U\cup W} - K_U$ whose reduced leave is a good s-chain of size $m+m'$ that contains exactly one pure edge. Then there exists a repacking of $\mathcal{P}$ whose reduced leave is a good s-chain that has a decomposition into an m-path and an m'-path such that the end vertices of the paths are twin in $K_{U\cup W} - K_U$.*

Proof. Let L be the reduced leave of $\mathcal{P}$ and note that $|E(L)| = m + m'$. Because L is good and contains exactly one pure edge, we can find some decomposition $\{H, L - H\}$ of L into two paths such that H has odd length and contains the pure edge, $L - H$ has even length, and each of the paths has both end vertices in W. Let $m^* \in \{m, m'\}$ and $P \in \{H, L - H\}$ such that $|E(P)| \geqslant m^*$ and $|E(P)| \equiv m^* \pmod 2$ (such an m^* and P exist because $|E(L)| = m + m'$). If $|E(P)| = m^*$ then we are done, so suppose $|E(P)| > m^*$. Let $p = |E(P)|$.

Case 1. Each cycle of L contains at most two edges of P. Then exactly $p - s$ cycles of the chain contain two edges of P and the rest contain one edge of P. Because L is good and both end vertices of P are in W, if C is a cycle of L that contains two edges of P then C is an end cycle of L, the link vertex of C is in W, and $C \cap P$ contains no pure edges. Thus, because $p > m^* \geqslant s$, it must be that $p = s + 2$ and $m^* = s$. Then $\{P', L - P'\}$, where P' is obtained from P by removing the end vertices and the incident edges, is a decomposition of L into an m-path and an m'-path such that both end vertices of each path are in U.

Case 2. There exists a cycle C in L such that $C \cap P$ is a path of length at least 3. Let $P_0 = [x_0, \ldots, x_r]$ be a subpath of P such that x_0 is an end vertex of P, P_0 contains no pure edge, and P_0 contains exactly two edges in $C \cap P$. If $C \cap P$ contains no pure edge then it is easy to see such a subpath exists. If $C \cap P$ contains the pure edge then (since L is good) C is an end cycle of L whose link vertex is in W and again there exists such a subpath. So we can apply Lemma 3.9 to obtain a repacking of $\mathcal{P}$ whose reduced leave L' is a good s-chain that has a decomposition $\{P', L' - P'\}$ into two paths such that P' has length $p-2$, P' has a pure edge if and only if P does, and both paths have end vertices in W. It is clear that by repeating this procedure we will eventually obtain a repacking of $\mathcal{P}$ whose reduced leave either has a decomposition into two paths which satisfies the criteria for Case 1 or has a decomposition into an m-path and an m'-path such that both end vertices of each path are in W. $\square$

Lemma 3.11. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even, and let M be a list of integers. Let m, m' and s be positive integers such that $s \geqslant 2$, $m+m'$ is odd, $m, m' \geqslant \max(s, 3)$, $m+m' \leqslant \min(2|U|+3, 2|W|+1, |U|+|W|)$, and $m + m' \leqslant 2|U| + 1$ if $3 \in \{m, m'\}$. Suppose there exists an (M)-packing $\mathcal{P}$ of $K_{U \cup W} - K_U$ whose reduced leave has size $m + m'$, contains exactly one pure edge, and is either a good s-chain or a good s-ring that, if $3 \in \{m, m'\}$, is not a 2-chain with link vertex in U. Then there exists a repacking of $\mathcal{P}$ whose reduced leave is the edge-disjoint union of an m-cycle and an m'-cycle.*

Proof. Let L be the reduced leave of $\mathcal{P}$. We first show that the result holds for $s = 2$. If L is a 2-chain then the result follows by Lemma 3.8. If L is a 2-ring then by our hypotheses and Lemma 3.6 there are twin vertices x and y in $K_{U \cup W} - K_U$ such that $\deg_L(x) \geqslant 4$, $y \notin V(L)$ and $x, y \in U$ if $3 \in \{m, m'\}$ (if $3 \in \{m, m'\}$ then apply Lemma 3.6(i), otherwise apply Lemma 3.6(iii)). Performing an (x, y)-switch results in a repacking of $\mathcal{P}$ whose reduced leave is a 2-chain with link vertex in W if $3 \in \{m, m'\}$, and the result follows by Lemma 3.8. So the result holds for $s = 2$ and it is sufficient to show, for each integer $s' \geqslant 3$, that if the result holds for $s = s' - 1$ then it holds for $s = s'$.

Case 1. Suppose that L is a good s'-chain. By Lemma 3.10 we can obtain a repacking of $\mathcal{P}$ with a reduced leave whose only component is a good s'-chain with a decomposition into paths of length m and m' whose end vertices are twin. Let $[x_0, x_1, \ldots, x_m]$ be the path of length m and let $\mathcal{P}'$ be the repacking of $\mathcal{P}$ obtained by performing the (x_0, x_m)-switch with origin x_1.

If the terminus of the switch is not x_{m-1}, then the reduced leave of $\mathcal{P}'$ is the edge-disjoint union of an m-cycle and an m'-cycle and we are done. If the terminus of the switch is x_{m-1}, then the reduced leave of $\mathcal{P}'$ is a good $(s'-1)$-ring that contains exactly one pure edge and the result follows by our inductive hypothesis.

Case 2. Suppose that L is a good s'-ring. Let A be the ring cycle of L that contains the pure edge in L and note that if s' is odd then both link vertices of A are in W. Let x and y be twin vertices in $K_{U \cup W} - K_U$ such that x is a link vertex in A, $x \in U$ if s' is even, and $y \notin V(L)$. Such a vertex y exists by Lemma 3.6(ii) because $m + m' \leqslant \min(2|U| + 3, 2|W| + 1, |U| + |W|)$, W contains two vertices of degree 4 in L if s' is odd, and U contains two vertices of degree 4 in L if s' is even (for then $s' \geqslant 4$). Let $\mathcal{P}'$ be the repacking of $\mathcal{P}$ obtained by performing an (x, y)-switch with origin in A. If the terminus of this switch is also in A, then the reduced leave of $\mathcal{P}'$ is a good s'-chain and we can proceed as in Case 1. Otherwise, the reduced leave of $\mathcal{P}'$ is a good $(s' - 1)$-ring and the result follows by our inductive hypothesis. □

Lemma 3.12. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even, and let M be a list of integers. Let m, m', k and t be positive integers such that $m, m' \geqslant \max(k + t - 1, 3)$, $m + m' \leqslant \min(2|U| + 3, 2|W| + 1, |U| + |W|)$, and $m + m' \leqslant 2|U| + 1$ if $3 \in \{m, m'\}$. Suppose there exists an (M)-packing $\mathcal{P}$ of $K_{U \cup W} - K_U$ with a reduced leave L of size $m + m'$ such that L contains exactly one pure edge and L has exactly k components, $k - 1$ of which are cycles and one of which is a good t-chain that, if $3 \in \{m, m'\}$, is not a 2-chain with link vertex in U. Then there exists a repacking of $\mathcal{P}$ whose reduced leave is the edge-disjoint union of an m-cycle and an m'-cycle.*

Proof. First note that, since L contains exactly one pure edge and L has a decomposition into cycles, $m + m'$ is odd. Without loss of generality let m be odd and m' be even.

By Lemma 3.11 it is sufficient to show that we can construct a repacking of $\mathcal{P}$ whose reduced leave is a good s-chain for some $s \in \{2, \ldots, k + t - 1\}$ and is not a 2-chain with link vertex in U if $m = 3$. If $k = 1$, then we are finished, so we can assume $k \geqslant 2$. By induction on k, it suffices to show that there is a repacking of $\mathcal{P}$ with a reduced leave with exactly $k - 1$ components, one of which is a good t'-chain for $t' \in \{t, t + 1\}$ that contains a link vertex in W if $m = 3$ and the remainder of which are cycles.

Let H be the component of L which is a good t-chain, and let C be a component of L such that C is a cycle and C contains the pure edge if H does not. Let H_1 and H_t be the end cycles of H where H_1 contains the pure edge if H does and the link vertex of H_1 is in W if $t \geqslant 3$.

Case 1. Suppose that either $t \geqslant 3$ or it is the case that $t = 2$, H_1 contains a pure edge and the link vertex of H is in W. Then let x and y be vertices such that $x \in V(H_t)$, x is not a link vertex of H, $y \in V(C)$, $x, y \in W$ if t is odd, and $x, y \in U$ if t is even. Let $\mathcal{P}'$ be a repacking of $\mathcal{P}$ obtained by performing an (x, y)-switch with origin in H_t. The reduced leave of $\mathcal{P}'$ has exactly $k - 1$ components, $k - 2$ of which are cycles and one of which is a good t'-chain, where $t' = t+1$ if the terminus of the switch is also in H_t and $t' = t$ otherwise. Also, H' has a link vertex in W because H does. So we are finished by our inductive hypothesis.

Case 2. Suppose that $t = 2$ and H contains no pure edge. Then C contains the pure edge. Let x_1 and x_2 be vertices such that $x_1 \in V(C) \cap W$, $x_2 \in V(H_1) \cap W$, and x_2 is not the link vertex of H. Let $\mathcal{P}'$ be a repacking of $\mathcal{P}$ obtained by performing an (x_1, x_2)-switch with origin in H_1 and let L' be the reduced leave of $\mathcal{P}'$. If the terminus of this switch is in C, then L' has exactly $k-1$ components, $k-2$ of which are cycles and one of which is a 2-chain with link vertex in W if $m = 3$. In this case we are finished by our inductive hypothesis. Otherwise the terminus of this switch is in H_1 and L' has exactly $k-1$ components, $k-2$ of which are cycles and one of which is a 3-chain H' such that one end cycle of H' contains a pure edge and has its link vertex in W. If H' is good, then we are again finished by our inductive hypothesis. Otherwise, it must be that both link vertices of H' are in W and we proceed as follows.

Let H'_1 and H'_3 be the end cycles of H' where H'_1 has the pure edge. Let $y_1, y_2 \in W$ be vertices such that y_1 is the link vertex in H'_3 and $y_2 \notin V(L')$ (note that y_2 exists by Lemma 3.6(ii)). Let $\mathcal{P}''$ be a repacking of $\mathcal{P}'$ obtained by performing a (y_1, y_2)-switch with origin in H'_3 and let L'' be the reduced leave of $\mathcal{P}''$. If the terminus of this switch is not in H'_3, then L'' has exactly $k-1$ components, $k-2$ of which are cycles and one of which is a 2-chain with link vertex in W. In this case we are finished by our inductive hypothesis. Otherwise, the terminus of this switch is in H'_3 and L'' has exactly k components, $k-1$ of which are cycles and one of which is a 2-chain that contains a pure edge and has its link vertex in W. In this case we can proceed as we did in Case 1.

Case 3. Suppose that $t = 2$, H_1 contains the pure edge and the link vertex of H is in U. Note that $m \geqslant 5$ by the hypotheses of the lemma. Let x be the link vertex of H and let y be a vertex in $V(C) \cap U$. Let $\mathcal{P}'$ be a repacking of $\mathcal{P}$ obtained by performing an (x, y)-switch with origin in H_2 and let L' be the reduced leave of $\mathcal{P}'$. If the terminus of this switch is in C or H_1, then L' has exactly $k-1$ components, $k-2$ of which are cycles and one of which is a 2-chain. In this case we are finished by our inductive hypothesis. Otherwise the terminus of this switch is in H_2 and L' has exactly k components, $k-1$ of which are cycles and one of which is a 2-chain that contains no pure edges. In this case we can proceed as we did in Case 2. $\square$

3.1.2 Proof of Lemma 3.3

Here we use Lemmas 2.17, 3.4 and 3.12 to prove Lemma 3.3 which was our main goal in this section. We first require two more simple results. Lemma 3.13 is an easy bound on the maximum number of components in the reduced leave of a packing, and Lemma 3.14 allows us to find a repacking whose reduced leave is a vertex-disjoint union of a single 2-chain and a collection of cycles. This subsection also contains Lemma 3.15, which is used only in the proof of Lemma 3.28.

Lemma 3.13. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even. If G is a subgraph of $K_{U\cup W} - K_U$ such that G contains μ pure edges, G has one vertex of degree 4, and each other vertex of G has degree 2, then G has at most $\left\lfloor \frac{1}{4}(|E(G)| + \mu)\right\rfloor - 1$ components.*

Proof. Because each vertex of G has even degree, G has a decomposition $\mathcal{D}$ into cycles. Since there are μ pure edges in G, at most μ cycles in $\mathcal{D}$ have length 3 and each other cycle in $\mathcal{D}$ has length at least 4. Thus $|E(G)| \geqslant 4(|\mathcal{D}|-\mu)+3\mu$ which implies $|\mathcal{D}| \leqslant \left\lfloor \frac{1}{4}(|E(G)| + \mu)\right\rfloor$. At least one component of G contains a vertex of degree 4 and hence contains two cycles, and each component of G contains at least one cycle. The result follows. □

In Lemma 3.14 and in some cases in the proof of Lemma 3.3 we require a packing whose reduced leave is the vertex-disjoint union of a 2-chain and a collection of cycles. In such a leave one vertex has degree 4 and the remaining vertices have degree 2. For an (M)-packing $\mathcal{P}$ of an even graph G we define

$$d(\mathcal{P}) = \frac{1}{2}\sum_{x\in V(L)}(\deg_L(x) - 2) = |E(L)| - |V(L)|,$$

where L is the reduced leave of $\mathcal{P}$. Note that $d(\mathcal{P}) \geqslant 0$ because L is even, and that if $d(\mathcal{P}) = 1$ then L has exactly one vertex of degree 4 and the remaining vertices have degree 2.

Lemma 3.14. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even, let M be a list of integers, and let $\mu \in \{1, 2\}$. Suppose there exists an (M)-packing $\mathcal{P}$ of $K_{U\cup W} - K_U$ with a reduced leave L such that $|E(L)| \leqslant \min(2(|U|+2), 2|W|+1, |U| + |W|)$, L has k components, L has exactly μ pure edges, and L has at least one vertex of degree at least 4. Then there exists a repacking $\mathcal{P}'$ of $\mathcal{P}$ with a reduced leave L' such that exactly one vertex x' of L' has degree 4, every other vertex of L' has degree 2, and L' has at most $k + d(\mathcal{P}) - 1$ components. Furthermore if $|E(L)| \leqslant 2|U| + 2$ and there is a vertex in W with degree at least 4 in L, then x' is in W.*

Proof. The proof is by induction on $d(\mathcal{P})$. Because L has at least one vertex of degree at least 4, $d(\mathcal{P}) \geqslant 1$. If $d(\mathcal{P}) = 1$, then we are finished immediately because one vertex of L has degree 4 and every other vertex has degree 2. So suppose that $d(\mathcal{P}) \geqslant 2$ and hence that L contains either at least two vertices of degree 4 or at least one vertex of degree at least 6.

Let $\mathcal{P}''$ be the repacking of $\mathcal{P}$ obtained by applying Lemma 3.5 with y and z chosen to be vertices in $U \cup W$ such that $\deg_L(y) \geqslant 4$, $z \notin V(L)$, $y, z \in U$ if such vertices exist in U, and $y, z \in W$ otherwise. These choices for y and z exist by Lemma 3.6(iii), and by Lemma 3.6(i) they will be in U unless $|E(L)| > 2|U| + 2$ or $\deg_L(x) = 2$ for all $x \in V(L) \cap U$. Note that $d(\mathcal{P}'') = d(\mathcal{P}) - 1$ and the reduced leave of $\mathcal{P}''$ has at most $k + 1$ components. Thus we can complete the proof by applying our inductive hypothesis. $\square$

Proof of Lemma 3.3. Note first that $m_1 + m_2 + h \equiv \mu \pmod 2$. If $\mu = 0$, then the result follows by Lemma 3.4. So suppose $\mu \in \{1, 2\}$.

Let U and W be disjoint sets of sizes u and $w = v - u$ and let $\mathcal{P}$ be a packing of $K_{U \cup W} - K_U$ satisfying the hypotheses of the lemma. Let L be the reduced leave of $\mathcal{P}$ and let k be the number of components of L (note that $k \leqslant 3$). Below we will sometimes wish to apply Lemma 2.17 or 3.12 with $m = h$ and $m' = m_1 + m_2$. Accordingly, we note that if $h = 3$ then $m_1 + m_2 + h \leqslant 2u + \mu$ because $m_1 + m_2 \leqslant 3h$, $m_1 + m_2 + h \equiv \mu \pmod 2$ and $u \geqslant 5$. We also note that if $\mu = 2$ then $m_1 + m_2 + h \leqslant 2w$ because $m_1 + m_2 + h \equiv \mu \pmod 2$.

Let C_1, C_2 and H be edge-disjoint cycles in L such that $V(H) = h$ and $|V(C_1)| + |V(C_2)| = m_1 + m_2$ (we do not assume that $|V(C_1)| = m_1$ and $|V(C_2)| = m_2$).

Case 1. Suppose that $k = 3$. Then the components of L are C_1, C_2 and H. Let x and y be vertices such that $x \in V(C_1) \cap W$ and $y \in V(C_2) \cap W$. By performing an (x, y)-switch we obtain a repacking of $\mathcal{P}$ whose reduced leave is either the edge-disjoint union of an h-cycle and an $(m_1 + m_2)$-cycle or the vertex-disjoint union of an h-cycle and a 2-chain of size $m_1 + m_2$ with link vertex in W. In the former case we are finished and in the latter case we apply Lemma 2.17 (if $\mu = 2$) or Lemma 3.12 (if $\mu = 1$) with $m = h$ and $m' = m_1 + m_2$.

Case 2. Suppose that $k \in \{1, 2\}$, that $(k, d(\mathcal{P})) \neq (1, m_1 + m_2)$ and that W contains a vertex of degree at least 4 in L if $h = 3$. Note that L must have a vertex of degree at least 4. Applying Lemma 3.14 to $\mathcal{P}$, we see that there is a repacking of $\mathcal{P}$ whose reduced leave L' is the vertex-disjoint union of a 2-chain and $k' - 1$ cycles for some $k' \leqslant k + d(\mathcal{P}) - 1$. Furthermore, the link vertex of the 2-chain is in W if $h = 3$. If we can show that $h, m_1 + m_2 \geqslant k' + 1$, then we can complete the proof by applying Lemma 2.17 or Lemma 3.12 with $m = h$ and $m' = m_1 + m_2$.

Case 2a. Suppose further that $h \leqslant m_1 + m_2$. Then it is sufficient to show that

$h \geqslant k' + 1$. By Lemma 3.13, $k' + 1 \leqslant \lfloor \frac{m_1+m_2+h+\mu}{4} \rfloor$. Because $m_1 + m_2 \leqslant 3h$ and $\mu \leqslant 2$, we have $\lfloor \frac{m_1+m_2+h+\mu}{4} \rfloor \leqslant h$. So $h \geqslant k' + 1$ as required.
Case 2b. Suppose further that $h > m_1 + m_2$. Then it is sufficient to show that $m_1 + m_2 \geqslant k' + 1$. If $k = 2$ then $d(\mathcal{P}) \leqslant \max(m_1, m_2)$ and if $k = 1$ then $d(\mathcal{P}) \leqslant m_1 + m_2$. So, because we have assumed that $(k, d(\mathcal{P})) \neq (1, m_1 + m_2)$, we have $k + d(\mathcal{P}) \leqslant m_1 + m_2$. Thus $m_1 + m_2 \geqslant k + d(\mathcal{P}) \geqslant k' + 1$ as required.
Case 3. Suppose that $(k, d(\mathcal{P})) = (1, m_1 + m_2)$. Then W contains more than one vertex of degree at least 4 in L. Also L has no cut vertex because L has h vertices and contains an h-cycle.

Let x and y be twin vertices in $K_{U \cup W} - K_U$ such that $\deg_L(x) \geqslant 4$ and $y \notin V(L)$ (such vertices exist by Lemma 3.6(iii)). Then let $\mathcal{P}^*$ be a repacking of $\mathcal{P}$ obtained by performing an (x, y)-switch and let k^* be the number of components in the reduced leave of $\mathcal{P}^*$. Note that $k^* = 1$ because L has no cut vertex and $d(\mathcal{P}^*) = d(\mathcal{P}) - 1 = m_1 + m_2 - 1$. Now we can proceed as we did in Case 2 (note that our argument in Case 2 did not depend upon L being the edge-disjoint union of an h-cycle, an m_1-cycle and an m_2-cycle).
Case 4. Suppose that $k \in \{1, 2\}$, $h = 3$ and that each vertex in $V(L) \cap W$ has degree 2 in L. Then $m_1 + m_2 \leqslant 9$ and it must be that $m_1 = m_2 = 4$ if $\mu = 1$ and $\{m_1, m_2\} \in \{\{3, 4\}, \{3, 6\}, \{5, 4\}\}$ if $\mu = 2$.
Case 4a. Suppose further that C_1 and C_2 are vertex-disjoint. Let $x \in V(C_1) \cap W$ and $y \in V(C_2) \cap W$, and let $\mathcal{P}'$ be the repacking of $\mathcal{P} \cup \{H\}$ obtained by performing an (x, y)-switch with origin in C_1. If the terminus of this switch is in C_2, then the reduced leave of $\mathcal{P}'$ is an $(m_1 + m_2)$-cycle and we can remove from $\mathcal{P}'$ a 3-cycle that contains exactly one pure edge to complete the proof. If the terminus of this switch is in C_1, then the reduced leave of $\mathcal{P}'$ is an (m_1, m_2)-chain with link vertex in W and we can remove from $\mathcal{P}'$ a 3-cycle that contains exactly one pure edge and then proceed as in Case 2.
Case 4b. Suppose further that C_1 and C_2 share at least one vertex (in U). By applying Lemma 3.5 once or twice to $\mathcal{P} \cup \{H\}$, we can obtain a repacking $\mathcal{P}'$ of $\mathcal{P} \cup \{H\}$ whose reduced leave L' is 2-regular. Note that L' is either an $(m_1 + m_2)$-cycle or the vertex-disjoint union of two cycles whose lengths add to $m_1 + m_2$. In either case we remove from $\mathcal{P}'$ a 3-cycle H' that contains exactly one pure edge. In the former case we are finished immediately and in the latter case we can proceed as in Case 1, 2 or 4a, depending on $V(L') \cap V(H')$. (If $V(L') \cap V(H') = \emptyset$ then proceed as in Case 1, if W contains a vertex of degree 4 in $L' \cup H'$ then proceed as in Case 2, and otherwise proceed as in Case 4a.) □

We conclude this section with the following result which will be used in the proof of Lemma 3.28 to obtain two cycles from a leave with a vertex of degree at least 4.

Lemma 3.15. *Let U and W be disjoint sets with $|U|$ odd and $|W|$ even, let M be a list of integers, and let $\mu \in \{1,2\}$. Let m and m' be positive integers such that m is odd, $m, m' \geqslant \max(\lfloor \frac{1}{4}(m+m') + \mu \rfloor, 3)$, $m + m' \leqslant \min(2|U|+4, 2|W|+1, |U|+|W|)$, and $m + m' \leqslant 2(|U|+1)$ if $3 \in \{m, m'\}$. Suppose there exists an (M)-packing $\mathcal{P}$ of $K_{U \cup W} - K_U$ with a reduced leave L of size $m+m'$ such that L has exactly μ pure edges, L has at least one vertex of degree at least 4 and, if $3 \in \{m, m'\}$, there is a vertex of degree at least 4 in $V(L) \cap W$. Then there exists a repacking of $\mathcal{P}$ whose reduced leave is the edge-disjoint union of an m-cycle and an m'-cycle.*

Proof. Note that $m + m' \equiv \mu \pmod 2$. The proof splits into two cases.
Case 1. Suppose that L has exactly one vertex x of degree 4 and every other vertex of L has degree 2. Note that, by the hypotheses of the lemma, x is in $V(L) \cap W$ if $3 \in \{m, m'\}$. Then L is the vertex-disjoint union of a 2-chain and $k-1$ cycles, where k is the number of components in L. So the result follows by Lemma 3.12 (if $\mu = 1$) or Lemma 2.17 (if $\mu = 2$). (Note that $m, m' \geqslant k+1$ since $k \leqslant \lfloor \frac{1}{4}(m+m') + \mu \rfloor - 1$ by Lemma 3.13.)
Case 2. Suppose that L has at least two vertices of degree at least 4, or one vertex of degree at least 6. Let $\mathcal{P}'$ be the repacking of $\mathcal{P}$ obtained by applying Lemma 3.14, and let L' be the reduced leave of $\mathcal{P}'$. Then L' has exactly one vertex of degree 4, every other vertex of L' has degree 2, and there is a vertex of degree 4 in $V(L) \cap W$ if $3 \in \{m, m'\}$. We can proceed as in Case 1. □

3.2 Base decompositions

The aim of this section is to prove Lemmas 3.22, 3.23 and 3.28. These lemmas share a common form. Under various technical conditions, they guarantee the existence of an $(N, 3^a, 4^b, 5^c, 6^d, k)$-decomposition of $K_{u+w} - K_u$ that includes cycles with lengths $(3^a, 4^b, 5^c, 6^d, k)$ that each contain at most one pure edge (where k is even and perhaps 0). In order to prove Theorem 3.1 we will then take a base decomposition provided by one of these lemmas and repeatedly apply Lemma 3.3 to produce a desired (M)-decomposition of $K_{u+w} - K_u$. Very roughly speaking, Lemma 3.22 will be used when M has few odd entries, Lemma 3.23 will be used when M has many large entries, and Lemma 3.28 will be used when M has few large entries.

In essence, Lemmas 3.23 and 3.28 are proved as follows. Consider $K_{u+w} - K_u$ as $K_{U,W} \cup K_W$, where U and W are disjoint sets of sizes u and w. For some entry m of N, we use Theorem 1.4 to find an $(N \setminus (m))$-packing $\mathcal{P}$ of K_W whose leave L has size $a+c+m-t$, where $t = uw - (2a+4b+4c+6d)$, $t = 0$ if $m = 0$ and $t \in \{2, \ldots, m-2\}$ if $m > 0$. We then use various other results to find a $(3^a, 4^b, 5^c, 6^d, k, m)$-decomposition of $K_{U,W} \cup L$ such that one cycle of length m contains $m-t$ edges of L and each other cycle contains one edge of L if it has odd length and no edges of L if it has even length. Lemma 3.22 is proved similarly except that we consider $K_{u+w} - K_u$ as $K_{U \setminus U_1, W} \cup K_{W \cup U_1}$, where U and W are disjoint sets of sizes u and w and $U_1 \subseteq U$ with $|U_1| = 1$.

Throughout this section we will make extensive use of some existing results on cycle decompositions of complete graphs and complete bipartite graphs that are stated in previous chapters, namely Theorems 1.4, 1.20 and 2.22. We also require the following notation. For a list $X = (x_1, \ldots, x_n)$, let $\sum X = \sum_{i=1}^{n} x_i$. For a list X and a sublist Y of X, let $X \setminus Y$ be the list obtained from X by removing the entries of Y. For a real number x we denote the greatest even integer less than or equal to x by $\lfloor x \rfloor_{\mathrm{e}}$ and the least even integer greater than or equal to x by $\lceil x \rceil_{\mathrm{e}}$.

3.2.1 Preliminary results

The results in this subsection are tools that we will use in the proofs of Lemmas 3.22, 3.23 and 3.28. Lemmas 3.16 and 3.17 provide cycle packings of the complete bipartite graph whose leaves have decompositions into two paths of specified lengths. Lemmas 3.18, 3.19 and 3.20 provide cycle packings of the union of the complete bipartite graph with one or more cycles. Lemma 3.21 allows us to decrease the number of 4-cycles and increase the number of 6-cycles in a packing.

Lemma 3.16. *Let U' and W be sets such that $|U'|$ and $|W|$ are even, let $m_1, \ldots, m_\tau$ be even integers such that $4 \leqslant m_1 \leqslant \cdots \leqslant m_\tau \leqslant 3m_{\tau-1}$ and $m_1 + \cdots + m_\tau = |U'||W|$. If $m_{\tau-1} + m_\tau \leqslant 2|U'|$ when $|U'| = |W|$ and $m_{\tau-1} + m_\tau \leqslant 2\min(|U'|, |W|) + 2$ otherwise, then*

(i) for all distinct $i, j \in \{1, \ldots, \tau\}$ there exists an $((m_1, \ldots, m_\tau) \setminus (m_i, m_j))$-packing of $K_{U',W}$ whose reduced leave has a decomposition into an m_i-path and an m_j-path whose end vertices are in W; and

(ii) for each $i \in \{1, \ldots, \tau\}$, there exists an $((m_1, \ldots, m_\tau) \setminus (m_i))$-packing of $K_{U',W}$ whose reduced leave has a decomposition into an $(m_i - 2)$-path and a 2-path whose end vertices are in W.

Proof. By Theorem 2.22, there exists an $(m_1, \ldots, m_\tau)$-decomposition $\mathcal{D}$ of $K_{U',W}$. We can remove an m_i-cycle from $\mathcal{D}$ to obtain the packing required by (ii), so it remains to prove (i). Let $\mathcal{P}$ be a packing of $K_{U',W}$ obtained from $\mathcal{D}$ by removing an m_i-cycle and an m_j-cycle. Assume that $m_i \leqslant m_j$. Let L be the reduced leave of $\mathcal{P}$. The proof divides into cases according to whether L is connected.

Case 1. Suppose that L is connected. Then L has at least one and at most m_i vertices of degree 4, and every other vertex of L has degree 2. Furthermore, if L has exactly m_i vertices of degree 4 then L has no cut vertex, since in this case L has exactly m_j vertices and contains an m_j-cycle. So it can be seen that, by applying [64, Lemma 3.4] and then [64, Lemma 3.5] to this packing, we can obtain an $((m_1, \ldots, m_\tau) \setminus (m_i, m_j))$-packing of $K_{U',W}$ with a reduced leave L' of size $m_i + m_j$ such that exactly one vertex of L' has degree 4, every other vertex of L' has degree 2, and L' has at most $m_i - 1$ components. Then, by applying [64, Lemma 3.2] we can obtain an $((m_1, \ldots, m_\tau) \setminus (m_i, m_j))$-packing of $K_{U',W}$ whose reduced leave has a decomposition into an m_i-path and an m_j-path whose end vertices are in W (note that $4 \leqslant m_i \leqslant m_j$).
Case 2. Suppose that L is not connected. Then L must consist of two vertex-disjoint cycles. Let $x, y \in U'$ such that x and y are in distinct cycles of L. By applying an (x, y)-switch we obtain an $((m_1, \ldots, m_\tau) \setminus (m_i, m_j))$-packing of $K_{U',W}$ whose reduced leave L' is either an $(m_i + m_j)$-cycle or an (m_i, m_j)-chain with link vertex in U'. In either case it is easy to see that L' has a decomposition into an m_i-path and an m_j-path whose end vertices are in W. $\square$

Lemma 3.17. *Let U' and W be sets such that $|U'|$ and $|W|$ are even and $|W| \geqslant 8$, and let ℓ and t be integers such that $\ell \in \{2, 4, \ldots, 12\}$ and $t \in \{6, 8, \ldots, |W| - 2\}$. Let M be a list of integers such that $m \in \{4, 6\}$ for all entries m in M and $(\sum M) + k + \ell + t = |U'||W|$, where $k = \lceil \frac{t+2}{3} \rceil_{\mathrm{e}}$ if $t \geqslant 12$ and $k = 0$ if $t \leqslant 10$. If $\max(k + 2, \ell, 8) + t \leqslant 2|U'| + 2$ and $(\ell, t, |U'|, |W|) \neq (12, 6, 8, 8)$, there exists an (M, k)-packing of $K_{U',W}$ whose reduced leave has a decomposition into an ℓ-path and a t-path whose end vertices are in W.*

Proof. If $\ell \in \{4, 6, \ldots, 12\}$, then apply Lemma 3.16(i) taking $m_1, \ldots, m_\tau$ as the list (M, k, ℓ, t) reordered to be nondecreasing and $(m_i, m_j) = (\ell, t)$. If $\ell = 2$, then apply Lemma 3.16(ii) taking $m_1, \ldots, m_\tau$ as the list $(M, k, t + 2)$ reordered to be nondecreasing and $m_i = t+2$. The condition that $m_\tau \leqslant 3m_{\tau-1}$ holds by our definition of k. If $|U'| < |W|$, then $m_{\tau-1} + m_\tau \leqslant 2|U'| + 2$ holds because $\max(k + 2, \ell, 8) + t \leqslant 2|U'| + 2$. If $|W| < |U'|$, it is routine to show that $m_{\tau-1} + m_\tau \leqslant 2|W| + 2$ holds by considering the cases $\ell = 2$ and $\ell \in \{4, 6, \ldots, 12\}$ separately. Similarly, if $|W| = |U'|$ then $m_{\tau-1} + m_\tau \leqslant 2|W|$ holds since $(\ell, t, |U'|, |W|) \neq (12, 6, 8, 8)$. $\square$

Lemma 3.18. *Let U' and W be sets with $|U'|, |W|$ even, and let a, c, m and t be nonnegative integers such that either*

(i) $(m, t) = (0, 0)$ *and* $a + c \in \{3, \ldots, |W|\}$; *or*

(ii) $t \in \{2, 4, \ldots, \lfloor m \rfloor_{\mathrm{e}} - 2\}$ *and* $a + c \in \{1, \ldots, |W| - m + \frac{t}{2} + 1\}$.

Suppose there is an (M)-packing $\mathcal{P}$ of $K_{U',W}$ with a reduced leave L such that $\deg_L(x) = 2$ for each $x \in V(L) \cap W$ and L is a union of edge-disjoint paths $P_0, \ldots, P_{a+c}$ such that

- *P_0 has length t, a of the paths $P_1, \ldots, P_{a+c}$ have length 2, and the remaining c have length 4;*
- *there are vertices $x_0, \ldots, x_{a+c} \in W$ such that, for $i \in \{1, \ldots, a+c\}$, the end vertices of P_i are x_{i-1} and x_i; and*
- *the end vertices of P_0 are x_0 and x_{a+c} (if $(m,t) = (0,0)$, $x_{a+c} = x_0$ and P_0 is trivial).*

Let C be an $(a+c+m-t)$-cycle such that $V(C) \subseteq W$ if $(m,t) = (0,0)$ and $V(C) \subseteq W \cup \{\alpha\}$ for some $\alpha \notin U' \cup W$ if $t > 0$ (note that $a+c+m-t \in \{0\} \cup \{3, \ldots, |W|\}$). Then there exists an $(M, 3^a, 5^c, m)$-decomposition $\mathcal{P}'$ of $K_{U',W} \cup C$ that, if $m > 0$, includes an m-cycle containing $m-t$ edges of C. Furthermore, if $|V(C)| + \frac{t-2}{2} \leqslant |W| - 1$ and $c \geqslant 1$, then $\mathcal{P}'$ includes a 5-cycle that has exactly one edge of K_W and has a vertex in $W \setminus V(C)$.

Proof. It follows from (i) and (ii) that $a+c+m-t \in \{3, \ldots, |W|\}$. Let C be such an $(a+c+m-t)$-cycle. By permuting vertices in $\mathcal{P}$, we can assume that $x_0x_1, x_1x_2, \ldots, x_{a+c-1}x_{a+c}$ are consecutive edges in C (note that $x_{a+c} = x_0$ if $(m,t) = (0,0)$ and that $|E(C)| - (a+c) \geqslant 2$ otherwise) and that no internal vertices of P_0 are in $V(C)$. To see that we can do this note that, if $t > 0$, then $\frac{t-2}{2} + (a+c+m-t) \leqslant |W|$ by (ii). Furthermore, if $|V(C)| + \frac{t-2}{2} \leqslant |W| - 1$ and $c \geqslant 1$, we can ensure that some path of length 4 in $\{P_1, \ldots, P_{a+c}\}$ has an internal vertex in $W \setminus V(C)$. Let P' be the $(m-t)$-path induced by the edges of C other than $x_0x_1, x_1x_2, \ldots, x_{a+c-1}x_{a+c}$ (if $(m,t) = (0,0)$, then $x_{a+c} = x_0$ and P' is trivial). Then $\{P_0 \cup P'\} \cup \{P_i \cup [x_{i-1}, x_i] : i = 1, \ldots, a+c\}$ is a $(3^a, 5^c, m)$-decomposition of $C \cup L$ and the result follows. □

Lemma 3.19. *Let U' and W be sets such that $|U'| \geqslant 2$ and $|W| \geqslant 4$ are even. Let a, b, c, d and m be nonnegative integers such that*

(i) $d = 0$ *if* $|U'| = 2$;

(ii) $2a + 4b + 4c + 6d + t = |U'||W|$ *where* $t \in \{0, 2, 4\}$;

(iii) $2a + 4c + t \leqslant 2|W|$; *and*

(iv) *either*

- $(m,t) = (0,0)$ *and* $a + c \in \{0\} \cup \{3, \ldots, |W|\}$; *or*
- $t \in \{2,4\}$, $m \in \{t+2, \ldots, |W|\}$ *and* $a+c \in \{1, \ldots, |W| - m + \frac{t}{2} + 1\}$.

Let C be an $(a+c+m-t)$-cycle such that $V(C) \subseteq W$ if $t=0$ and $V(C) \subseteq W \cup \{\alpha\}$ for some $\alpha \notin U' \cup W$ if $t>0$ (note that $a+c+m-t \in \{0\} \cup \{3,\dots,|W|\}$). Then there exists a $(3^a,4^b,5^c,6^d,m)$-decomposition of $K_{U',W} \cup C$ that, if $m>0$, includes an m-cycle containing $m-t$ edges of C.

Proof. If $(a,c,m,t)=(0,0,0,0)$, then the result follows from Theorem 1.20. Thus, using (iv), we may assume that $a+c+m-t \in \{3,\dots,|W|\}$ and $2a+4c+t \geqslant 4$.

Suppose there exists a $(4^b,6^d)$-packing $\mathcal{P}'$ of $K_{U',W}$ with a reduced leave L' such that L' is connected and $\deg_{L'}(x)=2$ for all $x \in V(L') \cap W$. Then $|E(L')| = 2a+4c+t$ by (ii). We claim that in this case L' has a suitable decomposition into paths so that we can complete the proof by applying Lemma 3.18 (with $M=(4^b,6^d)$) to $\mathcal{P}'$. If $|E(L')|=4$, then $(a,c,t)=(1,0,2)$ and L' is a 4-cycle and has a suitable decomposition into two 2-paths. If $|E(L')| \geqslant 6$ then, because L' is a connected even graph, it has a closed Eulerian trail. Because L' is bipartite and $\deg_{L'}(x)=2$ for all $x \in V(L') \cap W$, any subtrail of this trail that begins at a vertex in $V(L') \cap W$ and has length 2 or 4 is a path. Thus, a suitable decomposition of L' into a 2-paths, c 4-paths and a t-path exists. So it suffices to find a $(4^b,6^d)$-packing of $K_{U',W}$ with a reduced leave L' such that L' is connected and $\deg_{L'}(x)=2$ for all $x \in V(L') \cap W$.

By applying Theorem 1.20 and removing cycles, we can obtain a $(4^b,6^d)$-packing $\mathcal{P}''$ of $K_{U',W}$. We can do this because $|U'||W|-4b-6d \in \{0\} \cup \{4,6,\dots,|U'||W|\}$ by (ii) and (iv), and because $|U'||W|-4b-6d \equiv 0 \pmod 4$ when $|U'|=2$ by (i). Let L'' be the reduced leave of $\mathcal{P}''$.

Case 1. Suppose $\deg_{L''}(x)=2$ for all $x \in V(L'') \cap W$. If L'' is connected then we are done. Otherwise, let $y_1,y_2 \in V(L'') \cap U'$ such that y_1 is in a largest component of L'' and y_2 is in another component of L''. By performing a (y_1,y_2)-switch with origin adjacent to y_2 we obtain a repacking of $\mathcal{P}''$ whose reduced leave has a component larger than any component in L''. We can repeat this process until we obtain a repacking of $\mathcal{P}''$ whose reduced leave is connected.

Case 2. Suppose $\deg_{L''}(x) \geqslant 4$ for some $x \in V(L'') \cap W$. By repeatedly applying Lemma 3.5 we can obtain a repacking of $\mathcal{P}''$ whose reduced leave has no vertices of degree greater than 2 in W (note that $|E(L'')| \leqslant 2|W|$ by (iii)). Thus we can proceed as in Case 1 to complete the proof. □

The following is a method for packing 3-cycles and 5-cycles into the complete graph with a hole, where each cycle has exactly one pure edge. It will have a similar role to Lemma 2.20 in Chapter 2.

Lemma 3.20. *Let W be a set of even size $w \geqslant 6$, and let a and c be non-negative integers such that a is even and $(a,c) \neq (0,0)$. Let n and b be the integers such that $a+2c = nw-2b$ and $0 \leqslant b \leqslant \frac{w-2}{2}$, and let U' be a set* *such that* $U' \cap W = \emptyset$ *and* $|U'| = 2n$. *Let* $\ell_1, \ldots, \ell_n$ *be integers such that* $\ell_i \in \{\frac{w}{2}, \ldots, w\}$ *for* $i \in \{2, \ldots, n\}$, $\ell_1 \in \{\frac{1}{2}(w-2b), \ldots, w-2b\} \setminus \{1,2\}$ *and* $\ell_1 + \cdots + \ell_n = a + c$. *Then, for any edge-disjoint cycles* $C_1, \ldots, C_n$ *in* K_W *with lengths* $\ell_1, \ldots, \ell_n$, *there exists a* $(3^a, 5^c)$*-packing of* $K_{U',W} \cup C_1 \cup \cdots \cup C_n$ *whose reduced leave is a subgraph of* $K_{U',W}$ *isomorphic to* $K_{2,2b}$. *Furthermore, if* $a + 2c \not\equiv 2 \pmod{w}$, *and if* $c \not\equiv 2 \pmod{\frac{w}{2}}$ *when* $a = 0$, *then there do exist such integers* $\ell_1, \ldots, \ell_n$.

Proof. Suppose first that we are given a list $\ell_1, \ldots, \ell_n$ satisfying our hypotheses. Let $U' = \{p_1, \ldots, p_{2n}\}$. By Lemma 3.19, there is a $(3^{2\ell_i - w}, 5^{w-\ell_i})$-decomposition $\mathcal{D}_i$ of $K_{\{p_{2i-1}, p_{2i}\},W} \cup C_i$ for $i \in \{2, \ldots, n\}$. Let W_1 be a set of size $w - 2b$ such that $V(C_1) \subseteq W_1 \subseteq W$. Also by Lemma 3.19, there is a $(3^{2\ell_1 - w + 2b}, 5^{w-2b-\ell_1})$-decomposition $\mathcal{D}_1$ of $K_{\{p_1,p_2\},W_1} \cup C_1$. Using the facts that $nw - 2b = a + 2c$ and that $\ell_1 + \cdots + \ell_n = a + c$, it can be seen that $\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_n$ is a $(3^a, 5^c)$-packing of $K_{U',W} \cup C_1 \cup \cdots \cup C_n$. The reduced leave of this packing is $K_{\{p_1,p_2\},W \setminus W_1}$, which is isomorphic to $K_{2,2b}$.

Now suppose that $a + 2c \not\equiv 2 \pmod{w}$ and that $c \not\equiv 2 \pmod{\frac{w}{2}}$ if $a = 0$. Note that, because $a + 2c = nw - 2b$, the former implies that $w - 2b \neq 2$ and the latter implies that $w - 2b \neq 4$ if $a = 0$. Let $\ell_1 = w - 2b$ if $a \geqslant w - 2b$ and let $\ell_1 = \frac{1}{2}(w - 2b + a)$ if $a < w - 2b$. Then $\ell_1 \in \{\frac{1}{2}(w-2b), \ldots, w-2b\} \setminus \{1,2\}$. To show that there exist integers $\ell_2, \ldots, \ell_n$ such that $\ell_i \in \{\frac{w}{2}, \ldots, w\}$ for $i \in \{2, \ldots, n\}$ and $\ell_2 + \cdots + \ell_n = a + c - \ell_1$, and hence to complete the proof, it suffices to show that $\frac{w}{2}(n-1) \leqslant a + c - \ell_1 \leqslant w(n-1)$. If $a \geqslant w - 2b$, then $a + c - \ell_1 = a + c - w + 2b$ and

$$\tfrac{w}{2}(n-1) = \tfrac{1}{2}(a + 2c - w + 2b) \leqslant a + c - w + 2b \leqslant a + 2c - w + 2b = w(n-1),$$

where both equalities follow from $a + 2c = nw - 2b$, the first inequality follows because $a \geqslant w - 2b$ and the second inequality follows because $c \geqslant 0$. If $a < w - 2b$, then

$$a + c - \ell_1 = \tfrac{1}{2}(a + 2c - w + 2b) = \tfrac{w}{2}(n-1),$$

where the first equality follows because $\ell_1 = \frac{1}{2}(w - 2b + a)$ and the second equality follows because $a + 2c = nw - 2b$. □

Lemma 3.21. *Let* $\mathcal{P} = \{C_1, \ldots, C_r, X_1, \ldots, X_{3j}\}$ *be an* $(M, 4^{3j})$*-decomposition of an even graph* G *where* $X_1, \ldots, X_{3j}$ *are* 4*-cycles. If there is a set* S *of four vertices in* G *that are pairwise twin and pairwise nonadjacent such that* $|V(X_i) \cap S| = 2$ *for* $i \in \{1, \ldots, 3j\}$, *then there is an* $(M, 6^{2j})$*-decomposition* $\mathcal{P}' = \{C'_1, \ldots, C'_r, Y'_1, \ldots, Y'_{2j}\}$ *of* G *such that* $Y'_1, \ldots, Y'_{2j}$ *are* 6*-cycles and, for each* $i \in \{1, \ldots, r\}$, $V(C'_i) = \pi_i(V(C_i))$ *for some permutation* π_i *of* $V(G)$ *that fixes each vertex in* $V(G) \setminus S$. *Furthermore,* $|V(Y'_i) \cap S| = 3$ *for* $i \in \{1, \ldots, 2j\}$.

Proof. The result is trivial if $j = 0$, so we may assume that $j \geqslant 1$. We claim that, given any cycle packing $\mathcal{P}^\star$ of G whose reduced leave $L^\star$ is a subgraph of $K_{S,V(G)\setminus S}$ with exactly 12 edges, we can apply a sequence of switches on vertices in S to obtain a repacking whose reduced leave has a $(6,6)$-decomposition. If this is the case, then we can begin with $\mathcal{P}$ and j times repeat the process of removing three 4-cycles that each contain two vertices of S, applying such a sequence of switches, and adding the two 6-cycles in the resulting reduced leave to the resulting repacking. Because we only apply switches on vertices in S, it follows from Lemma 1.22 that we will be able to find appropriate 4-cycles at each stage, and the final result will be an $(M, 6^{2j})$-decomposition of G with the required properties. So it suffices to prove our claim.

Let $\mathcal{P}^\star$ be a cycle packing of G whose reduced leave $L^\star$ is a subgraph of $K_{S,V(G)\setminus S}$ with exactly 12 edges. Note that $L^\star$ is even because G is even. If $L^\star$ contains a 6-cycle, then $L^\star$ has a $(6,6)$-decomposition and we are finished. So we may suppose that $L^\star$ has no 6-cycle but has a $(4,8)$-decomposition or a $(4,4,4)$-decomposition.

Case 1. Suppose that no vertex in S has degree 6 in $L^\star$ and that $L^\star$ is connected. It is routine to check that $L^\star$ contains a path $[x_0, \ldots, x_6]$ of length 6 and a vertex $y \notin \{x_0, \ldots, x_6\}$ such that $x_0, x_6 \in S$ and x_0y and x_2y are edges in $L^\star$. By performing the (x_0, x_6)-switch with origin x_1 we obtain a repacking of $\mathcal{P}^\star$ whose reduced leave has a $(6,6)$-decomposition with each of the 6-cycles containing exactly three vertices from S and we are finished. (Note that $\{x_0, x_2, x_4, x_6\} = S$ and if the terminus of the switch is x_5 then the reduced leave contains the 6-cycle $(x_0, y, x_2, \ldots, x_5)$, and otherwise it contains the 6-cycle $(x_1, \ldots, x_6)$.)

Case 2. Suppose that no vertex in S has degree 6 in $L^\star$ and that $L^\star$ is disconnected. Then $L^\star$ is a vertex-disjoint union of a copy of $K_{2,4}$ and a 4-cycle. Let $x, y \in S$ be vertices such that $\deg_{L^\star}(x) = 4$ and $\deg_{L^\star}(y) = 2$. By performing an (x, y)-switch whose origin is adjacent to x we obtain a repacking of $\mathcal{P}^\star$ whose leave satisfies the conditions of Case 1 and we can proceed as we did in that case.

Case 3. Suppose that a vertex in S has degree at least 6 in $L^\star$. By repeatedly applying Lemma 3.5, we can obtain a repacking of $\mathcal{P}^\star$ whose leave either has a $(6,6)$-decomposition or satisfies the conditions of Case 1 or Case 2 (note that each application of Lemma 3.5 is simply a switch on vertices in S). In the former case we are finished and in the latter we can proceed as we did in Case 1 or Case 2. □

3.2.2 Lists with few odd entries

Lemma 3.22. *Let $u \geqslant 5$ and $w \geqslant 8$ be integers such that u is odd and w is even. Let N be a list of integers and let a, b, c and d be nonnegative integers such that the following hold.*

(i) $(\sum N) - t + a + c = \binom{w+1}{2}$, *where* $t \in \{0, 2, \dots, w-2\}$*;*

(ii) $2a + 4b + 4c + 6d + k + t = (u-1)w$, *where* $k = \lceil \frac{t+2}{3} \rceil_{\mathrm{e}}$ *if* $t \geqslant 12$ *and* $k = 0$ *otherwise;*

(iii) $3 \leqslant \ell \leqslant \min(u, w)$ *for each entry* ℓ *in* N*, and* $d = 0$ *if* $u = 5$*;*

(iv) if $t > 0$*, there is some entry* m *in* N *such that* $m \geqslant t + 2$*; and*

(v) either

- $a + c \geqslant 6$, $a + 2c \leqslant w$, *and* $b \geqslant 1$*; or*
- $a + c = 3$, $(m, t) \neq (w, 2)$, *and* $(a, c, t, u, w) \neq (0, 3, 6, 9, 8)$.

Then there exists an $(N, 3^a, 4^b, 5^c, 6^d, k)$*-decomposition of* $K_{u+w} - K_u$ *that includes cycles with lengths* $(3^a, 4^b, 5^c, 6^d, k)$ *that each contain at most one pure edge.*

Proof. Let U and W be disjoint sets of sizes u and w and let $U_1 \subseteq U$ with $|U_1| = 1$. Observe that $K_{U \cup W} - K_U = K_{U \setminus U_1, W} \cup K_{W \cup U_1}$. Let $m = 0$ if $t = 0$. We first choose integers a_2, a_3, c_2 and c_3. Let (a_3, c_3) be the leftmost pair from the appropriate row of the table below such that $a_3 \leqslant a$ and $c_3 \leqslant c$, and let $a_2 = a - a_3$ and $c_2 = c - c_3$. It is routine to check using (v) that some pair in the appropriate row will always satisfy these conditions.

case	(a_3, c_3)
$a + c = 3$	(a, c)
$a + c \geqslant 6$, $t > 0$, a even	$(0, 1), (2, 0)$
$a + c \geqslant 6$, $t > 0$, a odd	$(1, 0)$
$a + c \geqslant 6$, $t = 0$, a even	$(0, 0)$
$a + c \geqslant 6$, $t = 0$, a odd	$(3, 0), (1, 2)$

This choice ensures that a_2, a_3, c_2 and c_3 are nonnegative integers such that a_2 is even, $a_2 + a_3 = a$, $c_2 + c_3 = c$, $2a_3 + 4c_3 \leqslant 12$, $a_3 + c_3 \in \{1, 2, 3\}$ if $t > 0$, and $a_3 + c_3 \in \{0, 3\}$ if $t = 0$.

We now construct packings $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ as follows (we justify that these packings exist later).

- $\mathcal{P}_1$ is an $(N \setminus (m))$-packing of $K_{W \cup U_1}$. The reduced leave of $\mathcal{P}_1$ is $C^* \cup C^\dagger$, where C^* is an $(a_2 + c_2)$-cycle such that $U_1 \not\subseteq V(C^*)$ and $C^\dagger$ is an $(m - t + a_3 + c_3)$-cycle such that $U_1 \not\subseteq V(C^\dagger)$ if $t = 0$.

Let $U_2 \subseteq U \setminus U_1$ with $|U_2| = 0$ if $(a_2, c_2) = (0, 0)$ and $|U_2| = 2$ otherwise. Let $b_2 = \frac{1}{4}(|U_2|w - 2a_2 - 4c_2)$. By (v) and the choice of a_2 and c_2, we have $b_2 \in \{0, \ldots, \frac{w-4}{2}\}$.

- If $|U_2| = 2$, then $\mathcal{P}_2$ is a $(3^{a_2}, 4^{b_2}, 5^{c_2})$-decomposition of $K_{U_2,W} \cup C^*$ and, if $b_2 > 0$, the union of the 4-cycles in $\mathcal{P}_2$ is a copy of $K_{2,2b_2}$. If $|U_2| = 0$, then $\mathcal{P}_2 = \emptyset$.

- $\mathcal{P}_3$ is a $(3^{a_3}, 4^{b-b_2+3j}, 5^{c_3}, 6^{d-2j}, k, m)$-decomposition of $K_{U_3,W} \cup C^\dagger$, where $U_3 = U \setminus (U_1 \cup U_2)$ and

$$j = \begin{cases} 0 & \text{if } b \geqslant b_2, \\ \lceil \frac{1}{3}(b_2 - b) \rceil & \text{otherwise.} \end{cases}$$

  Furthermore, if $m > 0$, there is an m-cycle in $\mathcal{P}_3$ that contains $m - t$ edges of $C^\dagger$.

The union $\mathcal{P}' = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ will be an $(N, 3^a, 4^{b+3j}, 5^c, 6^{d-2j}, k)$-decomposition of $K_{U \cup W} - K_U$. Using (i), we can see that $\mathcal{P}'$ will include cycles with lengths N that contain all but $a+c$ edges of $K_{W \cup U_1}$ (the $\sum N - m$ edges in cycles in $\mathcal{P}_1$ are all in $K_{W \cup U_1}$, as are $m - t$ edges in an m-cycle in $\mathcal{P}_3$ if $m > 0$). So, because each odd cycle in $\mathcal{P}'$ must contain at least one edge of K_W and because K_W is a subgraph of $K_{W \cup U_1}$, $\mathcal{P}'$ will include cycles with lengths $(3^a, 4^{b+3j}, 5^c, 6^{d-2j}, k)$ that each contain at most one edge of K_W.

If $b \geqslant b_2$, then $j = 0$ and this will complete the proof. Otherwise $b_2 > b$ and we will be able to apply Lemma 3.21 to $\mathcal{P}'$ (with $S \subseteq U_2 \cup U_3$) to obtain a decomposition $\mathcal{P}$ with the required properties provided we can find $3j$ 4-cycles in $\mathcal{P}'$ that meet the hypotheses of Lemma 3.21. If $3j = b_2 - b$, we will be able to use $3j$ 4-cycles from $\mathcal{P}_2$. If $3j \in \{b_2 - b + 1, b_2 - b + 2\}$, then $b_2 \geqslant 3j + b - 2 \geqslant 3j - 1$ (since $b \geqslant 1$) and we will be able to use $3j - 1$ 4-cycles from $\mathcal{P}_2$ and any one 4-cycle from $\mathcal{P}_3$. (It must be the case that $b \geqslant 1$ by (v) because $b_2 > b$ implies $(a_2, c_2) \neq (0, 0)$ and $a + c \geqslant 6$ by our choices of b_2, a_2 and c_2.) Note that Lemma 3.21 ensures $\mathcal{P}$ will include cycles with lengths $(3^a, 4^b, 5^c, 6^d, k)$ that each contain at most one edge of K_W.

So it remains to establish the existence of the packings $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$. In what follows we will often use the facts that $w \geqslant 8$ and that either $(m, t) = (0, 0)$ or $t < m \leqslant w$ (the latter follows from (iv)).

Proof that $\mathcal{P}_1$ exists. First observe that $a_2 + c_2 \in \{0\} \cup \{3, \ldots, w\}$ and $m - t + a_3 + c_3 \in \{0\} \cup \{3, \ldots, w + 1\}$ by (iii), (iv), (v), our choice of a_3 and c_3 and the definition of m. Then, by Theorem 1.4, a packing with the required properties exists by (iii) and because

$$\begin{aligned}
&\textstyle\sum(N \setminus (m)) + |E(C^\dagger)| + |E(C^*)| \\
&= (\tbinom{w+1}{2} + t - (a+c) - m) + (m - t + a_3 + c_3) + (a_2 + c_2) \\
&= \tbinom{w+1}{2}
\end{aligned}$$

where the first equality follows by (i). If $t = 0$, then $|V(C^*)| + |V(C^\dagger)| = a + c \leqslant w$ by (v) and we can permute the vertices of this packing so that $U_1 \nsubseteq V(C^*) \cup V(C^\dagger)$. If $t > 0$, then $|V(C^*)| = a_2 + c_2 \leqslant w$ by (v) and we can permute the vertices of this packing so that $U_1 \nsubseteq V(C^*)$.

Proof that $\mathcal{P}_2$ exists. This is trivial if $(a_2, c_2) = (0, 0)$ and $|U_2| = 0$, so assume that $|U_2| = 2$ and $(a_2, c_2) \neq (0, 0)$. Then the definition of b_2 implies that $a_2 + 2c_2 = w - 2b_2$. So the existence of $\mathcal{P}_2$ follows immediately by Lemma 3.20 because $a_2 + c_2 \in \{\frac{1}{2}(w - 2b_2), \ldots, w - 2b_2\} \setminus \{1, 2\}$ by (v), and our choice of a_2 and c_2.

Proof that $\mathcal{P}_3$ exists. We will show that $\mathcal{P}_3$ exists using either Lemma 3.19 (if $t \in \{0, 2, 4\}$) or Lemmas 3.17 and 3.18 (if $t \geqslant 6$). Note that $|U_3| = u - |U_2| - 1 \in \{u - 3, u - 1\}$ and hence $|U_3| \geqslant 4$, except when $u = 5$ and $|U_2| = 2$. We first establish two useful facts.

(a) $\boldsymbol{b - b_2 + 3j \geqslant 0}$ and $\boldsymbol{d - 2j \geqslant 0}$. Obviously $b - b_2 + 3j \geqslant 0$ by the definition of j, and clearly $d - 2j \geqslant 0$ if $b \geqslant b_2$ and hence $j = 0$. So it remains to show that $d - 2j \geqslant 0$ when $b < b_2$. Then $j = \lceil \frac{1}{3}(b_2 - b) \rceil$. Observe that

$$2a_3 + 4b + 4c_3 + 6d + k + t = w|U_3| + 4b_2 \tag{3.2.1}$$

by (ii) and the definitions of a_3, c_3, b_2 and U_3. So it cannot be that $u = 5$ because then $b > b_2$ by (3.2.1) (since $d = 0$ by (iii), $|U_3| \geqslant 2$, $2a_3 + 4c_3 \leqslant 12$ and, by (iii) and (iv), $t \leqslant 2$ and $k = 0$). So assume that $u \geqslant 7$ and hence $|U_3| \geqslant 4$. Then it follows from (3.2.1) that $d \geqslant \frac{2}{3}(b_2 - b)$ using the facts that $2a_3 + 4c_3 \leqslant 12$ and $k \leqslant t \leqslant w - 2$. So we have $d - 2j \geqslant 0$.

(b) $\boldsymbol{2a_3 + 4(b - b_2 + 3j) + 4c_3 + 6(d - 2j) + k + t = |U_3|w}$. Observe that

$$\begin{aligned}
&2a_3 + 4(b - b_2 + 3j) + 4c_3 + 6(d - 2j) + k + t \\
&= (2a + 4b + 4c + 6d + k + t) - (2a_2 + 4b_2 + 4c_2) \\
&= |U_3|w
\end{aligned}$$

where the final equality follows by (ii) and because $2a_2 + 4b_2 + 4c_2 = |U_2|w$ by the definitions of U_2 and b_2.

Case 1. Suppose that $t \in \{0, 2, 4\}$. Then $k = 0$. We claim that $\mathcal{P}_3$ exists by Lemma 3.19. To see that we can apply Lemma 3.19, note that $|U_3| \geqslant 2$ and that $d = 0$ if $|U_3| = 2$ by (iii). Also, using $2a_3 + 4c_3 \leqslant 12$, $2a_3 + 4c_3 + t \leqslant 2w$. Finally, $a_3+c_3 \in \{0, 3\}$ if $t = 0$ and $a_3+c_3 \in \{1, \dots, w-m+\frac{t}{2}+1\}$ if $t \in \{2, 4\}$ by (v) and our choice of a_3 and c_3.

Case 2. Suppose that $t \geqslant 6$. Then $u \geqslant 9$ because $u \geqslant m \geqslant 8$ by (iii) and (iv). Also, $|U_3| \geqslant u - 3 \geqslant 6$. By Lemma 3.18, to show that $\mathcal{P}_3$ exists it suffices to find a $(4^{b-b_2+3j}, 6^{d-2j}, k)$-packing of $K_{U_3,W}$ whose reduced leave has a decomposition into a t-path and a $(2a_3 + 4c_3)$-path with end vertices in W (note that $1 \leqslant a_3 + c_3 \leqslant 3 \leqslant w - m + \frac{t}{2} + 1$). Also note that, since $K_{U_3,W}$ is an even graph, such a leave must be connected.

By Lemma 3.17, to find such a packing it suffices to show that $\max(k + 2, 2a_3 + 4c_3, 8) + t \leqslant 2|U_3| + 2$ and $(2a_3 + 4c_3, t, |U_3|, w) \neq (12, 6, 8, 8)$ (note that $2a_3 + 4c_3 \in \{2, 4, \dots, 12\}$). We have $\max(k + 2, 8) + t \leqslant 2|U_3| + 2$ because $t \leqslant u - 3$ (by (iii) and (iv)), $k \leqslant \frac{t+7}{3}$ and $|U_3| \geqslant u - 3$. We have $2a_3+4c_3+t \leqslant 2|U_3|+2$ because $t \leqslant u-3$, $2a_3+4c_3 \leqslant 12$ and either $2a_3+4c_3 \leqslant 6$ or $|U_3| = u - 1$ (by our choice of a_3 and c_3 and the definitions of U_2 and U_3). It follows directly from (v) that $(2a_3 + 4c_3, t, |U_3|, |W|) \neq (12, 6, 8, 8)$. □

3.2.3 Lists with many large entries

Lemma 3.23. *Let $u \geqslant 7$ and $w \geqslant 8$ be integers such that u is odd and w is even. Let N be a list of integers and let a, b, c and d be nonnegative integers such that the following conditions hold.*

- *(i)* $(\sum N) - t + a + c = \binom{w}{2}$, *where* $t \in \{2, 4, \dots, w - 2\}$;
- *(ii)* $2a + 4b + 4c + 6d + k + t = uw$, *where* $k = \lceil \frac{t+2}{3} \rceil_{\mathrm{e}}$ *if* $t \geqslant 12$ *and* $k = 0$ *otherwise;*
- *(iii)* $3 \leqslant \ell \leqslant \min(u, w)$ *for each entry* ℓ *in* N;
- *(iv)* $a \geqslant \frac{w}{2} + 1$, $b \geqslant 1$ *and* $c \leqslant 1$;
- *(v) there is some entry* $m \geqslant \max(t + 2, 7)$ *in* N *such that* $uw \geqslant (a + c)\lfloor m \rfloor_{\mathrm{e}}$ *if* $a + 2c > \frac{3w}{2} + 3$;
- *(vi)* $(m, t) \neq (w, 2)$, *and if* $a \geqslant \frac{w}{2} + 4$, $(m, t) \notin \{(w - 1, 2), (w, 4)\}$.

Then there exists an $(N, 3^a, 4^b, 5^c, 6^d, k)$-decomposition of $K_{u+w} - K_u$ that includes cycles with lengths $(3^a, 4^b, 5^c, 6^d, k)$ that each contain at most one pure edge.

Proof. Let U and W be disjoint sets of sizes u and w and observe that $K_{U \cup W} - K_U = K_{U,W} \cup K_W$. We first choose integers a_2, a_3, c_2 and c_3. Let (a_3, c_3) be the leftmost pair from the appropriate row of the table below such that $a_3 \leqslant a$, $c_3 \leqslant c$, and $a_2 + 2c_2 \not\equiv 2 \pmod{w}$ where $a_2 = a - \frac{w}{2} - a_3$ and $c_2 = c - c_3$. It is routine to check using (iv) that some pair in the appropriate row will always satisfy these conditions.

case	(a_3, c_3)
$a - \frac{w}{2}$ even	$(0,1), (2,0), (2,1), (4,0)$
$a - \frac{w}{2}$ odd	$(1,0), (1,1), (3,0)$

This choice ensures that a_2, a_3, c_2 and c_3 are nonnegative integers such that a_2 is even, $\frac{w}{2} + a_2 + a_3 = a$, $c_2 + c_3 = c$, $a_3 + c_3 \in \{1, 2, 3, 4\}$, and $2a_3 + 4c_3 \leqslant 8$.

We now construct packings $\mathcal{P}_0, \ldots, \mathcal{P}_3$ as follows (we justify that these packings exist later).

- $\mathcal{P}_0$ is an $(N \setminus (m))$-packing of $K_W - I$, where I is a 1-factor on vertex set W. The reduced leave of $\mathcal{P}_0$ is a union of cycles $C^\dagger \cup C_1 \cup \cdots \cup C_n$, where

  - n and b_2 are the nonnegative integers such that $a_2 + 2c_2 = nw - 2b_2$ and $0 \leqslant b_2 \leqslant \frac{w-4}{2}$ (note that $a_2 + 2c_2 \not\equiv 2 \pmod{w}$);
  - $C^\dagger$ is an $(m - t + a_3 + c_3)$-cycle;
  - if $a_2 + c_2 > 0$, $|V(C_i)| \in \{\frac{w}{2}, \ldots, w\}$ for $2 \leqslant i \leqslant n$, and $|V(C_1)| \in \{\frac{1}{2}(w - b_2), \ldots, w - b_2\} \setminus \{1, 2\}$;
  - $|V(C_1)| + \cdots + |V(C_n)| = a_2 + c_2$.

  The cycle lengths $|V(C_1)|, \ldots, |V(C_n)|$ exist by Lemma 3.20, noting that $c_2 \leqslant 1$ by (iv), that a_2 is even, and that $a_2 + 2c_2 \not\equiv 2 \pmod{w}$.

- $\mathcal{P}_1$ is a $(3^{w/2})$-decomposition of $K_{U_1,W} \cup I$ for some $U_1 \subseteq U$ with $|U_1| = 1$.

- $\mathcal{P}_2$ is a $(3^{a_2}, 4^{b_2}, 5^{c_2})$-decomposition of $K_{U_2,W} \cup C_1 \cup \cdots \cup C_n$, where $U_2 \subseteq U \setminus U_1$ with $|U_2| = 2n$ and, if $b_2 > 0$, the union of the 4-cycles in $\mathcal{P}_2$ is a copy of $K_{2,2b_2}$.

- $\mathcal{P}_3$ is a $(3^{a_3}, 4^{b-b_2+3j}, 5^{c_3}, 6^{d-2j}, k, m)$-decomposition of $K_{U_3,W} \cup C^\dagger$, where $U_3 = U \setminus (U_1 \cup U_2)$ and

$$j = \begin{cases} 0 & \text{if } b \geqslant b_2, \\ \lceil \frac{1}{3}(b_2 - b) \rceil & \text{otherwise.} \end{cases}$$

  Furthermore, if $m > 0$, there is an m-cycle in $\mathcal{P}_3$ that contains $m - t$ edges of $C^\dagger$.

The union $\mathcal{P}' = \mathcal{P}_0 \cup \cdots \cup \mathcal{P}_3$ will be an $(N, 3^a, 4^{b+3j}, 5^c, 6^{d-2j}, k)$-decomposition of $K_{U\cup W} - K_U$. Using (i), we can see that $\mathcal{P}'$ will include cycles with lengths N that contain all but $a+c$ edges of K_W (the $\sum N - m$ edges in cycles in $\mathcal{P}_0$ are all in K_W, as are $m-t$ edges in an m-cycle in $\mathcal{P}_3$ if $m > 0$). So, because each odd cycle in $\mathcal{P}'$ must contain at least one edge of K_W, $\mathcal{P}'$ will include cycles with lengths $(3^a, 4^{b+3j}, 5^c, 6^{d-2j}, k)$ that each contain at most one edge of K_W.

If $b \geqslant b_2$, then $j = 0$ and this will complete the proof. Otherwise $b_2 > b$ and we will be able to apply Lemma 3.21 to $\mathcal{P}'$ (with $S \subseteq U_2 \cup U_3$) to obtain a decomposition $\mathcal{P}$ of $K_{U\cup W} - K_U$ with the required properties provided we can find $3j$ 4-cycles in $\mathcal{P}'$ that meet the hypotheses of Lemma 3.21. If $3j = b_2 - b$, then $b_2 \geqslant 3j$ and we will be able to use $3j$ 4-cycles from $\mathcal{P}_2$. If $3j \in \{b_2 - b + 1, b_2 - b + 2\}$, then $b_2 \geqslant 3j - 1$ because $b \geqslant 1$ by (iv) and we will be able to use $3j - 1$ 4-cycles from $\mathcal{P}_2$ and any one 4-cycle from $\mathcal{P}_3$. Note that Lemma 3.21 ensures $\mathcal{P}$ will include cycles with lengths $(3^a, 4^b, 5^c, 6^d, k)$ that each contain at most one edge of K_W.

So it remains to establish the existence of the packings $\mathcal{P}_0, \ldots, \mathcal{P}_3$. It is straightforward to see that $\mathcal{P}_1$ exists.

Proof that $\mathcal{P}_0$ exists. First observe that $m - t + a_3 + c_3 \in \{3, \ldots, w\}$ by (iii), (v), (vi) and our choices of a_3 and c_3. Then, by Theorem 1.4, a packing with the required properties exists by (iii) and because

$$\begin{aligned}
&\textstyle\sum(N \setminus (m)) + |E(C^\dagger)| + |E(C_1)| + \cdots + |E(C_n)| \\
&= (\tbinom{w}{2} + t - (a+c) - m) + (m - t + a_3 + c_3) + (a_2 + c_2) \\
&= \tbinom{w}{2} - \tfrac{w}{2}.
\end{aligned}$$

The first equality follows by (i) and the definitions of $C^\dagger$ and $C_1, \ldots, C_n$. The second equality follows because $a_2 + a_3 + \frac{w}{2} = a$ and $c_2 + c_3 = c$.

Proof that $\mathcal{P}_2$ exists. This is trivial if $(a_2, c_2) = (0, 0)$. If $(a_2, c_2) \neq 0$, then this follows immediately by Lemma 3.20 because $|V(C_i)| \in \{\frac{w}{2}, \ldots, w\}$ for $2 \leqslant i \leqslant n$, $|V(C_1)| \in \{\frac{1}{2}(w - b_2), \ldots, w - b_2\} \setminus \{1, 2\}$ and $|V(C_1)| + \cdots + |V(C_n)| = a_2 + c_2$.

Proof that $\mathcal{P}_3$ exists. We will show that $\mathcal{P}_3$ exists using either Lemma 3.19 (if $t \in \{2, 4\}$) or Lemmas 3.17 and 3.18 (if $t \geqslant 6$). We first establish some useful facts. Recall that $|E(C^\dagger)| = m - t + a_3 + c_3$ and note that $m > t$.

(a) $\boldsymbol{b - b_2 + 3j \geqslant 0}$ and $\boldsymbol{d - 2j \geqslant 0}$. Obviously $b - b_2 + 3j \geqslant 0$ by the definition of j and we can establish that $d - 2j \geqslant 0$ by a similar argument to the one used in the proof of Lemma 3.22.

(b) $2a_3+4(b-b_2+3j)+4c_3+6(d-2j)+k+t=w|U_3|$. Observe that

$$\begin{aligned}&2a_3+4(b-b_2+3j)+4c_3+6(d-2j)+k+t\\&=2(a-a_2-\tfrac{w}{2})+4(b-b_2)+4(c-c_2)+6d+k+t\\&=(2a+4b+4c+6d+k+t)-(2a_2+4b_2+4c_2)-w\\&=w|U_3|.\end{aligned}$$

The first equality follows because $a_2+a_3+\frac{w}{2}=a$ and $c_2+c_3=c$. The final equality follows because $2a+4b+4c+6d+k+t=uw$ by (ii), $w|U_2|=2a_2+4b_2+4c_2$ by the definition of U_2, and $|U_3|=u-|U_2|-1$.

(c) $|U_3|\geqslant 4$. Recall $|U_3|=u-2n-1$. If $n\leqslant 1$, then $|U_3|\geqslant 4$ because $u\geqslant 7$. So suppose $n\geqslant 2$. By the definition of n, this implies that $a_2+2c_2\geqslant w+4$ and hence that $a+2c\geqslant\frac{3w}{2}+4$ because $a\geqslant a_2+\frac{w}{2}$ and $c\geqslant c_2$. Thus, using (v),

$$uw\geqslant(a+c)\lfloor m\rfloor_{\mathrm{e}}\geqslant(\tfrac{w}{2}+a_2+c_2)\lfloor m\rfloor_{\mathrm{e}}.$$

Now $2a_2+4c_2+4b_2=2nw$ by the definitions of n and b_2, and hence $a_2+c_2=nw-2b_2-c_2$. Observe that $c_2\leqslant 1$ by (iv) and so $a_2+c_2\geqslant nw-2b_2-1$. Using this fact, we have $uw\geqslant((n+\frac{1}{2})w-2b_2-1)\lfloor m\rfloor_{\mathrm{e}}$. Rearranging yields

$$u-2n-1\geqslant(n+\tfrac{1}{2})(\lfloor m\rfloor_{\mathrm{e}}-2)-\tfrac{1}{w}(2b_2+1)\lfloor m\rfloor_{\mathrm{e}}.$$

Because $b_2\leqslant\frac{w-4}{2}$ and hence $\frac{1}{w}(2b_2+1)<1$, we see that

$$u-2n-1>(n-\tfrac{1}{2})(\lfloor m\rfloor_{\mathrm{e}}-2)-2. \tag{3.2.2}$$

So because $n\geqslant 2$ and $\lfloor m\rfloor_{\mathrm{e}}\geqslant 6$ by (v), we have $|U_3|=u-2n-1\geqslant 4$.

Case 1. Suppose that $t\in\{2,4\}$. Then $k=0$. So $\mathcal{P}_3$ exists by Lemma 3.19 because $|U_3|\geqslant 4$, $a_3+c_3\in\{1,2,3,4\}$, $2a_3+4c_3\leqslant 8$, and $w-m+\frac{t}{2}+1\geqslant a_3+c_3$ by (vi) (by our choice of a_3 and c_3, $a\geqslant\frac{w}{2}+4$ if $a_3+c_3=4$).
Case 2. Suppose that $t\geqslant 6$. Then $u\geqslant 9$ because $u\geqslant m\geqslant 8$ by (iii) and (v). By Lemma 3.18, to show that $\mathcal{P}_3$ exists it suffices to find a $(4^{b-b_2+3j},6^{d-2j},k)$-packing of $K_{U_3,W}$ whose reduced leave has a decomposition into a t-path and a $(2a_3+4c_3)$-path (note that $a_3+c_3\in\{1,2,3,4\}$). Also note that, since $K_{U_3,W}$ is an even graph, such a leave must be connected.

Noting that $2a_3+4c_3\leqslant 8$, by Lemma 3.17, to find such a packing it suffices to show that $\max(k+2,8)+t\leqslant 2|U_3|+2$. Note that $2|U_3|+2=2u-4n$. If $n\in\{0,1\}$, this holds because $u\geqslant 9$, $k\leqslant\frac{t+7}{3}$, and $t\leqslant u-3$ by (iii) and (v). So suppose that $n\geqslant 2$. We have

$$2u-4n>(2n-1)(\lfloor m\rfloor_{\mathrm{e}}-2)-2\geqslant 3t-2,$$

where the first inequality follows by multiplying (3.2.2) by 2, and the second holds because $n \geqslant 2$ and $\lfloor m \rfloor_{\mathrm{e}} \geqslant t+2$ by (v). Thus $\max(k+2, 8)+t \leqslant 2u-4n$ holds because $t \geqslant 6$ and $k \leqslant \frac{t+7}{3}$. □

3.2.4 Lists with few large entries

Lemma 3.28 is the most intricate of our base decomposition lemmas and requires some more preliminary results. Lemma 3.24 is an edge-colouring result that is easily obtained by combining a theorem of Fournier [54] with the well-known result that any graph with chromatic index at most ℓ has a proper edge-colouring with ℓ colours such that the sizes of any two colour classes differ by at most one (see [74]). Lemma 3.25 will allow us to decompose the union of $K_{3,w}$ and a graph on the part of size w whose vertices have odd degrees. This is useful when a is small. Lemmas 3.26 and 3.27 are more results giving cycle packings of the union of a complete bipartite graph with one or more cycles. The hypotheses of Lemmas 3.26 and 3.27 both concern the quantity $\rho = 2a+4c+t$. This is the number of edges of $K_{U,W}$ that will be used in the m-cycle that contains $m-t$ edges of K_W, and the a 3-cycles and c 5-cycles that each contain one edge of K_W.

Lemma 3.24 ([54, 74])**.** *Let G be a graph with maximum degree ℓ. If the subgraph of G induced by the vertices of degree ℓ contains no cycle, then G has a proper edge-colouring with ℓ colours such that the sizes of any two colour classes differ by at most one.*

Lemma 3.25. *Let G be a graph with vertex set W such that $|E(G)| = \lceil \frac{3}{4}|W| \rceil$ and each vertex of G has degree 1 or 3. Let $\{p_1, p_2, p_3\}$ be a set of three vertices not in W and let $\alpha\beta$ be an edge of G. Then there exists a $(5^{\lfloor 3|W|/4 \rfloor})$-packing $\mathcal{P}$ of $K_{\{p_1,p_2,p_3\},W} \cup G$ such that each cycle in $\mathcal{P}$ contains exactly one edge from G and*

- *if $|W| \equiv 0 \pmod 4$, then the reduced leave of $\mathcal{P}$ is empty;*
- *if $|W| \equiv 2 \pmod 4$, then the reduced leave of $\mathcal{P}$ is the 3-cycle (p_3, α, β).*

Proof. Let $w = |W|$. Let $B = \{x \in W : \deg_G(x) = 3\}$ and note that $|B| = \lceil \frac{3w}{4} \rceil - \frac{w}{2}$.

Case 1. Suppose that $w \equiv 0 \pmod 4$. Then $|E(G)| = \frac{3w}{4}$. Let $E = E(G)$ and let $W' = \{v_e : e \in E\}$ be a set of $|E|$ vertices disjoint from $W \cup \{p_1, p_2, p_3\}$. Let H be the graph obtained from G by adding the vertices in W' and then replacing each edge $yz \in E$ with the two edges yv_{yz} and zv_{yz}.

Note that the maximum degree of H is 3, no two vertices of degree 3 are adjacent in H, and $|E(H)| = \frac{3w}{2}$. So by Lemma 3.24 there exists a proper 3-edge colouring γ of H with colour set $\{1, 2, 3\}$ such that $|\gamma^{-1}(i)| = \frac{1}{3}|E(H)| =$

$\frac{w}{2}$ for $i \in \{1,2,3\}$. For each vertex $x \in V(H)$ we denote by $\gamma(x)$ the set of colours assigned by γ to the edges incident with x. For $i \in \{1,2,3\}$, let $W'_i = \{x \in V(H) : \gamma(x) = \{1,2,3\} \setminus \{i\}\}$ and $A_i = \{x \in W : \gamma(x) = \{i\}\}$. Let $A = \{x \in W : \deg_G(x) = 1\}$.

We shall show that there is a bijection $f : W' \to A$ such that $f(W'_i) = A_i$ for each $i \in \{1,2,3\}$. Then

$$\mathcal{P} = \{(p_{\gamma(yv_{yz})}, y, z, p_{\gamma(zv_{yz})}, f(v_{yz})) : yz \in E\}$$

will form a packing of $K_{\{p_1,p_2,p_3\},W} \cup G$ with $|E|$ 5-cycles, each of which contains four edges of $K_{\{p_1,p_2,p_3\},W}$ and one edge in E. Thus the reduced leave of $\mathcal{P}$ will be empty. So it suffices to show that such a bijection f exists, and hence it suffices to show that $|W'_i| = |A_i|$ for $i \in \{1,2,3\}$.

Obviously $|W'_1| + |W'_2| + |W'_3| = |E|$. Because each edge of H is incident with exactly one vertex in W', we have $|\gamma^{-1}(k)| = |W'_i| + |W'_j|$ for $\{i,j,k\} = \{1,2,3\}$. So, because the colour classes of γ have equal size, it follows that $|W'_1| = |W'_2| = |W'_3| = \frac{1}{3}|E| = \frac{w}{4}$. Furthermore, for any $\{i,j,k\} = \{1,2,3\}$, we have $2|\gamma^{-1}(i)| = |A_i| + |W'_j| + |W'_k| + |B|$ by considering the total degree of the graph induced by the colour class $\gamma^{-1}(i)$. Solving for $|A_i|$, it follows that $|A_1| = |A_2| = |A_3| = \frac{1}{3}|A| = \frac{w}{4}$.

Case 2. Suppose that $w \equiv 2 \pmod 4$. Then $|E(G)| = \frac{3w+2}{4}$. The proof proceeds as in Case 1 with the following exceptions. We let $E = E(G) \setminus \{\alpha\beta\}$, so that $|E| = \frac{3w-2}{4}$ and again $|E(H)| = \frac{3w}{2}$. At most one pair of vertices of degree 3 are adjacent in H, so Lemma 3.24 can still be applied and again each colour class of γ has size $\frac{w}{2}$. We may assume without loss of generality that $\gamma(\alpha\beta) = 3$. The reduced leave of the packing $\mathcal{P}$ will be the 3-cycle $\{p_3, \alpha, \beta\}$. Because each edge of H except $\alpha\beta$ is incident with exactly one vertex in W', we find $|W'_1| = |W'_2| = \frac{w-2}{4}$ and $|W'_3| = \frac{w+2}{4}$. We deduce that $|A_1| = |A_2| = \frac{w-2}{4}$ and $|A_3| = \frac{w+2}{4}$. □

Lemma 3.26. *Let U' and W be sets such that $|U'| \geqslant 2$ and $|W| \geqslant 6$ are even, let $(m,t) \in \{(0,0),(4,2),(5,2),(6,2),(6,4)\}$, let a, b, c and d be nonnegative integers, and let $\rho = 2a + 4c + t$. Suppose that*

(i) $d = 0$ *if* $|U'| = 2$*;*

(ii) $\rho + 4b + 6d = |U'||W|$*;*

(iii) $\rho \in \{0\} \cup \{4,6,\dots,2|W|\}$*;*

(iv) $t = 2$ *when* $\rho = 4$*, and* $(a,c) \notin \{(0,2),(1,1)\}$ *when* $\rho \in \{6,8\}$ *and* $t = 0$*; and*

(v) if $t \in \{2,4\}$ *and* $\rho = 2|W| - 2i$ *for some* $i \in \{0,1,2\}$*, then* $m \leqslant c+t+i+1$*.*

Let C be an $(a+c+m-t)$-cycle such that $V(C) \subseteq W$ (note that $a+c+m-t \in \{0\} \cup \{3,\ldots,|W|\}$). Then there exists a $(3^a,4^b,5^c,6^d,m)$-decomposition of $K_{U',W} \cup C$ that, if $m>0$, includes an m-cycle containing $m-t$ edges of C.

Proof. The result follows immediately from Lemma 3.19. To see that hypothesis (iv) of Lemma 3.19 is satisfied when $t \in \{2,4\}$, it may help to note the following facts. If $t \in \{2,4\}$ and $\rho = 2|W|-2i$ for some $i \in \{0,1,2\}$, then $c+t+i+1 = |W| + \frac{t}{2} + 1 - a - c$ by the definition of i, and hence $m \leqslant c+t+i+1$ implies $|W| - m + \frac{t}{2} + 1 \geqslant a+c$. If $t \in \{2,4\}$ and $\rho \leqslant 2|W|-6$, then $a+c \leqslant |W| - 3 - \frac{t}{2}$ because $\rho \leqslant 2|W|-6$, and hence $a+c \leqslant |W| - m + \frac{t}{2} + 1$ because $(m,t) \in \{(4,2),(5,2),(6,2),(6,4)\}$. □

Lemma 3.27. *Let U' and W be sets with $|U'|,|W|$ even, $|U'| \geqslant 4$ and $|W| \geqslant 10$, let $(m,t) \in \{(0,0),(4,2),(5,2),(6,2),(6,4)\}$, and let a, b, c and d be nonnegative integers, and let $\rho = 2a+4c+t$. Suppose that*

(i) $\rho + 4b + 6d = |U'||W|$;

(ii) $\rho \in \{2|W|-4, 2|W|-2, \ldots, 4|W|\}$, $t=2$ *if* $\rho \in \{2|W|-4, 2|W|-2\}$, *and* $t \in \{2,4\}$ *if* $\rho = 2|W|$; *and*

(iii) if $\rho \geqslant 4|W|-6$ *and* $t \in \{2,4\}$ *then* $c \geqslant 3$.

Then there are integers $\ell_1,\ell_2 \in \{3,\ldots,w\}$ such that $\ell_1+\ell_2 = a+c+m-t$ and, for any edge-disjoint cycles C_1 and C_2 in K_W with lengths ℓ_1 and ℓ_2,

- *if $|U'| \geqslant 6$ or d is even, there exists a $(3^a,4^b,5^c,6^d,m)$-decomposition $\mathcal{P}$ of $K_{U',W} \cup C_1 \cup C_2$;*
- *if $|U'|=4$, d is odd and $c \geqslant 1$, there exists a $(3^a,4^b,5^{c-1},6^{d-1},m)$-packing $\mathcal{P}$ of $K_{U',W} \cup C_1 \cup C_2$ whose reduced leave contains exactly one edge of K_W and has a $(3,4,4)$-decomposition;*
- *if $|U'|=4$, d is odd and $c=0$, there exists a $(3^{a-1},4^b,6^{d-1},m)$-packing $\mathcal{P}$ of $K_{U',W} \cup C_1 \cup C_2$ whose reduced leave contains exactly one edge of K_W, has a $(4,5)$-decomposition, and has a vertex of degree 4 in W.*

Furthermore, if $m>0$, then in each case there is an m-cycle in $\mathcal{P}$ that contains $m-t$ edges of K_W (or C_2).

Proof. Let $w=|W|$. Let $U'_1 \subseteq U'$ with $|U'_1|=2$ and let $U'_2 = U' \setminus U'_1$. Let $\delta = 1$ if d is odd and $\delta = 0$ if d is even. Note that, by (i),

$$\rho + 2\delta \equiv 0 \pmod{4}. \tag{3.2.3}$$

We will select values for a_1, b_1, c_1, a_2, b_2, c_2 and d_2 such that $a_1+a_2+c_1+c_2 = a+c$ according to the following tables (the criteria for the cases are given below).

case	a_1	b_1	c_1
1	$\min(\lfloor a \rfloor_{\mathrm{e}}, w)$	0	$\frac{w-a_1}{2}$
2	$\min(\lfloor a \rfloor_{\mathrm{e}}, w-4)$	2	$\frac{w-a_1}{2}-2$
3	$\min(\lfloor a \rfloor_{\mathrm{e}}, w-4)$	2	$\frac{w-a_1}{2}-2$
4	$\min(\lfloor a+\delta \rfloor_{\mathrm{e}}, w)$	0	$\frac{w-a_1}{2}$
5	$\max(0, w-4-2c+2\delta)$	2	$\frac{w-a_1}{2}-2$
6	$w-2\delta$	δ	0
7	$w-4$	2	0

case	a_2	b_2	c_2	d_2
1	$a-a_1$	b	$c-c_1$	d
2	$a-a_1$	$b-2$	$c-c_1$	d
3	$a-a_1$	$b+1$	$c-c_1$	$d-2$
4	$a+\delta-a_1$	$b+\frac{3(d-\delta)}{2}+2\delta$	$c-\delta-c_1$	0
5	$a+\delta-a_1$	$b+\frac{3(d-\delta)}{2}+2\delta-2$	$c-\delta-c_1$	0
6	$a-\delta-a_1$	$b+\frac{3(d-\delta)}{2}$	δ	0
7	$a-\delta-a_1$	$b+\frac{3(d-\delta)}{2}+\delta-2$	δ	0

We will apply Lemma 3.26 to show that a_1+c_1 and a_2+c_2+m-t are in $\{3,\ldots,w\}$ and that, for any edge-disjoint cycles C_1 and C_2 in K_W with lengths a_1+c_1 and a_2+c_2+m-t, there is a $(3^{a_1},4^{b_1},5^{c_1})$-packing $\mathcal{P}_1$ of $K_{U_1',W}\cup C_1$ and a $(3^{a_2},4^{b_2},5^{c_2},6^{d_2},m)$-packing $\mathcal{P}_2$ of $K_{U_2',W}\cup C_2$ from which we can obtain a packing with the required properties. In each case the fact that the hypotheses of Lemma 3.26 are satisfied when constructing $\mathcal{P}_1$ and $\mathcal{P}_2$ can be deduced from (3.2.3), the hypotheses of this lemma and the criteria of the relevant case. In particular, we use $w \geqslant 10$ frequently. For brevity, let $\rho_1 = 2a_1+4c_1$ and $\rho_2 = 2a_2+4c_2+t$. For each case, we now detail the criteria for the case, explain how a packing with the required properties can be obtained from $\mathcal{P}_1\cup\mathcal{P}_2$, and justify some of the less obvious deductions required to see that the hypotheses of Lemma 3.26 are satisfied. To show that hypotheses (iii), (iv) and (v) of Lemma 3.26 are satisfied in constructing $\mathcal{P}_2$, it suffices to show that

$$\rho_2 \geqslant 4,\ t=2 \text{ if } \rho_2=4, \text{ and } (a_2,c_2)\notin\{(0,2),(1,1)\} \text{ if } \rho_2\in\{6,8\} \text{ and } t=0, \tag{3.2.4}$$

and

$$\rho_2 \leqslant 2w, \text{ and } m \leqslant c_2+t+i+1 \text{ when } t\in\{2,4\} \text{ and } i\in\{0,1,2\}, \tag{3.2.5}$$

where i is the integer such that $\rho_2 = 2w - 2i$.
Case 1: $|U'| \geqslant 6$, $\rho \geqslant 2w+8$, and $t \in \{2,4\}$ if $\rho = 2w+8$. In this case $\mathcal{P}_1 \cup \mathcal{P}_2$ is itself a packing with the required properties. Note that $\rho_1 = 2w$ and $\rho_2 = \rho - 2w$. We have $c_2 \geqslant 0$ because either $c_1 = 0$ or $a_2 \in \{0,1\}$ and $2a_2 + 4c_2 = \rho_2 - t \geqslant 4$. We have (3.2.4) by the criteria for this case. We have (3.2.5) by (iii) when $c_1 = 0$ and because

$$c_2 + t + i + 1 = \tfrac{1}{4}(\rho_2 - 2a_2 - t) + t + i + 1 \geqslant \tfrac{2w+2i+3t+4-2a_2}{4} \geqslant 7 \quad \text{for } t \in \{2,4\}$$

when $a_2 \in \{0,1\}$.
Case 2: $|U'| \geqslant 6$, $\rho \leqslant 2w+8$, $t = 0$ if $\rho = 2w+8$, and $b \geqslant 2$. In this case $\mathcal{P}_1 \cup \mathcal{P}_2$ is itself a packing with the required properties. Note that $\rho_1 = 2w - 8$ and $\rho_2 = \rho - 2w + 8$. We have $c_2 \geqslant 0$ because either $c_1 = 0$ or $a_2 \in \{0,1\}$ and $2a_2 + 4c_2 = \rho_2 - t \geqslant 4$ by (ii). We have (3.2.4) by (ii). We have (3.2.5) by the criteria for this case.
Case 3: $|U'| \geqslant 6$, $\rho \leqslant 2w+8$, $t = 0$ if $\rho = 2w+8$, and $b \in \{0,1\}$. Lemma 3.21 can be applied to $\mathcal{P}_1 \cup \mathcal{P}_2$ (using the two 4-cycles in $\mathcal{P}_1$ and any one 4-cycle in $\mathcal{P}_2$) to obtain a packing with the required properties. We have $d_2 \geqslant 0$ because $4b + 6d = |U'|w - \rho \geqslant 4w - 8$ by (i) and the criteria for this case. We have $c_2 \geqslant 0$, (3.2.4) and (3.2.5) by similar arguments to those in Case 2.
Case 4: $|U'| = 4$, $c \geqslant 1$, $\rho \geqslant 2w+8$, and $t \in \{2,4\}$ if $\rho \in \{2w+8, 2w+10\}$. Lemma 3.21 can be applied to $\mathcal{P}_1 \cup \mathcal{P}_2$ (using any $\frac{3}{2}(d-\delta)$ 4-cycles in $\mathcal{P}_1 \cup \mathcal{P}_2$) to obtain a $(3^{a+\delta}, 4^{b+2\delta}, 5^{c-\delta}, 6^{d-\delta}, m)$-decomposition of $K_{U',W} \cup C_1 \cup C_2$. If $\delta = 0$ this completes the proof and if $\delta = 1$ we can remove two 4-cycles and a 3-cycle to obtain a packing with the required properties. Note that $\rho_1 = 2w$ and $\rho_2 = \rho - 2w - 2\delta$. We have $c_2 \geqslant 0$ because either $c_1 = 0$ or $a_2 \in \{0,1\}$ and $2a_2 + 4c_2 = \rho_2 - t \geqslant 4$ (note that if $\delta = 1$, then $\rho \geqslant 2w+10$ by (3.2.3)). We have (3.2.4) by the criteria for this case. We have (3.2.5) by similar arguments to those in Case 1.
Case 5: $|U'| = 4$, $c \geqslant 1$, $\rho \leqslant 2w+10$, and $t = 0$ if $\rho \in \{2w+8, 2w+10\}$. Lemma 3.21 can be applied to $\mathcal{P}_1 \cup \mathcal{P}_2$ (using any $\frac{3}{2}(d-\delta)$ 4-cycles in $\mathcal{P}_1 \cup \mathcal{P}_2$) to obtain a $(3^{a+\delta}, 4^{b+2\delta}, 5^{c-\delta}, 6^{d-\delta}, m)$-decomposition of $K_{U',W} \cup C_1 \cup C_2$. If $\delta = 0$ this completes the proof and if $\delta = 1$ we can remove two 4-cycles and a 3-cycle to obtain a packing with the required properties. Note that $\rho_1 = 2w - 8$ and $\rho_2 = \rho - 2w + 8 - 2\delta$. We have $a_2 \geqslant 0$ because $2a + 4c = \rho - t \geqslant 2w - 6$ by (ii) and hence $a + 2c \geqslant w - 3$. Further, $a_2 \geqslant \delta$ when $\rho \geqslant 2w - 2$. We have $b_2 \geqslant 0$ because $4b + 6d = 4w - \rho \geqslant 2w - 10$ by (i) and the criteria for this case and hence $b \geqslant \frac{w-3d-5}{2}$. We have $c_2 \geqslant 0$ because either $c \geqslant \frac{w}{2} - 2 + \delta$ and $c_1 = \frac{w}{2} - 2$ or $c_1 = c - \delta$. We have (3.2.4) by (ii) (note that if $\delta = 1$, then $\rho \geqslant 2w - 2$ by (3.2.3) and that $a_2 \geqslant \delta$ when $\rho \geqslant 2w - 2$). We have (3.2.5) by the criteria for this case.

Case 6: $|U'| = 4$, $c = 0$, and $\rho \geqslant 2w + 8$. If $\delta = 1$, then we can ensure that the 4-cycle in $\mathcal{P}_1$ and the 5-cycle in $\mathcal{P}_2$ with one edge of K_W share at least one vertex in W. We will justify this below. Lemma 3.21 can be applied to $\mathcal{P}_1 \cup \mathcal{P}_2$ (using $\frac{3}{2}(d-\delta)$ 4-cycles in $\mathcal{P}_2$) to obtain a $(3^{a-\delta}, 4^{b+\delta}, 5^{\delta}, 6^{d-\delta}, m)$-decomposition of $K_{U',W} \cup C_1 \cup C_2$ in which, if $\delta = 1$, a 4-cycle with no edges of K_W and a 5-cycle with one edge of K_W share a vertex in W. If $\delta = 0$ this completes the proof and if $\delta = 1$ we can remove the 4-cycle and 5-cycle from the decomposition to obtain a packing with the required properties. Note that $\rho_1 = 2w - 4\delta$ and $\rho_2 = \rho - 2w + 6\delta$. We have $a_2 \geqslant 0$ because $2a = \rho - t \geqslant 2w + 4$ by the criteria for this case and hence $a \geqslant w + 2$. We have (3.2.4) by the criteria for this case. We have (3.2.5) because $\rho \leqslant 4w - 6\delta$ by (i) and because $\rho \leqslant 4w - 8$ when $t \in \{2, 4\}$ by (iii) (note that if $\delta = 1$, then $\rho \leqslant 4w - 10$ by (3.2.3)).

It remains to show that, if $\delta = 1$, then we can ensure that the 4-cycle X in $\mathcal{P}_1$ and the 5-cycle Y in $\mathcal{P}_2$ with one edge of K_W share at least one vertex in W. Note that $V(X) \cap W = W \setminus V(C_1)$. When $V(C_2) \nsubseteq V(C_1)$, we can permute the vertices of $\mathcal{P}_2$ so that the edge of Y in K_W is incident with a vertex in $V(C_2) \setminus V(C_1)$ and hence in $V(X)$. When $V(C_2) \subseteq V(C_1)$, noting that $|V(C_2)| \leqslant w - 2$ and $t \leqslant 4$, we can ensure that Y has a vertex in $W \setminus V(C_2)$. (This can be seen by directly applying Lemma 3.18 to construct $\mathcal{P}_2$. The hypotheses of Lemma 3.18 are satisfied since the reduced leave of a (4^{b_2})-packing of $K_{U'_2,W}$ is a copy of $K_{2,w-2b_2}$, which clearly has the required path decomposition.) We can then permute the vertices of $\mathcal{P}_2$ so that this vertex of Y is in $W \setminus V(C_1)$ and hence in $V(X)$.

Case 7: $|U'| = 4$, $c = 0$, $\rho \leqslant 2w + 6$. A packing with the required properties can be obtained from $\mathcal{P}_1 \cup \mathcal{P}_2$ as in Case 6. Note that $\rho_1 = 2w - 8$ and $\rho_2 = \rho - 2w + 8 + 2\delta$. We have $b_2 \geqslant 0$ because $4b + 6d = 4w - \rho \geqslant 2w - 6$ by (i) and the criteria for this case and hence $b \geqslant \frac{w-3d-3}{2}$. We have $a_2 \geqslant 0$ by similar arguments to those in Case 5. We have (3.2.4) by (ii). We have $\rho_2 \leqslant 16$ by the criteria for this case, so (3.2.5) holds. $\square$

Lemma 3.28. *Let $u \geqslant 5$ and $w \geqslant 10$ be integers such that u is odd and w is even. Let N be a list of integers and let a, b, c and d be nonnegative integers such that the following conditions hold.*

(i) $(\sum N) - t + a + c = \binom{w}{2}$, *where* $t \in \{0, 2, 4\}$;

(ii) $2a + 4b + 4c + 6d + t = uw$;

(iii) $3 \leqslant \ell \leqslant \min(u, w)$ *for each entry* ℓ *in* N, *and* $d = 0$ *if* $u = 5$;

(iv) *either* $a \geqslant \frac{w}{2}$ *and* $a + c \geqslant \frac{w}{2} + 3$, *or* $c \geqslant \frac{3w}{4}$ *and* $a + c \geqslant \frac{3w}{4} + 4$;

(v) *if* $b + d \leqslant 2$ *and* $t \in \{2, 4\}$, *then* $a \in \{0, 1, 2, 3, 4, \frac{w}{2}, \frac{w}{2} + 1, \frac{w}{2} + 2, \frac{w}{2} + 3\}$;

(vi) if $t \in \{2,4\}$, there is some entry m in N such that $(m,t) \in \{(4,2),(5,2),(6,2),(6,4)\}$.

Then there exists an $(N, 3^a, 4^b, 5^c, 6^d)$-decomposition of $K_{u+w} - K_u$ that includes cycles with lengths $(3^a, 4^b, 5^c, 6^d)$ that each contain at most one pure edge.

Proof. Let U and W be disjoint sets of sizes u and w and observe that $K_{U \cup W} - K_U = K_{U,W} \cup K_W$. Let $m = 0$ if $t = 0$. We first choose disjoint subsets U_1 and U_2 of U and nonnegative integers a_1, a_2, a_3, c_1, c_2 and c_3. Let a_1, c_1 and $|U_1|$ be given as follows.

case	a_1	c_1	$\|U_1\|$
$a \geqslant \frac{w}{2}$	$\frac{w}{2}$	0	1
$a < \frac{w}{2}$ and $w \equiv 0 \pmod 4$	0	$\frac{3w}{4}$	3
$1 \leqslant a < \frac{w}{2}$ and $w \equiv 2 \pmod 4$	1	$\frac{3w-2}{4}$	3
$a = 0$ and $w \equiv 2 \pmod 4$	0	$\frac{3w-2}{4}$	3

Using (iv), we see that $a_1 \leqslant a$ and $c_1 \leqslant c$. Further, let a', c', b_3 and d_3 be given as follows.

case	a'	c'	b_3	d_3
$a = 0$, $w \equiv 2 \pmod 4$, d is even	1	$c - c_1 - 2$	$b + 1$	d
$a = 0$, $w \equiv 2 \pmod 4$, d is odd	0	$c - c_1 - 1$	$b + 2$	$d - 1$
$a \geqslant 1$ or $w \equiv 0 \pmod 4$	$a - a_1$	$c - c_1$	b	d

Let $\rho = 2a' + 4c' + t$. Using (iv) we see that a', c', b_3 and d_3 are nonnegative. Let $|U_2|$ and ρ_3 be the nonnegative even integers that satisfy the conditions given below.

case	conditions
$(u, \|U_1\|) = (5,3)$ or $\rho \leqslant 8$	$\rho = \|U_2\|w + \rho_3$, $\|U_2\| = 0$
$10 \leqslant \rho \leqslant (u - \|U_1\| - 4)w + 8$	$\rho = \|U_2\|w + \rho_3$, $\rho_3 \in \{10, 12, \ldots, 2w+8\}$
$\rho \geqslant (u - \|U_1\| - 4)w + 10$, $(u, \|U_1\|) \neq (5,3)$	$\rho = \|U_2\|w + \rho_3$, $\|U_2\| = u - \|U_1\| - 4$

Note that $|U_2| \in \{0, 2, \ldots, u - |U_1| - 4\}$ unless $(u, |U_1|) = (5,3)$ and $|U_2| = 0$. Now let

$$a_2 = \min(\lfloor a' \rfloor_{\mathrm{e}}, \tfrac{1}{2}|U_2|w), \quad c_2 = \tfrac{1}{4}(|U_2|w - 2a_2), \quad a_3 = a' - a_2, \quad c_3 = c' - c_2.$$

By our definitions, $2a_2 + 4c_2 = |U_2|w$ and $2a_3 + 4c_3 + t = \rho_3$. Clearly, a_2, c_2 and a_3 are nonnegative. When $\rho_3 < 10$ we have $|U_2| = 0$ and $(a_3, c_3) = (a', c')$. Thus it follows from (iv) and the definitions of a' and c' that either $\rho_3 \geqslant 10$ or $a_3 + c_3 \geqslant 3$. Furthermore, c_3 is nonnegative because $c_2 = 0$ when $a_2 = \frac{1}{2}|U_2|w$, and $a_3 \in \{0,1\}$ and $2a_3 + 4c_3 + t = \rho_3 \geqslant 6$ when $a_2 = \lfloor a' \rfloor_{\mathrm{e}}$. It may be that

$a_1 + a_2 + a_3 \neq a$ or $c_1 + c_2 + c_3 \neq c$. However, $\mathcal{P}_3$ (defined below) will be produced by applying Lemma 3.26 or 3.27 with $(a, b, c, d) = (a_3, b_3, c_3, d_3)$ and then possibly removing cycles. Observe that ρ_3, w and t satisfy one of the following

$$\rho_3 \leqslant 2w,\ t \in \{0, 4\} \text{ if } \rho_3 \in \{2w-4, 2w-2\}, \text{ and } t = 0 \text{ if } \rho_3 = 2w. \tag{3.2.6}$$

$$\rho_3 \geqslant 2w-4,\ t = 2 \text{ if } \rho_3 \in \{2w-4, 2w-2\}, \text{ and } t \in \{2, 4\} \text{ if } \rho_3 = 2w. \tag{3.2.7}$$

We now construct packings $\mathcal{P}_0, \ldots, \mathcal{P}_3$ as follows (we later show how these packings produce the required decomposition, and justify that they exist).

- $\mathcal{P}_0$ is an $(N \setminus (m))$-packing of $K_W - I$, where I is a 1-factor on vertex set W. The reduced leave of $\mathcal{P}_0$ is the edge-disjoint union of cycles $C^\star, C_1, \ldots, C_n, C_1^\dagger, C_2^\dagger$, where
  - $C^\star$ is trivial if $a \geqslant \frac{w}{2}$ and $|E(C^\star)| = \lceil \frac{w}{4} \rceil$ otherwise; and
  - $n = \frac{|U_2|}{2}$, $|E(C_i)| \in \{\frac{w}{2}, \ldots, w\}$ for $1 \leqslant i \leqslant n$, and $\sum_{i=1}^{n} |E(C_i)| = a_2 + c_2$; and
  - $|E(C_1^\dagger)| + |E(C_2^\dagger)| = m - t + a_3 + c_3$.

  The cycle lengths $|E(C_1)|, \ldots, |E(C_n)|$ will be given by Lemma 3.20 (note that $a_2 + 2c_2 = nw$). The cycle lengths $|E(C_1^\dagger)|$ and $|E(C_2^\dagger)|$ will be given by Lemma 3.26 or 3.27.

- $\mathcal{P}_1$ is a $(3^{a_1}, 5^{c_1})$-packing of $K_{U_1,W} \cup I \cup C^\star$. The reduced leave L_1 of $\mathcal{P}_1$ is a 3-cycle if $a = 0$ and $w \equiv 2 \pmod 4$ and is trivial otherwise.

- $\mathcal{P}_2$ is a $(3^{a_2}, 5^{c_2})$-decomposition of $K_{U_2,W} \cup C_1 \cup \cdots \cup C_n$.

- $\mathcal{P}_3$ is a packing of $K_{U_3,W} \cup C_1^\dagger \cup C_2^\dagger$ that, if $m > 0$, includes an m-cycle containing $m - t$ edges of K_W, where $U_3 = U \setminus (U_1 \cup U_2)$, with a reduced leave L_3. The properties of $\mathcal{P}_3$ and L_3 divide according to the following cases. The cases are mutually exclusive because d_3 is defined so as to be even when $a = 0$ and $w \equiv 2 \pmod 4$.

  Case 1: $a = 0$, $w \equiv 2 \pmod 4$, and d is even.
  Then $\mathcal{P}_3$ is a $(3^{a_3-1}, 4^{b_3-1}, 5^{c_3}, 6^{d_3}, m)$-packing, L_3 has exactly one pure edge, L_3 has a $(3, 4)$-decomposition, and $L_1 \cup L_3$ has a vertex of degree 4;

  Case 2: $a = 0$, $w \equiv 2 \pmod 4$, and d is odd.
  Then $\mathcal{P}_3$ is a $(3^{a_3}, 4^{b_3-2}, 5^{c_3}, 6^{d_3}, m)$-packing, L_3 has no pure edges, L_3 has a $(4, 4)$-decomposition, and $L_1 \cup L_3$ has a vertex of degree 4;

Case 3: (3.2.7) **holds, $|U_3| = 4$, d_3 is odd, and $c_3 \geqslant 1$.**
Then $\mathcal{P}_3$ is a $(3^{a_3}, 4^{b_3}, 5^{c_3-1}, 6^{d_3-1}, m)$-packing, L_3 has exactly one pure edge, and L_3 has a $(3, 4, 4)$-decomposition;

Case 4: (3.2.7) **holds, $|U_3| = 4$, d_3 is odd, and $c_3 = 0$.**
Then $\mathcal{P}_3$ is a $(3^{a_3-1}, 4^{b_3}, 5^{c_3}, 6^{d_3-1}, m)$-packing, L_3 has exactly one pure edge, L_3 has a $(4, 5)$-decomposition, and there is a vertex in W with degree 4 in L_3;

Case 5: otherwise.
Then $\mathcal{P}_3$ is a $(3^{a_3}, 4^{b_3}, 5^{c_3}, 6^{d_3}, m)$-decomposition and L_3 is trivial.

Let $\mathcal{P}' = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$. Then $\mathcal{P}'$ is a packing of $K_{U\cup W} - K_U$ with reduced leave $L_1 \cup L_3$. If we are in Case 5 then $\mathcal{P}'$ is an $(N, 3^a, 4^b, 5^c, 6^d)$-decomposition with the required properties. Otherwise we can obtain an $(N, 3^a, 4^b, 5^c, 6^d)$-decomposition $\mathcal{P}$ of $K_{U\cup W} - K_U$ with the required properties by applying Lemma 3.15 with m and m' as per the following table. That $\mathcal{P}'$ is an $(N, 3^a, 4^b, 5^c, 6^d)$-decomposition in Case 5, and that the entries in the second and third columns of the table are correct, can be checked using the definitions of $\mathcal{P}_0, \ldots, \mathcal{P}_3$, a', c', a_3, b_3, c_3 and d_3.

Using (i), we can see that $\mathcal{P}'$ includes cycles with lengths N that contain all but $a + c$ edges of K_W (the $\sum N - m$ edges in cycles in $\mathcal{P}_0$ are all in K_W, as are $m - t$ edges in an m-cycle in $\mathcal{P}_3$ if $m > 0$). The same is true of $\mathcal{P}$, since Lemma 3.15 yields a repacking. So, because each odd cycle in $\mathcal{P}$ contains at least one edge of K_W, $\mathcal{P}$ includes cycles with lengths $(3^a, 4^b, 5^c, 6^d)$ that each contain at most one edge of K_W.

case	cycle type of $\mathcal{P}'$	size of $L_1 \cup L_3$	(m, m')
1	$(N, 3^a, 4^b, 5^{c-2}, 6^d)$	10	$(5, 5)$
2	$(N, 3^a, 4^b, 5^{c-1}, 6^{d-1})$	11	$(5, 6)$
3	$(N, 3^a, 4^b, 5^{c-1}, 6^{d-1})$	11	$(5, 6)$
4	$(N, 3^{a-1}, 4^b, 5^c, 6^{d-1})$	9	$(3, 6)$

So it remains to establish the existence of the packings $\mathcal{P}_0, \ldots, \mathcal{P}_3$. We first establish three useful facts.

(a) $\rho_3 + 4b_3 + 6d_3 = |U_3|w$. It follows from the definitions of a', c', b_3 and d_3 that $2a' + 4b_3 + 4c' + 6d_3 + t = 2a + 4b + 4c + 6d + t - |U_1|w$. It follows from the definitions of a_2 and c_2 that $2a_2 + 4c_2 = |U_2|w$. Thus, because $a_3 = a' - a_2$ and $c_3 = c' - c_2$, we have $2a_3 + 4b_3 + 4c_3 + 6d_3 + t = (u - |U_1| - |U_2|)w$ by (ii).

(b) If $|U_3| = 2$, $t \in \{2, 4\}$ and $\rho_3 = 2w - 2i$, then $6 \leqslant c_3 + t + i + 1$. Because $|U_3| = 2$, $(u, |U_1|) = (5, 3)$ by the definition of U_2 and thus $a \leqslant \frac{w}{2} - 1$ by the definition of U_1. So from $2a_3 + 4c_3 + t = 2w - 2i$ we deduce $c_3 \geqslant \frac{w+2-t-2i}{4}$ and hence $c_3 + t + i + 1 \geqslant \frac{w+3t+2i+6}{4}$. Because $w \geqslant 10$ and $t \geqslant 2$, the result follows.

(c) $\rho_3 \leqslant 4w$ and, if $\rho_3 \geqslant 4w-6$ and $t \in \{2,4\}$, then $c_3 \geqslant 3$. If $\rho_3 \leqslant 2w+8$, then $\rho_3 < 4w-6$ (note $w \geqslant 10$). If $\rho_3 > 2w+8$, then $|U_2| = u-|U_1|-4$ and $|U_3| = 4$ by the definitions of U_2 and U_3. So $\rho_3 + 4b_3 + 6d_3 = 4w$ by (a) and hence $\rho_3 \leqslant 4w$. Furthermore, if we now suppose that $\rho_3 \geqslant 4w-6$, then $4b_3+6d_3 \leqslant 6$ and so $b_3+d_3 \leqslant 1$. Then $b+d \leqslant 2$, because $b \leqslant b_3$ and $d \leqslant d_3+1$. So by (v), $a' \leqslant 4$ and hence $a_3 \leqslant 4$. Because $2a_3 + 4c_3 + t = \rho_3 \geqslant 4w-6$, it follows that $c_3 \geqslant 6$.

Proof that $\mathcal{P}_0$ exists. First observe that $3 \leqslant \lceil \frac{w}{4} \rceil \leqslant w$ because $w \geqslant 10$. We choose lengths $|E(C_1)|, \dots, |E(C_n)|$ with the required properties, which exist by Lemma 3.20 because $a_2+2c_2 \equiv 0 \pmod{w}$. If $|U_3| = 2$ or (3.2.6) holds, then $a_3+c_3+m-t \in \{0\} \cup \{3,\dots,w\}$ by Lemma 3.26 with $(a,b,c,d) = (a_3,b_3,c_3,d_3)$ and $U' = U_3$ (the hypotheses are satisfied by (iii), (a), (b), and because either $\rho_3 \geqslant 10$ or $a_3+c_3 \geqslant 3$) and we let $|E(C_1^\dagger)| = a_3+c_3+m-t$ and $|E(C_2^\dagger)| = 0$. If $|U_3| \geqslant 4$ and (3.2.7) holds, then we let $|E(C_1^\dagger)|$ and $|E(C_2^\dagger)|$ be the cycle lengths given by Lemma 3.27 with $(a,b,c,d) = (a_3,b_3,c_3,d_3)$ and $U' = U_3$ (the hypotheses are satisfied by (a) and (c)). Then, by Theorem 1.4, a packing with the required properties exists by (iii) and because

$$
\begin{aligned}
&\textstyle\sum(N \setminus (m)) + |E(C^\star)| + |E(C_1^\dagger)| + |E(C_2^\dagger)| + |E(C_1)| + \dots + |E(C_n)| \\
&= (\tbinom{w}{2} + t - m - a - c) + |E(C^\star)| + (m - t + a_3 + c_3) + (a_2 + c_2) \\
&= \tbinom{w}{2} - (a+c) + (a'+c') + |E(C^\star)| \\
&= \tbinom{w}{2} - \tfrac{w}{2}.
\end{aligned}
$$

The first equality holds by (i) and the definitions of $C_1, \dots, C_n, C_1^\dagger, C_2^\dagger$. The second equality holds by the definitions of a_3 and c_3. The final equality holds because it follows from the definitions of a', c' and $C^\star$ that $a + c - a' - c' = \frac{w}{2} + |E(C^\star)|$.

Proof that $\mathcal{P}_2$ exists. This follows immediately by Lemma 3.20 because $|E(C_i)| \in \{\frac{w}{2}, \dots, w\}$ for $1 \leqslant i \leqslant n$ and $\sum_{i=1}^{n} |E(C_i)| = a_2 + c_2$.

Proof that $\mathcal{P}_3$ exists. We established above that if $|U_3| = 2$ or (3.2.6) holds, then $C_2^\dagger$ is trivial and we can apply Lemma 3.26 with $(a,b,c,d) = (a_3,b_3,c_3,d_3)$, $U' = U_3$ and $C = C_1^\dagger$. Also, we established that if $|U_3| \geqslant 4$ and (3.2.7) holds, then we can apply Lemma 3.27 with $(a,b,c,d) = (a_3,b_3,c_3,d_3)$, $U' = U_3$ and $(C_1, C_2) = (C_1^\dagger, C_2^\dagger)$. Let $\mathcal{P}_3'$ be the packing produced by applying the appropriate lemma. In Cases 3, 4 and 5, $\mathcal{P}_3'$ is itself a packing with the required properties. In Case 1, we can obtain a packing with the required properties by removing a 3-cycle and a 4-cycle from $\mathcal{P}_3'$ (note that $a_3 \geqslant 1$ and $b_3 \geqslant 1$ in this case). In Case 2, we can obtain a packing with the required properties by removing two 4-cycles from $\mathcal{P}_3'$ (note that $b_3 \geqslant 2$ in this case). In Cases 1 and 2 we will ensure that $L_1 \cup L_3$ has a vertex of degree 4 when we construct $\mathcal{P}_1$

below.
Proof that $\mathcal{P}_1$ exists. If $a \geqslant \frac{w}{2}$, then $(a_1, c_1) = (\frac{w}{2}, 0)$, $C^\star$ is trivial and a packing with the required properties clearly exists. So we may assume that $a < \frac{w}{2}$. By Lemma 3.25 there is a packing $\mathcal{P}'_1$ of $K_{U_1,W} \cup I \cup C^\star$ with $\lfloor \frac{3w}{4} \rfloor$ 5-cycles with a reduced leave L'_1 such that L'_1 is trivial when $w \equiv 0 \pmod 4$, L'_1 is a 3-cycle when $w \equiv 2 \pmod 4$, and L'_1 shares a vertex with L_3 when $w \equiv 2 \pmod 4$ and $a = 0$. If $w \equiv 0 \pmod 4$ or $a = 0$, then $\mathcal{P}'_1$ is a packing with the required properties. If $1 \leqslant a < \frac{w}{2}$ and $w \equiv 2 \pmod 4$, then $\mathcal{P}'_1 \cup \{L_1\}$ is a packing with the required properties. □

3.3 Proof of Theorem 3.1

This section contains the proof of the main result for the chapter. Lemma 3.29 dispenses with the case where the sum of odd entries in the list $(m_1, \ldots, m_\tau)$ is small. In this case we can obtain the required decomposition using known cycle decomposition results for the complete graph and the complete bipartite graph. The remaining cases of Theorem 3.1 are proved by repeatedly applying Lemma 3.3 to base the decompositions given by Lemmas 3.22, 3.23 and 3.28.

Lemma 3.29. *Let $u \geqslant 5$ and v be odd integers such that $v \geqslant u + 4$, and let $m_1, \ldots, m_\tau$ be a nondecreasing list such that the following hold*

(i) $m_1 \geqslant 3$ and $m_\tau \leqslant \min(u, v - u)$;

(ii) $m_1 + \cdots + m_\tau = \binom{v}{2} - \binom{u}{2}$; and

(iii) the sum of odd entries in $m_1, \ldots, m_\tau$ is at most $\frac{(v-u)(v-u-2)}{2}$.

Then there exists an $(m_1, \ldots, m_\tau)$-decomposition of $K_v - K_u$.

Proof. Let $w = v - u$ and let U and W be disjoint sets of size u and w respectively. Let $U_1 \subseteq U$ such that $|U_1| = 1$, and let $M = m_1, \ldots, m_\tau$. We will form an (M)-decomposition of $K_{U \cup W} - K_U$ from a packing $\mathcal{P}_0$ of $K_{W \cup U_1}$ and a packing $\mathcal{P}_1$ of $K_{U \setminus U_1, W}$.

Let $n_1, \ldots, n_s$ where $n_1 \leqslant \cdots \leqslant n_s$ be the sublist of M containing all of its even entries. Note that $n_1 + \cdots + n_s \geqslant \binom{u+w}{2} - \binom{u}{2} - \frac{w(w-2)}{2} = uw + \frac{w}{2}$ by (ii) and (iii). Let s' be the largest element of $\{1, \ldots, s\}$ such that $n_{s'} \leqslant 3n_{s'-1}$. Observe that $n_1 + \cdots + n_{s'} > w(u - 1)$ because

$$n_{s'+1} + \cdots + n_s < \sum_{i=0}^{\infty} \frac{n_s}{3^i} \leqslant \sum_{i=0}^{\infty} \frac{w}{3^i} < \frac{3w}{2}$$

where the first inequality follows because $n_i < \frac{1}{3} n_{i+1}$ for each $i \in \{s', \ldots, s-1\}$, and the second inequality follows because $n_s \leqslant w$ by (i).

We now define a sublist M_1 of $(n_1, \dots, n_{s'})$ as follows. Begin with M_1 empty. Iteratively apply the following procedure: while there is an entry x of $(n_1, \dots, n_{s'}) \setminus M_1$ such that $\sum M_1 + x \leqslant w(u-1)$, add the largest such entry to M_1. When no such entry exists, terminate the procedure and fix M_1. Let $M_0 = M \setminus M_1$, let t be the integer such that $\sum M_1 + t = w(u-1)$. Because $n_1 + \dots + n_{s'} > w(u-1)$, this procedure will terminate and M_1 will be a proper sublist of $(n_1, \dots, n_{s'})$. Thus, the smallest even entry in M_0 is $n_{s''}$ for some $s'' \in \{1, \dots, s'\}$. Also, t is even because $\sum M_1$ and $w(u-1)$ are even. We establish three more useful facts.

(a) $\boldsymbol{\sum M_1 = w(u-1) - t}$ and $\boldsymbol{\sum M_0 = \binom{w+1}{2} + t}$. The former follows from the definition of t and the latter follows from the former by the definition of M_0 and by (ii).

(b) $\boldsymbol{t \leqslant n_{s''} - 2}$ and there are at least two even entries in $\boldsymbol{M_0}$. If t were at least $n_{s''}$, then another even entry of $(n_1, \dots, n_{s'}) \setminus M_1$ would have been added to M_1 before the procedure terminated. So $t \leqslant n_{s''} - 2$. Because $n_1 + \dots + n_s \geqslant uw + \frac{w}{2}$, the even entries in M_0 sum to at least $\frac{3w}{2} + t$ and hence there are at least two by (i).

(c) $\boldsymbol{(n_{s'-u+1}, \dots, n_{s'})}$ is a sublist of $\boldsymbol{M_1}$ and $\boldsymbol{t \leqslant w - 4}$. Because $n_{s'} \leqslant w$ by (i), the first $u-1$ entries added to M_1 are $n_{s'}, n_{s'-1}, \dots, n_{s'-u+1}$. Thus, if $t = w-2$, then $n_{s''} = w$ by (b) and it would follow that $M_1 = (w^{u-1})$ and $t = 0$.

If $t = 0$, then an (M)-decomposition of $K_{U \cup W} - K_U$ is given by $\mathcal{P}_0 \cup \mathcal{P}_1$, where $\mathcal{P}_0$ is an (M_0)-decomposition of $K_{W \cup U_1}$, and $\mathcal{P}_1$ is an (M_1)-decomposition of $K_{U \setminus U_1, W}$. Noting (a), (c) and (i), we see that $\mathcal{P}_0$ exists by Theorem 1.4 and $\mathcal{P}_1$ exists by Theorem 2.22. Thus we can assume that $t \in \{2, 4, \dots, w-4\}$.

We now define integers p, $p^\dagger$, b and $b^\dagger$ and (possibly empty) lists M_0' and M_1'. We will then show that there exists an $(M_0 \setminus M_0')$-packing $\mathcal{P}_0$ of $K_{W \cup U_1}$ whose reduced leave L_0 is the edge-disjoint union of a p-path and a $p^\dagger$-path, an $(M_1 \setminus M_1')$-packing $\mathcal{P}_1$ of $K_{U \setminus U_1, W}$ whose reduced leave L_1 is the edge-disjoint union of a b-path and a $b^\dagger$-path such that there exists an (M_0', M_1')-decomposition $\mathcal{P}_2$ of $L_0 \cup L_1$. This will suffice to complete the proof as $\mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_2$ will be an (M)-decomposition of $K_{U \cup W} - K_U$.

- If there is an entry q in M_0 that is at least $t+3$, then let $M_0' = (q)$ and let $M_1' = (r)$ where r is the smallest entry of M_1. Let $b = t+2$, $b^\dagger = r-2$, $p = q - t - 2$ and $p^\dagger = 2$.

- If $t \geqslant 4$ and each entry in M_0 is at most $t+2$, then M_0 contains at least two entries equal to $t+2$ by (b). Let $M_0' = (t+2, t+2)$, and let M_1' be empty. Let $b = 2$, $b^\dagger = t-2$, $p = t$ and $p^\dagger = 4$.

- If $t = 2$ and each entry in M_0 is at most 4, then M_0 contains at least two entries equal to 4 by (b). Let $M_0' = (4,4)$ and let $M_1' = (r)$ where r is the smallest entry of M_1. Let $b = 4$, $b^\dagger = r-2$, $p = 4$, and $p^\dagger = 2$.

In each case note that $p + p^\dagger = \sum M_0' - t$ and $b + b^\dagger = \sum M_1' + t$. Hence $\sum(M_0 \setminus M_0') + p + p^\dagger = \binom{w+1}{2}$ and $\sum(M_1 \setminus M_1') + b + b^\dagger = w(u-1)$ by (a).

Proof that $\mathcal{P}_1$ exists. Using (a), (b), (c) and (i), it can be checked that by Lemma 3.16 there is an $(M_1 \setminus M_1')$-packing of $K_{U \setminus U_1, W}$ whose reduced leave has a decomposition into a b-path B and a $b^\dagger$-path $B^\dagger$ with end vertices x and y in W (apply Lemma 3.16(ii) with $m_i = b + b^\dagger$ if $2 \in \{b, b^\dagger\}$ and Lemma 3.16(i) with $m_i = b$ and $m_j = b^\dagger$ otherwise).

Proof that $\mathcal{P}_0$ exists. First suppose that $M_0' \neq (4,4)$. Using (a), (c) and (i), there is an $(M_0 \setminus M_0', p + p^\dagger)$-decomposition of $K_{W \cup U_1}$ by Theorem 1.4 (in each case $3 \leqslant p + p^\dagger \leqslant w + 1$). Let $\mathcal{P}_0$ be the result of removing a $p + p^\dagger$ cycle from this decomposition and permuting vertices so that the reduced leave of the resulting packing is the edge-disjoint union of paths P and $P^\dagger$ with end vertices x and y such that $V(P) \cap V(B) = V(P^\dagger) \cap V(B^\dagger) = \{x, y\}$. This relabelling is possible provided that $|V(B) \cap V(B^\dagger) \cap W| + p + p^\dagger - 2$, $|V(B) \cap W| + p - 1$ and $|V(B^\dagger) \cap W| + p^\dagger - 1$ are each at most $w + 1$ (for a proof of this, see [64, Lemma 5.2]). These inequalities can be checked using (c) and the facts that $|V(B) \cap V(B^\dagger) \cap W| \leqslant |V(B) \cap W| = \frac{b+2}{2}$ and $|V(B^\dagger) \cap W| = \frac{b^\dagger+2}{2}$.

Now suppose that $M_0' = (4,4)$. We form $\mathcal{P}_0$ as above, except that we permute vertices so that $V(P) \cap V(B) = \{x, y, z\}$ where $V(B) \cap W = \{x, y, z\}$ and z is not adjacent to x or y in P.

In each case the properties of B, $B^\dagger$, P and $P^\dagger$ ensure that there is an (M_0', M_1')-decomposition $\mathcal{P}_2$ of $L_0 \cup L_1$. □

We introduce some more notation. For a list M and an integer m we let $\nu_m(M)$ denote the number of entries of M that are equal to m. For a list M, let $\nu(M)$ denote the total number of entries of M and let $\nu_{\mathrm{o}}(M)$ denote the number of odd entries of M. For a nondecreasing list $M = (m_1, \ldots, m_s)$, we say that M is *well-behaved* if $m_s \leqslant 3m_{s-1}$.

We say that a list R is a *refinement* of an integer $m \geqslant 3$ if $\sum R = m$, each entry of R is at least 3 and at most one entry of R is odd. For any integer $m \geqslant 3$ the list (m) is a refinement of m. We say that a list R is a refinement of a list $M = (m_1, \ldots, m_s)$ if R can be reordered as $(R_1, \ldots, R_s)$ where R_i is a refinement of m_i for each $i \in \{1, \ldots, s\}$. The fact that $\nu_{\mathrm{o}}(R) = \nu_{\mathrm{o}}(M)$ is crucial and we will use it frequently. The *basic refinement* of an integer $m \geqslant 3$

is R, where

$$R = \begin{cases} (4^{m/4}), & \text{if } m \equiv 0 \pmod 4; \\ (3, 4^{(m-9)/4}, 6), & \text{if } m \equiv 1 \pmod 4 \text{ and } m \geqslant 9; \\ (4^{(m-6)/4}, 6), & \text{if } m \equiv 2 \pmod 4; \\ (3, 4^{(m-3)/4}), & \text{if } m \equiv 3 \pmod 4; \\ (5), & \text{if } m = 5. \end{cases}$$

We say that a list R is the basic refinement of a list $M = (m_1, \ldots, m_s)$ if R can be reordered as $(R_1, \ldots, R_s)$ where R_i is the basic refinement of m_i for each $i \in \{1, \ldots, s\}$. Note that for an even integer ℓ, the basic refinement is equal to the list $\mathcal{R}_\ell$ as defined in Section 2.2.

Lemma 3.30 shows how Lemma 3.3 can be repeatedly applied to our base decompositions to obtain the decompositions required by Theorem 3.1.

Lemma 3.30. *Let $u \geqslant 5$ and v be integers such that $v \geqslant u+4$, let N and $Z = (z_1, \ldots, z_q)$ be nondecreasing lists of integers such that $z_q \leqslant \min(u, v-u, 3z_{q-1})$, and let R be a refinement of Z. If there exists an (N, R)-decomposition of $K_v - K_u$ that includes cycles with lengths R that each contain at most one pure edge, then there exists an (N, Z)-decomposition of $K_v - K_u$.*

Proof. Assume that there exists such an (N, R)-decomposition $\mathcal{D}$ of $K_v - K_u$. Let ℓ be the number of entries in R. Note that $\ell \geqslant q$, and that if $\ell = q$, then $R = Z$ and the result is obviously true. So suppose that $\ell > q$. By induction, it suffices to show that there is an (N, R')-decomposition $\mathcal{D}'$ of $K_v - K_u$ where R' is a refinement of Z with $\ell - 1$ entries and $\mathcal{D}'$ contains cycles with lengths R' that each contain at most one pure edge. Let $R_1, R_2, \ldots, R_q$ be a reordering of R so that R_i is a refinement of z_i for $i \in \{1, \ldots, q\}$.

Case 1. Suppose that there is exactly one list R_i in $R_1, \ldots, R_q$ such that $\nu(R_i) \geqslant 2$. Let $R_i = a_1, \ldots, a_{\nu(R_i)}$ and let $j = q$ if $i \neq q$ and $j = q-1$ if $i = q$. Let C_1, C_2 and C_3 be cycles in $\mathcal{D}$ of lengths a_1, a_2 and z_j that each contain at most one pure edge. We can obtain a decomposition $\mathcal{D}'$ with the required properties by applying Lemma 3.3 to $\mathcal{D} \setminus \{C_1, C_2, C_3\}$ with $h = z_j$, $m_1 = a_1$ and $m_2 = a_2$. We have $m_1 + m_2 + h \leqslant z_i + z_j \leqslant 2\min(u, v-u)$ from our hypotheses. We have $m_1 + m_2 \leqslant 3h$ because either $a_1 + a_2 \leqslant z_i \leqslant z_q = h$ or $(i, j) = (q, q-1)$, in which case $a_1 + a_2 \leqslant z_q \leqslant 3z_{q-1}$ by our hypotheses.

Case 2. Suppose that there are at least two lists in $R_1, \ldots, R_q$ that each have at least two entries. Let r be the largest entry in $R_1, \ldots, R_q$ and let $i \in \{1, \ldots, q\}$ such that r is an entry of R_i. Let j be an element of $\{1, \ldots, q\} \setminus \{i\}$ such that $\nu(R_j) \geqslant 2$ and let $R_j = a_1, \ldots, a_{\nu(R_j)}$. Let C_1, C_2 and C_3 be cycles in $\mathcal{D}$ of lengths a_1, a_2 and r that each contain at most one pure edge. We can obtain a decomposition $\mathcal{D}'$ with the required properties by applying Lemma 3.3 to $\mathcal{D} \setminus \{C_1, C_2, C_3\}$ with $h = r$, $m_1 = a_1$ and $m_2 = a_2$. We have

$m_1 + m_2 + h \leqslant z_i + z_j \leqslant 2\min(u, v-u)$ from our hypotheses. We have $m_1 + m_2 \leqslant 3h$ because $a_1, a_2 \leqslant r$. $\square$

We are now ready to prove the main result of this chapter.

Proof of Theorem 3.1. If there exists an $(m_1, \ldots, m_\tau)$-decomposition of $K_v - K_u$ then (i)–(iv) hold by Lemma 3.2. So it remains to show, for any integers u and v with $v - u \geqslant 10$ and list $M = (m_1, \ldots, m_\tau)$, that if $m_\tau \leqslant \min(u, v-u, 3m_{\tau-1})$ and (i)–(iv) hold, then there exists an (M)-decomposition of $K_v - K_u$.

If $m_1 = m_2 = \cdots = m_\tau$ then the result follows by Theorem 2.2 (m_i odd) or Theorem 1.14 (m_i even, see [64]). If $u = 1$, there is an (M)-decomposition of K_v by Theorem 1.4 and $K_v = K_v - K_1$. If $u = 3$, there is an $(M, 3)$-decomposition of K_v by Theorem 1.4 and deleting the edges of a 3-cycle produces an (M)-decomposition of $K_v - K_3$. If the sum of odd entries in M is at most $\frac{(v-u)(v-u-2)}{2}$, then the result follows by Lemma 3.29. Thus we can suppose that $m_1 < m_\tau$, $u \geqslant 5$, and the sum of odd entries in M is greater than $\frac{(v-u)(v-u-2)}{2}$.

We will proceed as follows. First we choose a sublist Z of M such that Z is well-behaved. Then we define a refinement $R = (3^a, 4^b, 5^c, 6^d, k)$ of Z such that a, b, c, d and $M \setminus Z$ satisfy the hypotheses of Lemma 3.22, 3.23 or 3.28 (R is not always the basic refinement of Z but it is always 'close' to it). The appropriate lemma will then yield an $(M \setminus Z, R)$-decomposition $\mathcal{D}$ of $K_v - K_u$ that contains cycles with lengths R that each contain at most one pure edge. Applying Lemma 3.30 will then produce an (M)-decomposition of $K_v - K_u$. So it remains to define Z and R, and to show that the hypotheses of Lemma 3.22, 3.23 or 3.28 are satisfied. In each of the following cases we will also specify the entry m that is used in applications of Lemma 3.22, 3.23 and 3.28.

Let $w = v - u$. Throughout this proof we employ some notational shorthand concerning lists. For a list X, a set S and an integer x, we write $x \in X$ if at least one entry of X is equal to x, $X \subseteq S$ if each entry of X is an element of S, and $\max_{\mathrm{e}}(X)$ for the largest even entry of X. For a sublist $X = x_1, \ldots, x_s$ of M, we define $\sum_{\mathrm{e}} X = \sum_{i=1}^{s} \lfloor x_i \rfloor_{\mathrm{e}}$ and $t(X) = uw - \sum_{\mathrm{e}} X$. Note that $\sum_{\mathrm{e}} X$ can also be written as $\sum X - \nu_{\mathrm{o}}(X)$. Then $\sum_{\mathrm{e}} X = uw - t(X)$ and, by (iii), $\sum(M \setminus X) = \binom{w}{2} - \nu_{\mathrm{o}}(X) + t(X)$. Clearly $t(X)$ is always even. We abbreviate $t(Z)$ to t.

The proof splits into three cases, depending on $\nu_{\mathrm{o}}(M)$ and $\nu_5(M)$.

Case 1. Suppose that $\nu_{\mathrm{o}}(M) - \nu_5(M) \geqslant \frac{w}{2} + 3$. We aim to satisfy the hypotheses of either Lemma 3.23 (in Case 1a) or Lemma 3.28 (in Case 1b). We choose Z and m according to the following procedure.

1. Let Z'' be the list consisting of the $\frac{w}{2} + 3$ largest odd entries of M that are not equal to 5.

Each entry in M is at most $\min(u,w) \leqslant u$, so $\sum_{\mathrm{e}} Z'' \leqslant (u-1)(\frac{w}{2}+3)$ and hence $t(Z'') \geqslant \frac{(u+1)(w-6)}{2} + 6 \geqslant 2u+8$. Below, this will imply that $\nu(Z' \setminus Z'') \geqslant 2$.

2. Begin with $Z' = Z''$ and repeatedly add the largest entry of $M \setminus Z'$ to Z', until $M \setminus Z'$ is empty or Z' satisfies $t(Z') \leqslant \max(M \setminus Z') - 2$.

   It follows from this definition that $t(Z') \geqslant 0$ (note that $t(Z')$ is even). Because $\sum(M \setminus Z') = \binom{w}{2} - \nu_{\mathrm{o}}(Z') + t(Z')$, it follows from (iv) that $t(Z') = 0$ if $M \setminus Z'$ is empty. If $t(Z') \geqslant 2$, then $t(Z') \leqslant \max(M \setminus Z') - 2 \leqslant w-2$ and each entry in $Z' \setminus Z''$ is at least $\max(M \setminus Z')$.

3.1 If $t(Z') \neq 2$, then let $Z = Z'$ and let $m = 0$ if $t = 0$ and $m = \max(M \setminus Z')$ if $t > 0$. Note that $t = t(Z')$.

3.2 If $t(Z') = 2$ and $M \setminus Z' \nsubseteq \{3, w-1, w\}$, then let $Z = Z'$ and let m be the largest entry in $M \setminus Z'$ such that $4 \leqslant m \leqslant w-2$. Note that $t = 2$.

3.3 If $t(Z') = 2$, $M \setminus Z' \subseteq \{3, w-1, w\}$ and $3 \in M \setminus Z'$, then let $Z = (Z', 3)$ and let $m = 0$. Note that $t = 0$.

3.4 If $t(Z') = 2$, $M \setminus Z' \subseteq \{w-1, w\}$ and $w \in M \setminus Z'$, then let $Z = Z' \setminus (\min(Z''))$, and let $m = w$. Note that $t = 2 + \lfloor \min(Z'') \rfloor_{\mathrm{e}}$.

   We have $\min(Z'') \leqslant w-3$ and hence $t \leqslant w-2$. Otherwise, because $Z' \setminus Z'' \subseteq \{w\}$, $Z' = (w^i, (w-1)^{w/2+3})$ for some i and, by the definition of $t(Z')$, $t(Z') \equiv 6 \pmod{w}$. This contradicts $t(Z') = 2$.

3.5 If $t(Z') = 2$ and $M \setminus Z' \subseteq \{w-1\}$, then let $Z = (Z' \setminus (w), w-1)$, and let $m = w-1$. Note that $t = 4$.

   There is a w in Z' for otherwise $Z' \subseteq \{w-1\}$ and hence $M \subseteq \{w-1\}$ which contradicts $m_1 < m_\tau$. Further, because $t(Z') = 2$ and $Z' = (w^i, (w-1)^j)$ for some i and j, we have $j \equiv 1 \pmod{\frac{w}{2}}$ and hence can deduce from $\sum(M \setminus Z') = \binom{w}{2} - \nu_{\mathrm{o}}(Z') + t(Z')$ that $M \setminus Z' = ((w-1)^h)$ for some $h \equiv -1 \pmod{\frac{w}{2}}$. Thus $h \geqslant 2$ and so $m \in M \setminus Z$.

We first show that Z is well-behaved. For $i \in \{\tau-1, \tau\}$, because $\nu(Z' \setminus Z'') \geqslant 2$, if m_i is not added to Z'' in step 1 then it is added to Z' in step 2. So unless our procedure terminates at step 3.5, $(m_{\tau-1}, m_\tau)$ is a sublist of Z. If the procedure terminates at step 3.5, then $Z \subseteq \{w-1, w\}$, and it follows that Z is well-behaved.

Let $k = \lceil \frac{t+2}{3} \rceil_{\mathrm{e}}$ if $t \geqslant 12$ and $k = 0$ otherwise. We now note some important properties that hold for any refinement $(3^a, 4^b, 5^c, 6^d, k)$ of Z.

(a) $2a + 4b + 4c + 6d + k + t = uw$ and $\sum(M \setminus Z) + a + c = \binom{w}{2} + t$. These properties follow because $\sum_{\mathrm{e}} Z = uw - t$ and $\sum(M \setminus Z) = \binom{w}{2} - \nu_{\mathrm{o}}(Z) + t$.

(b) Either $(m, t) = (0, 0)$, or $t \geqslant 2$ and $t + 2 \leqslant m \leqslant w$. This is easy to check in each case.

(c) $a + c \geqslant \frac{w}{2} + 2$ and, if $m < w$, $a + c \geqslant \frac{w}{2} + 3$. This follows because $a + c = \nu_{\mathrm{o}}(Z)$, $\nu_{\mathrm{o}}(Z'') = \frac{w}{2} + 3$ and either Z'' is a sublist of Z or $Z'' \setminus (\min(Z''))$ is a sublist of Z (the latter occurs only in case 3.4 when $m = w$).

(d) $m \in M \setminus Z$ if $m > 0$, and $\min(Z \setminus Y) \geqslant m$, where $Y = Z'' \setminus (\min(Z''))$ if the procedure terminates at step 3.4 and $Y = Z''$ otherwise. This is clear if $m = 0$, so we may suppose that $m > 0$ and, by (b), that $t \geqslant 2$. It is easy to check in each case that $m \in M \setminus Z$ and also that $m \in M \setminus Z'$, and it follows that $\max(M \setminus Z') \geqslant m$. We noted $\min(Z' \setminus Z'') \geqslant \max(M \setminus Z')$ after step 2. If the procedure terminated at a step other than 3.5, then $\min(Z \setminus Y) \geqslant \min(Z' \setminus Z'')$ and the statement holds (note it did not terminate at step 3.3 because $m > 0$). If the procedure terminated at step 3.5, then $Z \setminus Y \subseteq \{w-1, w\}$ and $m = w - 1$, so the statement holds.

Case 1a. Suppose that $m \geqslant 7$. Then $t \geqslant 2$ by (b). Let $x = \max(Z)$. Let $(3^a, 4^b, 5^c, 6^d)$ be the basic refinement of $(Z \setminus (x), x - k)$ if $x - k \neq 9$ and the basic refinement of $(Z \setminus (x), 4, 5)$ if $x - k = 9$. Note that, by (b) and (d), $x \geqslant m \geqslant t + 2$. Thus $x - k = x \geqslant 7$ if $t \leqslant 10$ and, if $t \geqslant 12$, $x - k \geqslant t + 2 - \lceil \frac{t+2}{3} \rceil_{\mathrm{e}} \geqslant 8$. It can be seen that a, b, c, d and $M \setminus Z$ satisfy the conditions of Lemma 3.23 using (a) – (d) and the following facts.

- $c \in \{0, 1\}$ and $a = \nu_{\mathrm{o}}(Z) - c \geqslant \frac{w}{2} + 1$. We have $c = 0$ if $x - k \neq 9$ and $c = 1$ if $x - k = 9$ because $5 \notin Y$ by the definition of Y, $5 \notin Z \setminus Y$ by (d), and $x - k \neq 5$. Then $a = \nu_{\mathrm{o}}(Z) - c \geqslant \frac{w}{2} + 1$ using (c).
- $b \geqslant 1$. This is obvious if $x - k = 9$. If $x - k \neq 9$, the basic refinement of $x - k$ contains a 4 (recall that $x - k \geqslant 7$).
- Either $a \leqslant \frac{w}{2} + 3$ or $uw \geqslant (a + c)\lfloor m \rfloor_{\mathrm{e}}$. The former holds if $\min(Z'') < m$ because then each odd entry in $M \setminus Z''$ is less than m by the definition of Z'' and so every entry of $Z \setminus Y$ is even by (d). The latter holds if $\min(Y) \geqslant m$ because then $\min(Z) \geqslant m$ by (d) and so $(a + c)\lfloor m \rfloor_{\mathrm{e}} = \nu_{\mathrm{o}}(Z)\lfloor m \rfloor_{\mathrm{e}} \leqslant \sum_{\mathrm{e}} Z \leqslant uw - t$.
- $(m, t) \neq (w, 2)$ and, if $a \geqslant \frac{w}{2} + 4$, then $(m, t) \notin \{(w - 1, 2), (w, 4)\}$. Clearly $(m, t) \neq (w, 2)$ and $(m, t) \neq (w - 1, 2)$ because if $t = 2$, then the procedure terminated at step 3.2 and $m \leqslant w - 2$. If $(m, t) = (w, 4)$, then $Z \setminus Y \subseteq \{w\}$ by (d) and hence $a \leqslant \nu_{\mathrm{o}}(Z) = \nu_{\mathrm{o}}(Y) \leqslant \frac{w}{2} + 3$.

Case 1b. Suppose that $m \leqslant 6$. Then $t \in \{0, 2, 4\}$ by (b) and $k = 0$. Let $(3^a, 4^b, 5^c, 6^d)$ be the basic refinement of Z. It can be seen that a, b, c, d and $M \setminus Z$ satisfy the conditions of Lemma 3.28 using (a) – (d) and the following facts.

- If $u = 5$ then $m_\tau \leqslant 5$ by our hypotheses so $d = 0$.
- $a \geqslant \frac{w}{2} + 2$ because $\nu_{\mathrm{o}}(Y) \geqslant \frac{w}{2} + 2$, $5 \notin Y$, and Y is a sublist of Z.
- If $b + d \leqslant 2$ and $t \in \{2, 4\}$, then $a \leqslant \frac{w}{2} + 3$. Because $b + d \leqslant 2$, $Z \setminus (y_1, y_2) \subseteq \{3, 5\}$ for some $y_1, y_2 \in Z$ (note that the basic refinement of any integer in $\{3, \ldots, w\} \setminus \{3, 5\}$ contains a 4 or a 6). For $i \in \{1, 2\}$, if $y_i \geqslant 7$ and y_i is odd, then $y_i \in Y$ and $y_i \notin Z \setminus Y$. Because $t \geqslant 2$, $3 \notin Z \setminus Y$, using (b) and (d). Thus any odd entries in $Z \setminus Y$ are 5s, and $a \leqslant \frac{w}{2} + 3$.

Case 2. Suppose that $\nu_{\mathrm{o}}(M) - \nu_5(M) < \frac{w}{2} + 3$, $\nu_5(M) \geqslant w$ and $\nu_{\mathrm{o}}(M) \geqslant w + 4$. We aim to satisfy the hypotheses of Lemma 3.28. We choose Z and m according to the following procedure.

1. Begin with $Z'' = (5^{\lceil 3w/4 \rceil})$ and add the four largest odd entries of $M \setminus (5^w)$ to Z''. Note that $\nu_5(M \setminus Z'') \geqslant \lfloor \frac{w}{4} \rfloor$ because $\nu_5(M) \geqslant w$.

   Because each odd entry in M is at most $\min(u, w-1) \leqslant u$, $\sum_{\mathrm{e}} Z'' \leqslant 3w + 2 + 4(u-1)$ and hence $t(Z'') \geqslant (u-3)(w-4) - 10$. This implies that $t(Z'') > 0$ (note that $u \geqslant 5$ and $w \geqslant 10$).

2. Begin with $Z' = Z''$ and repeatedly add the largest entry of $M \setminus Z'$ to Z', until $M \setminus Z'$ is empty or Z' satisfies $t(Z') \leqslant \max(M \setminus Z') - 2$.

   It follows from this definition that $t(Z') \geqslant 0$ (note that $t(Z')$ is even). Because $\sum(M \setminus Z') = \binom{w}{2} - \nu_{\mathrm{o}}(Z') + t(Z')$, it follows from (iv) that $t(Z') = 0$ if $M \setminus Z'$ is empty. If $t(Z') \geqslant 2$, then $t(Z') \leqslant \max(M \setminus Z') - 2 \leqslant w - 2$ and each entry in $Z' \setminus Z''$ is at least $\max(M \setminus Z')$.

3.1 If $M \setminus Z'$ is empty or $\max(M \setminus Z') \leqslant 6$, then let $Z = Z'$. Let $m = 0$ if $t = 0$ and $m = \max(M \setminus Z)$ if $t > 0$. Note that $t = t(Z') \leqslant 4$.

3.2 If $\max(M \setminus Z') > 6$, then let $Z = (Z', 5^i)$ where $i = \lfloor \frac{t(Z')}{4} \rfloor$ and let $m = 0$ if $t = 0$ and $m = 5$ if $t = 2$. Observe that $t = 0$ if $t(Z') \equiv 0 \pmod 4$ and $t = 2$ if $t(Z') \equiv 2 \pmod 4$.

   To see that Z is a sublist of M note that

$$\nu_5(M \setminus Z') = \nu_5(M \setminus Z'') \geqslant \lfloor \tfrac{w}{4} \rfloor \geqslant \lfloor \tfrac{t(Z')}{4} \rfloor = i$$

because $\min(Z' \backslash Z'') \geqslant \max(M \backslash Z') > 6$ and $t(Z') \leqslant w-2$. Furthermore, $\lfloor \frac{w}{4} \rfloor > i$ when $t(Z') \equiv 2 \pmod 4$ and so $5 \in M \setminus Z$ when $t = 2$.

We first show that Z is well-behaved. This is obvious if $u \leqslant 7$ and hence $m_\tau \leqslant 7$ by our hypotheses. If $u \geqslant 9$ then $t(Z'') \geqslant (u-3)(w-4) - 10 > 2w$ and $\nu(Z' \setminus Z'') \geqslant 2$. Thus, for $i \in \{\tau - 1, \tau\}$ if m_i is not added to Z'' in step 1 then it is added to Z' in step 2. So $(m_{\tau-1}, m_\tau)$ is a sublist of Z and Z is well-behaved.

Let $(3^a, 4^b, 5^c, 6^d)$ be the basic refinement of Z. It can be seen using arguments similar to those of Case 1b that a, b, c, d and $M \setminus Z$ satisfy the hypotheses of Lemma 3.28. Note that $c \geqslant \frac{3w}{4}$ and $a + c \geqslant \frac{3w}{4} + 4$ because Z'' is a sublist of Z. If $b + d \leqslant 2$ and $t \geqslant 2$, then $a \leqslant 4$ because, by arguments similar to those used in Case 1b, $(Z \setminus (y_1, y_2)) \subseteq \{3, 5\}$ for some $y_1, y_2 \in Z$ and the only odd entries in $Z \setminus Z''$ are 5s.

Case 3. Suppose that $\nu_{\mathrm{o}}(M) - \nu_5(M) < \frac{w}{2} + 3$ and that either $\nu_{\mathrm{o}}(M) < w + 4$ or $\nu_5(M) < w$. We aim to satisfy the hypotheses of Lemma 3.22. Accordingly we redefine $t(X)$ and t. For a sublist X of M, we define $t(X) = (u-1)w - \sum_{\mathrm{e}} X$. Then $\sum_{\mathrm{e}} X = (u-1)w - t(X)$ and, by (iii), $\sum(M \backslash X) = \binom{w+1}{2} - \nu_{\mathrm{o}}(X) + t(X)$. Again, $t(X)$ is always even and we abbreviate $t(Z)$ to t. Observe that $\nu_{\mathrm{o}}(M) > \frac{w-2}{2} \geqslant 3$ since the sum of odd entries in M is greater than $\frac{w(w-2)}{2}$. Let σ be the sum of the $\nu_{\mathrm{o}}(M) - 3$ smallest odd entries in M.

Case 3a. Suppose further that $\sigma \leqslant \binom{w+1}{2}$. We choose Z and m according to the following procedure.

1. Let Z'' be a list consisting of the largest three odd entries of M.

   We have $\sum_{\mathrm{e}} Z'' \leqslant 3(u-1)$ and hence $t(Z'') \geqslant (u-1)w - 3(u-1) \geqslant 7u - 7 > 5u$. Below, this will imply that $\nu(Z \setminus Z'') \geqslant 5$.

2. Begin with $Z = Z''$ and repeatedly add the largest even entry of $M \backslash Z$ to Z, until $M \backslash Z$ contains no even entries or until Z satisfies $t \leqslant \max_{\mathrm{e}}(M \setminus Z) - 2$. Let $m = 0$ if $t = 0$, $m = \max(M \setminus Z)$ if $t \geqslant 4$, and let m be an entry of $M \setminus Z$ such that $4 \leqslant m \leqslant w - 1$ if $t = 2$ (we show below that such an entry exists).

   It follows from this definition that $t \geqslant 0$. To show that a suitable choice of m exists when $t = 2$, suppose otherwise that $M \setminus Z \subseteq \{3, w\}$. Then, the sum of the odd entries in M is at most $3(w-1) + 3(\frac{w}{2} - 1) = \frac{9w}{2} - 6$, because each of the three odd entries in Z is at most $w - 1$, each odd entry in $M \setminus Z$ is a 3, and $\nu_{\mathrm{o}}(M) - \nu_5(M) \leqslant \frac{w}{2} + 2$. This contradicts our assumption that the sum of the odd entries in M is greater than $\frac{w(w-2)}{2}$ (note that $w \geqslant 10$).

For $i \in \{\tau-1, \tau\}$, because $\nu(Z \setminus Z'') \geqslant 2$, if m_i is not added to Z'' in step 1 then it is added to Z in step 2. Thus $(m_{\tau-1}, m_\tau)$ is a sublist of Z and Z is well-behaved. Let $k = \lceil \frac{t+2}{3} \rceil_{\mathrm{e}}$ if $t \geqslant 12$ and $k = 0$ otherwise. Let $(3^a, 4^b, 5^c, 6^d)$ be the basic refinement of $(Z \setminus (m_\tau), m_\tau - k)$. If $m_\tau \leqslant 6$, then $m_\tau - k = m_\tau$. If $m_\tau \geqslant 7$ then, as in Case 1a, $m_\tau - k \geqslant 7$. Using arguments similar to those in the previous cases and the following facts we can see that a, b, c, d and $M \setminus Z$ satisfy the conditions of Lemma 3.22.

- $a + c = \nu_{\mathrm{o}}(Z) = 3$ and $(m, t) \neq (w, 2)$. The former follows from the definition of Z (note that $m_\tau - k$ and m_τ have the same parity). The latter follows from our choice of m.

- $m \geqslant t+2$ if $t > 0$. If $t = 2$ this is obvious by our choice of m, so we may suppose that $t \geqslant 4$. We have $\sum(M \setminus Z) = \binom{w+1}{2} - \nu_{\mathrm{o}}(Z) + t > \binom{w+1}{2}$ because $\nu_{\mathrm{o}}(Z) = 3$ and $t \geqslant 4$. However, the sum of the odd entries in $M \setminus Z$ is σ, and $\sigma \leqslant \binom{w+1}{2}$. Thus $M \setminus Z$ has an even entry and we have $m \geqslant \max_{\mathrm{e}}(M \setminus Z) \geqslant t + 2$ by the definition of Z.

Case 3b. Suppose further that $\sigma > \binom{w+1}{2}$. We choose Z and m according to the following procedure.

1. Let Z'' be a sublist of M with maximum sum subject to the constraints that $\nu(Z'') = \nu_{\mathrm{o}}(Z'') = 6$ and $\nu_5(Z'') \leqslant 1$.

   Such a sublist exists because $\nu_{\mathrm{o}}(M) \geqslant 6$ and $\nu_{\mathrm{o}}(M) - \nu_5(M) \geqslant 5$. These facts must hold because $\sigma > \binom{w+1}{2}$ and since either $\nu_{\mathrm{o}}(M) \leqslant w + 3$ or $\nu_5(M) \leqslant w - 1$ by the criteria for Case 3. We have $\sum_{\mathrm{e}} Z'' \leqslant 6(u-1)$ and hence $t(Z'') \geqslant (u-1)w - 6(u-1) > 3u$. Below, this will imply that $\nu(Z \setminus Z'') \geqslant 3$.

2. Begin with $Z = Z''$ and repeatedly add to Z the largest entry of $M \setminus Z$ not equal to 5 until Z satisfies $t \leqslant \max(M \setminus Z) - 2$. Let $m = \max(M \setminus Z)$ if $t > 0$ and let $m = 0$ if $t = 0$.

   It follows from this definition that $t \geqslant 0$. This process terminates with $M \setminus Z \nsubseteq \{5\}$, since $\sum(M \setminus Z) = \binom{w+1}{2} - \nu_{\mathrm{o}}(Z) + t \geqslant \frac{11w}{2} - (\frac{w}{2} + 2) - 1 \geqslant 5w - 3$ and $5\nu_5(M \setminus Z) \leqslant 5\nu_5(M) \leqslant 5(w-1)$, using the criteria for Case 3 and the fact that $\nu_5(Z) = \nu_5(Z'') \in \{0, 1\}$. Note that $t \leqslant \max(M \setminus Z) - 2 \leqslant w - 2$.

We must have $m_\tau \geqslant 7$ for otherwise, by the criteria for Case 3, $\sigma \leqslant 3(\frac{w}{2} + 2) + 5(w - 4) = \frac{13w}{2} - 14$, contradicting $\sigma > \binom{w+1}{2}$ (note $w \geqslant 10$). So, for $i \in \{\tau - 1, \tau\}$, because $\nu(Z \setminus Z'') \geqslant 2$, if m_i is not added to Z'' in step 1 then it is added to Z in step 2 (note that if $m_{\tau-1} = 5$ then $m_{\tau-1} \in Z''$). Thus $(m_{\tau-1}, m_\tau)$ is a sublist of Z and Z is well-behaved.

Let $k = \lceil \frac{t+2}{3} \rceil_{\mathrm{e}}$ if $t \geqslant 12$ and $k = 0$ otherwise. Let $(3^a, 4^b, 5^c, 6^d)$ be the basic refinement of $(Z \setminus (m_\tau), m_\tau - k)$ if $m_\tau - k \neq 9$ and the basic refinement of $(Z \setminus (m_\tau), 4, 5)$ if $m_\tau - k = 9$. As in Case 1a, $m_\tau - k \geqslant 7$. Using arguments similar to those in the previous cases and the following facts we can see that a, b, c, d and $M \setminus Z$ satisfy the conditions of Lemma 3.22.

- $a + 2c \leqslant w$. We have $a + 2c \leqslant \nu_{\mathrm{o}}(Z) + \nu_5(Z) + 1$ (equality occurs when $m_\tau - k = 9$). Thus, $a + 2c \leqslant \frac{w}{2} + 5 \leqslant w$ by the definition of Z and because $\nu_{\mathrm{o}}(M) - \nu_5(M) \leqslant \frac{w}{2} + 2$ and $\nu_5(Z) = \nu_5(Z'') \leqslant 1$.
- $a + c \geqslant 6$ because $a + c = \nu_{\mathrm{o}}(Z) \geqslant \nu_{\mathrm{o}}(Z'') = 6$.
- $b \geqslant 1$ as in Case 1a. $\square$

Chapter 4

Cycle Packings of λK_v

The main result of this chapter provides necessary and sufficient conditions for the existence of packings of the complete multigraph with cycles of arbitrary specified lengths. Our proof relies on Theorem 1.9 for cycle decompositions of the complete multigraph [31, 32] and on techniques for constructing packings of the complete graph with uniform length cycles [27, 63].

For positive integers λ and v, λK_v is the complete multigraph with λ parallel edges between each pair of v distinct vertices. For $\lambda \geqslant 2$, λK_v contains 2-cycles (pairs of parallel edges). For vertices x and y in a multigraph G, the *multiplicity* of xy is the number of edges in G which have x and y as their endpoints, denoted $\mu_G(xy)$. If $\mu_G(xy) \leqslant 1$ for all pairs of vertices in $V(G)$ then we say that G is a *simple* graph. A multigraph is said to be *even* if every vertex has even degree and is said to be *odd* if every vertex has odd degree. Definitions for decompositions and packings of graphs extend naturally to multigraphs, in particular a cycle decomposition of λK_v is a partition of the $\lambda\binom{v}{2}$ edges of λK_v into cycles.

Theorem 4.1. *Let $m_1, \ldots, m_\tau$ be a list and let λ and v be positive integers. Then there exists an $(m_1, \ldots, m_\tau)$-packing of λK_v if and only if*

(i) $2 \leqslant m_1, \ldots, m_\tau \leqslant v$;

(ii) $m_1 + \cdots + m_\tau = \lambda\binom{v}{2} - \delta$, *where δ is a nonnegative integer such that $\delta \neq 1$ when $\lambda(v-1)$ is even, $\delta \neq 2$ when $\lambda = 1$, and $\delta \geqslant \frac{v}{2}$ when $\lambda(v-1)$ is odd;*

(iii) $\sum_{m_i=2} m_i \leqslant \begin{cases} (\lambda-1)\binom{v}{2} - 2 & \text{if } \lambda \text{ and } v \text{ are odd and } \delta = 2, \\ (\lambda-1)\binom{v}{2} & \text{if } \lambda \text{ is odd; and} \end{cases}$

(iv) $m_\tau \leqslant \begin{cases} \frac{\lambda}{2}\binom{v}{2} - \tau + 2 & \text{if } \lambda \text{ is even and } \delta = 0, \\ \frac{\lambda}{2}\binom{v}{2} - \tau + 1 & \text{if } \lambda \text{ is even and } 2 \leqslant \delta < m_\tau. \end{cases}$

The necessity of conditions (i)–(iv) follows from Theorem 1.9 as we now show.

Lemma 4.2. *Let $m_1, \dots, m_\tau$ be a nondecreasing list and let λ and v be positive integers. If there exists an $(m_1, \dots, m_\tau)$-packing of λK_v then*

(i) $2 \leqslant m_1, \dots, m_\tau \leqslant v$;

(ii) $m_1 + \cdots + m_\tau = \lambda\binom{v}{2} - \delta$, *where δ is a nonnegative integer such that $\delta \neq 1$ when $\lambda(v-1)$ is even, $\delta \neq 2$ when $\lambda = 1$, and $\delta \geqslant \frac{v}{2}$ when $\lambda(v-1)$ is odd;*

(iii) $\sum_{m_i=2} m_i \leqslant \begin{cases} (\lambda-1)\binom{v}{2} - 2 & \text{if } \lambda \text{ and } v \text{ are odd and } \delta = 2, \\ (\lambda-1)\binom{v}{2} & \text{if } \lambda \text{ is odd; and} \end{cases}$

(iv) $m_\tau \leqslant \begin{cases} \frac{\lambda}{2}\binom{v}{2} - \tau + 2 & \text{if } \lambda \text{ is even and } \delta = 0, \\ \frac{\lambda}{2}\binom{v}{2} - \tau + 1 & \text{if } \lambda \text{ is even and } 2 \leqslant \delta < m_\tau. \end{cases}$

Proof. Suppose there exists an $(m_1, \dots, m_\tau)$-packing $\mathcal{P}$ of λK_v with leave L. Condition (i) is obvious. The degree of each vertex in λK_v is $\lambda(v-1)$, so if $\lambda(v-1)$ is even then L is an even multigraph and if $\lambda(v-1)$ is odd then L is an odd multigraph. Hence (ii) follows because an even graph cannot have a single edge, an even simple graph cannot have two edges, and an odd graph on v vertices has at least $\frac{v}{2}$ edges. To see that condition (iii) holds, note that there are at most $\lfloor\frac{\lambda}{2}\rfloor\binom{v}{2}$ edge-disjoint 2-cycles in λK_v. Furthermore, note that if λ and v are both odd and $\delta = 2$ then L is a 2-cycle (because L is an even multigraph and has two edges). If λ is even and $\delta = 0$ then (iv) follows directly from Theorem 1.9, so suppose λ is even and $2 \leqslant \delta < m_\tau$. Then L contains at least one cycle so there exists an $(m_1, \dots, m_\tau, M)$-decomposition of λK_v for some list M containing at least one entry. So (iv) follows from Theorem 1.9. □

It remains to prove the sufficiency of conditions (i)–(iv) in Theorem 4.1. We first adapt some tools that were crucial to the proof of Theorem 1.8 to packings of the complete multigraph. These are given in Section 4.1.

4.1 Preliminary results

In order to prove Lemmas 4.4 and 4.5 we require the following cycle switching lemma for cycle packings of multigraphs. Lemma 4.3 is closely related to the cycle switching method applied to simple graphs in Chapters 2 and 3 (see Lemma 1.22).

Lemma 4.3. *Let v and λ be positive integers, let M be a list of integers, let $\mathcal{P}$ be an (M)-packing of λK_v, let L be the leave of $\mathcal{P}$, let α and β be distinct vertices of L, and let π be the transposition $(\alpha\beta)$. Let E be a subset of $E(L)$ such that, for each vertex $x \in V(L) \setminus \{\alpha, \beta\}$, E contains precisely* $\max(0, \mu_L(x\alpha) - \mu_L(x\beta))$ *edges with endpoints x and α, and precisely* $\max(0, \mu_L(x\beta) - \mu_L(x\alpha))$ *edges with endpoints x and β (so E may contain multiple edges with the same endpoints), and E contains no other edges. Then there exists a partition of E into pairs such that for each pair $\{x_1y_1, x_2y_2\}$ of the partition, there exists an (M)-packing $\mathcal{P}'$ of λK_v with leave $L' = (L - \{x_1y_1, x_2y_2\}) + \{\pi(x_1)\pi(y_1), \pi(x_2)\pi(y_2)\}$.*

Furthermore, if $\mathcal{P} = \{C_1, \ldots, C_t\}$, then $\mathcal{P}' = \{C_1', \ldots, C_t'\}$ where for $i \in \{1, \ldots, t\}$, C_i' is a cycle of the same length as C_i such that for $i \in \{1, \ldots, t\}$

- *If neither α nor β is in $V(C_i)$, then $C_i' = C_i$;*
- *If exactly one of α and β is in $V(C_i)$, then $C_i' = C_i$ or $C_i' = \pi(C_i)$; and*
- *If both α and β are in $V(C_i)$, then $C_i' = Q_i \cup Q_i^*$ where $Q_i = P_i$ or $\pi(P_i)$, $Q_i^* = P_i^*$ or $\pi(P_i^*)$, and P_i and P_i^* are the two paths from α to β in C_i.*

When $\lambda(v-1)$ is even, Lemma 4.3 is identical to the original version of the result [31, Lemma 2.1] so this case has already been proved. Lemma 4.3, as it is stated here, differs from the original version [31, Lemma 2.1] in that here $\mathcal{P}$ is a cycle packing of λK_v regardless of the parity of $\lambda(v-1)$, whereas when $\lambda(v-1)$ is odd [31, Lemma 2.1] concerns a cycle packing of $\lambda K_v - I$, where I is a 1-factor of λK_v. However, in this case the proof of Lemma 4.3 follows from very similar arguments to those used in the corresponding case of the original proof.

In applying Lemma 4.3 we say that we are performing the (α, β)-switch with origin x and terminus y (where $\{x_1, y_1, x_2, y_2\} \subseteq \{\alpha, \beta, x, y\}$). Note that x_1y_1 and x_2y_2 may be parallel edges, in which case $x = y$.

For integers $p \geqslant 2$ and $q \geqslant 1$, a (p, q)-*lasso* is the union of a p-cycle and a q-path such that the cycle and the path share exactly one vertex and that vertex is an end-vertex of the path. The order of a (p, q)-lasso is $p + q$. A (p, q)-lasso with cycle $(x_1, x_2, \ldots, x_p)$ and path $[x_p, y_1, y_2, \ldots, y_q]$ is denoted by $(x_1, x_2, \ldots, x_p)[x_p, y_1, y_2, \ldots, y_q]$. A *chord* of a cycle is an edge which is incident with two vertices of the cycle but is not in the cycle. Note that a chord may be an edge parallel to an edge of the cycle.

The following lemmas extend results for packings of the complete (simple) graph [27] to packings of the complete multigraph.

Lemma 4.4. *Let v, s and λ be positive integers such that $s \geqslant 3$, and let M be a list of integers. Suppose there exists an (M)-packing $\mathcal{P}$ of λK_v whose leave* *contains a lasso of order at least $s+2$ and suppose that if s is even then the cycle of the lasso has even length. Then there exists an (M, s)-packing of λK_v.*

Proof. Let L be the leave of $\mathcal{P}$. Suppose that L contains a (p, q)-lasso $(x_1, x_2, \ldots, x_p)[x_p, y_1, y_2, \ldots, y_q]$ such that $p + q \geqslant s + 2$ and p is even if s is even. If L contains an s-cycle then we add it to the packing to complete the proof, so assume L does not contain an s-cycle and hence $p \neq s$.
Case 1. Suppose $2 \leqslant p < s$ and either $p = 2$ or $p \equiv s \pmod 2$. We can assume that $p + q = s + 2$ since L contains a $(p, s + 2 - p)$-lasso.

Let L' be the leave of the packing $\mathcal{P}'$ obtained from $\mathcal{P}$ by applying the (x_1, y_{q-1})-switch with origin x_2 (note that $\mu_L(x_2y_{q-1}) = 0$ for otherwise L contains an s-cycle). If the terminus of the switch is not y_{q-2} then L' contains an s-cycle which completes the proof (recall that $s = p + q - 2$). Otherwise, the terminus of the switch is y_{q-2} and L' contains a (q, p)-lasso $(x'_1, x'_2, \ldots, x'_q)[x'_q, y'_1, y'_2, \ldots, y'_p]$. If $p = 2$ then L' contains an s-cycle which completes the proof, so assume L' contains no s-cycle and $p \geqslant 3$.

Let L'' be the leave of the packing $\mathcal{P}''$ obtained from $\mathcal{P}'$ by applying the (x'_2, y'_p)-switch with origin x'_3 (note that $\mu_{L'}(x'_3y'_p) = 0$ for otherwise L' contains an s-cycle). If the terminus of this switch is not y'_{p-1} then L'' contains an s-cycle which completes the proof (recall that $s = p+q-2$). Otherwise, the terminus of the switch is y'_{p-1} and L'' contains a $(p + 2, q - 2)$-lasso, so since $p < s$ and $p \equiv s \pmod 2$, the result follows by repeating the procedure described in this case.
Case 2. Suppose $3 \leqslant p < s$ and $p \not\equiv s \pmod 2$. As above, assume $p+q = s+2$. Then s is odd, $p \geqslant 4$ is even and q is odd by our hypotheses.

Let L' be the leave of the packing $\mathcal{P}'$ obtained from $\mathcal{P}$ by applying the (x_2, y_q)-switch with origin x_3 (note that $\mu_L(x_3y_q) = 0$ for otherwise L contains an s-cycle). If the terminus of the switch is not y_{q-1} then L' contains an s-cycle which completes the proof. Otherwise, the terminus of the switch is y_{q-1} and L' contains a $(q + 2, p - 2)$-lasso. Note that $q + 2 \leqslant s$ (because $p + q = s + 2$ and $p \geqslant 4$) and $q + 2 \equiv s \pmod 2$. If $q + 2 = s$ then this completes the proof, otherwise we can proceed as in Case 1.
Case 3. Suppose $3 \leqslant s < p$. Let L' be the leave of the packing $\mathcal{P}'$ obtained from $\mathcal{P}$ by applying the (x_{p-s+1}, y_1)-switch with origin x_{p-s+2} (note that $\mu_L(x_{p-s+2}y_1) = 0$ for otherwise L contains an s-cycle). If the terminus of the switch is not x_p then L' contains an s-cycle which completes the proof. Otherwise, L' contains a $(p - s + 2, q + s - 2)$-lasso. By repeating this process we obtain an (M)-packing of λK_v whose leave contains a $(p', p + q - p')$-lasso such that $2 \leqslant p' \leqslant s$. Thus we can proceed as in Case 1 or Case 2. □

Lemma 4.5. *Let v, s and λ be positive integers with $s \geqslant 3$, and let M be a list of integers. Suppose there exists an (M)-packing of λK_v whose leave L has* *a component H containing an $(s+1)$-cycle with a chord. Then there exists an (M)-packing of λK_v with a leave L' such that $E(L') = (E(L) \setminus E(H)) \cup E(H')$, where H' is a graph with $V(H') = V(H)$ and $|E(H')| = |E(H)|$ which contains an $(s,1)$-lasso. Furthermore, $\deg_{H'}(x) \geqslant \deg_H(x)$ for each vertex x in the s-cycle of this lasso.*

Proof. Let $(x_1, \ldots, x_{s+1})$ be an $(s+1)$-cycle in H with chord x_1x_e for some $e \in \{2, 3, \ldots, s-1\}$ (note that L may not be a simple graph). If H contains an $(s,1)$-lasso then we are finished immediately, so suppose otherwise. If $e = 2$, then perform the (x_3, x_2)-switch with origin x_4 (note that $\mu_L(x_2x_4) = 0$ because H contains no $(s,1)$-lasso). The leave of the resulting packing contains the $(s,1)$-lasso $(x_4, \ldots, x_{s+1}, x_1, x_2)[x_2, x_3]$, and $\deg_{H'}(x_i) \geqslant \deg_H(x_i)$ for $i \in \{1, \ldots, s+1\} \setminus \{3\}$. If $e = 3$, then H contains an $(s,1)$-lasso which completes the proof.

So suppose $e \geqslant 4$ and let $\mathcal{P}^*$ be the packing with leave L^* obtained from $\mathcal{P}$ by applying the (x_{e-1}, x_e)-switch with origin x_{e-2} (note that $\mu_L(x_{e-2}x_e) = 0$ for otherwise L contains an $(s,1)$-lasso). If the terminus of the switch is not x_{e+1} then $E(L^*) = (E(L) \setminus E(H)) \cup E(H^*)$, where H^* is a graph with $V(H^*) = V(H)$ and $|E(H^*)| = |E(H)|$ which contains the $(s,1)$-lasso $(x_{e+1}, \ldots, x_{s+1}, x_1, \ldots, x_{e-2}, x_e)[x_e, x_{e-1}]$. Also note that $\deg_{H^*}(x_e) \geqslant \deg_H(x_e)$ and $\deg_{H^*}(x_i) = \deg_H(x_i)$ for $i \in \{1, \ldots, s+1\} \setminus \{e, e-1\}$. Otherwise, the terminus of the switch is x_{e+1} and $E(L^*) = (E(L) \setminus E(H)) \cup E(H^*)$, where H^* is a graph with $V(H^*) = V(H)$ and $|E(H^*)| = |E(H)|$ which contains an $(s+1)$-cycle $(x_1^*, \ldots, x_{s+1}^*)$ with chord $x_1^*x_{e-1}^*$. Furthermore, the degree of each vertex in this $(s+1)$-cycle remains unchanged in H'. The result follows by repeating this process. □

4.2 Main result

Theorem 4.1 is established for λ odd and λ even in Lemmas 4.6 and 4.7 respectively. These results rely on using Lemmas 4.4 and 4.5 to modify cycle packings of λK_v obtained via Theorem 1.9.

Lemma 4.6. *Let $m_1, \ldots, m_\tau$ be a list and let λ and v be positive integers, with λ odd, then there exists an $(m_1, \ldots, m_\tau)$-packing of λK_v if and only if*

(i) $2 \leqslant m_1, \ldots, m_\tau \leqslant v$;

(ii) $m_1 + \cdots + m_\tau = \lambda\binom{v}{2} - \delta$, *where δ is a nonnegative integer such that $\delta \neq 1$, $(\lambda, \delta) \neq (1, 2)$, and if v is even then $\delta \geqslant \frac{v}{2}$; and*

(iii) $\sum_{m_i=2} m_i \leqslant \begin{cases} (\lambda-1)\binom{v}{2} - 2 & \text{if } v \text{ is odd and } \delta = 2, \\ (\lambda-1)\binom{v}{2} & \text{otherwise.} \end{cases}$

Proof. If there exists an $(m_1, \ldots, m_\tau)$-packing $\mathcal{P}$ of λK_v, then conditions (i)–(iii) hold by Lemma 4.2. So it remains to show that if λ, v and $m_1, \ldots, m_\tau$ satisfy (i)–(iii), then there is an $(m_1, \ldots, m_\tau)$-packing of λK_v.

Let $\varepsilon = \delta$ if v is odd, and $\varepsilon = \delta - \frac{v}{2}$ if v is even. If $\varepsilon = 0$ then the result follows by Theorem 1.9, so suppose $\varepsilon \geqslant 1$ and note that if v is odd then $\varepsilon \neq 1$ and $(\lambda, \varepsilon) \neq (1, 2)$.

Case 1. Suppose v is odd or $\varepsilon \geqslant 3$. Note that if v is odd and $\varepsilon = 2$ then $2 + \sum_{m_i=2} m_i \leqslant (\lambda - 1)\binom{v}{2}$ by (iii).

We show that there exists a list N such that $2 \leqslant n \leqslant v$ for all $n \in N$, $\sum N = \varepsilon$ and $\sum_{n \in N, n=2} n + \sum_{m_j=2} m_j \leqslant (\lambda - 1)\binom{v}{2}$. If this list exists, then by Theorem 1.9 there exists an $(m_1, \ldots, m_\tau, N)$-decomposition $\mathcal{D}$ of λK_v (if v is odd) or $\lambda K_v - I$ (if v is even), where I is a 1-factor of λK_v. We obtain the required packing by removing cycles of lengths N from $\mathcal{D}$.

We first consider the cases when $v \in \{2, 3\}$. If $v = 2$, then ε is even by (i) and (ii) and there exists a 2-cycle decomposition of $\lambda K_2 - I$. If $v = 3$ and ε is even, then $m_i = 3$ for some $i \in \{1, \ldots, \tau\}$ by (i) and (ii). Then $\varepsilon + \sum_{m_i=2} m_i \leqslant (\lambda - 1)\binom{v}{2}$ by (ii) and we take $N = (2^{\varepsilon/2})$. If $v = 3$ and ε is odd then $\varepsilon - 3 + \sum_{m_i=2} m_i \leqslant (\lambda - 1)\binom{v}{2}$ by (ii) and we take $N = (2^{(\varepsilon-3)/2}, 3)$. In each of these cases we can see that there exists an $(m_1, \ldots, m_\tau, N)$-decomposition of λK_v since the hypotheses of Theorem 1.9 are satisfied by (i)–(iii).

Now assume $v \geqslant 4$ and let q and r be nonnegative integers such that $\varepsilon = vq + r$ and $0 \leqslant r < v$. If $q = 0$ or $r \notin \{1, 2\}$ then we take $N = (r, v^q)$. If $q \geqslant 1$ and $r \in \{1, 2\}$ then $N = (3, v-3+r, v^{q-1})$ (note that either $v-3+r \geqslant 3$, or $v = 4$ and $r = 1$). If $\varepsilon = 2$ or $(v, r) = (4, 1)$, then N contains exactly one entry equal to 2 and otherwise $n \geqslant 3$ for all $n \in N$. By the hypotheses of this case if $\varepsilon = 2$ then $2 + \sum_{m_i=2} m_i \leqslant (\lambda - 1)\binom{v}{2}$. Further, if $v = 4$ and $\varepsilon = 4q + 1$ for some $q \geqslant 1$ then (i) and (ii) imply that $m_i = 3$ for some $i \in \{1, \ldots, \tau\}$ so again $2 + \sum_{m_i=2} m_i \leqslant (\lambda - 1)\binom{v}{2}$. We can therefore see that there exists an $(m_1, \ldots, m_\tau, N)$-decomposition of λK_v since the hypotheses of Theorem 1.9 are satisfied by (i)–(iii) and the fact that $\sum N = \varepsilon$.

Case 2. Suppose v is even and $\varepsilon \in \{1, 2\}$. Let $M = m_1, \ldots, m_\tau$ and let m be the least odd entry in M if M contains an odd entry, otherwise let m be the least entry in M such that $m \geqslant 4$ (such an entry exists by (iii)). Note that if $\varepsilon = 1$ then by (ii) M contains an odd entry and m is odd.

Case 2a. Suppose $m + \varepsilon \leqslant v$. By Theorem 1.9 there exists an $(M \setminus (m), m+\varepsilon)$-decomposition $\mathcal{D}$ of $\lambda K_v - I$, where I is a 1-factor of λK_v. Let $\mathcal{P}$ be the $(M \setminus (m))$-packing of λK_v that is obtained by removing an $(m + \varepsilon)$-cycle from $\mathcal{D}$. Let L be the leave of $\mathcal{P}$ and note that L consists of an $(m + \varepsilon)$-cycle and the 1-factor I.

If L contains an $(m + \varepsilon, 1)$-lasso then we apply Lemma 4.4 to $\mathcal{P}$ with $s = m$ to complete the proof. The hypotheses of Lemma 4.4 are satisfied because $\varepsilon + 1 \geqslant 2$, and if m is even then M contains no odd entries so $\varepsilon = 2$ by (ii).

So suppose L does not contain an $(m+\varepsilon, 1)$-lasso. Then $m+\varepsilon$ is even and L contains a component H such that H is the union of an $(m+\varepsilon)$-cycle and a 1-factor on $V(H)$. We apply Lemma 4.5 to $\mathcal{P}$ with $s = m+\varepsilon-1$ to obtain an $(M \setminus (m))$-packing $\mathcal{P}'$ of λK_v whose leave L' contains a component H' on $m+\varepsilon$ vertices that has $\frac{3}{2}(m+\varepsilon)$ edges and contains an $(m+\varepsilon-1, 1)$-lasso. If $\varepsilon = 1$ then we can add the m-cycle of this lasso to $\mathcal{P}'$ to complete the proof. Otherwise $\varepsilon = 2$ and H' contains an $(m+1)$-cycle with a chord because $m \geqslant 3$ and any vertex in this cycle has degree at least 3 (note that $\deg_H(x) = 3$ for each $x \in V(H)$). Then we can apply Lemma 4.5 with $s = m$ to $\mathcal{P}'$ to obtain an $(M \setminus (m))$-packing $\mathcal{P}''$ of λK_v whose leave contains an $(m, 1)$-lasso. We add the m-cycle of this lasso to $\mathcal{P}''$ to complete the proof.

Case 2b. Suppose $m+\varepsilon > v$. Then $m \geqslant v-1$ and $\varepsilon = 2$ (note that ε is even if $m = v$).

If $m = v$ then $m_i \in \{2, v\}$ for all $i \in \{1, \ldots, \tau\}$, so $\lambda\binom{v}{2} - \frac{v}{2} \equiv 2 + \sum_{m_i=2} m_i \pmod{v}$ by (ii) and hence $2 + \sum_{m_i=2} m_i \leqslant (\lambda-1)\binom{v}{2}$ by (iii). Then by Theorem 1.9 there exists an $(M, 2)$-decomposition $\mathcal{D}$ of $\lambda K_v - I$. We remove a 2-cycle from $\mathcal{D}$ to complete the proof.

So suppose that $m = v-1$. Since ε is even, M contains an even number of odd entries, so at least two entries of M are equal to $v-1$. Let $\mathcal{D}_0$ be an $(M \setminus ((v-1)^2), v^2)$-decomposition of $\lambda K_v - I$ which exists by Theorem 1.9. Let $\mathcal{P}_0$ be the $(M \setminus ((v-1)^2), v)$-packing of λK_v formed by removing a v-cycle from $\mathcal{D}_0$. The leave L_0 of $\mathcal{P}_0$ is the union of a v-cycle and the 1-factor I. Let $\mathcal{P}_1$ be the packing obtained by applying Lemma 4.5 to $\mathcal{P}_0$ with $s = v-1$. Then the leave of $\mathcal{P}_1$ contains a $(v-1, 1)$-lasso. We add the $(v-1)$-cycle of this lasso to $\mathcal{P}_1$ and remove a v-cycle to obtain an $(M \setminus (v-1))$-packing $\mathcal{P}_2$ of λK_v. The leave of $\mathcal{P}_2$ has size $3\frac{v}{2} + 1$.

By applying Lemma 4.5 to $\mathcal{P}_2$ with $s = v-1$ we obtain an $(M \setminus (v-1))$-packing $\mathcal{P}_3$ of λK_v whose leave contains a $(v-1, 1)$-lasso. We add the $(v-1)$-cycle of this lasso to $\mathcal{P}_3$ to complete the proof. □

Lemma 4.7. *Let $m_1, \ldots, m_\tau$ be a nondecreasing list and let λ and v be positive integers with λ even. Then there exists an $(m_1, \ldots, m_\tau)$-packing of λK_v if and only if*

(i) $2 \leqslant m_1, \ldots, m_\tau \leqslant v$;

(ii) $m_1 + \cdots + m_\tau = \lambda\binom{v}{2} - \delta$, *where δ is a nonnegative integer such that $\delta \neq 1$; and*

(iii) $m_\tau \leqslant \begin{cases} \frac{\lambda}{2}\binom{v}{2} - \tau + 2 & \text{if } \delta = 0, \\ \frac{\lambda}{2}\binom{v}{2} - \tau + 1 & \text{if } 2 \leqslant \delta < m_\tau. \end{cases}$

Proof. If there exists an $(m_1, \dots, m_\tau)$-packing $\mathcal{P}$ of λK_v with leave L, then conditions (i)–(iii) hold by Lemma 4.2. So it remains to show that if λ, v and $m_1, \dots, m_\tau$ satisfy (i)–(iii), then there exists an $(m_1, \dots, m_\tau)$-packing of λK_v. If $\delta = 0$ then the result follows immediately from Theorem 1.9, so suppose $\delta \geqslant 2$.

Let

$$N = \begin{cases} (\delta) & \text{if } 2 \leqslant \delta < m_\tau, \\ (2^{(\delta - m_\tau)/2}, m_\tau) & \text{if } \delta \geqslant m_\tau \text{ and } \delta \equiv m_\tau \pmod 2, \\ (2^{(\delta - m_\tau + 1)/2}, m_\tau - 1) & \text{if } \delta \geqslant m_\tau \text{ and } \delta \not\equiv m_\tau \pmod 2. \end{cases}$$

Note that in each case $\sum N = \delta$. We show that there exists an $(m_1, \dots, m_\tau, N)$-decomposition $\mathcal{D}$ of λK_v because the hypotheses of Theorem 1.9 are satisfied by (i)–(iii) and the definition of N. The required packing is then obtained by removing cycles of lengths N from $\mathcal{D}$.

Let s be the number of entries in N. Let $M = m_1, \dots, m_\tau$. First observe that $\sum M + \sum N = \lambda\binom{v}{2}$ by (ii) and since $\sum N = \delta$. By (i) and the definition of N it also holds that $2 \leqslant n \leqslant m_\tau \leqslant v$ for all $n \in N$. If $2 \leqslant \delta < m_\tau$, then $m_\tau \leqslant \frac{\lambda}{2}\binom{v}{2} - \tau - s + 2$ by (iii) and since $s = 1$. If $\delta \geqslant m_\tau$, then because $\sum M \geqslant m_\tau + 2(\tau - 1)$ and $\sum N \geqslant m_\tau - 1 + 2(s-1)$, it follows that

$$\begin{aligned} \tfrac{\lambda}{2}\tbinom{v}{2} - \tau - s + 2 &= \tfrac{1}{2}(\textstyle\sum M + \sum N) - \tau - s + 2 \\ &\geqslant \tfrac{1}{2}(m_\tau + 2(\tau - 1) + m_\tau - 1 + 2(s-1)) - \tau - s + 2 \\ &= m_\tau - \tfrac{1}{2}. \end{aligned}$$

Therefore $\max(N, M) = m_\tau \leqslant \frac{\lambda}{2}\binom{v}{2} - \tau - s + 2$ because $\frac{\lambda}{2}\binom{v}{2} - \tau - s + 2$ is an integer. So by Theorem 1.9 we can see that there exists an (M, N)-decomposition of λK_v which completes the proof. $\square$

Chapter 5

Conclusion

This thesis makes significant progress on the problem of generalising the Doyen-Wilson Theorem, a topic which has generated a lot of interest in the literature since the proof of the original theorem in 1973. Specifically, we obtain new results for embedding odd cycle systems and decomposing the complete graph with a hole into cycles. The main results of Chapter 2 are in response to Problems 1.23 and 1.24, and the main result of Chapter 3 responds to Problem 1.25. Finally, Chapter 4 contains a complete solution to Problem 1.26.

Our main results in Chapter 2 stem from Theorem 2.2 and concern m-cycle decompositions when $m \geqslant 9$ is odd. These results give a complete solution to Problem 1.23 for when an m-cycle system of order u can be embedded in one of order v if $m \leqslant 15$, if m is a prime power, or if $v - u \geqslant m + 1$. Further, a complete solution to Problem 1.24 for m-cycle decompositions of $K_v - K_u$ is given when $m \leqslant 15$, or if $u \geqslant m - 2$ and $v - u \geqslant m + 1$ both hold. As is implied by Corollary 2.3, the remaining cases could be completely solved if, for each odd $m \geqslant 17$, a finite number of example cases are found.

Problem 5.1. *For odd $m \geqslant 17$, let u and v be integers such that $u < v$ and the conditions in Lemma 1.11 hold. Does there exist an m-cycle decomposition of $K_v - K_u$ when*

(i) $u < m - 2$ and $v < \omega_m(u) + m + 1$; or

(ii) $m - 2 \leqslant u \leqslant \frac{(m-1)(m-2)}{2}$ and $v \leqslant u + m - 1$?

Recall that for integers u and m, $\omega_m(u)$ is the least integer such that $(u, \omega_m(u))$ is m-admissible.

An interesting problem related to Problem 1.24 is when the complete multigraph with a hole $(\lambda+\mu)K_v - \lambda K_u$ can be decomposed into m-cycles. There has been some interest in the literature around this problem, beginning with results for enclosings of triple systems [19]. While there are some results known for m

even [9], very little is known for m odd when $m > 3$. The following problem is therefore a potential application for Theorem 2.2 and the techniques used in Chapter 2.

Problem 5.2. *Given an odd integer $m \geqslant 3$ and positive integers λ and μ, for which values of u and v does there exist an m-cycle decomposition of the multigraph $(\lambda + \mu)K_v - \lambda K_u$?*

Many of the known results are restricted to cases when there are one or two vertices outside the hole and m is small [8, 10, 77]. More general results are known for m even when $m + 2 \leqslant \min(u, v - u)$ [9].

Using a similar methodology to Chapter 2, results in Chapter 3 extend Theorem 2.2 to decompositions of the complete graph with a hole into arbitrary specified length cycles. Theorem 3.1 gives a solution to Problem 1.25 when there is an upper bound on the length of the longest cycle in the decomposition. Extending Theorem 2.2 in this manner echoes the progress that was made for decompositions of the complete graph, where the uniform length cycle case was solved first and then extended to cycles of arbitrary lengths (see Theorems 1.2 and 1.4). Some of the tools used to obtain Theorem 3.1 rely on the assumption that there are at least ten vertices outside the hole. With more work along similar lines to the methods in Chapter 3 this small gap could be filled, providing an answer to the following question.

Problem 5.3. *Given a list of integers M, for which values of u and v such that $v - u \leqslant 8$ and $\max(M) \leqslant \min(u, v - u)$ does there exist an (M)-decomposition of $K_v - K_u$?*

A key method in proving the results in Chapters 2 and 3 is the cycle switching technique described by Lemma 1.22. Cycle switching has been shown to be an effective method for obtaining cycle decomposition results of graphs containing large sets of pairwise twin vertices. This has been demonstrated previously for the complete graph and the complete bipartite graph [29, 33, 64]. The results in Chapters 2 and 3 build tools for applying cycle switching to decompositions of the complete graph with a hole.

The main tool developed by cycle switching in Chapters 2 and 3 is Lemma 3.3 which, under certain conditions, adds the lengths of two cycle together to form a single, longer cycle. The upper bounds on the cycle lengths in the decompositions given by Theorems 2.2 and 3.1 are required in order to apply Lemma 3.3 or similar merging results. As we saw in Chapter 1, in the proof of Theorem 1.4 for cycle decompositions of the complete graph, a 'merging' lemma was complemented with an 'equalising' lemma [29]. This enabled a complete solution to Conjecture 1.3, whereas either of these methods on its own was limited in the results it could obtain. Developing other tools for modifying cycle packings of the complete graph with a hole could be beneficial for resolving some of the cases of Problem 5.1.

In spite of the limitations of cycle switching, the known results have been significantly improved as a result of applying this technique to the complete graph with a hole. It is also possible that cycle switching methods could be extended to obtain results for cycle decompositions of other graphs whose vertices can be partitioned into sets of pairwise twin vertices. For example this raises the following question concerning the complete multipartite graph.

Problem 5.4. *Given an integer $m \geqslant 3$, for which lists of integers $a_1, \ldots, a_r$ does there exist an m-cycle decomposition of the complete multipartite graph $K_{a_1,\ldots,a_r}$ with r parts of sizes $a_1, \ldots, a_r$?*

There are numerous partial results for Problem 5.4, including results for when m is small [43, 65]. Other results include cases when the parts are equal sizes [86, 87, 88] or $a_i \geqslant m+2$ for $i \in \{1, \ldots, r\}$ [64]. Also see the survey [20].

Results in Chapters 2 and 3 rely on applying cycle switching methods to existing decompositions of the complete graph with a hole. Therefore finding decompositions of the complete graph with a hole into short cycles is an important aspect of the methodology. These decompositions are given in Sections 2.2 and 3.2. The majority of m-cycles in the decompositions of $K_v - K_u$ given in Chapter 2 contain either one or m pure edges, and a similar approach is taken to cycles in the decompositions given in Chapter 3. This approach enabled the use of cycle switching techniques to merge cycle lengths and obtain longer cycles in the decomposition. However, developing constructions for decompositions of $K_v - K_u$ containing cycles with a mix of pure and cross edges could be useful for obtaining decompositions into cycles of length greater than $\min(u, v-u)$. This could then resolve some cases of Problem 5.1. The base decompositions in Chapter 3 could also be built on to find results for the following problem.

Problem 5.5. *Given a list of integers M such that each entry of M is in $\{3, 4, 5, 6\}$, for which values of u and v does there exist an (M)-decomposition of $K_v - K_u$?*

First note that if $u \in \{1, 3\}$ then the solution to Problem 5.5 follows directly from Theorem 1.4, and by removing a 3-cycle from the decomposition of K_v if $u = 3$. If $\max(M) \leqslant \min(u, v-u)$ then the solution to Problem 5.5 is given by Theorem 3.1 or the solution to Problem 5.3. Further note that if $v = u$ then the decomposition is trivial, and if $v = u + 2$ then $M = (3^1, 4^{(u-1)/2})$ by Lemma 3.2 so the decomposition obviously exists. Thus we can assume that $u \geqslant 5$, $v - u \geqslant 4$ and $\max(M) > \min(u, v-u)$, so the following are the remaining cases for a solution to Problem 5.5.

Case 1. $u \geqslant 5$, $v = u + 4$ and $\max(M) \in \{5, 6\}$.
Case 2. $u = 5$, $v \geqslant 11$ and $\max(M) = 6$.

Case 1 can be simplified by noting that, by Lemma 3.2(iv), there are at most six odd cycles in a cycle decomposition of $K_{u+4} - K_u$. Lemma 3.21 could also be used to simplify the results of both cases.

The main result of Chapter 4 is a complete solution to Problem 1.26. That is, Theorem 4.1 states the conditions for when there exists a packing of the complete multigraph with cycles of specified lengths. The proof of Theorem 4.1 applies the multigraph version of cycle switching to modify known cycle decompositions of the complete multigraph. Following a complete solution to Problem 1.26, it is natural to ask whether there exist packings of the complete multigraph with other graphs. The following question is one such example.

Problem 5.6. *For positive integers λ and v, for which lists of integers $s_1, \ldots, s_\tau$ does there exist a packing of λK_v with stars of sizes $s_1, \ldots, s_\tau$?*

A solution to Problem 5.6 is known when $\lambda = 1$ [70]. If $s_i = s$ for $i \in \{1, \ldots, \tau\}$, then the problem has also been solved whenever $s \mid \lambda\binom{v}{2}$ [92] (see also [11, 22, 60]). However, in general, Problem 5.6 is still an open problem for $\lambda \geqslant 2$.

Results in this thesis make substantial progress towards generalising the Doyen-Wilson Theorem to odd cycle systems, and decomposing the complete graph with a hole into cycles. Despite the strength of the results given here, there are still unsolved cases for Problems 1.23–1.25. Moreover, the cycle switching and base decomposition methods used to obtain the results in this thesis also give rise to several interesting open problems.

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Appendix A

For each of the following values of m, u and v, let $V = \{1, \ldots, v\}$ and $U = \{1, \ldots, u\}$. Then $\mathcal{D}$ is an m-cycle decomposition of $K_V - K_U$.

The algorithm used to obtain these decompositions begins by finding a random cycle decomposition of $K_v - K_u$, with cycles of any length. The next step of the algorithm is to apply cycle switches to improve the decomposition. The algorithm terminates when all cycles have length m or if there is no solution after a given number of iterations. In the latter case solutions were found by running the algorithm again on a new initial decomposition.

Missing decompositions for Theorem 2.2.

(m,u,v)	$\mathcal{D}$
(9,7,21)	{(6, 9, 18, 20, 8, 21, 12, 19, 13), (4, 8, 15, 7, 18, 14, 17, 10, 19), (3, 14, 8, 10, 7, 16, 21, 6, 19), (2, 12, 14, 20, 13, 3, 9, 21, 18), (1, 12, 18, 6, 11, 20, 2, 21, 13), (1, 11, 12, 5, 8, 13, 10, 2, 14), (4, 11, 17, 7, 9, 16, 18, 15, 13), (3, 8, 11, 10, 5, 14, 7, 13, 16), (1, 15, 11, 21, 10, 12, 17, 3, 18), (1, 10, 4, 14, 11, 7, 20, 17, 16), (6, 14, 9, 8, 18, 13, 12, 15, 16), (2, 11, 5, 16, 10, 15, 21, 17, 13), (4, 16, 14, 13, 11, 18, 19, 9, 20), (5, 15, 14, 10, 20, 12, 7, 21, 19), (1, 19, 7, 8, 17, 15, 3, 21, 20), (1, 8, 12, 16, 2, 9, 17, 5, 21), (4, 12, 9, 13, 5, 18, 10, 6, 17), (3, 10, 9, 5, 20, 19, 8, 6, 12), (2, 17, 18, 4, 15, 6, 20, 16, 19), (3, 11, 9, 4, 21, 14, 19, 15, 20), (1, 9, 15, 2, 8, 16, 11, 19, 17)}
(11, 9, 25)	{(6, 19, 17, 15, 12, 7, 22, 8, 25, 21, 23), (1, 16, 13, 4, 17, 2, 23, 3, 10, 11, 18), (2, 16, 15, 6, 10, 20, 4, 25, 24, 9, 19), (1, 11, 2, 10, 18, 21, 14, 8, 16, 19, 22), (1, 19, 4, 24, 14, 2, 22, 11, 12, 6, 21), (4, 12, 18, 8, 21, 9, 11, 17, 20, 25, 14), (3, 13, 8, 15, 25, 23, 11, 20, 16, 9, 22), (1, 13, 21, 5, 15, 7, 19, 14, 23, 9, 25), (3, 12, 13, 15, 21, 7, 20, 9, 14, 18, 25), (2, 18, 6, 11, 14, 13, 19, 10, 16, 4, 21), (5, 19, 18, 13, 23, 10, 21, 12, 24, 8, 20), (3, 15, 10, 8, 17, 18, 7, 14, 6, 24, 21), (8, 19, 25, 13, 24, 17, 21, 16, 14, 15, 23), (3, 14, 20, 23, 19, 21, 11, 5, 18, 24, 16), (3, 19, 15, 24, 7, 23, 12, 17, 10, 22, 20), (3, 17, 25, 11, 19, 12, 20, 24, 22, 16, 18), (5, 13, 11, 16, 17, 14, 22, 21, 20, 6, 25), (5, 16, 25, 10, 14, 12, 8, 11, 15, 22, 23), (1, 12, 5, 17, 23, 4, 18, 22, 25, 2, 20), (1, 10, 5, 22, 13, 17, 7, 25, 12, 2, 15), (1, 17, 6, 13, 9, 12, 10, 4, 11, 3, 24), (2, 13, 7, 16, 12, 22, 4, 15, 20, 19, 24), (1, 14, 5, 24, 11, 7, 10, 9, 15, 18, 23), (6, 16, 23, 24, 10, 13, 20, 18, 9, 17, 22)}
(13, 11, 29)	{(1, 19, 24, 7, 13, 6, 25, 27, 28, 10, 22, 29, 23), (1, 14, 17, 3, 27, 11, 20, 7, 28, 4, 19, 21, 16), (5, 19, 7, 15, 17, 16, 13, 29, 27, 14, 20, 24, 22), (1, 24, 15, 20, 5, 14, 3, 23, 6, 18, 27, 4, 26), (8, 13, 24, 23, 10, 14, 16, 29, 20, 26, 15, 11, 18), (1, 20, 2, 17, 23, 11, 21, 3, 16, 22, 4, 18, 28), (8, 23, 26, 27, 12, 13, 17, 24, 14, 19, 28, 20, 25), (3, 25, 22, 23, 5, 24, 12, 10, 18, 15, 16, 8, 26), (2, 13, 27, 19, 18, 16, 24, 6, 15, 3, 20, 4, 14), (1, 13, 19, 22, 3, 12, 14, 18, 26, 28, 23, 27, 21), (2, 19, 16, 28, 6, 14, 7, 12, 11, 17, 18, 5, 21), (5, 12, 20, 9, 29, 7, 26, 21, 24, 11, 25, 10, 17), (6, 21, 29, 28, 25, 19, 11, 26, 10, 20, 22, 17, 27), (1, 12, 15, 4, 17, 20, 18, 3, 13, 5, 29, 2, 25), (5, 15, 10, 13, 20, 27, 7, 16, 25, 18, 22, 14, 26), (2, 23, 20, 6, 22, 13, 18, 21, 4, 25, 14, 11, 28), (2, 15, 22, 8, 17, 28, 9, 25, 5, 27, 16, 26, 24), (2, 12, 4, 24, 10, 29, 8, 27, 22, 21, 25, 7, 18), (1, 18, 24, 8, 15, 27, 10, 19, 29, 26, 17, 9, 22), (4, 23, 12, 28, 5, 16, 6, 26, 13, 14, 9, 15, 29), (1, 15, 23, 14, 28, 8, 19, 26, 9, 21, 13, 11, 29), (7, 17, 21, 14, 8, 12, 19, 9, 27, 24, 29, 25, 23), (3, 19, 20, 21, 12, 26, 25, 15, 28, 13, 23, 9, 24), (2, 16, 9, 18, 12, 17, 6, 19, 15, 21, 28, 22, 26), (1, 17, 25, 13, 15, 14, 29, 12, 16, 11, 22, 2, 27), (3, 28, 24, 25, 12, 22, 7, 21, 10, 16, 23, 18, 29), (4, 13, 9, 12, 6, 29, 17, 19, 23, 21, 8, 20, 16)}

(15, 13, 33)	{(7, 23, 33, 24, 28, 30, 8, 15, 27, 14, 21, 20, 31, 13, 29), (2, 25, 3, 30, 20, 24, 23, 15, 12, 17, 32, 5, 26, 22, 31), (1, 19, 20, 4, 25, 31, 9, 17, 6, 21, 8, 16, 5, 18, 33), (2, 29, 32, 27, 21, 18, 17, 13, 22, 9, 19, 8, 20, 6, 30), (2, 26, 33, 3, 24, 17, 31, 23, 20, 14, 30, 22, 29, 6, 28), (3, 23, 14, 32, 20, 11, 25, 9, 26, 10, 27, 8, 24, 5, 29), (2, 17, 4, 14, 12, 18, 11, 31, 7, 24, 32, 13, 33, 6, 19), (3, 16, 11, 23, 12, 21, 7, 22, 8, 31, 30, 15, 18, 13, 28), (1, 14, 33, 22, 4, 27, 19, 30, 29, 28, 5, 31, 12, 26, 21), (6, 15, 11, 27, 12, 28, 17, 16, 29, 26, 8, 23, 30, 7, 18), (7, 14, 28, 33, 30, 25, 27, 17, 20, 29, 10, 31, 16, 15, 32), (1, 16, 19, 33, 15, 9, 32, 21, 3, 17, 25, 14, 24, 11, 28), (4, 29, 24, 6, 26, 19, 21, 5, 25, 20, 13, 23, 18, 10, 30), (2, 15, 7, 28, 32, 8, 29, 12, 30, 11, 26, 3, 22, 6, 23), (4, 18, 30, 9, 14, 15, 29, 11, 22, 21, 28, 20, 33, 27, 23), (1, 15, 17, 11, 19, 12, 16, 7, 25, 6, 31, 3, 14, 2, 18), (2, 21, 31, 4, 19, 28, 10, 25, 32, 12, 22, 15, 3, 18, 27), (1, 20, 26, 14, 31, 27, 28, 25, 33, 29, 23, 21, 16, 30, 32), (4, 15, 20, 16, 14, 17, 26, 25, 18, 29, 27, 9, 24, 30, 21), (2, 20, 10, 15, 26, 23, 28, 9, 18, 24, 25, 13, 27, 22, 32), (1, 17, 7, 33, 4, 32, 6, 16, 9, 23, 19, 3, 27, 24, 26), (1, 24, 16, 18, 14, 11, 21, 33, 10, 23, 17, 19, 15, 5, 30), (5, 20, 18, 31, 29, 17, 21, 13, 15, 24, 19, 10, 22, 16, 33), (1, 29, 9, 21, 10, 14, 19, 7, 20, 12, 33, 2, 22, 28, 31), (2, 16, 23, 32, 3, 20, 9, 33, 8, 17, 5, 19, 25, 12, 24), (13, 24, 31, 15, 28, 16, 27, 20, 22, 25, 29, 19, 32, 26, 30), (1, 23, 22, 24, 10, 32, 33, 31, 19, 18, 8, 28, 26, 7, 27), (4, 16, 26, 27, 6, 14, 29, 21, 15, 25, 23, 5, 22, 18, 28), (1, 22, 14, 5, 27, 30, 17, 33, 11, 32, 31, 26, 13, 16, 25), (4, 24, 21, 25, 8, 14, 13, 19, 22, 17, 10, 16, 32, 18, 26)}

Missing decompositions for Theorem 2.4.

(m,u,v)	$\mathcal{D}$
(5, 11)	{(1, 8, 11, 9, 7, 2, 6, 3, 10), (1, 6, 10, 4, 9, 8, 3, 11, 7), (1, 9, 10, 8, 7, 5, 6, 4, 11), (2, 9, 6, 11, 5, 8, 4, 7, 10), (2, 8, 6, 7, 3, 9, 5, 10, 11)}
(9, 5, 17)	{(5, 6, 15, 17, 16, 9, 12, 8, 7), (2, 6, 9, 13, 15, 11, 16, 12, 7), (4, 9, 5, 14, 12, 13, 11, 10, 16), (1, 10, 14, 17, 3, 11, 5, 15, 16), (2, 10, 5, 12, 6, 3, 9, 11, 17), (1, 7, 14, 16, 8, 10, 13, 5, 17), (1, 14, 8, 13, 6, 16, 7, 4, 15), (3, 10, 9, 8, 17, 4, 14, 11, 12), (6, 7, 10, 12, 17, 9, 15, 8, 11), (1, 9, 14, 2, 16, 5, 8, 4, 13), (2, 8, 6, 17, 13, 3, 14, 15, 12), (1, 11, 2, 15, 7, 17, 10, 4, 12), (2, 9, 7, 3, 15, 10, 6, 14, 13), (1, 6, 4, 11, 7, 13, 16, 3, 8)}
(9, 7, 25)	{(4, 13, 6, 16, 22, 11, 20, 9, 21), (1, 11, 5, 12, 2, 19, 14, 6, 24), (5, 10, 9, 18, 21, 6, 17, 8, 25), (3, 8, 10, 24, 21, 12, 22, 5, 19), (2, 11, 3, 25, 24, 14, 7, 19, 17), (6, 12, 11, 25, 19, 23, 10, 14, 22), (2, 8, 23, 6, 10, 13, 16, 20, 21), (1, 9, 8, 12, 18, 22, 15, 3, 21), (2, 20, 17, 10, 21, 15, 4, 12, 25), (1, 10, 7, 12, 15, 24, 9, 14, 20), (2, 13, 19, 20, 18, 3, 9, 7, 15), (6, 11, 8, 18, 24, 23, 9, 15, 20), (1, 8, 19, 4, 20, 22, 25, 15, 14), (3, 14, 16, 18, 5, 24, 7, 20, 23), (4, 9, 17, 13, 21, 25, 16, 7, 23), (2, 9, 25, 6, 15, 11, 24, 3, 16), (7, 11, 16, 12, 10, 18, 14, 17, 22), (11, 17, 12, 23, 13, 20, 24, 16, 19), (1, 13, 24, 17, 25, 4, 11, 21, 19), (10, 16, 15, 19, 22, 21, 14, 25, 20), (1, 15, 10, 19, 12, 14, 11, 13, 25), (4, 17, 23, 18, 15, 5, 9, 19, 24), (5, 16, 23, 15, 13, 14, 8, 21, 17), (2, 18, 7, 8, 13, 3, 20, 12, 24), (4, 8, 22, 10, 11, 9, 13, 5, 14), (1, 17, 7, 21, 5, 20, 8, 24, 22), (1, 16, 17, 3, 22, 2, 23, 25, 18), (6, 8, 16, 9, 22, 23, 11, 18, 19), (1, 12, 13, 22, 4, 10, 2, 14, 23), (4, 16, 21, 23, 5, 8, 15, 17, 18), (3, 10, 25, 7, 13, 18, 6, 9, 12)}
(9, 11, 17)	{(1, 13, 2, 16, 9, 12, 4, 14, 15), (7, 16, 8, 13, 9, 14, 11, 15, 17), (4, 15, 6, 12, 13, 17, 14, 5, 16), (3, 14, 16, 10, 12, 11, 17, 9, 15), (1, 12, 2, 17, 16, 3, 13, 7, 14), (5, 12, 8, 15, 13, 10, 14, 6, 17), (1, 16, 6, 13, 5, 15, 12, 3, 17), (2, 14, 12, 17, 4, 13, 11, 16, 15), (7, 12, 16, 13, 14, 8, 17, 10, 15)}
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