Transversals, indivisible plexes and partitions of latin squares

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Declaration

I declare that this thesis is my own work and that this thesis contains no material which has been accepted for the award of any other degree or diploma in any university or other institution. To the best of my knowledge this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

Judith Egan
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Abstract

A k-plex in a latin square of order n is a selection of kn entries in which each row, column and symbol occurs exactly k times. A 1-plex is also called a transversal. A k-plex is indivisible if it contains no c-plex for $0 < c < k$. Two or more plexes are parallel if no two of them share an entry. This thesis is the first major study giving general results on indivisible k-plexes for $k > 1$.

For $n \notin \{2, 6\}$, the existence of a latin square of order n with a partition into 1-plexes was shown by Bose, Shrikhande and Parker. A natural extension of the result to k-plexes is the following. We prove that if k is a proper divisor of n, then there exists a latin square of order n composed of parallel indivisible k-plexes.

Define $\kappa(n)$ to be the largest integer k such that some latin square of order n contains an indivisible k-plex. A conjecture by Rodney says that every latin square contains a 2-plex, which implies that $\kappa(n) < n$. We show that for all $n > 2$, there exists a latin square of order n with two parallel indivisible $\lfloor \frac{n}{2} \rfloor$-plexes. This proves that $\kappa(n) \geqslant \lfloor \frac{n}{2} \rfloor$ for all $n > 2$. We report on extensive computations of the indivisible partitions of the latin squares of order $n \leqslant 9$. Up to order 8 we count the number of indivisible partitions of every type.

Due to Bose, Shrikhande and Parker, and Finney for $n = 6$, we know that there exists a latin square for each order $n > 2$ which possesses a k-plex for all $0 \leqslant k \leqslant n$. Wanless showed that there are latin squares of every even order which contain no odd plexes, where an odd plex is a k-plex such that k is odd. It has been conjectured by Rodney, and by Wanless, that every latin square has a set of $\lfloor \frac{n}{2} \rfloor$ parallel 2-plexes which implies that every latin square of odd order has a k-plex for each $0 \leqslant k \leqslant n$. We show that among the latin squares of even order there are many other possibilities concerning the existence of odd k-plexes. We prove that for all even $n > 2$, there exists a latin square of order n which has no k-plex for any odd $k < \lfloor \frac{n}{4} \rfloor$, but does have a k-plex for every other $k \leqslant \frac{1}{2}n$.

A result by Wanless and Webb is that, for all $n > 3$, there exists a latin square of order n with at least one entry not in any transversal. Such a latin square is called a confirmed bachelor and illustrates a restriction on the transversals in a latin square. In our study of transversals we consider confirmed bachelor latin squares in greater detail and other types of restrictions that might occur. We present a concise

alternative proof of the result by Wanless and Webb. A main result is that for all even $n \geqslant 10$, except perhaps if n is a power of 2, there exists a latin square of order n that possesses a transversal, but every transversal coincides on a single entry. A theorem by Wanless, which was prompted by our data, shows that there exist arbitrarily large latin squares of odd order in which the proportion of entries not in a transversal is asymptotic to one ninth. We report on computations of parallel transversals in the latin squares of order 9. Thus we prove that the above mentioned conjectures by Rodney and Wanless are true for order 9, and that every latin square of order 9 has at least 3 parallel transversals. Our computations suggest that the constructions which prove our theorems illustrate rare behaviour in large latin squares. The computations also give further evidence in support of a conjecture by Ryser that every latin square of odd order has a transversal.

Many of the theorems in this thesis rely on a simple but powerful lemma. The lemma gives a necessary condition for the existence of k-plexes. To show the existence of k-plexes we use constructive techniques. It remains open to establish a sufficient condition for the existence of k-plexes in latin squares.

Keywords: latin square, transversal, plex, duplex, triplex, quadruplex, indivisible plex, indivisible partition, orthogonal partition, bachelor square, mutually orthogonal latin squares, partial latin square, protoplex, homogeneous partial latin square, latin trade, homogenous latin trade, rainbow factor, graph factorisation, quasigroup, complete mapping, orthomorphism.

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Chapter 1

Preliminary

1.1 Introduction

A *latin square* of *order* n is an $n \times n$ matrix in which n distinct symbols are arranged so that each symbol occurs once in each row, and once in each column. The unbordered Cayley table of a finite group is an example of a latin square. The very popular Sudoku puzzle found in contemporary media, when completed, is an example of a latin square of order 9.

A *transversal* in a latin square of order n is a selection of n entries in which each row, column and symbol occur once.

A k*-plex* in a latin square is a selection of entries in which each row, column and symbol occurs exactly k times. A 1-plex is usually called a transversal. In statements of a general nature we may simply refer to a *plex* or the plural form 'plexes'. We call a k-plex an *odd plex* or an *even plex* if k is odd or even, respectively. Various names for specific cases of k-plexes have been used. Statistical literature sometimes refers to a transversal as a *directrix* and uses the terms *duplex, triplex* and *quadruplex* for a 2-plex, 3-plex and 4-plex respectively. Some other names, k*-stagger* [20] and k*-transversal* [39], are discussed in [150] where the name k-plex is first adopted in the current context. The name k-transversal is used with a different meaning in [46, p453] and [48, p8]. A k-plex is mentioned, but not named, as one of three generalisations of transversals which are described in [48, p33].

This thesis is about the existence and non-existence of plexes within latin squares and their substructures. The major part of the author's original contribution lies in the following results:

1. Theorem 3.2 showing existence of latin squares with k-plexes for some but not all odd values of k.

2. Theorem 4.1 showing existence of latin squares composed of disjoint plexes that contain no smaller plex.

3. Theorem 4.2 showing latin squares with relatively large plexes that contain no smaller plex.

4. Theorem 5.2 showing latin squares with transversals coinciding on a single entry.

5. Extensive computations of plexes and partitions of latin squares of order less than or equal to 9. See Sections 4.5 and 5.4.

Interest in transversals in latin squares began around 1782 with the work of Euler [71]. In the 1930's, the statistical work of Fisher and Yates (e.g. [80]) promoted the value of latin squares in the design and analysis of scientific experiments, stimulating additional interest in latin squares and transversals. Work in the area of group theory and quasigroup theory has progressed our understanding of transversals through study of their algebraic equivalent. The very recent resolution of a longstanding conjecture by Hall and Paige [95] brings news about plexes in latin squares that are based on groups. Independent of the original motivation, transversals have been studied for the greater part of the last century. However, a number of very basic questions about them remain a challenge to mathematicians.

One important unresolved question concerns the next conjecture. The conjecture is attributed to Ryser, 1967 [39, p143].

Conjecture 1.1. *Every latin square of odd order has a transversal.*

The study of k-plexes for $k > 1$ spans a much shorter time period than the case $k = 1$. Originally motivated by statistical application, studies of k-plexes for $k = 2, 3, 4$ in small latin squares appear in the literature from 1945 [76]. One of the earliest general statements about k-plexes, for $k > 1$, is the next conjecture which is attributed to Rodney, 1994 [39, p143],[55], [132].

Conjecture 1.2. *Every latin square contains a* 2*-plex.*

Several general results for k-plexes for $k > 1$ appear in [150]. Some earlier results about k-plexes originated from the study of partial latin squares and their possible completion to a latin square [20, 45]. Extending on Wanless's work in [150], the author's recent results in [65] motivate the particular direction and approach of this research. Indeed we continue from that point to consolidate and develop results obtained earlier, and explore questions arising. Several chapters of this thesis incorporate, as appropriately indicated, work submitted in [65] because it is necessary for the exposition of the current results.

The remaining sections of this introductory chapter introduce essential definitions, other motivating problems, and review literature relevant to this study. We report on various different approaches that have been used to tackle Conjectures 1.1 and 1.2. For further explanation of latin square terminology and concepts we refer the reader to standard reference texts by Dénes and Keedwell [46, 48]. The small amount of graph theory and group theory terminology that we use accords with the standard usage. Any other assumed terminology is likely to be found in the Handbook of Combinatorial Designs [39].

In Chapter 2 we introduce a key lemma along with the infinite families of latin squares and notation used in our work in Chapters 3, 4 and 5. The content of these three main chapters is outlined during this first chapter as we review related work. Further research problems and concluding remarks can be found at the end of Chapters 3, 4 and 5.

1.2 Equivalence of latin squares

The following equivalence relations are of fundamental importance in the theory of latin squares [46].

If L and L' are latin squares of order n and it is possible to transform L to L' by permuting the rows, permuting the columns and permuting symbols, then L' is said to be *isotopic* to L'. The ordered triple of permutations which transforms L to L' is an *isotopy* or *isotopism.* The set of all latin squares isotopic to L form the *isotopy class* of L. An *autotopy* of L is an isotopy which preserves L. The set of autotopies of L forms the *autotopy group* of L. The cardinality of an isotopy class of order n is $n!^3$ divided by the order of its autotopy group.

Another important equivalence relation is *conjugacy.* If L' can be obtained from L by permuting the roles of the rows, columns and symbols, then L' is said to be a *conjugate* of L. Since permutation of the three roles generates a group isomorphic to S_3, the symmetric group of order 3, the number of distinct conjugates of a latin square is always, $1, 2, 3$ or 6, the index of a subgroup of S_3.

Two latin squares L and L' are *main class isotopic*, or *paratopic*, if L is isotopic to any conjugate of L'. A map consisting of an isotopy and a permutation which transforms a latin square to one of its conjugates is a *paratopy*, or *paratopism.*

The set of all conjugates of the latin squares in an isotopy class is known by several names; *main class*, *paratopy class* or *species.* We use the name species. The number of transversals in a latin square is well known to be a species invariant and it follows from similar reasoning that so is the number of k-plexes in a latin square. The action of the permutations, whether an isotopy or a paratopy, on a latin square L can be thought of as a relabelling of the rows, columns and symbols of L [46, p7]. Thus, such permutations cannot alter the existence of a transversal, a k-plex, or any other similar substructure of a latin square.

A latin square that is equal to its transpose is called *symmetric*. A latin square is *semi-symmetric* if three of its conjugates are equal.

1.3 Plexes, indivisible plexes and partitions

The classical notion of a latin square is a matrix. Equivalently, we may consider a latin square as a set of ordered triples. Set terminology and notation is an advantage in our later workings.

A set L of n^2 ordered triples $(x, y, z) \in \mathcal{I}(L)^3$, where $\mathcal{I}(L)$ is a set of cardinality n, and no two distinct elements of L agree in more than one coordinate, defines a latin square of order n. The set $\mathcal{I}(L)$ is the *index set* of L and we say that L is *indexed* by $\mathcal{I}(L)$. If $(x, y, z) \in L$, then the corresponding matrix has the symbol, or *entry*, z at the intersection of row x and column y. Typically, we will set $\mathcal{I}(L) = \{0, 1, \ldots, n-1\}$ to facilitate calculations in the cyclic group $(\mathbb{Z}_n, +)$, denoted by $\mathbb{Z}_n$. A subset of a latin square which itself is a latin square is called a *subsquare*. A subsquare of order 2 is called an *intercalate*.

In set terminology, a *k-plex* is a subset of a latin square consisting of kn entries in which each row, column and symbol occurs k times. Our definition of a k-plex differs from the first usage in [150] as we insist on containment in some latin square of order n. This is further explained in Section 1.12 where we give an alternative definition of a k-plex as a partial latin square.

We call a set of plexes *parallel*, or disjoint, if no two of them share a common element. The union of an a-plex and a parallel b-plex in a latin square L yields an $(a+b)$-plex of L. The reverse process, that is *dividing* an $(a+b)$-plex into an a-plex parallel to a b-plex, is not necessarily possible. If a k-plex contains no c-plex for $0 < c < k$ then it is said to be *indivisible*.

If K is a k-plex in L, it follows that the *complement*, that is $L \setminus K$, is an $(n-k)$-plex. Together, these plexes form a *$(k, n-k)$-partition*. More generally, a $(k_1^{\alpha_1}, k_2^{\alpha_2}, \ldots)$-partition is a partition with α_1 parts which are k_1-plexes, α_2 parts which are k_2-plexes, etc. The tuple $(k_1^{\alpha_1}, k_2^{\alpha_2}, \ldots)$ is called the *type* of a partition. It is of particular interest when all parts are of the same size. We call a $(k^{n/k})$-partition a *k-partition*. If each plex in a partition is indivisible, then we say that the partition itself is indivisible. Under set union of two or more parts, the indivisible partitions of a latin square produce all other partitions admitting divisible plexes. Divisible plexes have also been called degenerate plexes [78, 84].

Conjecture 1.3, stated next, is stronger than Conjectures 1.1 and 1.2. Dougherty [55] states Conjecture 1.3 as a revision, apparently by Rodney, of the earlier Conjecture 1.2. Independently, Wanless states Conjecture 1.3 in [150].

Conjecture 1.3. *Every latin square of order n contains $\lfloor n/2 \rfloor$ parallel 2-plexes.*

Wanless [150] proved, by computation, that Conjecture 1.3 is true for $n \leqslant 8$. In Chapter 5 we report on computations proving that Conjecture 1.3 is true for $n = 9$.

Conjecture 1.3 implies, by union of parts, that every latin square of odd order possesses a k-plex for each $k \in \{0, 1, \dots, n\}$, and that no latin square will fail to contain a k-plex for any even value of k in that range.

The *plex range*, of a latin square L, is the set of values of k for which a k-plex is contained in L. Let R be the plex range of a latin square L of order n. Then L is said to have a *complete* plex range if $R = \{0, 1, \dots, n\}$, an *even* plex range if $R = \{0, 2, 4, \dots, n\}$, and a *mixed* plex range if R is neither complete nor even. (Only squares of even order can have an even plex range.)

Existence of a 1-partition implies a complete plex range but the converse does not hold. For example, the shading in (1.1) identifies a $(1^2, 2^2)$-partition so, by an appropriate union of plexes from the partition, this latin square has a complete plex range. However it does not have a 1-partition.

$$\begin{pmatrix} 5 & 1 & 2 & 3 & 4 & 0 \\ 1 & 0 & 3 & 2 & 5 & 4 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 4 & 5 & 0 & 1 & 2 & 3 \\ 0 & 2 & 1 & 4 & 3 & 5 \end{pmatrix} \tag{1.1}$$

Among the small latin squares of even order, Finney [78, 79] for order 6 and Wanless [150] for order 8, report the existence of latin squares with a 3-plex but no transversal, hence latin squares with a mixed plex range. The following conjecture by Wanless [150] remains open.

Conjecture 1.4. *For all even $n > 4$, there exists a latin square of order n with a 3-plex but no transversal.*

In Chapter 3 we show that, for all even $n > 2$, there exists a latin square of order n that has no k-plex for any odd $k < \lfloor \frac{n}{4} \rfloor$, but does have a k-plex for every other $k \leqslant \frac{1}{2}n$. A corollary is the existence, in the general case, of latin squares which possess a mixed plex range. Latin squares with a complete or even plex range are shown in the next sections by Corollaries 1.17 and 1.28.

Conjecture 1.3, along with evidence from small order studies, including our own in Chapters 4 and 5, suggests that all latin squares are 'highly divisible' in the sense that they possess many different partitions.

Problem 1.5. For any given n, define $\kappa(n)$ to be the largest k such that there exists a latin square of order n containing an indivisible k-plex. What is the asymptotic behaviour of $\kappa(n)$?

A major goal of this thesis is to better understand $\kappa(n)$.

Conjectures 1.2 and 1.3 imply that, for all $n > 2$, $\kappa(n) < n$. A result in [65], and restated in this thesis as Lemma 4.8, says that, for all odd $k > 1$, there exists a latin square of order $2k$ with an indivisible k-partition. In Chapter 4 we will show that $\kappa(n) \geqslant \lfloor \frac{n}{2} \rfloor$ by showing two parallel $\lfloor \frac{n}{2} \rfloor$-plexes in a latin square for each order $n > 2$. In Section 4.5, we establish $\kappa(n)$ exactly for $n \leqslant 8$ and show that $\kappa(9)$ is either 6 or 7.

As noted in [154], the existence of proportionally large indivisible plexes, such as those mentioned above, may be surprising. In 1996, Dougherty [55] reports the next conjecture which was made in response to Conjecture 1.2.

Conjecture 1.6. *If $k > 2$, then any k-plex in a latin square can be partitioned into an a-plex and a b-plex for some a and b with $a + b = k$.*

Dougherty was clearly unaware that indivisible 3-plexes in latin squares of order 5 and 6 had been identified by Finney in [77, 78]. Also, Drisko is attributed with a counterexample of order 5 at Dougherty's homepage [54]. Beyond our results about $\kappa(n)$, the data on indivisible partitions in Section 4.5 suggests a prevalence of indivisible plexes in latin squares which may also be surprising.

1.4 Similar concepts in designs and graphs

The problems that we consider are related to a number of problems that have been studied in a variety of combinatorial contexts. Some examples connected with combinatorial designs and graphs are mentioned next. We do not present a thorough coverage. Examples in some other contexts will be mentioned in later sections.

Together a k-plex and its complementary $(n-k)$-plex are examples of an *orthogonal partition* of a latin square. Much early literature about k-plexes in small latin squares is expressed using this terminology [76–78]. A precise definition is not important to us but we caution that orthogonality of partitions has been defined in different ways [7, 127]. Another example of an orthogonal partition of a latin square is the obvious partition by its rows, columns or symbols (e.g.[7]). Much interest in orthogonal partitions, especially those with a regular part size which we discuss in Section 1.7, stems from their extensive application in the design and analysis of experiments [48, Ch.10]. Bailey [7] and Gilliland [88] both consider orthogonal partitions in a general setting, of which latin squares are a particular example.

Whenever we refer to a partition of a latin square it can be assumed that we mean a partition into parallel plexes. Finding a partition of a latin square L is equivalent to finding a frequency square orthogonal to L, where frequency square and orthogonality are as defined in [39, p465-466]. The non-existence of a latin square of order 6 with a

1-partition motivated the first studies of 2-plexes by Finney [76–78]. Various further designs involving latin squares of order 6 are discussed in [82–86, 93].

Any latin square yields a transversal design, by the method described in [39, p161]. A k-plex in a latin square corresponds to a k-parallel class of the transversal design. Non-separable k-parallel classes in Steiner triple systems have been investigated [40, p418]. An indivisible k-plex in a latin square corresponds to a non-separable k-parallel class in the associated design. Abel et al. [1] use a 2-plex, of a finite group with no transversal, to construct a class of generalised Bhaskar Rao designs.

In graph theory a latin square of order n may be described in several different ways. For example, a proper edge colouring of the complete bipartite graph $K_{n,n}$ with n colours. In this context each colour represents a symbol in the latin square. A transversal is represented in the graph by a 1-factor (or *perfect matching*) in which each edge has a distinct colour. A matching whose edges have distinct colours is called a *rainbow* [158], a *rainbow matching* [2], an *orthogonal matching* [141] or *multicolored matching* [3]. A k-plex is a k-factor of $K_{n,n}$ in which each of the n colours occurs on k edges and a k-partition is a k-factorisation into factors of this type.

Graph factorisation problems have been well studied [39, 126, 148]. One of the earliest results of graph theory, by Petersen in 1891, says that every regular graph of even degree has a 2-factorisation [52, p39]. The corresponding result for k-plexes in latin squares remains open and is the subject of Conjecture 1.3.

The term k-plex in graph theory may take a different meaning to our usage. In the study of graphs of social networks the name k-plex has been used to described a generalised clique [135]. Prior usage of the name k-plex for that purpose was not known in 2002 when it was first adopted in the context relevant to this thesis.

1.5 Partial transversals

Another generalisation of a transversal in a latin square is the following.

A *partial transversal* of *size* or *length* m is a set of m elements of a latin square such that no row, column or symbol occurs more than once.

The following conjecture by Brualdi [46, p103] also remains open after many decades.

Conjecture 1.7. *Every latin square of order n has a partial transversal of size $n-1$.*

In 1975, Stein [139] conjectured that the same is true for a more general structure which he calls an *equi-n-square*.

Conjecture 1.8. *Every $n \times n$ matrix in which each of n symbols occurs n times has a partial transversal of size $n-1$.*

Erdős and Spencer [70] proved that an $n \times n$ matrix in which no symbol occurs more than $(n-1)/16$ times has a transversal.

There have been errors in several published results concerning Conjecture 1.7. A fatal flaw in a published proof of Conjecture 1.7, by Derienko [51], is shown by Cameron and Wanless [23]. A recent article by Hatami and Shor [98] explains a "bug" in [136] which was subsequently duplicated in [87]. Hatami and Shor [98] conclude that the following theorem, with a corrected higher constant, is proven and that their method cannot achieve a bound better than $n - O(\log^2 n)$.

Theorem 1.9. *Every latin square of order n has a partial transversal of length at least $n - 11.053 \log^2 n$.*

The theorem improves on earlier bounds $(2n+3)/3$ by Koksma [107], $3n/4$ by Drake [56] and then $n - \sqrt{n}$ by Woolbright [157] and independently by Brouwer, de Vries and Wieringa [17].

Zaker [159] uses a graph setting to show that the computational complexity of finding a partial transversal of maximum size in a 2-plex is an NP-complete problem.

Cameron and Wanless [23] show that every latin square contains a set of n entries such that every row and column occurs once and no symbol occurs more than twice.

Akabari and Alipour [3] showed that the number of partial transversals of length $n-1$ is even.

Aharoni and Berger [2] further generalise Conjectures 1.1, 1.7 and 1.8 in the context of hypergraphs.

With some consideration as to an appropriate definition, one might also ask questions about partial k-plexes. A requirement that no row, column or symbol occurs more than k times does not imply, for $k > 1$, that such a set of elements has cardinality divisible by k. An idea of length m corresponding to k entries in each of m rows (or columns) such that no row, column or symbol occurs more than k times is perhaps the clearest analogy. There is considerable interest in the case where every row, column or symbol contains either 0 or k entries, and the set also satisfies the definition of a *latin trade*. We will discuss such latin trades in Section 1.13.

1.6 Sets of permutations

Let S_n denote the set of all distinct permutations of $[n] = \{1, 2, \dots, n\}$. Let $\bar{x} = x_1 x_2 \dots x_n$ and $\bar{y} = y_1 y_2 \dots y_n$ be distinct permutations in S_n. Then $\bar{x}$ and $\bar{y}$ are said to *agree* in *position* $i \in [n]$ if $x_i = y_i$. The rows of any latin square identify a *sharply transitive* set of permutations. That is, the rows in L, considered as a set of permutations, form a subset of S_n, and for any $i, j \in [n]$, exactly one permutation in

L takes i to position j. Every pair of rows of a latin square agree in zero positions, and a set of rows of a latin square is maximal in size with regard to this property.

An alternative approach to Conjectures 1.1 and 1.7 emerged in an unpublished manuscript by Kézdy and Snevily [105]. Results of [105] are recorded in [23] and can also be found in [130]. We shall see from their work, that Conjectures 1.1 and 1.7 might be solved by answering the following.

Problem 1.10. Let $S \subset S_n$. What is the minimum $|S|$ such that every permutation in S_n agrees with some member of S in at least two positions?

More generally, let $f(n,s)$ be the minimum size of $S \subseteq S_n$ such that every permutation of S_n agrees in at least s positions with some element of S. For example, as every pair of distinct permutations in S_n disagrees in at least two positions we have $f(n, n-1) = |S_n| = n!$. The function $f(n,s)$ has a nice interpretation as a graph. Define the graph $G_{n,s} = (V, E)$, where V consists of the elements of S_n and E consists of the pairs of vertices which agree in at least s places. Then $f(n,s)$ is equal to the size of the smallest dominating set in $G_{n,s}$.

For $s = 2$, the following is called the Kézdy-Snevily Conjecture [23].

Conjecture 1.11. *If n is even then $f(n,2) = n$. If n is odd then $f(n,2) > n$.*

Using Hall's Theorem [96], Cameron and Ku [22], and Kézdy and Snevily [105], determined the exact value of $f(n,1)$:

Theorem 1.12. *For all n, $f(n,1) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.*

The next result, due to Cameron and Wanless [23], gives bounds on $f(n,2)$.

Theorem 1.13. *For all $n > 2$, $\left\lfloor \frac{n}{2} \right\rfloor + 2 \leqslant f(n,2) \leqslant \frac{4}{3}n + O(1)$.*

A latin square has a transversal if and only if there is a permutation in S_n that agrees in exactly one position with each of its rows. Thus, $f(n,2) > n$ only if every latin square of order n has a transversal. Hence, Conjecture 1.1 is implied by Conjecture 1.11. The existence of latin squares with no transversal shows that, for even n, $f(n,2) \leqslant n$.

Proof that Conjecture 1.11 implies Conjecture 1.7 is given in [23]. The idea of the proof is that given L contradictory to Conjecture 1.7, then every element of S_n must either agree in three positions with some row in L, or agree with two rows of L in at least two places. Append the symbol $n+1$ to each of the rows of L to give a set $S' \subset S_{n+1}$. It is then argued that any permutation in S_{n+1} agrees at least twice with each member of S' which shows that $f(n+1,2) \leqslant n$, in contradiction to Conjecture 1.11.

For $n \in \{5,7,9\}$, $f(n,2) \leqslant n+1$ follows from the examples given in [23]. These were found by adding a permutation that agrees with every transversal of the latin square (considered as a permutation) to the rows of a latin square. Keevash and Ku [104] provide constructions that imply bounds on $f(n, s)$ for general s.

Let $g(n, s)$ be the minimum size of a maximal set of permutations of S_n such that every two of them agree in *fewer than* s positions. For example, $g(n, 1) = n$ since, by Theorem 1.34 in Section 1.12, every latin rectangle can be extended to a latin square. The following conjecture made by Quistorff in 1999 can be found in [130], although using different terminology.

Conjecture 1.14. *If n is even then $g(n, 2) = n$. If n is odd then $g(n, 2) > n$.*

Quistorff [130] shows that Conjecture 1.11 implies Conjecture 1.14, which itself implies Conjectures 1.1 and 1.7.

Much of the literature about sets of permutations is expressed in terms of Hamming distance and covering radius of the permutation space. We refer the reader to [23], and the surveys [21, 130], for further coverage of the ideas mentioned in this section and their various connections with problems in coding theory.

1.7 Orthogonal latin squares and k-partitions

Two latin squares L_1 and L_2 of order n are said to be *orthogonal mates* if the n^2 ordered pairs in the matrix obtained by superimposition of L_1 and L_2 are distinct. It follows that these n^2 ordered pairs are distinct if and only if the positions of a fixed symbol in L_1 correspond to the positions occupied by a transversal in L_2. Thus follows a simple but important consequence.

Theorem 1.15. *A latin square has an orthogonal mate if and only if it has a decomposition into disjoint transversals.*

We often defer to traditional phrasing of the concept. Clearly, the decomposition in Theorem 1.15 is a 1-partition. A set of latin squares that are pairwise orthogonal mates is called a set of *mutually orthogonal latin squares* (MOLS).

In 1782 Euler conjectured that if $n \equiv 2 \bmod 4$, then there is no pair of MOLS of order n [71]. A 1960 disproof of Euler's conjecture is contained in the next, famous, theorem by Bose, Shrikhande and Parker. The proof unfolded over several papers. Parker [124] showed the case $n = 10$, Bose and Shrikhande [14, 15] proved $n = 22$ and infinitely many other cases for $n \equiv 2 \bmod 4$. A combined effort [16] proved the theorem.

Theorem 1.16. *For all n except 2 and 6, there is a pair of MOLS of order n.*

As recorded in [150], Theorem 1.16 together with an example of order 6 such as (1.1), show the following.

Corollary 1.17. *For all $n \neq 2$, there exists a latin square of order n with a complete plex range.*

In 1896, Moore was the first to show that the maximum cardinality of a set of MOLS of order n is $n-1$, and that this upper bound is achieved if n is a prime power [120]. A set of MOLS which achieves the upper bound is called a *complete* set of MOLS. Bose proved that a complete set of MOLS is equivalent to an affine plane of order n [13]. A prominent open problem is the following [113, p38].

Conjecture 1.18. *There exist a complete set of MOLS of order n if and only if n is a prime power.*

The next result, by Bruck and Ryser [18], resolves many cases of Conjecture 1.18.

Theorem 1.19. *If $n \equiv 1$ or $2 \bmod 4$ and there exists a complete set of MOLS of order n, then there exist integers a and b such that $n = a^2 + b^2$.*

Even for relatively small n, such as $n = 10$, it remains open to show the exact maximum cardinality of a set of MOLS. Moreover, for $n = 10$ a set of 3 MOLS has not been found despite considerable effort by many people over many decades. See the list of references in [119]. A computational result by Lam, Thiel and Swiercz [112] proved that there is no complete set of MOLS of order 10. Combining their result with a result by Shrikhande [137], shows that the maximum number of MOLS of order 10 is between 2 and 6 inclusive. Several attempts have been made to find 3 MOLS of order 10 by computer search. The most recent, by McKay, Meynert and Myrvold [119], concluded that if there is a set of 3 MOLS of order 10, then each latin square in the set has a trivial paratopy group.

There are many other important open problems, equivalences and a huge amount of interest in orthogonal latin squares and their applications which we do not detail here. For example, see the reference texts [39], [46], [48] and [113], and the numerous references therein. We will apply the valuable result of Theorem 1.16 in our work but our attention is strongly focused toward indivisible k-plexes and latin squares with a restriction of their transversals which results in the absence of a 1-partition.

Other authors have studied latin squares with a closely related restriction (on transversals) that still permits the existence of an orthogonal mate. Danziger, Wanless and Webb [44] introduce the notion of a Δ-*crimped* latin square, a latin square that contains two elements γ_1 and γ_2 such that there is a transversal through γ_1 but every such transversal includes γ_2 as well. This concept is defined more precisely in [44] using a version of our Lemma 2.1 in Chapter 2, which is crucial to their result. The key observation of [44] is that a Δ-*crimped* latin square cannot be a member of a set of 3 MOLS.

A k-maxMOLS(n) is a set of k MOLS of order n that is maximal in the sense that it is not contained in any set of $k + 1$ MOLS.

Problem 1.20. For which integers k and n do k-maxMOLS(n) exist?

For $k = 1$, an answer to the problem is given by Theorem 1.23 in the next section. A 1-maxMOLS, has no orthogonal mate and is called a *bachelor* latin square. A latin square is called *monogamous* if it has an orthogonal mate, but is not in any set of 3 MOLS. Thus a monogamous latin square and its orthogonal mate are 2-maxMOLS. Danziger, Wanless and Webb [44] show the existence of Δ-*crimped* latin squares of order $n \notin \{1, 2, 4\}$. By finding orthogonal mates for many of these latin squares they prove the existence of 2-maxMOLS for all $n > 6$, except perhaps when $n = 2p$ for a prime $p \geqslant 11$. Prior work on the 2-maxMOLS problem includes the work of Drake, van Rees and Wallis [56, 57, 142].

Conjecture 1.21. *For all $n \notin \{1, 2, 4, 5, 6\}$, there exist 2-maxMOLS of order n.*

Conjecture 1.21 is stated in [44] and by the results stated there, they conclude that, for $n < 100$, Conjecture 1.21 is only open for $n \in \{22, 26, 34, 38, 58, 62, 74, 86, 94\}$. The status of Problem 1.20 for other values of k is summarised in [39, p190].

The question of the existence of a latin square with a 1-partition is answered by Theorem 1.16. An analogous question for k-plexes is the following.

Problem 1.22. For which integers n and k does there exist a latin square of order n with an indivisible k-partition?

We present an almost complete answer to Problem 1.22 in Chapter 4. We do not rule out the possibility that there exists some indivisible k-plex for $k = n$. In other words, an indivisible latin square. Note that Problem 1.22 without the condition "indivisible" is easily answered. Theorem 1.16 along with (1.1) yield a k-partition in a latin square of order mk for each order mk except the two cases for $k = 1$ which are specifically excluded by Theorem 1.16.

1.8 Bachelor latin squares

A *bachelor latin square* is a latin square with no orthogonal mate. An entry of a latin square is called *transversal-free* if no transversal includes the entry. A *confirmed bachelor square* is a latin square that has at least one transversal-free entry.

The question of the existence of a bachelor latin square for each even order n was resolved by Euler in 1779 (e.g. [46, p445]). For $n \in \{1, 3\}$ it is easily found that there is no bachelor latin square of order n. In 1944, Mann showed existence of a bachelor for each order $n > 3$ when $n \equiv 1 \bmod 4$ [117]. In general, for orders $n \equiv 3 \bmod 4$, the question of existence of bachelor squares remained open until 2006. The next theorem is due to Wanless and Webb [155].

Theorem 1.23. *For any positive integer $n \notin \{1, 3\}$, there exists a confirmed bachelor square of order n.*

The proof of Theorem 1.23 illustrates the base case of a useful lemma, stated in Chapter 2, and one which underpins the main substance of Chapters 3, 4 and 5. A description of the infinite family used to prove Theorem 1.23 is in Section 2.5.2.

Independently, and using a variation of the same technique, Evans [73] showed a similar result, namely, the existence of a bachelor latin square for each order $n \notin \{1, 3\}$.

We present a very simple alternative proof of Theorem 1.23 for odd $n > 3$ in Section 4.3.

In Chapter 4 we extend the above mentioned result by Mann ([117]) to show that very large parts of a latin square may fail to contain not only transversals, but any odd plexes, and that this happens for each order n.

In Chapter 5 we investigate several types of restrictions they may exist on the transversals in a latin square. We consider, for example, the maximum number of transversals in a partition and the number of transversal-free entries in a latin square. We will see that an absence of transversals is not necessary to the existence of some strong restrictions. Our main result is that for all even $n \geqslant 10$, except perhaps if n is a power of 2, there exists a latin square of order n that possesses a transversal, but every transversal coincides on a single entry. A theorem by Wanless will show that there exist arbitrarily large latin squares of odd order in which the proportion of entries not in a transversal is asymptotic to one ninth.

In 1990 van Rees [142] conjectured the following.

Conjecture 1.24. *The proportion of latin squares of order n which are bachelor squares is asymptotically equal to* 1.

The conjecture was based upon evidence of the latin squares of order $n \leqslant 7$. More recent views expressed in [118] and [155] say that the opposite may be true. Evidence for slightly larger orders [119] suggests that the proportion of bachelor latin squares of order n may be rapidly diminishing.

Many of the families of latin squares that we will define in the next chapter are confirmed bachelor families. Our families will be 'close' to some well known families with block structure, in the sense that not many entries differ. In the next sections we review known results for plexes in latin squares with a well defined structure.

1.9 Step type latin squares

A latin square is said to be of *q-step type* if it is of order mq and can be represented by a matrix of $q \times q$ blocks $B_{i,j}$ as follows:

$$\begin{pmatrix} B_{0,0} & B_{0,1} & \dots & B_{0,m-1} \\ B_{1,0} & B_{1,1} & \dots & B_{1,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m-1,0} & B_{m-1,1} & \dots & B_{m-1,m-1} \end{pmatrix},$$

where each block $B_{i,j}$ is a latin subsquare of order q and two blocks $B_{i,j}$ and $B_{i',j'}$ contain the same symbols if and only if $i + j \equiv i' + j' \bmod m$.

Euler first showed in 1779 that a cyclic (or 1-step) latin square of even order has no transversals. Later, Maillet in 1894 proved that if m is even and q is odd, then a q-step type latin square has no transversals [46, p446]. As the next theorem by Wanless records, Maillet's result extends to k-plexes [150].

Theorem 1.25. *If m is even and q is odd then a q-step type latin square has no odd plex.*

In Chapter 2 we generalise Theorem 1.25 to obtain Lemma 2.1 which gives a necessary condition for k-plexes, and is important to our work following. In the next section, an algebraic interpretation of Theorem 1.25 is stated by Theorem 1.29.

1.10 Quasigroups, loops and groups

Much of the theory of latin squares, and transversals, is stated in the context of quasigroups (e.g. [46]).

A *quasigroup* $(Q, \cdot)$ is a set Q on which the binary operation $\cdot$ satisfies (i) for all $a, b \in Q$ there is a unique $x \in Q$ such that $a \cdot x = b$; and (ii) for all $a, b \in Q$ there is a unique $y \in Q$ such that $y \cdot a = b$. It follows that the unbordered Cayley table of a quasigroup is a latin square. Conversely, for any latin square L there exists some quasigroup Q whose operation $\cdot$ is specified by L. A *loop* is a quasigroup with an identity element. An associative loop is a group. A result about the existence of plexes in all loops is true for all latin squares. This follows from the statements in Section 1.2 and considering that, for every latin square, there exists an isotopic latin square which has a natural order on the entries in its first row and first column. Such a latin square is said to be in *reduced* form [113, p4].

The algebraic equivalent of a transversal in a quasigroup is called a complete mapping and was first described by Mann [116]. A *complete mapping* of a quasigroup $(Q, \cdot)$ is a permutation f of Q such that the mapping $x \to g(x)$ where $g(x) = x \cdot f(x)$ is also a permutation of Q. Hence the set of ordered triples $\{(x, f(x), g(x) = x \cdot f(x))\}$ form a transversal. The permutation $g(x)$ is called an *orthomorphism*, terminology introduced in [99].

It is well known [48, p7] that any transversal in a group table can be translated by multiplication of its elements by some $g \in G$ to obtain a disjoint transversal. Varying g across all non identity elements of the group results in $n - 1$ disjoint translates of the first transversal. Thus;

Theorem 1.26. *If the Cayley table of a group of order n has a transversal then it has n disjoint transversals.*

An analogous statement to Theorem 1.26 for k-plexes is false [150]. For any k, multiplication on the elements of a k-plex by a fixed non identity element of the group will yield some k-plex, but one that is not necessarily parallel to the original when $k > 1$.

The next theorem is due to Wanless [150].

Theorem 1.27. *Let G be a group of finite order n with a non-trivial cyclic Sylow 2-subgroup. The Cayley table of G contains no k-plex for any odd k but has a 2-partition and hence contains a k-plex for every even k in the interval $0 \leqslant k \leqslant n$.*

Thus, the Cayley table of $\mathbb{Z}_n$ justifies the next statement.

Corollary 1.28. *For all even n, there exists a latin square of order n which has an even plex range.*

The proof of Theorem 1.27 makes use of the fact that for any normal subgroup N in G, the quotient G/N presents a block structure in the Cayley table. This means that the absence of odd plexes, when G has a non-trivial cyclic Sylow 2-subgroup, follows from Theorem 1.25. The next statement shows Theorem 1.25 in the context of loops. It is due to Pula [128].

Theorem 1.29. *If Q is a loop whose Cayley table has an odd plex, then Q has no $N \trianglelefteq Q$ such that $|N|$ is odd and $Q/N \cong \mathbb{Z}_{2^m}$ for $m \geqslant 1$.*

The normality condition for a loop in Theorem 1.29 follows similar notion as it does for groups. A definition is given in [125, p13].

In Corollary 1.33, we see that Theorem 1.29 also holds for more general objects than odd plexes.

Exciting news regarding plexes in group tables is the resolution of a conjecture by Hall and Paige from 1955 [95]. The result, due to the work of many, has now been stated as a theorem in the literature, for example in [128].

Theorem 1.30. *If G is a finite group, then the following conditions are equivalent:*

(A) G has a transversal,

(B) the Sylow 2-subgroups of G are trivial or non-cyclic,

(C) there is a permutation of the elements of G which yields a trivial product. That is $g_1, g_2, \ldots, g_n$, such that $g_1 g_2 \ldots g_n = 1$ where 1 denotes the identity element of G.

We summarise the developments which lead to Theorem 1.30. First, Paige in 1951 showed that (A) $\implies$ (C) [122]. In 1955, Hall and Paige proved that (A) $\implies$ (B) and showed that the converse holds for all soluble, symmetric and alternating groups. They conjectured that the converse holds for all finite groups and that statement is referred to as the Hall-Paige Conjecture [95]. In 1989, Dénes and Keedwell showed that for non-soluble groups (B) $\implies$ (C) [47]. They observed that proving the Hall-Paige Conjecture was thus equivalent to proving that all non-soluble groups have complete mappings. In 2003, Vaughan-Lee and Wanless [144] gave an elementary proof of (B) $\implies$ (C); the initial proof in [47] invoked the Feit-Thompson Theorem. Other works up to 1992 are reported by Evans in [72]. Work by Dalla Volta and Gavioli [145–147] stated properties of a minimal counterexample to the conjecture. In 2009, Wilcox showed that any minimal counterexample to the conjecture must be simple [156], and thus reduced the possible candidates to the Tits group or a sporadic simple group, some of which had already been proven to possess transversals. Final cases to complete the proof, other than the Janko group J_4, are shown by Evans [74]. The last case, J_4, is reportedly proven in correspondence cited in [74] but has not yet been published.

It follows from Theorems 1.26 and 1.30 that Theorem 1.27 completely categorises the plex ranges for finite groups. That is, if G has a non-trivial cyclic Sylow 2-subgroup then its Cayley table has an even plex range, and otherwise it has a complete plex range.

Corollary 1.31. *Conjecture 1.3 is true for any Cayley table of a finite group.*

Recent work by Pula [128] investigates products of elements in a loop and a framework for non-associative analogues of the Hall-Paige Conjecture. The remainder of this section concerns the article [128].

For a finite loop Q, let $P^k(Q)$ be the set of elements that can be represented as a product containing each element of Q precisely k times. Using this notation, condition (C) of Theorem 1.30 says that $1 \in P^1(G)$. The notation $A(Q)$ denotes the associator subloop of Q, the smallest normal subloop of Q such that $Q/A(Q)$ is a group. So when Q is a group, $A(Q) = \{1\}$.

A class of finite loops $\mathcal{Q}$ is said to satisfy the *HP-condition* if for all $Q \in \mathcal{Q}$, the following conditions are equivalent:

(A*) the Cayley table of Q has a transversal, (B*) there is no $N \trianglelefteq Q$ such that $|N|$ is odd and $Q/N \cong \mathbb{Z}_{2^m}$ for $m \geqslant 1$, and

(C*) the set $P^1(Q)$ intersects $A(Q)$.

Thus, Theorem 1.29 shows that (A*) $\implies$ (B*) for all loops. Pula shows that (B*) $\iff$ (C*) holds for all loops. The results by Pula are actually shown to hold for a wider class of structures than k-plexes, as follows.

A *row k-plex* of a latin square is a collection of triples representing each row exactly k times. A subset $C = \{(x_i, y_i, z_i) : 1 \leqslant i \leqslant m\} \subseteq L$ is call *column-entry regular*, or just *regular* for short, if for each symbol s we have $|\{i : y_i = s\}| = |\{i : z_i = s\}|$. That is, s appears as a symbol the same number of times it appears as a column. We denote by C_r, the multiset of indices appearing as rows in C. For example, if C is a k-plex then C_r contains precisely k copies of each index. The next theorem and comments following are also from [128].

Theorem 1.32. *If C is a regular subset of the Cayley table of Q, then $P^1(C_r)$ intersects $A(Q)$. In particular, if Q has a k-plex, or just a regular row k-plex, then $P^k(Q)$ intersects $A(Q)$.*

Corollary 1.33. *If a loop fails to satisfy (B^*) then it has no regular row odd plexes. Thus, Theorem 1.29 also applies to regular row odd plexes in Q.*

However, the HP-condition does not hold for all loops. The following example has no transversal but satisfies condition (B^*). The shading shows a regular row transversal.

Q	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	4	3	6	5
3	3	5	1	6	2	4
4	4	6	2	5	1	3
5	5	3	6	2	4	1
6	6	4	5	1	3	2

(1.2)

In Chapter 3 we show latin squares, for each even order $n \geqslant 8$, which also fail the HP-condition.

For all odd ordered loops, condition (B*) is always true and so the implication (B*) $\implies$ (A*) is equivalent to Conjecture 1.1. More generally, the question of whether a loop has a 2-plex if and only if $P^2(Q)$ intersects $A(Q)$ would resolve Conjecture 1.2. The condition $P^2(Q)$ intersects $A(Q)$ is proven for all loops [128].

1.11 Diagonally cyclic latin squares

We continue to review results relating to latin squares with a defined structure.

A latin square L is a *diagonally cyclic latin square* or DCLS if $(x, y, z) \in L \iff (x+1, y+1, z+1) \in L$ where $\mathcal{I}(L) = \{0, 1, \ldots, n-1\}$ and calculations are in $\mathbb{Z}_n$. Thus any DCLS is completely determined by a single row or column. We say that row 0 generates a DCLS if and only if $f(i) = 0 \cdot i$ and $g(i) = f(i) - i$ are permutations of $\{0, 1, \ldots, n-1\}$. Thus, f and g guarantee distinct symbols in the rows and columns, respectively. Also, f is a complete mapping of the elements of $\mathbb{Z}_n$ so we see, by the definition of a complete mapping, that every DCLS of order n corresponds to a transversal of the Cayley table of $\mathbb{Z}_n$. Conversely, given a transversal of the Cayley table of $\mathbb{Z}_n$, say $T = \big\{(x, y, x+y)\big\}$, then row 0 of a DCLS is determined by $f(y) = 0 \cdot y = x+y$ for each element of T. If n is even, by Theorem 1.25, the Cayley table of $\mathbb{Z}_n$ has no transversal so there is no DCLS of even order. More directly, if n is even then $\sum_{i=1}^{n-1} f(i) \equiv \frac{1}{2}n$ but $\sum_{i=1}^{n-1} g(i) = 0$ [91].

DCLSs have a long history of study in connection with orthogonal latin squares and combinatorial designs. We refer the reader to [152] for further results and detailed review of the history of investigation into DCLSs and their applications.

It follows immediately from the definition that a DCLS has the property that every diagonal consists of a *cyclic transversal* or *cyclically generated transversal.* That is, for a fixed $\alpha \in \mathcal{I}(L)$, the *diagonal* $D_\alpha = \big\{(x, \alpha + x, 0 \cdot \alpha + x) \ : \ x \in \mathcal{I}(L)\big\}$ is a transversal. In the next section we consider problems about sets of cyclically generated transversals.

A DCLS is a latin square possessing a cyclic automorphism of order n. More generally, as described in [152], a *bordered diagonally cyclic latin square* of type B_b includes latin squares with a cyclic automorphism of order $n-b$. A B_0 type is a DCLS. Collectively, the B_b type latin squares are named Parker squares [152], in recognition of Parker's work on constructing sets of orthogonal latin squares involving these types of latin squares [123]. Parker latin squares are useful to us in Chapter 4 where we consider plexes in latin squares that contain subsquares.

1.12 Partial latin squares and completability

In Section 1.3 we defined a k-plex as a subset of a latin square. We may also consider a k-plex as a partial latin square. Indeed, some of the earliest results about k-plexes, other than small order studies, appear in the context of partial latin squares [4], [20], [45]. The definition of a PLS and information in this section is useful in Sections 4.2.2 and Sections 4.4 where we are interested in extending plexes into larger latin squares.

A *partial latin square* (PLS) of *order* n is a subset P of $R \times C \times S$, where each of R, C and S is a set of cardinality n, such that no two distinct elements of P agree in more than one coordinate. Often $R = C = S$ and we then refer to this set as the *index set* of P, and denote it by $\mathcal{I}(P)$. A PLS P of order n with $|P| = n^2$ is a latin square. The *size* of a PLS P is the cardinality of P.

A PLS of order n is *completable* if it is a subset of some latin square of order n.

Colbourn [38] showed that the decision problem *Is a PLS of order n completable?* is an NP-complete problem.

Classical results of M. Hall [94] and Ryser [133] apply Hall's Theorem [96] to state necessary and sufficient conditions for the extension of a latin rectangle to a latin square. The result of [133], stated next, is known as Ryser's Theorem.

Theorem 1.34. *An $r \times s$ latin rectangle R with entries from n symbols is completable to a latin square of order n if and only if each symbol occurs in R at least $r + s - n$ times.*

Another classical result is the following, due to Evans [75].

Theorem 1.35. *For any integers t and $n \geqslant 2t$, a $t \times t$ latin subsquare is completable to a latin square of order n.*

In answer to a well known problem posed in 1960 by Evans [75], which is later referred to as the Evans Conjecture, Smetaniuk [138] proved that any PLS of order n and size at most $n - 1$ is completable. Independently, Andersen and Hilton [6] obtained the same result. Earlier work by Häggkvist [92] proved the case for $n \geqslant 1111$. The question of completability again arises by insisting that further structure is imposed on the PLS. For example, the next conjecture made in 1980 by Häggkvist is stated in [50].

Conjecture 1.36. *Any PLS of order nr whose elements occur in $(n - 1)$ disjoint subsquares of order r is completable.*

Conjecture 1.36 is true for $r = 1$, by proof of the Evans Conjecture, and for $r = 3$ by Denley and Häggkvist [50]. Other special cases are proven in [49] and [109]. Further interest, see for example the survey [53], includes PLSs which are uncompletable or have precisely one completion, and which are critical with respect to this property. The latter are called *critical sets* and are closely connected with *latin trades* which we will discuss in the next section. We first mention some results and problems concerning PLSs of the kind closest to our interest.

A *k-plex* of *order n* is a completable PLS of order n and size kn and with the property that each row, column and symbol occurs k times.

Example (1.3) is a PLS of order 5 and size 10 such that each row, column and symbol occurs twice. However, by inspection, it is not completable and is therefore *not* a 2-plex.

$$\begin{pmatrix} 0 & 1 & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & 3 & \cdot \\ \cdot & \cdot & \cdot & 4 & 2 \\ \cdot & \cdot & 4 & \cdot & 3 \end{pmatrix} \tag{1.3}$$

Following terminology introduced in [19], we call such PLSs, which may or may not be completable, *protoplexes* and define them as follows. A *k-protoplex of order n* is a PLS of order n in which each row, column and symbol occurs exactly k times. Thus, a k-plex is equivalent to a completable k-protoplex. The definition of k-plex that we use follows [19, 67] and is a restriction of the original definition in [150]. In defining a k-plex as a PLS, Wanless [150] allowed that a k-plex be not necessarily completable; that is, a k-protoplex. Our definition of a protoplex is very similar to that of a *homogeneous PLS* which we discuss in Section 1.13. However, unlike protoplexes, homogeneous PLSs are allowed to have empty rows and columns.

The following problem is posed by Donovan in [53].

Problem 1.37. For which integers k and n is there an uncompletable k-protoplex of order n?

A partial answer to Problem 1.37 is given next, by Wanless [150].

Theorem 1.38. *If $1 < k < n$ and $k > \frac{1}{4}n$, then there exists an uncompletable k-protoplex of order n.*

Conjecture 1.39, stated next, can be found in Daykin and Häggkvist [45]. They proved that if $n = 16m$, where m is a positive integer and $k \leqslant \sqrt{n}/128$, then a k-protoplex is completable. In 1996 Burton [20] showed by computation that all 2-protoplexes of order 8 are completable. Having evidence of uncompletable 2-protoplexes for smaller order he also states the conjecture.

Conjecture 1.39. *For $k \leqslant \frac{1}{4}n$ every k-protoplex of order n is completable, hence is a k-plex of order n.*

The next partial extension result, due to Burton [20], can be found in [150].

Theorem 1.40. *For $k \leqslant \frac{1}{4}n$, every k-protoplex of order n is contained in a $(k+1)$-protoplex of order n.*

If Conjecture 1.39 is true, then the k-protoplex in Theorem 1.40 is a k-plex.

In Section 4.2.2 we further consider completability of k-protoplexes of order $2k$ and thereby derive results about indivisible plexes and partitions. The existence of arbitrarily large indivisible plexes follows from the next theorem by Wanless [150].

Theorem 1.41. *For arbitrary k and $n \geqslant k^2$, there exists an indivisible k-protoplex of order n.*

Our results in Chapter 4 realise a suggestion in [150] that the bound k^2 in Theorem 1.41 might be improved. It will follow from our results that, for $k > 1$, there exist indivisible k-plexes (hence indivisible k-protoplexes) of order $n \in \{2k, 2k+1\}$.

The next problem posed by Alspach and Heinrich in 1990 [4], in connection with their work on combinatorial designs, concerns protoplexes consisting of parallel transversals.

Problem 1.42. For each k, does there exist an integer $N(k)$ such that any k-protoplex consisting of k transversals of order $n \geqslant N(k)$ is completable?

Grüttmüller [91] poses the following restriction of Problem 1.42 to diagonally cyclic latin squares.

Problem 1.43. Does there exist for each k an odd integer $C(k)$ such that any k-protoplex of odd order $n \geqslant C(k)$ and consisting of k cyclically generated transversals is completable to a DCLS?

An *idempotent* latin square is a latin square L such that $x \cdot x = x$ for all $x \in \mathcal{I}(L)$. As noted in [91], the existence of an idempotent latin square for all $n \neq 2$ shows that $N(1) = 3$. Grüttmüller [91] proved that:

Theorem 1.44. *For all $k \geqslant 1$, $N(k) \geqslant 4k - 1$.*

Theorem 1.45. *For all $k \geqslant 3$, $C(k) \geqslant 3k - 1$.*

Grüttmüller further proved that $C(2) = 3$ [90]. Computational results [91] prove that a PLS P, consisting of k cyclically generated transversals, is completable to a DCLS if $2 \leqslant k \leqslant 7$ and n is an odd integer in the range $3k - 1 \leqslant n \leqslant 21$. Also, [91] shows that P is completable (non-cyclically) for $k = 2, 3$ or 4 and arbitrary n such that $4k - 1 \leqslant n \leqslant 21$. This supports the following conjectures by Grüttmüller [91].

Conjecture 1.46. *For $k \geqslant 3$, $N(k) = 4k - 1$.*

Conjecture 1.47. *For $k \geqslant 3$, $C(k) = 3k - 1$.*

As noted [34], it has not been shown that $N(2)$ exists. Cavenagh, Hämäläinen and Nelson [34] recently proved the following which may suggest that $C(3) = 9$.

Theorem 1.48. *For prime $p > 7$, a PLS consisting of three cyclically generated transversals of order p is completable to a DCLS.*

As noted in Section 1.11, a DCLS of order n consists of n disjoint cyclically generated transversals and there is a bijection between the set of DCLSs of order n and transversals in the Cayley table of $\mathbb{Z}_n$. Similarly, an equivalent statement to Theorem 1.48 is that every partial transversal of size 3 in the Cayley table of $\mathbb{Z}_p$, for prime $p > 7$, can be extended to a transversal.

1.13 Latin trades and homogeneous partial latin squares

A *critical set* is a uniquely completable PLS P of order n such that every proper subset of P has at least two different completions to a latin square of order n. For a given order, finding the set of achievable sizes of critical sets and the smallest and largest such sizes are among questions of interest e.g. [53]. It has been conjectured by Bate and van Rees [9] that the smallest size of a critical set of a latin square of order n is $\lfloor n^2/4 \rfloor$. We refer the interested reader to a survey by Keedwell [103] for more detailed information on the study of critical sets. It is an area of research closely connected with the study of latin trades.

A PLS T is said to be a *latin trade* if $T \neq \emptyset$ and there exists a PLS T' with *size* $|T| = |T'|$, and for each $(x, y, z) \in T$ there exists a unique $x' \neq x$, $y' \neq y$ and $z' \neq z$ such that $(x', y, z) \in T'$, $(x, y', z) \in T'$, and $(x, y, z') \in T'$. The pair (T, T') is called a *latin bitrade*.

A latin trade T may be thought of as a non-empty subset of a latin square which may be replaced with another set T' to obtain a new latin square of the same order. That is, $T \subset L$ is a latin trade if there exists a PLS T' with $T \cap T' = \emptyset$ and such that $(L \setminus T) \cup T'$ is a latin square. Thus a bitrade identifies the different elements of two latin squares of the same order. Some early studies, by Drápal and Kepka, used the name *exchangeable partial groupoids* to describe latin bitrades [63, 64].

A latin trade is *minimal* if no proper subset of it is a latin trade. Much attention has focused on minimal latin trades because of their close connection with critical sets. That well known connection (e.g. [30]) is as follows.

Lemma 1.49. *If C is a critical set of a latin square L, then every latin trade $T \subseteq L$ must intersect L in at least one element. For each element (x, y, z) in C there exists a unique minimal latin trade $T \subseteq L$ such that $T \cap L = (x, y, z)$.*

A summary of the theory of latin bitrades and their applications is contained in a survey by Cavenagh [30].

Latin trades can be used in the generation of random latin squares [33] and the possibility of compact storage for large catalogues of latin squares is discussed in [151].

As mentioned in [30], latin trades have been used to obtain existence proofs like Theorem 1.23 by [155]. Indeed, most of the work in our later chapters will use a latin trade in the Cayley table of an Abelian group. The trades that we employ are designed to work well with Lemma 2.1 which is stated in the next chapter. Some of them will closely resemble, or could be adapted from, those described in [27] and which involve 3 rows of a latin square. Otherwise they might be obtained using a technique referred to as cycle switching in [151]. Of course, as noted in [29], any latin square may be thought of as involving some latin trade with another latin square of the same order.

Recent studies of latin trades have identified several interesting connections between latin trades, geometry, topology and permutations. We mention these, briefly, as we next review the rather extensive literature on latin trades. Much of this work is very recent. A reader who is new to the topic may find it helpful to also refer to the survey [30]. Information in this section is presented for context only and it will not be used in our later work.

In contrast to our own designs of latin trades which will be detailed in the next chapter, those latin trades which themselves closest resemble a k-plex are the following. Our review will focus on homogeneous latin trades.

A *k-homogeneous latin trade* for $k \geqslant 2$ is a latin trade which intersects each row, each column and each symbol of the latin square either 0 or k times. Hence the size of a *k-homogeneous latin trade* is equal to km for some integer m.

A minimal 2-homogeneous latin trade is an intercalate and any 2-homogeneous latin trade is necessarily a union of intercalates.

According to Cavenagh [29], one reason for interest in k-homogeneous latin trades is that they often have, compared with other minimal latin trades, large size with respect to the order of the latin square they are contained in and so may be useful in locating small critical sets. Cavenagh also notes that the k-homogeneous property is invariant under conjugacies, and that k-homogeneous latin trades often partition into disjoint partial transversals and thus have implications for partial orthogonality.

Construction of k-homogeneous latin trades for each $k \geqslant 2$ is shown by Bean et al. [10] but these are not required to be minimal.

Cavenagh, Donovan and Drápal [24] use a geometric method, a hexagonal packing of the Euclidean plane with circles where arcs identify a labelling of rows, columns and symbols, to prove existence of 3-homogeneous latin trades of size $3m$ for each $m \geqslant 3$. It is further shown by Cavenagh [29] that the construction in [24] determines all 3-homogeneous latin trades. Thus, in contrast to 2-homogeneous latin trades:

Theorem 1.50. *Every 3-homogeneous latin trade is composed of 3 disjoint partial transversals.*

It is now known, for example [30], that Theorem 1.50 is implied by an earlier and more general result from 1983 by Negami [121]. Negami's result states that any 6-connected graph with genus 1 is uniquely embeddable on the torus. Other recent proofs of Theorem 1.50 appear in [89, 97, 115]. Grannell, Griggs and Knor [89] point to an even earlier result from 1973, by Altshuler [5], and present an exposition of the relationship of the classification problem for 3-homogeneous latin trades with two other combinatorial classification problems. The alternative proof of Theorem 1.50 in [89] is based on the work of Altshuler and Negami.

Another geometric construction, due to Cavenagh, Donovan and Drápal, involves rectangular packings of the plane with unit circles, where labels are derived by specified intersections and line segments [25]. The authors show 4-homogeneous latin trades of size $8m$ for each integer $m \geqslant 4$.

Cavenagh, Donovan and Yazici [31] construct minimal k-homogeneous latin trades of size km for $k \geqslant 3$ and $m \geqslant 1.75k^2 + 3$. It is noted in [31] that these trades partition into disjoint transversals.

Although every latin trade can be completed to some latin square, a latin trade of order n need not be a completable PLS (of order n). We will discuss the *embedding order* of a latin trade later in this section. The question of whether or not a latin trade embeds into a latin square of some specific form, for example the Cayley table of an Abelian group, is certainly of further interest to other researchers and in our own pursuits. Cavenagh [28] gives a construction method using the trades of [24] to prove that:

Theorem 1.51. *For even m there exists a 3-homogeneous latin trade of size $3m^2$ in the Cayley table of $(\mathbb{Z}_2)^m$.*

A well known result by Drápal [58] is the following.

Theorem 1.52. *Let m and n be positive integers. Suppose that we can partition the area of an equilateral triangle of side n into m smaller (integer-sided) equilateral triangles, such that each vertex of a triangle occurs as the vertex of at most 3 of the smaller triangles. Then there exists a spherical latin bitrade (T, T') of size m such that T embeds into the Cayley table of $\mathbb{Z}_n$.*

We use the terminology and notation of [35] to explain the adjective "spherical" in the last sentence of the theorem.

Each row x of a latin bitrade (W, B) defines a permutation ψ_x of the symbols of row x, where $\psi_x(z) = z'$ if and only if $(x, y, z) \in W$ and $(x, y, z') \in B$ for some y. If ψ_x is a single cycle, then we say that the row x is *separated*. Similarly, we may determine that each column and symbol of the bitrade is either separated or non-separated. If every row, column and symbol is separated, then we say that the latin bitrade (W, B) is *separated*.

Any non-separated bitrade can be transformed into a separated bitrade by sending offending entries of (W, B), say in a smaller cycle of ψ_x, to an empty row. For an example see [30] or [35]. Performing a similar operation, the column and symbol cycles can also be separated. Thus, in some sense, all latin bitrades are considered by the class of separated bitrades.

A latin bitrade (W, B) is said to *primary*, or *connected*, if there exists no latin bitrades (W', B') and (W'', B'') such that $W \cap W'' = \emptyset$, $W = W' \cup W''$ and $B = B' \cup B''$.

Let $R_W = \{r_x : (x,y,z) \in W\}$, $C_W = \{c_y : (x,y,z) \in W\}$ and $S_W = \{s_z : (x,y,z) \in W\}$. Clearly $R_W = R_B$, and similarly for the columns and symbols.

Given a separated, connected latin bitrade (W,B), we may construct a *triangulation* $\mathcal{G}_{W,B}$ whose vertex set is $R_W \cup C_W \cup S_W$ and whose edges are pairs of vertices corresponding to some triple $(x,y,z) \in W$, or equivalently within some triple of B. If we define white and black faces of $\mathcal{G}_{W,B}$ to be the triples from W and B, respectively, then $\mathcal{G}_{W,B}$ is a face 2-colourable triangulation of some surface [59].

We now orient the edges $\mathcal{G}_{W,B}$ so that each white face contains directed edges from a row to a column, from a column to a symbol and from a symbol to a row. The subgraph induced by the vertices of a face, black or white, has both in-degree and out-degree of each vertex equal to 1. This shows that the surface in which $\mathcal{G}_{W,B}$ is embedded is orientable. Therefore, we may associate with $\mathcal{G}_{W,B}$ the non-negative integer value g for the genus of $\mathcal{G}_{W,B}$ given by Euler's genus formula:

$$\begin{aligned} g &= \tfrac{1}{2}(2 - f + e - v) \\ &= \tfrac{1}{2}(2 - (|W| + |B|) + 3|W| - (|R_W| + |C_W| + |S_W|)) \\ &= \tfrac{1}{2}(2 + |W| - |R_W| - |C_W| - |S_W|). \end{aligned}$$

Thus, we may now explain fully the statement of Theorem 1.52. A *spherical* or *planar* latin bitrade is a latin bitrade of genus 0. A *spherical* latin trade W is a latin trade for which there exists some disjoint mate B such that (W,B) is a spherical latin bitrade. In general, there may exist more than one disjoint mate for W but Cavenagh and Wanless show in [35] that if W is spherical, then the spherical bitrade (W,B) is uniquely determined. In fact they show that any other disjoint mate B' of W is neither primary nor separated. (Hence an associated genus does not exist.)

Lefevre et al. [114] construct, for each genus $g \geqslant 0$, a latin bitrade of smallest possible size and also a minimal latin bitrade of size $8g + 8$.

A main result of Cavenagh and Wanless [35], and one independently shown using an entirely different method by Drápal, Hämäläinen and Kala [61], is the following.

Theorem 1.53. *Every spherical latin trade can be embedded in a finite Abelian group.*

Cavenagh and Wanless [35] also show that, for every $g \geqslant 1$, there is a latin trade of genus g which cannot be embedded in any group, and another latin trade of genus g which can be embedded in the Cayley table of $\mathbb{Z}_n$ for some n.

Drápal, Hämäläinen and Kala consider methods for generating dissections of equilateral triangles to arrive at their proof of Theorem 1.53 in [61]. Their approach was motivated by a recent finding of Cavenagh and Lisoněk [26] that planar Eulerian triangulations are equivalent to spherical latin bitrades. In other words, spherical latin bitrades are equivalent to cubic 3-connected bipartite planar graphs.

Drápal, Hämäläinen and Rosendorf [62] report on the computer generation of spherical bitrades up to size 24. An earlier computer enumeration and classification of small latin trades up to size 19, by Wanless [153], also determines the respective *embedding order*, the smallest order n such that a latin square of order n contains the latin trade. Species representatives of each latin trade are available at the author's homepage [149].

Several of the recent results on latin trades refer to Drápal's representation of a latin bitrade as a set of three permutations. The idea, which we explain below, is now published in [60] but was earlier circulated at a workshop in Prague in 2003. Also [32] provides proof of the main concepts. Some further terminology is required.

Let (W, B) be a connected and separated latin bitrade. Define the map $\beta_d : W \to B$ where $d \in \{1, 2, 3\}$ and $\beta_d(w_1, w_2, w_3) = (b_1, b_2, b_3)$ implies that $w_d \neq b_d$ and $w_i = b_i$ for $i \neq d$. In other words, with two coordinates fixed, the (different) d-th coordinate identifies a unique triple in B. In addition, let $\tau_1, \tau_2, \tau_3 : W \to W$ be given by $\tau_1 = \beta_2\beta_3^{-1}$, $\tau_2 = \beta_3\beta_1^{-1}$ and $\tau_3 = \beta_1\beta_2^{-1}$. Note that τ_i leaves the i-th coordinate of a triple fixed. The three permutations $\{\tau_1, \tau_2, \tau_3\}$ are called the *permutation representation* of (W, B).

In the reverse direction, a set of permutations with particular properties defines a latin bitrade. Let $\mathrm{Mov}(\pi)$ denote the set of points that a permutation π acts on. Let τ_1, τ_2, τ_3 be permutations and let $\Omega = \mathrm{Mov}(\tau_1) \cup \mathrm{Mov}(\tau_2) \cup \mathrm{Mov}(\tau_3)$. Define four properties:

(T1) $\tau_1\tau_2\tau_3 = 1$,

(T2) if ρ_i is a cycle of τ_i and ρ_j is a cycle of τ_j, then $|\mathrm{Mov}(\rho_\mathrm{i}) \cap \mathrm{Mov}(\rho_\mathrm{j})| \leqslant 1$ for $1 \leqslant i < j \leqslant 3$,

(T3) τ_1, τ_2 and τ_3 have no fixed points,

(T4) the group $\langle \tau_1, \tau_2, \tau_3 \rangle$ is transitive on Ω.

The next theorem, by Drápal [60], states how these properties relate to latin bitrades.

Theorem 1.54. *A latin bitrade (T, T') is equivalent (up to isotopism) to three permutations τ_1, τ_2 and τ_3 acting on a set Ω and satisfying properties* (T1), (T2) *and* (T3)*. If property* (T4) *is also satisfied then (T, T') is primary. If $\{\tau_1, \tau_2, \tau_3\}$ satisfy* (T1), (T2) *and* (T3) *then let $\mathcal{A}_i = \{\rho : \rho \text{ is a cycle of } \tau_i\}$. Then, a separated latin bitrade (T, T') of size $|\Omega|$, and with $|\mathcal{A}_1|$ rows, $|\mathcal{A}_2|$ columns and $|\mathcal{A}_3|$ symbols is specified by*

$$T = \big\{(\rho_1, \rho_2, \rho_3) : \ \rho_i \in (A)_i \text{ and the } \rho_i \text{ all act on a common point of } \Omega\big\},$$

and

$$T' = \big\{(\rho_1, \rho_2, \rho_3) : \ \rho_i \in (A)_i \text{ and } x, x' \text{ and } x'' \text{ are distinct points of } \Omega \text{ such that } x\rho_1 = x', x\rho_2 = x'' \text{ and } x''\rho_3 = x\big\}.$$

Lefevre, Donovan and Drápal [115] use a permutation representation to provide a shorter proof, than originally shown by [24] and [25], of the existence of 3 and 4-homogeneous latin trades. The authors [115] also give a further construction for 4-homogeneous latin trades.

Again via a permutation representation, Cavenagh, Drápal and Hämäläinen [32] show how some latin bitrades may be directly derived from groups. They study the special case when the set Ω is the set of elements of a group G and the group acts on itself by right translation.

Let G be a finite group and let a, b and c be distinct non-identity elements of G. Let $A = \langle a \rangle$, $B = \langle b \rangle$ and $C = \langle c \rangle$. Define three properties:

(G1) $abc = 1$,

(G2) $|A \cap B| = |A \cap C| = |B \cap C| = 1$,

(G3) $\langle a, b, c \rangle = G$.

The main tool of [32], and used in conjunction with a number of special conditions to yield particular properties, is the following.

Theorem 1.55. *If properties* (G1) *and* (G2) *are satisfied, then a latin bitrade* $(T^\circ, T^\star)$ *of size* $|G|$ *is given by*

$$T^\circ = \big\{(gA, gB, gC) \ : \ g \in G\big\}, \quad T^\star = \big\{(gA, gB, ga^{-1}C) \ : \ g \in G\big\}.$$

If (G3) is also satisfied then $(T^\circ, T^\star)$ *is primary.*

Some further properties of the bitrade thus constructed are next listed [32].

- $(T^\circ, T^\star)$ has $|G \colon A|$ rows each with $|A|$ entries, $|G \colon B|$ columns each with $|B|$ entries and $|G \colon C|$ symbols each occurring $|C|$ times.
- Thus if $|A| = |B| = |C| = k$, then T° (hence $T^\star$) are k-homogeneous latin trades.
- T° is equivalent to the following sets.

$$\big\{(g_1A, g_2B, g_3C) \ : \ g_1, g_2, g_3 \in G \text{ and } |g_1A \cap g_2B \cap g_3C| = 1\big\}, \text{ and}$$
$$\big\{(g_1A, g_2B, g_3C) \ : \ g_1, g_2, g_3 \in G \text{ and } g_1A \cap g_2B = \{g_3\}\big\}.$$

Orthogonality and minimality can also be encoded, however we omit the detail for these properties. In one example, using a non-Abelian group of order p^3 where p is an odd prime, the authors [32] show that for each prime p there exists a minimal primary p-homogeneous latin bitrade of size p^3. In general they prove the existence of minimal, k-homogeneous latin trades for each odd $k \geqslant 3$.

Behrooz, Bagheri and Mahmoodian [12] give a recursive construction for k-homogeneous latin bitrades in their proof of the next theorem. They also prove the existence of a k-homogeneous latin bitrade of size km for several more cases; for all odd k and $m \geqslant k$, for all $m \geqslant k$ where $9 \leqslant k \leqslant 37$, and a special case when k is even.

Theorem 1.56. *Let $m \geqslant k$ and $m' \geqslant k'$. If there exists a k-homogeneous latin bitrade of size km and a k'-homogeneous latin bitrade of size $k'm'$, then there exists a kk'-homogeneous latin bitrade of size $kmk'm'$.*

The next theorem, by Cavenagh and Wanless [36], gives the spectrum of possible sizes of k-homogeneous latin bitrades. It proves a conjecture in [12].

Theorem 1.57. *For all $k \geqslant 3$, a k-homogeneous latin bitrade of size s exists if and only if k divides s and $s \geqslant k^2$.*

The authors [36] conjecture that Theorem 1.57 completely determines all possible sizes of primary k-homogeneous latin bitrades. The proof of Theorem 1.57 shows a connection between k-homogeneous latin trades and transversals in the Cayley table of $\mathbb{Z}_n$. We mention this further in the next section.

In connection with Problem 1.5 we ask the following.

Problem 1.58. For which integers n and k is there an indivisible k-plex of order n that is a (k-homogeneous) latin trade?

Unfortunately, other than Theorem 1.50 and the obvious indivisibility of 2-homogeneous latin trades, classification according to divisibility is not readily found in the literature. Many of the constructions are elaborate and their proofs are non trivial. Moreover, as illustrated by Theorem 1.55, one generic construction may yield various examples. Incidentally, we observe that a minimal 4-homogeneous latin trade given by Lemma 40 of [25] contains 2 disjoint transversals. It is not obvious that similar divisibility would hold in general as any mapping described by their Remark 38 yields a 4-homogeneous latin trade. An example, given in [25] and [30], of a 4-homogeneous latin trade of size 32 in the Cayley table of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is divisible. Our data in Section 4.5 shows that there is no indivisible 4-plex in the Cayley table of this group.

1.14 The number of plexes and partitions

1.14.1 Small order studies

As mentioned in Section 1.2, the number of k-plexes and partitions in a latin square is a species invariant. Thus, to consider all plexes among the latin squares of order n, we need only consider a list of species representatives of order n. However, except for $n \leqslant 8$, such a list is either very large or unavailable. We rely on a program used in [119] to generate the isotopy or species representatives of order 9. This program was also used in [118] to prove Conjecture 1.1 for the latin squares of order $n \leqslant 9$. We refer the reader to [119] for a history and details of the investigation of the number of latin squares of small order. The text [101] also gives information on algorithmic issues relevant to the classification of latin squares.

Detailed investigations of the plexes and partitions of latin squares of order $n \leqslant 6$ are given by the collected works of Finney [76–79], Freeman [84–86] and Johnston and Fullerton [100]. Several of the early works on plexes for $k > 1$ are well prior to modern technologies so these reports would have taken an impressive number of human hours to compute.

Killgrove et al. [106] give data on the number of transversals and 2-plexes in latin squares of order 6 and 7. Wanless [150] includes the first enumerative data on plexes for species of order 8 and identifies the maximum number of disjoint transversals in each species.

Data on the number of transversals in species of order 9, group tables of order $n \leqslant 23$, and turn-squares of order 14 is presented in [118].

A research problem [101, p265] suggests a computational check of Conjectures 1.1 and 1.2 for order 9. As noted above, Conjecture 1.1 for order 9 was shown in [118]. Our results in Section 4.5.3 will verify Conjecture 1.2 for order 9.

In Section 4.5 we report on extensive computations of indivisible plexes and partitions in latin squares of order $n \leqslant 9$. Although several earlier reports in the literature consider plexes in small latin squares, and as mentioned in Section 1.3, Finney [77, 78] found indivisible plexes, the value of $\kappa(n)$ for small n has not been explicitly computed. We compute $\kappa(n)$ exactly for $n \leqslant 8$ and prove that $\kappa(9) \in \{6, 7\}$. We enumerate every different type of indivisible partition for every species representative of order $n \leqslant 8$.

In Section 5.4 we report on further computations concerning transversals in latin squares of order $n \leqslant 9$. We will report on the number of transversal-free entries and the number of transversals in a partition. As mentioned in Section 1.3, we will prove Conjecture 1.3 for order 9.

Another useful outcome of our computations is finding latin squares with interesting properties. Generalisation of a small example might lead to proving existence in general of an interesting property. For example, the proof of Theorem 5.6 in the final chapter developed in this manner. The data also provides a solid resource for testing ideas and the development of further theory.

1.14.2 General cases

Conjecture 1.1 is attributed to Ryser but is a weakening of his original statement [134]. Ryser originally conjectured that the number of transversals of a latin square is odd if and only if the latin square is of odd order. The original statement is disproved by the existence of latin squares of odd order, for example of order 7, with an even number of transversals. However, the latin squares of even order satisfy Ryser's original statement as proved by Balasubramanian [8].

Theorem 1.59. *The number of transversals in a latin square of even order is even.*

As observed in [150], Theorem 1.59 does not generalise in an obvious way to k-plexes. In Section 4.5, we will see that there exist latin squares of even order that possess an odd number of k-plexes for $k > 1$.

Following notation in [118], we define

$$T(n) = \max\big\{\, t :\ \text{some latin square of order } n \text{ possesses } t \text{ transversals}\big\}.$$

McKay, McLeod and Wanless [118] proved the following.

Theorem 1.60. *For $n \geqslant 5$,*

$$15^{n/5} \leqslant T(n) \leqslant c^n \sqrt{n}\, n!$$

where $c = \sqrt{\frac{3-\sqrt{3}}{6}} e^{\sqrt{3}/6} \approx 0.61354.$

Also following [118], define

$$t(n) = \min\big\{\, t :\ \text{some latin square of order } n \text{ possesses } t \text{ transversals}\big\}.$$

As Conjecture 1.1 is open, $t(n) \geqslant 0$. We know by Theorem 1.25 that $t(n) = 0$ for all even n. For odd n, understanding $t(n)$ is apparently difficult. As follows, there are some results about the number of transversals in the Cayley table of $\mathbb{Z}_n$ for odd n.

For odd $n \geqslant 3$, let t_n denote the number of transversals in the Cayley table of $\mathbb{Z}_n$.

Vardi [143] conjectured the following. The conjecture was made in connection with study of an equivalent problem known as the *toroidal n-queens problem.* The equivalence is explained in [154].

Conjecture 1.61. *There are constants c_1, c_2 such that $0 < c_1 < c_2 < 1$ and*

$$c_1^n n! \leqslant t_n \leqslant c_2^n n!$$

Clark and Lewis [37] make the following conjecture.

Conjecture 1.62.

$$t_n \geqslant n(n-2)(n-4)\dots(3)(1).$$

Cavenagh and Wanless [36] showed that t_n is at least exponential in n. Combining their result with the upper bound given by Theorem 1.60 proves the next theorem.

Theorem 1.63. *If n is sufficiently large then*

$$(3.246)^n < t_n \leqslant (0.614)^n n!.$$

Theorem 1.63 improves on some earlier bounds given by Cooper and Kovalenko [41, 43, 108] and Rivin, Vardi and Zimmermann [131]. Estimates of t_n are also given by Cooper et al. [42]. More recently, Kuznetsov [110, 111] use a fast computational method to estimate t_n for considerably large n, up to $n = 205$.

Other results, which we next mention, concern the divisors of t_n, and more generally, divisors of the number of transversals in latin squares with some specified structure.

The next two theorems are due to McKay, McLeod and Wanless [118].

Theorem 1.64. *The number of transversals in a symmetric latin square of order n is congruent to n mod 2.*

Theorem 1.65. *If G is a group of order $n \not\equiv 1 \bmod 3$, then the number of transversals in the Cayley table of G is divisible by 3.*

Let $z_n = t_n/n$. It should be clear that z_n is the number of transversals through a fixed entry in the Cayley table of $\mathbb{Z}_n$ (n is odd) and is independent of the entry chosen.

The following three theorems by Stones and Wanless [140] follow from their study of orthomorphisms of $\mathbb{Z}_n$.

Theorem 1.66. *For odd n*

$$z_n \equiv \begin{cases} -2 \bmod n & \text{if } n \text{ is prime,} \\ 0 \bmod n & \text{otherwise.} \end{cases}$$

Theorem 1.67. *If n is a prime of the form $2 \times 3^k + 1$, then $z_n \equiv 1 \bmod 3$.*

Theorem 1.68. *For odd n*

$$z_n \equiv \begin{cases} 0 \bmod 3 & \text{if } n \not\equiv 1 \bmod 3 \text{ and } n \geqslant 5, \\ \zeta(n) \bmod 3 & \text{if } n \equiv 1 \bmod 3. \end{cases}$$

where $\zeta(n)$ is the number of partitions of $\{1, 2, \dots, n\}$ into parts of size 3 in which each part has sum divisible by n.

Finally, on the subject of transversals in the Cayley table of $\mathbb{Z}_n$, we mention next a result about pairwise intersection of its transversals.

Theorem 1.69. *Let $I(\mathbb{Z}_n)$ denote the set of integers i for which there exist transversals T and T' in the Cayley table of $\mathbb{Z}_n$ such that $|T \cap T'| = i$. For each odd $n \neq 5$, $I(\mathbb{Z}_n) = \{i : 0 \leqslant i \leqslant n-3\} \cup \{n\}$, while $I(\mathbb{Z}_5) = \{0, 1, 5\}$.*

Theorem 1.69 is due to Cavenagh and Wanless [36] who used it to prove Theorem 1.57. A key observation of [36], and one which exploits the equivalence of transversals in the Cayley table of $\mathbb{Z}_n$ and diagonally cyclic latin squares, is the following.

Lemma 1.70. *Let T and T' be two transversals in the Cayley table of $\mathbb{Z}_n$ such that $T \cap T' = n - k$. Then there exists a k-homogeneous latin bitrade of size nk.*

All of the general results mentioned in this section concern transversals in latin squares. For $k > 1$, there are no known general enumerative results for k-plexes. We will give a divisibility result for k-plexes in latin squares that are group tables in Section 4.5. One might hope that publication of the data in that section will inspire some serious results for plexes. We note that, for $k = 1$, proving the upper bound on $T(n)$ involved considerable work [118]. Generalising results such as Theorems 1.60 and 1.63 to k-plexes may be difficult.

Chapter 2

A key lemma, definitions and latin families

2.1 Introduction

In this chapter we introduce a key lemma, along with the infinite families of latin squares and notation which we will use in our proofs in forthcoming chapters. The families have been designed for a productive application of Lemma 2.1. They are collected together here to facilitate comparison between their structure and to give a central reference location for related results and comments which cross several chapters. When working with a latin family defined in this chapter, we will also assume the named sets and notation conventions which are associated with the family. We suggest the reader should refer back to this chapter to obtain the definition of a latin family as and when it is first introduced in our workings. For example, only the families $\mathcal{P}_n$ and $\mathcal{Q}_n$ in Section 2.4.1 are needed for Chapter 3.

2.2 A necessary condition for plexes

Let G be an Abelian group and let L be a latin square of order $|G|$ where $\mathcal{I}(L) = G$.

The function $\Delta : L \to G$ is given by

$$\Delta(e) = z - x - y \text{ for each } e = (x, y, z) \in L.$$

Lemma 2.1. *Let G be an Abelian group with identity ε and let L be a latin square indexed by G. If K is a k-plex in L, then*

$$\sum_{e \in K} \Delta(e) = -k \sum_{g \in G} g = \begin{cases} \omega & \text{if } k \text{ is odd and } G \text{ has a unique involution } \omega, \\ \varepsilon & \text{otherwise.} \end{cases}$$

Proof. By definition, each row, column and symbol occurs k times in K, so that

$$\sum_{e\in K}\Delta(e) = \sum_{(x,y,z)\in K}\Delta(x,y,z) = k\sum_{z\in G} z - k\sum_{x\in G} x - k\sum_{y\in G} y = -k\sum_{g\in G} g.$$

If G has a unique involution ω, which happens precisely when G has non-trivial cyclic Sylow 2-subgroups, then $\sum_{g\in G} g = \omega$. Otherwise $\sum_{g\in G} g = \varepsilon$ (see, e.g. [46, p.34]). The result follows. $\square$

For several of the latin families to follow we will set $\mathcal{I}(L) = \mathbb{Z}_n$ so that, by Lemma 2.1,

$$\sum_{e\in K}\Delta(e) \equiv \begin{cases} \frac{1}{2}n \bmod n & \text{if } k \text{ is odd and } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

The simplest case of Lemma 2.1, as above when $G = \mathbb{Z}_n$, is due to Egan and Wanless [65, 67] and independently to Evans [73]. The case $k = 1$ was applied in [155] to prove Theorem 1.23.

Lemma 2.1 and variants of it have also proved useful in [19, 44, 73, 129].

2.3 Notation

In usage of the notation below we rely on context to make clear which latin square L we are considering.

Define r_x to be the set of elements in row x of L. That is,

$$r_x = \big\{(x,y,z) \in L :\ y,z \in \mathcal{I}(L)\big\}. \tag{2.1}$$

To specify $X \subset L$ we identify, for each row of L, the columns in X, as follows.

$$\mathrm{col}(x) = \big\{y \in \mathcal{I}(L) :\ (x,y,z) \in X \text{ for some } z \in \mathcal{I}(L)\big\}. \tag{2.2}$$

As mentioned in Section 1.13, the latin families that we will use involve some latin trade with the Cayley table of a finite Abelian group G, with an identity element ε. The set Δ_*, as we next define, identifies the actual trade with the Cayley table of G. When applying Lemma 2.1 we focus on Δ_*, or perhaps Δ_i for some $i \in G$.

$$\Delta_* = \big\{e \in L :\ \Delta(e) \neq \varepsilon\big\}, \qquad \Delta_i = \big\{e \in \Delta_* :\ \Delta(e) = i\big\}. \tag{2.3}$$

2.4 Latin families of even order

For each member L of our families of even order we set $\mathcal{I}(L) = \mathbb{Z}_n$. We define L using notation $L[x, y] = z$, where all calculations are performed modulo n. We assume that each $i \in \mathcal{I}(L)$ is the least non-negative residue in its congruence class.

We partition $\mathcal{I}(L)$ by parity, as follows.

$$E = \{0, 2, 4, \ldots, n-2\} \text{ and } F = \{1, 3, 5, \ldots, n-1\}. \tag{2.4}$$

2.4.1 $\mathcal{P}_n$, $\mathcal{Q}_n$ and $\mathcal{R}_n$

Families $\mathcal{P}_n$ and $\mathcal{Q}_n$ originate in [65, 67]. They will be used in Chapters 3 and 4. The family $\mathcal{R}_n$ appears in [19]. It is used in Chapter 4.

Definition 2.2 (Latin square $\mathcal{P}_n$)**.** For $n = 4q = 2h$, define

$$\mathcal{P}_n[x, y] = \begin{cases} x + y + 2 & \text{if } x = n-3 \text{ and } y \in F, \\ x + y - 2 & \text{if } x = n-1 \text{ and } y \in F, \\ x + y & \text{otherwise.} \end{cases} \tag{2.5}$$

The smallest member $\mathcal{P}_4$ is isotopic to the Cayley table of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

In the case when $L = \mathcal{P}_n$, it is immediate from (2.5) that

$$\begin{aligned} \Delta_* &= \Delta_{-2} \cup \Delta_2, \text{ where,} \\ \Delta_2 &= \big\{(x, y, z) \in r_{n-3} : \ y \in F\big\}, \\ \Delta_{-2} &= \big\{(x, y, z) \in r_{n-1} : \ y \in F\big\}. \end{aligned}$$

Less formally, construction of $\mathcal{P}_n$ is evident by the table below of the value $\Delta(e)$ for each $e = (x, y, z) \in \Delta_*$.

$$\begin{array}{c|ccccc} & 1 & 3 & 5 & \cdots & n-1 \\ \hline n-3 & 2 & 2 & 2 & \cdots & 2 \\ n-1 & -2 & -2 & -2 & \cdots & -2 \end{array} \tag{2.6}$$

Definition 2.3 (Latin square $\mathcal{R}_n$)**.** For $n = 4q = 2h$, define

$$\mathcal{R}_n[x, y] = \begin{cases} x + y - 4 & \text{if } x = 3 \text{ and } y \in F, \\ x + y + 4 & \text{if } x = n-1 \text{ and } y \in F, \\ x + y & \text{otherwise.} \end{cases} \tag{2.7}$$

For $L = \mathcal{R}_n$,

$$\begin{aligned}
&\text{if } n = 4,\ \Delta_* = \emptyset \text{ (since } \mathcal{R}_4 \text{ is the Cayley table of } \mathbb{Z}_4).\\
&\text{Otherwise,}\\
&\quad \Delta_* = \Delta_{-4} \cup \Delta_4, \text{ where,}\\
&\quad \Delta_{-4} = \big\{(x,y,z) \in r_3 \,:\, y \in F\big\},\\
&\quad \Delta_4 = \big\{(x,y,z) \in r_{n-1} \,:\, y \in F\big\}.
\end{aligned}$$

For $\mathcal{R}_n$, the value $\Delta(e)$ for each $e \in \Delta_*$ is as follows.

$$\begin{array}{c|ccccc}
 & 1 & 3 & 5 & \cdots & n-1 \\ \hline
3 & -4 & -4 & -4 & \cdots & -4 \\
n-1 & 4 & 4 & 4 & \cdots & 4
\end{array} \tag{2.8}$$

Definition 2.4 (Latin square $\mathcal{Q}_n$)**.** For $n = 2h$ where $h \geqslant 3$ is odd, define

$$\mathcal{Q}_n[x,y] = \begin{cases} n-1 & \text{if } x = y = 0 \text{ or } x = y = n-1, \\ 0 & \text{if } \{x,y\} = \{0, n-1\}, \\ x+y-2 & \text{if } x = 1 \text{ and } y \in F, \\ x+y+2 & \text{if } x = n-1 \text{ and } y \in F \setminus \{n-1\}, \\ x+y & \text{otherwise.} \end{cases} \tag{2.9}$$

If $L = \mathcal{Q}_n$, then

$$\begin{aligned}
\Delta_* &= \Delta_{-2} \cup \Delta_2 \cup I_{\mathcal{Q}} \text{ where } I_{\mathcal{Q}} = \Delta_{-1} \cup \Delta_1, \qquad (2.10)\\
\Delta_{-2} &= \big\{(x,y,z) \in r_1 \,:\, y \in F\big\},\\
\Delta_2 &= \big\{(x,y,z) \in r_{n-1} \,:\, y \in F \setminus \{n-1\}\big\},\\
\Delta_{-1} &= \big\{(\,0,\,0,\,n-1)\big\},\\
\Delta_1 &= \big\{(\,0,\,n-1,\,0\,), (n-1,\,0,\,0\,), (n-1, n-1, n-1)\big\}.
\end{aligned}$$

For $\mathcal{Q}_n$, the value $\Delta(e)$ for each $e \in \Delta_*$ is as follows.

$$\begin{array}{c|ccccccc}
 & 0 & 1 & 3 & 5 & \cdots & n-3 & n-1 \\ \hline
0 & -1 & & & & \cdots & & 1 \\
1 & & -2 & -2 & -2 & \cdots & -2 & -2 \\
n-1 & 1 & 2 & 2 & 2 & \cdots & 2 & 1
\end{array} \tag{2.11}$$

Example 2.5. When we illustrate a member of $\mathcal{P}_n$, $\mathcal{Q}_n$ or $\mathcal{R}_n$ we adopt the convention of writing the rows and columns in the order $0, 2, 4, \ldots, n-2,\ 1, 3, 5, \ldots, n-1$. The marks $_*$ in (2.12) indicate elements of Δ_* in $\mathcal{Q}_{14}$. The circled entries identify $I_{\mathcal{Q}}$. The darker shaded cells show a 3-plex K and lighter shaded cells identify an 8-plex named J. It should be apparent that $\mathcal{Q}_{14} \setminus (K \cup J)$ is another 3-plex, so we have at hand a

$(3^2, 8)$-partition of this latin square.

	0	2	4	6	8	10	12	1	3	5	7	9	11	13
0	13_*	2	4	6	8	10	12	1	3	5	7	9	11	0_*
2	2	4	6	8	10	12	0	3	5	7	9	11	13	1
4	4	6	8	10	12	0	2	5	7	9	11	13	1	3
6	6	8	10	12	0	2	4	7	9	11	13	1	3	5
8	8	10	12	0	2	4	6	9	11	13	1	3	5	7
10	10	12	0	2	4	6	8	11	13	1	3	5	7	9
12	12	0	2	4	6	8	10	13	1	3	5	7	9	11
1	1	3	5	7	9	11	13	0_*	2_*	4_*	6_*	8_*	10_*	12_*
3	3	5	7	9	11	13	1	4	6	8	10	12	0	2
5	5	7	9	11	13	1	3	6	8	10	12	0	2	4
7	7	9	11	13	1	3	5	8	10	12	0	2	4	6
9	9	11	13	1	3	5	7	10	12	0	2	4	6	8
11	11	13	1	3	5	7	9	12	0	2	4	6	8	10
13	0_*	1	3	5	7	9	11	2_*	4_*	6_*	8_*	10_*	12_*	13_*

(2.12)

The illustrated 3-plex K is given by

$$\begin{array}{rclrcl} \mathrm{col}(0) &=& \{0,2,9\}, & \mathrm{col}(1) &=& \{1,3,5\}, \\ \mathrm{col}(2) &=& \{2,6,11\}, & \mathrm{col}(3) &=& \{0,12,5\}, \\ \mathrm{col}(4) &=& \{4,8,13\}, & \mathrm{col}(5) &=& \{0,2,7\}, \\ \mathrm{col}(6) &=& \{6,8,1\}, & \mathrm{col}(7) &=& \{4,9,13\}, \\ \mathrm{col}(8) &=& \{10,3,7\}, & \mathrm{col}(9) &=& \{4,1,11\}, \\ \mathrm{col}(10) &=& \{10,5,9\}, & \mathrm{col}(11) &=& \{6,3,13\}, \\ \mathrm{col}(12) &=& \{12,7,11\}, & \mathrm{col}(13) &=& \{8,10,12\}. \end{array} \tag{2.13}$$

In Chapter 3 we give a generalised definition of J.

2.4.2 $\mathcal{V}_{n,j}$

The family $\mathcal{V}_{n,j}$ is used in Chapter 4. These latin squares are turn-squares such as studied by Parker; see for example [118].

Definition 2.6 (Latin square $\mathcal{V}_{n,j}$)**.** For even $n \geqslant 6$ and $j \equiv 0 \bmod 4$, let $u = j/4$, $h = n/2$ and $U = \{0, 1, 2, \ldots, u-1\}$. Define

$$\mathcal{V}_{n,j}[x,y] = \begin{cases} x+y+h & \text{if for some } v \in U, \\ & (x,y) \in \big\{(0,2v),(0,h+2v),(h,2v),(h,h+2v)\big\}, \\ x+y & \text{otherwise.} \end{cases} \tag{2.14}$$

If $L = \mathcal{V}_{n,j}$, then

$$\Delta_* = \Delta_h \text{ and } |\Delta_*| = j. \tag{2.15}$$

2.4.3 $\mathcal{U}_n$

The family $\mathcal{U}_n$ will be used in Chapter 5.

Definition 2.7 (Latin square $\mathcal{U}_n$)**.** For $n = 2mq$ where q is odd, $q \geqslant 3$ and $m = 2^t$ for $t \geqslant 1$. Let $M = \{m, 3m, 5m, \ldots, n-m\}$. Define

$$\mathcal{U}_n[x,y] = \begin{cases} n-m & \text{if } \{x,y\} = \{0, n-\frac{1}{2}m\} \text{ or } \{n-m\}, \\ n-\frac{1}{2}m & \text{if } \{x,y\} = \{0\} \text{ or } \{n-\frac{1}{2}m\}, \\ 0 & \text{if } \{x,y\} = \{0, n-m\}, \\ y+mq-2m & \text{if } q > 3,\ x = m \text{ and } y \in M, \\ y-m & \text{if } x = mq-2m \text{ and } y \in M, \\ y+m & \text{if } x = n-m \text{ and } y \in M \setminus \{n-m\}, \\ x+y & \text{otherwise.} \end{cases} \tag{2.16}$$

For $L = \mathcal{U}_n$,

$$\Delta_* = \Delta_{-\frac{1}{2}m} \cup \Delta_{\frac{1}{2}m} \cup \Delta_m \cup \Delta_{2m} \cup \Delta_{m(1-q)} \cup \Delta_{m(q-3)}, \text{ where,} \tag{2.17}$$

$$\begin{aligned}
\Delta_{\frac{1}{2}m} &= \big\{(n-\tfrac{1}{2}m,\, n-\tfrac{1}{2}m,\, n-\tfrac{1}{2}m)\big\}, \\
\Delta_{-\frac{1}{2}m} &= \big\{(\,0,\, 0,\, n-\tfrac{1}{2}m), (\,0,\, n-\tfrac{1}{2}m,\, n-m\,), (n-\tfrac{1}{2}m,\, 0,\, n-m\,)\big\}, \\
\Delta_m &= \big\{(\,0,\, n-m,\, 0\,), (n-m,\, 0,\, 0\,), (n-m,\, n-m,\, n-m)\big\}, \\
\Delta_{2m} &= \begin{cases} \big\{(x,y,z) \in r_m \cup r_{n-m} \setminus \Delta_m \,:\, y \in M\big\} & \text{if } q = 5, \\ \big\{(x,y,z) \in r_{n-m} \setminus \Delta_m \,:\, y \in M\big\} & \text{otherwise,} \end{cases} \\
\Delta_{m(1-q)} &= \big\{(x,y,z) \in r_{m(q-2)} \,:\, y \in M\big\}, \\
\Delta_{m(q-3)} &= \begin{cases} \emptyset & \text{if } q = 3, \\ \Delta_{2m} & \text{if } q = 5, \\ \big\{(x,y,z) \in r_m \,:\, y \in M\big\} & \text{otherwise.} \end{cases}
\end{aligned}$$

For $\mathcal{U}_n$, we table below the value $\Delta(e)$ for each $e \in \Delta_*$.

	0	m	$3m$	$5m$	$\cdots$	$n-3m$	$n-m$	$n-\frac{1}{2}m$
0	$-\frac{1}{2}m$						m	$-\frac{1}{2}m$
m		$m(q-3)$	$m(q-3)$	$m(q-3)$	$\cdots$	$m(q-3)$	$m(q-3)$	
$m(q-2)$		$m(1-q)$	$m(1-q)$	$m(1-q)$	$\cdots$	$m(1-q)$	$m(1-q)$	
$n-m$	m	$2m$	$2m$	$2m$	$\cdots$	$2m$	m	
$n-\frac{1}{2}m$	$-\frac{1}{2}m$							$\frac{1}{2}m$

(2.18)

2.4.4 $\mathcal{A}_n$ and $\mathcal{B}_n$

Families $\mathcal{A}_n$ and $\mathcal{B}_n$ will be used in Chapter 5.

Definition 2.8 (Latin square $\mathcal{A}_n$)**.** For $n = 16d$ where $d \geqslant 1$, define

$$\mathcal{A}_n[x,y] = \begin{cases} 0 & \text{if } (x,y) \in \big\{(d,0),\ (0,15d)\big\}, \\ d & \text{if } (x,y) \in \big\{(4d,0),\ (d,13d)\big\}, \\ 4d & \text{if } (x,y) \in \big\{(4d,10d),\ (10d,0)\big\}, \\ 10d & \text{if } (x,y) \in \big\{(0,0),\ (10d,10d)\big\}, \\ 14d & \text{if } (x,y) \in \big\{(0,10d),\ (d,14d),\ (4d,13d)\big\}, \\ 15d & \text{if } (x,y) \in \big\{(0,14d),\ (d,15d)\big\}, \\ x+y & \text{otherwise.} \end{cases} \tag{2.19}$$

For $\mathcal{A}_n$, the value of $\Delta(e)$ for each $e \in \Delta_*$ is tabled below.

	0	$10d$	$13d$	$14d$	$15d$
0	$-6d$	$4d$		d	d
d	$-d$		$3d$	$-d$	$-d$
$4d$	$-3d$	$6d$	$-3d$		
$10d$	$-6d$	$6d$			

(2.20)

Definition 2.9 (Latin square $\mathcal{B}_n$)**.** For $n = 2h$ where $h \geqslant 5$ is an odd integer, define

$$\mathcal{B}_n[x,y] = \begin{cases} x+y+1 & \text{if } x=0 \text{ and } y \in \{n-2, n-1\}, \\ x+y-1 & \text{if } x=1 \text{ and } y \in \{0, n-2, n-1\}, \\ x+y+3 & \text{if } x=1 \text{ and } y = n-3, \\ x+y-3 & \text{if } x=4 \text{ and } y \in \{0, n-3\}, \\ x+y+4 & \text{if } x=0 \text{ and } y \in \{0,4,8,\dots,n-6\}, \\ x+y-4 & \text{if } x=4 \text{ and } y \in \{4,8,\dots,n-6\}, \\ x+y+h-7 & \text{if } n>14,\ x=6 \text{ and } y \in E, \\ x+y-h+7 & \text{if } n>14,\ x=h-1 \text{ and } y \in E \\ x+y & \text{otherwise.} \end{cases} \tag{2.21}$$

For $\mathcal{B}_n$, there are 8 elements $e \in \Delta_*$ for which $\Delta(e)$ is an odd value:

$$\begin{array}{c|cccc} & 0 & n-2 & n-3 & n-1 \\ \hline 0 & & 1 & & 1 \\ 4 & -3 & & -3 & \\ 1 & -1 & -1 & 3 & -1 \end{array} \tag{2.22}$$

For a even, $\Delta_a \subset \mathcal{B}_n$ is contained as follows:

$$\begin{array}{ll} \text{If } n = 22, & \Delta_4 \subset (r_0 \cup r_6) \text{ and } \Delta_{-4} \subset (r_4 \cup r_{h-1}). \\ \text{Otherwise,} & \Delta_4 \subset r_0 \text{ and } \Delta_{-4} \subset r_4. \\ \text{If } n > 14 \text{ and } n \neq 22, & \Delta_{h-7} \subset r_6 \text{ and } \Delta_{h+7} \subset r_{h-1}. \end{array} \tag{2.23}$$

2.5 Latin families of odd order

2.5.1 $\mathcal{H}_n$

The family $\mathcal{H}_n$ will be used in Chapter 4.

Definition 2.10 (Latin square $\mathcal{H}_n$)**.** For odd $n \geqslant 5$, we define the latin square $\mathcal{H}_n$ of order n and indexed by $\mathbb{Z}_n$.

$$\mathcal{H}_n[x,y] = \begin{cases} 0 & \text{if } (x,y) \in \big\{(1,0),(2,n-1)\big\}, \\ 1 & \text{if } (x,y) \in \big\{(0,0),(1,n-1)\big\}, \\ y+2 & \text{if } x=0 \text{ and } y \in \{1,3,5,\dots,n-2\}, \\ y & \text{if } x=2 \text{ and } y \in \{1,3,5,\dots,n-2\}, \\ x+y & \text{otherwise.} \end{cases} \tag{2.24}$$

For $\mathcal{H}_n$,

$$\begin{aligned}
\Delta_* &= \Delta_1 \cup \Delta_{-1} \cup \Delta_2 \cup \Delta_{-2}, \text{ where,}\\
\Delta_1 &= \big\{(\,0,\,0,\,1\,),(1,\,n-1,\,1\,)\big\},\\
\Delta_{-1} &= \big\{(\,1,\,0,\,0\,),(2,\,n-1,\,0\,)\big\},\\
\Delta_2 &= \big\{(x,y,z)\in r_0 \,:\, y\in\{1,3,5,\dots,n-2\}\big\},\\
\Delta_{-2} &= \big\{(x,y,z)\in r_2 \,:\, y\in\{1,3,5,\dots,n-2\}\big\}.
\end{aligned}$$

For $\mathcal{H}_n$, we table below the value $\Delta(e)$ for each $e \in \Delta_*$.

$$\begin{array}{c|ccccccc}
 & 0 & 1 & 3 & 5 & \cdots & n-2 & n-1\\
\hline
0 & 1 & 2 & 2 & 2 & \cdots & 2 & \\
1 & -1 & & & & & & 1\\
2 & & -2 & -2 & -2 & \cdots & -2 & -1
\end{array} \tag{2.25}$$

2.5.2 $\mathcal{W}_n$

The next family, which we name $\mathcal{W}_n$, is due to Wanless and Webb [155]. It was used to prove Theorem 1.23 for all odd $n > 3$. We consider $\mathcal{W}_n$ in Chapter 5 and will give data for its small members in Chapters 4 and 5. For $n = 5$, $\mathcal{W}_n$ and $\mathcal{H}_n$ are of the same species.

Definition 2.11 (Latin square $\mathcal{W}_n$)**.** For odd $n \geqslant 5$, we define the latin square $\mathcal{W}_n$ of order n and indexed by $\mathbb{Z}_n$.

For $n \equiv 1 \bmod 4$,

$$\mathcal{W}_n[x,y] = \begin{cases}
0 & \text{if } (x,y) \in \big\{(0,1),\big(\frac{n-3}{2},0\big),\big(n-1,\frac{n+3}{2}\big)\big\},\\
1 & \text{if } (x,y) \in \big\{(0,0),(n-1,1)\big\},\\
x+2 & \text{if } y=0 \text{ and } x \in \big\{1,3,5,\dots,\frac{n-7}{2}\big\},\\
x & \text{if } y=2 \text{ and } x \in \big\{1,3,5,\dots,\frac{n-7}{2}\big\} \cup \big\{\frac{n-3}{2}\big\},\\
\frac{n+1}{2} & \text{if } (x,y) \in \big\{\big(\frac{n-3}{2},\frac{n+3}{2}\big),(n-1,2)\big\},\\
x+y & \text{otherwise.}
\end{cases} \tag{2.26}$$

For $n \equiv 3 \bmod 4$,

$$\mathcal{W}_n[x,y] = \begin{cases} 0 & \text{if } (x,y) \in \big\{(0,1), (\frac{n-1}{2}, 0), (n-1, \frac{n+1}{2})\big\}, \\ 1 & \text{if } (x,y) \in \big\{(0,0), (n-1,1)\big\}, \\ x+2 & \text{if } y = 0 \text{ and } x \in \big\{1,3,5,\dots,\frac{n-5}{2}\big\}, \\ x & \text{if } y = 2 \text{ and } x \in \big\{1,3,5,\dots,\frac{n-5}{2}\big\}, \\ \frac{n-1}{2} & \text{if } (x,y) \in \big\{\big(\frac{n-1}{2}, \frac{n+1}{2}\big), (n-1,2)\big\}, \\ x+y & \text{otherwise.} \end{cases} \tag{2.27}$$

For $\mathcal{W}_n$, we table below the value $\Delta(e)$ for each $e \in \Delta_*$. The table on the left is for $n \equiv 1 \bmod 4$. The table on the right is for $n \equiv 3 \bmod 4$.

	0	1	2	$\frac{n+3}{2}$
0	1	-1		
1	2		-2	
3	2		-2	
5	2		-2	
$\vdots$	$\vdots$		$\vdots$	
$\frac{n-7}{2}$	2		-2	
$\frac{n-3}{2}$	$\frac{n+3}{2}$		-2	$\frac{n+1}{2}$
$n-1$		1	$\frac{n-1}{2}$	$\frac{n-1}{2}$

	0	1	2	$\frac{n+1}{2}$
0	1	-1		
1	2		-2	
3	2		-2	
5	2		-2	
$\vdots$	$\vdots$		$\vdots$	
$\frac{n-5}{2}$	2		-2	
$\frac{n-1}{2}$	$\frac{n+1}{2}$			$\frac{n-1}{2}$
$n-1$		1	$\frac{n-3}{2}$	$\frac{n+1}{2}$

(2.28)

2.5.3 $\mathcal{O}_{m,k}$

The family $\mathcal{O}_{m,k}$ will be used in Chapter 4.

Definition 2.12 (Latin square $\mathcal{O}_{m,k}$)**.** For odd integers, $m \geqslant 3$ and $k \geqslant 3$, we define $\mathcal{O}_{m,k}$ of order mk and indexed by $\mathbb{Z}_m \oplus \mathbb{Z}_k$. We test inequalities by assuming that c is the least non-negative residue in its congruence class.

$$\mathcal{O}_{m,k}[(a,b),(c,d)] =$$

$$\begin{cases} (\,a+c-1,\ b+d+1\,) & \text{if } a+c=0,\ b=0 \text{ and } c<\lfloor m/2\rfloor, \\ (\,a+c-1,\ b+d\,) & \text{if } a+c=0,\ b=0 \text{ and } c=\lfloor m/2\rfloor, \\ (\,a+c-1,\ b+d-1\,) & \text{if } a+c=0,\ b=0 \text{ and } c>\lfloor m/2\rfloor, \\ (\,a+c+1,\ b+d\,) & \text{if } a+c=m-1 \text{ and } b=0, \\ (\,a+c,\ b+d-1\,) & \text{if } a+c=m-1,\ b=1 \text{ and } c<\lfloor m/2\rfloor, \\ (\,a+c,\ b+d+1\,) & \text{if } a+c=m-1,\ b=k-1 \text{ and } c>\lfloor m/2\rfloor, \\ (\,a+c,\ b+d\,) & \text{otherwise.} \end{cases}$$

Example 2.13. For $\mathcal{O}_{3,3}$ we display in (2.29) the abbreviated ordered pairs $\Delta(e)$ for each $e\in\Delta_*$. For example, $e=\big((0,0),(0,2),(2,0)\big)\in\mathcal{O}_{3,3}$, with $\Delta(e)=(2,1)\neq(0,0)$. Hence, Δ value 21 is at row 00, column 02. Shading indicates a 3-partition of $\mathcal{O}_{3,3}$ for which a definition is later given, by (4.16) in Chapter 4.

	00	01	02	10	11	12	20	21	22
00	21	21	21				10	10	10
01									
02							01	01	01
10				10	10	10	22	22	22
11									
12									
20	10	10	10	20	20	20			
21	02	02	02						
22									

(2.29)

2.5.4 $\mathcal{D}_{3m}$

The family $\mathcal{D}_{3m}$ will be used in Chapter 5. This construction is due to Ian Wanless but, as explained in Chapter 5, was motivated by joint data.

Definition 2.14 (Latin square $\mathcal{D}_{3m}$)**.** For odd $m\geqslant 3$, we define the latin square $\mathcal{D}_{3m}$ of order $3m$ and indexed by $\mathbb{Z}_3\oplus\mathbb{Z}_m$.

$$
\mathcal{D}[(a,b),(c,d)] =
\begin{cases}
(\,1,\ d\,) & \text{if } (a=b=c=0) \text{ or } (a=2,\ b=0 \text{ and } c=1),\\
(\,0,\ d\,) & \text{if } (a=b=0 \text{ and } c=1) \text{ or } (a=c=2 \text{ and } b=0),\\
(\,0,\ d+1\,) & \text{if } a=1 \text{ and } b=c=0,\\
(\,1,\ d+1\,) & \text{if } a=1,\ b=0 \text{ and } c=2,\\
(\,0,\ d\,) & \text{if } a=c=0 \text{ and } b=1,\\
(\,1,\ b+d+1\,) & \text{if } a=c=2 \text{ and } b\neq 0,\\
(\,a+c,\ b+d\,) & \text{otherwise.}
\end{cases}
\tag{2.30}
$$

Let $\ell = m-1$. For each $e \in \Delta_* \subset \mathcal{D}_{3m}$, we display below the abbreviated ordered pairs $\Delta(e)$. The shaded region shows an $(m-1)\times m$ subrectangle of $\mathcal{D}_{3m}$. In Chapter 5 we show that this subrectangle consists of transversal-free entries.

	00	01	02	…	0ℓ	10	11	12	…	1ℓ	20	21	22	…	2ℓ
00	10	10	10	…	10	20	20	20	…	20					
01	0ℓ	0ℓ	0ℓ	…	0ℓ										
02															
⋮															
0ℓ															
10	21	21	21	…	21						11	11	11	…	11
11															
12															
⋮															
1ℓ															
20						10	10	10	…	10	20	20	20	…	20
21											01	01	01	…	01
22											01	01	01	…	01
⋮											⋮	⋮	⋮	⋱	⋮
2ℓ											01	01	01	…	01

(2.31)

Chapter 3

Latin squares with no small odd plexes

3.1 Introduction

The main results of this chapter appear in [67].

Latin squares with no small odd plexes were first shown in [65]:

Theorem 3.1 ([65])**.** *For all even $n > 2$, there exists a latin square of order n which has no k-plex for any odd $k < \lfloor \frac{n}{4} \rfloor$, but does have a k-plex for all even k and some odd $k \leqslant \frac{1}{2}n$.*

We will prove the following stronger statement.

Theorem 3.2. *For all even $n > 2$, there exists a latin square of order n which has no k-plex for any odd $k < \lfloor \frac{n}{4} \rfloor$, but does have a k-plex for every other $k \leqslant \frac{1}{2}n$.*

Theorem 3.2 will follow immediately from Theorems 3.15 and 3.18. In fact we will analyse the latin squares used to prove Theorem 3.2 in sufficient detail to describe all their possible types of partitions. We use the latin families $\mathcal{P}_n$ and $\mathcal{Q}_n$, and associated notation from Chapter 2.

3.2 Necessary conditions

We begin by showing that $\mathcal{P}_n$ and $\mathcal{Q}_n$ have no small odd plexes.

Lemma 3.3 ([65])**.** *Let k be an odd integer. If $k < \frac{1}{4}n$ or $k > \frac{3}{4}n$, then the latin square $\mathcal{P}_n$ has no k-plex.*

Proof. Assume that $K \subset \mathcal{P}_n$ is a k-plex for some odd k. By Lemma 2.1,

$$\sum_{e \in K \cap \Delta_*} \Delta(e) \equiv \tfrac{1}{2}n \bmod n. \tag{3.1}$$

Since K cannot satisfy (3.1) without inclusion of either $\frac{1}{4}n$ elements of Δ_{-2} or $\frac{1}{4}n$ elements of Δ_2, there must be a row in K with at least $\frac{1}{4}n$ elements. As n is even, the complement of K is an $(n-k)$-plex with $(n-k)$ odd. Hence $n-k \geqslant \frac{1}{4}n$, and so $k \leqslant \frac{3}{4}n$. □

Corollary 3.4. *If q is even, then a partition of the latin square $\mathcal{P}_{4q}$ has at most two odd plexes.*

Proof. Any partition has an even number of odd plexes, as $4q$ is even. Since q is even, Lemma 3.3 implies that an odd k-plex has $k \geqslant q+1$. Thus, four or more odd plexes in a partition is impossible. □

Lemma 3.5 ([65])**.** *Let k be an odd integer. If $k < \frac{1}{4}(n-2)$ or $k > \frac{1}{4}(3n+2)$, then the latin square $\mathcal{Q}_n$ has no k-plex.*

Proof. Assume that $K \subset \mathcal{Q}_n$ is a k-plex for some odd k. Then, by Lemma 2.1, (3.1) holds. Since $h = \frac{1}{2}n$ is odd, it is immediate that K contains an odd number of elements of $I_\mathcal{Q}$ as defined by (2.10). We consider two cases:

Case 1: $\sum_{e \in K \cap I_\mathcal{Q}} \Delta(e) = \pm 1$.
Then (3.1) requires that $|\Delta_{-2} \cap K| \geqslant \frac{1}{2}(h-1)$ or $|\Delta_2 \cap K| \geqslant \frac{1}{2}(h-1)$.

Case 2: $\sum_{e \in K \cap I_\mathcal{Q}} \Delta(e) = 3$.
Then (3.1) requires that either (i) $|\Delta_{-2} \cap K| \geqslant \frac{1}{2}(h+3)$, or (ii)$|\Delta_2 \cap K| \geqslant \frac{1}{2}(h-3)$. However, in subcase (ii), K also has two elements of $I_\mathcal{Q}$ in r_{n-1}.

Hence each possibility implies the existence of at least $\frac{1}{2}(h-1) = \frac{1}{4}(n-2)$ elements in the same row of K. As n is even, the complement of K is an $(n-k)$-plex with $(n-k)$ odd. Thus $n-k \geqslant \frac{1}{4}(n-2)$, so $k \leqslant \frac{1}{4}(3n+2)$. □

Plexes in squares of order six have been studied in detail. Our square $\mathcal{Q}_6$ belongs to species I in Fisher and Yates' catalogue [81]. Hence, due to Finney [78], we know that $\mathcal{Q}_6$ has a $(1^2, 2^2)$-partition but no $(1^4, 2)$-partition. Also, none of its eight distinct transversals intersect $I_\mathcal{Q} \cap r_5$. More generally:

Lemma 3.6. *Let $h \equiv 3 \bmod 4$ and $K \subset \mathcal{Q}_{2h}$ be a k-plex where $k = \frac{1}{2}(h-1)$. Then K contains precisely one element of $I_\mathcal{Q}$ and that element must be in r_0.*

Proof. It is clear from Lemma 3.5 that k is the minimal odd value for which a k-plex is possible. Assume that $|I_\mathcal{Q} \cap r_{n-1} \cap K| > 0$. By Lemma 2.1 we have two cases:

Case 1: $\sum_{e\in K\cap I_{\mathcal{Q}}} \Delta(e) = 3$.
Then $|I_{\mathcal{Q}} \cap r_{n-1} \cap K| = 2$ so Lemma 2.1 implies that either $|\Delta_{-2} \cap K| \geqslant \frac{1}{2}(h+3)$ or $|\Delta_2 \cap K| + |I_{\mathcal{Q}} \cap r_{n-1} \cap K| \geqslant \frac{1}{2}(h-3) + 2 = \frac{1}{2}(h+1)$.

Case 2: $\sum_{e\in K\cap I_{\mathcal{Q}}} \Delta(e) = 1$.
Then $|I_{\mathcal{Q}} \cap r_{n-1} \cap K| \geqslant 1$ and Lemma 2.1 implies that either $|\Delta_{-2} \cap K| \geqslant \frac{1}{2}(h+1)$, or, $|\Delta_2 \cap K| + |I_{\mathcal{Q}} \cap r_{n-1} \cap K| \geqslant \frac{1}{2}(h-1) + 1 = \frac{1}{2}(h+1)$.

Hence in all cases, K has at least $\frac{1}{2}(h+1) > k$ elements in either r_1 or r_{n-1} which is a contradiction. Since $|I_{\mathcal{Q}} \cap K|$ is odd, K contains precisely one element of $I_{\mathcal{Q}}$ in r_0. $\square$

Lemma 3.7. *A partition of the latin square $\mathcal{Q}_{2h}$ has at most two odd plexes.*

Proof. The latin square is of even order so the number of odd plexes in the partition is even. By Lemma 3.5, the number is not more than four. We now show that four is impossible. Assume there is a $(k_1, k_2, \dots, k_m)$-partition where k_i is odd for $i = 1, 2, 3, 4$. We have two cases:

Case 1: $h \equiv 1 \bmod 4$.
Since $\frac{1}{4}(n-2)$ is even, Lemma 3.5 implies that $k_i \geqslant \frac{1}{4}(n+2) > \frac{1}{4}n$ for $i = 1, 2, 3, 4$ which is impossible.

Case 2: $h \equiv 3 \bmod 4$.
It follows from Lemma 3.6 that at most two of the k_i are minimal, say k_1 and k_2. Therefore $k_3, k_4 \geqslant k_1 + 2 = \frac{1}{2}(h+3)$. However, then we have $\sum_{i=1}^{4} k_i \geqslant n+2$ which is a contradiction. $\square$

3.3 Partitions of $\mathcal{P}_n$

In this section we show partitions of $\mathcal{P}_n$ and determine its plex range. Throughout this section we assume that $n = 2h = 4q$ for an integer $q \geqslant 1$.

We first show the even values in the plex range of $\mathcal{P}_n$.

Lemma 3.8 ([65])**.** *The latin square $\mathcal{P}_n$ has a 2-partition and hence has a k-plex for all even k in the interval $0 \leqslant k \leqslant n$.*

Proof. For each $a \in E$, a 2-plex J_a is given by

$$\mathrm{col}(x) = \begin{cases} \{x+a,\ x+1+a\} & \text{if } x \in E, \\ \{x-3+a,\ x+a\} & \text{if } x \in F \text{ and } x < n-3, \\ \{x-3+a,\ x+2+a\} & \text{if } x = n-3, \\ \{x-3+a,\ x-2+a\} & \text{if } x = n-1. \end{cases}$$

$\square$

Next is a partition with odd plexes.

Lemma 3.9 ([65]). *The latin square $\mathcal{P}_n$ has an $(h-1, h+1)$-partition.*

Proof. An $(h-1)$-plex K is given by

$$\begin{aligned}
\mathrm{col}(n-2) &= E \setminus \{h\},\\
\mathrm{col}(h-2) &= F \setminus \{h+1\},\\
\mathrm{col}(1) &= \{y \in E : y \geqslant h\} \cup \{y \in F : y < h-2\},\\
\mathrm{col}(n-1) &= \{y \in E : y < h-2\} \cup \{y \in F : y > h\}.
\end{aligned}$$

For $x \in \{0, 2, 4, \dots, h-4\}$,

$$\begin{aligned}
\mathrm{col}(n-4-x) &= E \setminus \{n-2-x\},\\
\mathrm{col}(h-4-x) &= F \setminus \{n-1-x\},\\
\mathrm{col}(h+1+x) &= E \setminus \{x\},\\
\mathrm{col}(3+x) &= F \setminus \{1+x\}.
\end{aligned}$$

It is routine to check that K is an $(h-1)$-plex. $\square$

The next corollary follows from Lemma 3.3.

Corollary 3.10 ([65]). *For all $n \geqslant 8$, the latin square $\mathcal{P}_n$ has a mixed plex range.*

We point out that the latin square $\mathcal{P}_4$ is of the same species as the Cayley table of $\mathbb{Z}_2 \times \mathbb{Z}_2$. It has a complete plex range.

Our next two lemmas use partitions of the plexes constructed in Lemma 3.9 to create smaller plexes.

Lemma 3.11. *The $(h+1)$-plex in $\mathcal{P}_n$ defined in the proof of Lemma 3.9 contains a set of $\lfloor \frac{n}{8} \rfloor$ parallel 2-plexes.*

Proof. Let $m = 4\big(\lfloor \frac{n}{8} \rfloor - 1\big)$. For each $a \in \{0, 4, 8, \dots, m\}$, a 2-plex J_a is given by

$$\mathrm{col}(x) = \begin{cases}
\{n-4+x-a,\ n-2+x-a\} & \text{if } x \in E \text{ and } x < h,\\
\{1+x-a,\ 3+x-a\} & \text{if } x \in E \text{ and } x \geqslant h,\\
\{h-5+x-a,\ h-3+x-a\} & \text{if } x \in F \text{ and } x < h,\\
\{x-h-a,\ 2+x-h-a\} & \text{if } x \in F \text{ and } h < x < n-3,\\
\{2+x-h-a,\ 4+x-h-a\} & \text{if } x = n-3,\\
\{x-2-h-a,\ x-h-a\} & \text{if } x = n-1.
\end{cases}$$

$\square$

Lemma 3.12. *If q is even, then the $(h-1)$-plex in $\mathcal{P}_n$ defined in the proof of Lemma 3.9 contains a set of $\frac{n}{8}-1$ parallel 2-plexes.*

Proof. Let $m = 4\big(\frac{n}{8}-2\big) = h-8$. For each $a \in \{0,4,8,\ldots,m\}$ we show a 2-plex J_a:

For $x \in \{0,4,8,\ldots,h-4\}$,

$$
\begin{aligned}
\mathrm{col}(x) &= \{1+x-a,\ 3+x-a\},\\
\mathrm{col}(2+x) &= \{h+1+x-a,\ h+3+x-a\},\\
\mathrm{col}(h+x) &= \{x-a,\ 2+x-a\},\\
\mathrm{col}(h+2+x) &= \{h+x-a,\ h+2+x-a\},\\
\mathrm{col}(1+x) &= \{h-5+x-a,\ h-3+x-a\},\\
\mathrm{col}(3+x) &= \{n-5+x-a,\ n-3+x-a\},\\
\mathrm{col}(h+1+x) &= \{h-4+x-a,\ h-2+x-a\},\\
\mathrm{col}(h+3+x) &= \{n-4+x-a,\ n-2+x-a\}.
\end{aligned}
$$

□

We can now say which odd plexes exist in $\mathcal{P}_{4q}$ when q is even.

Lemma 3.13. *If q is even, then the latin square $\mathcal{P}_{4q}$ has a $\big(2^{q-1},(q+1)^2\big)$-partition and hence has a k-plex for all odd k in the interval $\frac{1}{4}n+1 \leqslant k \leqslant \frac{3}{4}n-1$.*

Proof. We have the following refinements of the $(h-1,h+1)$-partition of Lemma 3.9. Lemma 3.11 gives a division of the $(h+1)$-plex into $\frac{n}{8}=\frac{1}{2}q$ parallel 2-plexes and a parallel (odd) $(q+1)$-plex. Lemma 3.12 gives a division of the $(h-1)$-plex into $\frac{n}{8}-1=\frac{1}{2}q-1$ parallel 2-plexes and a parallel $(q+1)$-plex. Thus we have the claimed partition. The union of a $(q+1)$-plex with as many of the 2-plexes as required forms a k-plex, for each odd k in the interval $\frac{1}{4}n+1 \leqslant k \leqslant \frac{3}{4}n-1$. □

In (3.2) we illustrate the $(2^3,5^2)$-partition of $\mathcal{P}_{16}$ in Lemma 3.13. The circled elements show the 2-plex given by Lemma 3.12. It is contained within the (dark) 7-plex defined in the proof of Lemma 3.9. The two 2-plexes from Lemma 3.11 are shown by paler shading. The marks $_*$ identify elements of Δ_*.

	0	2	4	6	8	10	12	14	1	3	5	7	9	11	13	15
0	0	2	4	6	8	10	12	14	1	3	5	7	9	11	13	15
2	2	4	6	8	10	12	14	0	3	5	7	9	11	13	15	1
4	4	6	8	10	12	14	0	2	5	7	9	11	13	15	1	3
6	6	8	10	12	14	0	2	4	7	9	11	13	15	1	3	5
8	8	10	12	14	0	2	4	6	9	11	13	15	1	3	5	7
10	10	12	14	0	2	4	6	8	11	13	15	1	3	5	7	9
12	12	14	0	2	4	6	8	10	13	15	1	3	5	7	9	11
14	14	0	2	4	6	8	10	12	15	1	3	5	7	9	11	13
1	1	3	5	7	9	11	13	15	2	4	6	8	10	12	14	0
3	3	5	7	9	11	13	15	1	4	6	8	10	12	14	0	2
5	5	7	9	11	13	15	1	3	6	8	10	12	14	0	2	4
7	7	9	11	13	15	1	3	5	8	10	12	14	0	2	4	6
9	9	11	13	15	1	3	5	7	10	12	14	0	2	4	6	8
11	11	13	15	1	3	5	7	9	12	14	0	2	4	6	8	10
13	13	15	1	3	5	7	9	11	0_*	2_*	4_*	6_*	8_*	10_*	12_*	14_*
15	15	1	3	5	7	9	11	13	14_*	0_*	2_*	4_*	6_*	8_*	10_*	12_*

(3.2)

We have seen that when q is even a partition of $\mathcal{P}_{4q}$ has at most two odd plexes. In contrast, when q is odd $\mathcal{P}_{4q}$ has a partition containing four odd plexes.

Lemma 3.14. *If q is odd, then the latin square $\mathcal{P}_n$ has an h-partition in which each h-plex can be divided in the following ways:*

1. *into q parallel 2-plexes,*
2. *into two parallel q-plexes.*

Thus, when q is odd, $\mathcal{P}_{4q}$ has a q-partition and a $(2^q, q^2)$-partition. Hence, when q is odd, $\mathcal{P}_n$ has a k-plex for all odd k in the interval $\frac{1}{4}n \leqslant k \leqslant \frac{3}{4}n$.

Proof. We first show two parallel h-plexes, H and H', each of which is formed by the union of q parallel 2-plexes. We will then divide each of H and H' into two parallel q-plexes.

We partition the sets E and F as follows:

$$\begin{aligned} E_0 &= \{i \in E \,:\, i \equiv 0 \bmod 4\}, \quad E_2 = \{i \in E \,:\, i \equiv 2 \bmod 4\}, \\ F_1 &= \{i \in F \,:\, i \equiv 1 \bmod 4\}, \quad F_3 = \{i \in F \,:\, i \equiv 3 \bmod 4\}. \end{aligned}$$

We use the parallel 2-plexes J_a as defined in the proof of Lemma 3.8. Let $H = \bigcup_{a\in E_0} J_a$ and $H' = \bigcup_{a\in E_2} J_a$. Now H and H' each contain q parallel 2-plexes and together they form an h-partition.

We need to show that H and H' each contain a q-plex.

A q-plex $K_0 \subset H$ is given by

$$\mathrm{col}(x) = \begin{cases} E_0 & \text{if } (x \in E_0 \text{ and } x \geqslant h-2) \text{ or } (x \in F_3 \text{ and } x < h), \\ E_2 & \text{if } (x \in E_2 \text{ and } x < h) \text{ or } (x \in F_1 \text{ and } x \geqslant h-1), \\ F_1 & \text{if } (x \in E_0 \cup F_1 \text{ and } x < h-2) \text{ or } x = n-1, \\ F_3 & \text{if } (x \in E_2 \text{ and } x \geqslant h) \text{ or } (x \in F_3 \text{ and } h < x < n-1). \end{cases}$$

A q-plex $K_2 \subset H'$ is given by

$$\mathrm{col}(x) = \begin{cases} E_2 & \text{if } (x \in E_0 \text{ and } x \geqslant h-2) \text{ or } (x \in F_3 \text{ and } x < h), \\ E_0 & \text{if } (x \in E_2 \text{ and } x < h) \text{ or } (x \in F_1 \text{ and } x \geqslant h-1), \\ F_3 & \text{if } (x \in E_0 \cup F_1 \text{ and } x < h-2) \text{ or } x = n-1, \\ F_1 & \text{if } (x \in E_2 \text{ and } x \geqslant h) \text{ or } (x \in F_3 \text{ and } h < x < n-1). \end{cases}$$

It is routine to check that each of the above q-plexes is contained in its respective h-plex. It follows that $K_1 = H \setminus K_0$ and $K_3 = H' \setminus K_2$ are also q-plexes so we have divided the h-partition into a q-partition.

Dividing H in one way, and H' in the other, gives a $(2^q, q^2)$-partition. Using this partition, the union of one q-plex with as many of the 2-plexes as required yields a k-plex, for each odd k in the interval $\frac{1}{4}n \leqslant k \leqslant \frac{3}{4}n$. □

Combining the above results, we can now establish exactly which partitions of $\mathcal{P}_n$ are possible.

Theorem 3.15. *The plex range of the latin square $\mathcal{P}_n$ is the set*

$$\left\{\tfrac{1}{4}n \leqslant k \leqslant \tfrac{3}{4}n \,:\, k \text{ is odd}\right\} \cup \left\{0, 2, 4, \ldots, n\right\}.$$

Moreover, $\mathcal{P}_n$ possesses a partition of every type consistent with its plex range.

Proof. Any type of partition which has no odd plexes can be obtained by the union of plexes from the 2-partition of Lemma 3.8.

If $n \equiv 0 \mod 8$, then a $(q+1)$-plex is the smallest odd plex permitted by Lemma 3.3. A partition with odd plexes has exactly two odd plexes, by Lemma 3.4, so it is of a type which can be generated by the $\left(2^{q-1}, (q+1)^2\right)$-partition of Lemma 3.13.

In the case that $n \equiv 4 \bmod 8$, a partition with four odd plexes is a q-partition. A partition with precisely two odd plexes can, for all possible partition types, be formed by the union of plexes from a $(2^q, q^2)$-partition. Thus the partitions of Lemma 3.14 generate every possible partition type for odd plexes in this case. $\square$

3.4 Partitions of $\mathcal{Q}_n$

In this section we investigate partitions of $\mathcal{Q}_n$ and determine its plex range. We assume throughout this section that $n = 2h$ for an odd integer $h \geqslant 3$.

Lemma 3.16 ([65])**.** *The latin square $\mathcal{Q}_n$ has a 2-partition and hence has a k-plex for all even k in the interval $0 \leqslant k \leqslant n$.*

Proof. For each $a \in E$, a 2-plex J_a is given by

$$\mathrm{col}(x) = \begin{cases} \{x+a,\ x-1+a\} & \text{if } x \in E, \\ \{x+1+a,\ x-2+a\} & \text{if } x \in F \setminus \{1, n-1\}, \\ \{x+1+a,\ x-4+a\} & \text{if } x = 1, \\ \{x+1+a,\ x+a\} & \text{if } x = n-1. \end{cases}$$

$\square$

Notice that the 2-plex J_0 defined in the proof of Lemma 3.16 contains all of $I_{\mathcal{Q}}$. Thus, it follows from Lemma 2.1 that $\mathcal{Q}_n \setminus J_0$ is an $(n-2)$-plex which contains no odd plexes. We now show that $\mathcal{Q}_n$ itself contains many odd plexes. In the next lemma, the specified value of k is the smallest odd value that is not prohibited by Lemma 3.5.

Lemma 3.17. *The latin square $\mathcal{Q}_{2h}$ has a $(k^2, 2^{h-k})$-partition where $k = 2\lfloor \frac{h}{4} \rfloor + 1$.*

Proof. We will define an even plex J which is the union of $h-k$ parallel 2-plexes. Then, we will show an odd k-plex K, contained in the $2k$-plex $\mathcal{Q}_{2h} \setminus J$, thus collectively giving a $(k^2, 2^{h-k})$-partition.

Let $m = 2(h-k-1)$. For each $a \in \{0, 2, 4, \ldots, m\}$, a 2-plex J_a is given by

$$\mathrm{col}(x) = \begin{cases} \{x+1+a,\ x-2-a\} & \text{if } x \in E \text{ or } (x \in F \setminus \{1, n-1\} \text{ and } a < m), \\ \{x+1+a,\ x-4-a\} & \text{if } x = 1 \text{ and } a < m, \\ \{x+1+a,\ x-a\} & \text{if } x = n-1 \text{ and } a < m, \\ \{x+1+a,\ n-1\} & \text{if } x = 1 \text{ and } a = m, \\ \{x+1+a,\ x\} & \text{if } x \in F \setminus \{1, n-1\} \text{ and } a = m, \\ \{x+1+a,\ 1\} & \text{if } x = n-1 \text{ and } a = m. \end{cases}$$

Then $J = \bigcup_{a \in \{0,2,4,\ldots,m\}} J_a$. (An illustration of $J \subset \mathcal{Q}_{14}$ was given in Example 2.5.)

Next, we show the odd k-plex K. Let $t = \left\lfloor \frac{h}{4} \right\rfloor$, and let

$$\begin{aligned} T_x &= \{x+2, x+4, x+6, \dots, x+2t\}, \\ U_x &= \{n-1-4t+x, n+1-4t+x, n+3-4t+x, \dots, n-3-2t+x\}, \\ V_x &= \{h+x+4, h+x+8, h+x+12, \dots, h+x+4t\}, \\ W_x &= \{x+2, x+6, x+10, \dots, x-2+4t\}. \end{aligned}$$

We specify K in two cases.

Case 1: $h \equiv 1 \bmod 4$

For $x \in E$,

$$\operatorname{col}(x) = \begin{cases} T_x \cup \{x\} \cup U_x & \text{if } x \leqslant 2t \text{ or } x \geqslant h+2t+1, \\ T_x \cup \{x-1\} \cup U_x & \text{otherwise,} \end{cases}$$

and for $x \in F$,

$$\operatorname{col}(x) = \begin{cases} \{0\} \cup \{1, 3, 5, \dots, h-2\} & \text{if } x = 1, \\ \{h-1, h+1, h+3, \dots, n-2\} & \text{if } x = n-1, \\ V_x \cup W_x \cup \{x+4t\} & \text{if } h-2t \leqslant x \leqslant h+2t, \\ \{h+x\} \cup V_x \cup W_x & \text{otherwise.} \end{cases}$$

Case 2: $h \equiv 3 \bmod 4$

In the case $h = 7$, we rely on (2.13). For $h \neq 7$, we define K as follows.

For $x \in E$,

$$\operatorname{col}(x) = \begin{cases} T_x \cup \{x\} \cup U_x & \text{if } x \leqslant h-1, \\ T_x \cup \{x-1\} \cup U_x & \text{otherwise,} \end{cases}$$

and for $x \in F$,

$$\operatorname{col}(x) = \begin{cases} \{1, 3, 5, \dots, h-2\} & \text{if } x = 1, \\ \{h+x+2\} \cup V_x \cup W_x & \text{if } 3 \leqslant x \leqslant h-4, \\ V_x \cup \{h+x+10\} \cup W_x & \text{if } x = h-2, \\ \big(V_x \setminus \{h+x+8\}\big) \cup \{h+x+10, x+2+4t\} \cup W_x & \text{if } h \leqslant x \leqslant n-7, \\ V_x \cup W_x \cup \{x+2+4t\} & \text{if } n-5 \leqslant x \leqslant n-3, \\ \{h+1, h+3, h+5, \dots, n-2\} & \text{if } x = n-1. \end{cases}$$

$\square$

Combining the above results, we can now establish exactly which partitions of $\mathcal{Q}_n$ are possible.

Theorem 3.18. *The plex range of the latin square $\mathcal{Q}_n$ is the set*

$$\left\{\tfrac{1}{4}(n-2) \leqslant k \leqslant \tfrac{1}{4}(3n+2) : \ k \text{ is odd}\right\} \cup \{0, 2, 4, \dots, n\}.$$

Moreover, $\mathcal{Q}_n$ possesses a partition of every type consistent with its plex range.

Proof. The 2-partition of Lemma 3.16 generates an example of a partition of any type that has no odd plex. A partition with odd plexes has exactly two odd plexes by Lemma 3.7. Therefore, we can obtain a partition of any type with odd plexes (except those ruled out by Lemma 3.5) by using the partition described in Lemma 3.17. □

Corollary 3.19 ([65])**.** *If $n \geqslant 10$, then the latin square $\mathcal{Q}_n$ has a mixed plex range.*

The latin square illustrated in (1.1) is $\mathcal{Q}_6$. It has a complete plex range.

3.5 Concluding remarks

Theorem 3.2 shows existence of latin squares of even order with a plex range which is quite distinct from those which had previously been described, either in general or from small order studies. We summarise in Table 3.1 the latin squares of order $n \leqslant 9$ by the named plex ranges. The data for the table for orders $n \leqslant 8$ is due to the work of several authors: Finney [76–78], Freeman [84–86], Johnston and Fullerton [100], Killgrove et al. [106] and Wanless [150]. For order 9, the data is shown by our work in Chapter 5.

Table 3.1: Species of order $n \leqslant 9$ according to plex range.

Order n	Plex range			Number of species
	Complete	Even	Mixed	
1	1			1
2		1		1
3	1			1
4	1	1		2
5	2			2
6	6	4	2	12
7	147			147
8	283624	1	32	283657
9	19270853541			19270853541

The 34 species, of order $n \leqslant 9$ with a mixed plex range, consist of latin squares with a 3-plex but no transversal. Conjecture 1.4 says that a latin square with this property exists for every even order $n > 4$, and while that particular problem remains open, we are now seeing examples, such as $\mathcal{P}_{16}$ and $\mathcal{Q}_{18}$, which have neither a 3-plex nor a 1-plex, but do have a 5-plex. Wanless [150] observes that the only latin squares of order $n \leqslant 8$, without the maximum number of parallel 3-plexes, are the step-type latin squares of Theorem 1.25. He asks whether or not other larger latin squares can fail to possess a 3-plex. A question raised by Pula [129] is the following.

Problem 3.20. Do there exist latin squares outside of those explained by Theorem 1.25 which fail to contain any odd plexes?

The example in (1.2) is a latin square of order 6 which fails the HP-condition, as described in Section 1.10, but it does possess a 3-plex. The latin squares $\mathcal{P}_n$ and $\mathcal{Q}_n$ offer further examples of latin squares which fail the same condition. For $n \geqslant 8$, the families $\mathcal{P}_n$ and $\mathcal{Q}_n$ satisfy condition (B^*) but fail to contain transversals.

The construction of $\mathcal{P}_n$ and $\mathcal{Q}_n$ could conceivably be adapted to produce examples of latin squares with other plex ranges. For example, the family $\mathcal{U}_n$ which we use in Chapter 5 has some similarity in its construction with $\mathcal{Q}_n$, but $\mathcal{U}_n$ does possess transversals. On the other hand, it is not obvious how an argument relying on Lemma 2.1 might show a latin square in which the smallest odd k-plex is larger than that shown by $\mathcal{P}_n$ and $\mathcal{Q}_n$.

Problem 3.21. What is the maximum value of n/k such that there exists a latin square L of order n that possesses an odd k-plex but L has no c-plex for any odd $c < k$?

Whenever k is the smallest odd value in the plex range of a latin square of even order, every k-plex in the latin square is indivisible. Theorems 3.15 and 3.18 prove the next statement which answers a question raised in [150].

Theorem 3.22. *For all odd $k > 1$ and each $n \in \{4k-4,\ 4k-2,\ 4k,\ 4k+2\}$, there exists a latin square of order n that has a k-plex but every k-plex is indivisible.*

Problem 3.23. For $k > 2$, does there exist a latin square L that possesses a k-plex and every k-plex in L is indivisible but k is not the least odd value in the plex range of L?

Conjecture 1.3 implies that an example, if it exists, is of even order and k is odd.

It would be interesting to know if a latin square can have a plex range substantially different from examples seen thus far.

Problem 3.24. Does there exist a latin square which has an a-plex and a c-plex but does not have a b-plex for odd integers $a < b < c$?

An analogous problem, concerning k-factors of graphs, is answered in the negative by Katerinis [102].

The latin squares $\mathcal{P}_n$ and $\mathcal{Q}_n$ contain relatively large indivisible k-plexes, in the sense that k grows linearly with the order of the square. In Section 4.2.1 we will show indivisible $(\frac{1}{2}n)$-partitions in these families. Theorems 3.15 and 3.18 were shown by identifying partitions which, with a small case exception, are indivisible partitions. In the spirit of the next chapter, we record the following. Theorem 3.25 follows immediately from the plex range of $\mathcal{P}_{4q}$.

Theorem 3.25. *The following partitions of $\mathcal{P}_{4q}$ are indivisible:*

1. *For $q > 1$, the 2-partition of Lemma* 3.8.
2. *For odd $q > 1$, the 2-partition of Lemma* 3.14.
3. *For odd $q > 1$, the $(2^q, q^2)$-partition of Lemma* 3.14.
4. *For odd q, the q-partition of Lemma* 3.14.
5. *For even q, the $\left(2^{q-1}, (q+1)^2\right)$-partition of Lemma* 3.13.

The constraint $q > 1$ is necessary for Parts 1, 2 and 3 of Theorem 3.25. In Section 4.5 we see that $\mathcal{P}_4$ (isotopic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$) possesses no indivisible 2-plex.

Theorem 3.26. *The following partitions of $\mathcal{Q}_{2h}$ are indivisible:*

1. *The 2-partition of Lemma* 3.16.
2. *For $k = 2\lfloor \frac{h}{4} \rfloor + 1$, the $(k^2, 2^{h-k})$-partition of Lemma* 3.17.

Justification of the theorem for $\mathcal{Q}_6$, which has transversals, follows from the observation immediately after the proof of Lemma 3.16. Otherwise, the theorem follows immediately from the plex range of $\mathcal{Q}_{2h}$.

Chapter 4

Indivisible plexes and partitions

4.1 Introduction

Motivated by Problem 1.5, we continue to investigate indivisible partitions of the latin families used in Chapter 3. In Section 4.2 we will show that, for all even $n > 2$, there exists a latin square of order n that can be decomposed into two parallel indivisible $(\frac{1}{2}n)$-plexes. We then investigate ways of embedding such plexes into larger latin squares.

In Section 4.3 we find indivisible partitions, by direct construction, in latin squares of odd order. Collecting our results for even and odd orders together, we prove our two main theorems, the first of which addresses Problem 1.22.

Theorem 4.1. *If k divides n where $n > 2$, $k < n$ and $(n, k) \neq (6, 1)$, then there exists a latin square of order n with an indivisible k-partition.*

Theorem 4.2 addresses Problem 1.5.

Theorem 4.2. *For all $n > 2$, there exists a latin square of order n with two parallel indivisible $\lfloor \frac{n}{2} \rfloor$-plexes. Hence, for all $n > 2$, $\kappa(n) \geqslant \lfloor \frac{n}{2} \rfloor$.*

In Section 4.5, we report on computations of $\kappa(n)$ and indivisible partitions and plexes in latin squares of small order. Beforehand, in Section 4.4, we offer explanation for the absence of odd plexes in some examples of interest, including the largest indivisible plex that we found for order 9.

4.2 Latin squares of even order

The results in this section appear in [19].

4.2.1 Indivisible partitions of $\mathcal{P}_{2h}$, $\mathcal{Q}_{2h}$ and $\mathcal{R}_{2h}$

In this section we prove the following theorem.

Theorem 4.3. *For all integers $k \geqslant 2$, there exists a latin square of order $2k$ which contains two parallel indivisible k-plexes.*

Theorem 4.3 will follow immediately from Lemmas 4.4, 4.5 and 4.8.

We prove Theorem 4.3 in three cases. In each case, we identify an h-partition of an appropriate latin square and then prove that each of its two parts is indivisible.

Throughout, assume that $h = \frac{1}{2}n$ where n is the order of the latin square in context; either $\mathcal{P}_n$, $\mathcal{Q}_n$ or $\mathcal{R}_n$.

For our work in this section we introduce the following notation. Let K be a k-plex in L and let $X \subseteq L$. Define

$$X^K = |X \cap K|.$$

Lemma 4.4. *For all odd q, the latin square $\mathcal{R}_{4q}$ has an indivisible h-partition.*

Proof. It is easy to check that $\mathcal{R}_4$ has an indivisible 2-partition. For $q > 1$, an h-plex $H \subset \mathcal{R}_{4q}$ is given by

$$\mathrm{col}(x) = \begin{cases} E & \text{if } x < h, \\ F & \text{otherwise.} \end{cases} \tag{4.1}$$

Note that since q is odd, $h \equiv 2 \bmod 4$. We now show that H is indivisible. Since $\Delta_4^H = h$ and $\Delta_{-4}^H = 0$, a j-plex J contained in H has $j = \Delta_4^J = \Delta_*^J$. If j is odd, then Lemma 2.1 says that $4j \equiv h \bmod 4q$. However $4j \equiv 0 \bmod 4$ and $h \equiv 2 \bmod 4$, so we can conclude that j is not odd. On the other hand, if j is even then Lemma 2.1 says that $4j \equiv 0 \bmod 4q$ and thus, since $j \neq q$, we have $j \in \{0, h\}$. Hence H is indivisible. Similarly, $H' = \mathcal{R}_{4q} \setminus H$ is indivisible. □

The previous lemma proves Theorem 4.3 when $k \equiv 2 \bmod 4$. The next lemma proves the case $k \equiv 0 \bmod 4$.

Lemma 4.5. *For all even q, the latin square $\mathcal{P}_{4q}$ has an indivisible h-partition.*

Proof. An h-plex $H \subset \mathcal{P}_{4q}$ is given by

$$\mathrm{col}(x) = \begin{cases} E & \text{if } (x \in E \text{ and } x < h) \text{ or } (x \in F \text{ and } h - 1 \leqslant x < n - 1), \\ F & \text{otherwise.} \end{cases} \tag{4.2}$$

Note that since q is even, $h \equiv 0 \bmod 4$. We have $\Delta_{-2}^H = h$ and $\Delta_2^H = 0$. We rely on Lemma 2.1 and follow the process of the proof of Lemma 4.4. □

As an aside to our proof of Theorem 4.3, note that (4.2) defines an h-partition of $\mathcal{P}_{4q}$ regardless of the parity of q. If q is odd, then the h-partition consists of plexes which contain no even plex, but do, as easily verified, contain q-plexes. On the other hand, a slightly different h-plex of $\mathcal{P}_{4q}$, given by (4.3) below, puts all of Δ_* into one plex. The result is an h-partition in which both parts contain only even plexes.

$$\mathrm{col}(x) = \begin{cases} E & \text{if } x < h, \\ F & \text{otherwise.} \end{cases} \tag{4.3}$$

We do not know if $\mathcal{P}_{4q}$ has an indivisible h-partition when q is odd, although we did find h-partitions in which one part is indivisible.

We now return to the proof of Theorem 4.3. In the third and final case we rely on conditions other than Lemma 2.1.

The structure of the latin square $\mathcal{Q}_n$ of order $n = 2h$ may be thought of as an h-step type latin square (see Section 1.9) which has been *corrupted* by one element in each block. More specifically,

$$\mathcal{Q}_n = \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix},$$

where each of B_{00} and B_{11} is an $h \times h$ latin subsquare based on the symbols of E, except that each has precisely one occurrence of a symbol from F, and each of B_{01} and B_{10} is an $h \times h$ latin subsquare based on the symbols of F, except that each has precisely one occurrence of a symbol from E. Thus, there are four *corrupted entries* in $\mathcal{Q}_n$. It is a classical observation (e.g. see a theorem by H. B. Mann [46, p448]) that if there are c corrupted entries in one block of an h-step type latin square of order $2h$, then there are c corrupted entries in each block of the square. We will call such a square a *c-corrupted h-step type* latin square.

We now consider the number of elements in a k-plex found in the $h \times h$ blocks of a c-corrupted h-step type latin square.

Lemma 4.6 ([65]). *Suppose that L is a c-corrupted h-step type latin square of order $n = 2h$ where $L = \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix}$ and c entries of B_{00} have symbols from F, and the remaining $h^2 - c$ entries of B_{00} have symbols from E. If $K \subseteq L$ is a k-plex, then the following conditions are satisfied.*

1. *$B_{00}^K = B_{11}^K$, and $B_{01}^K = B_{10}^K$.*
2. *$-2c \leqslant B_{00}^K - B_{01}^K \leqslant 2c$.*
3. *If both k and h are odd, then $-2c < B_{00}^K - B_{01}^K < 2c$.*

Proof. Each of the h rows of $B_{00} \cup B_{01}$ and each of the h columns of $B_{01} \cup B_{11}$ is represented k times in K. Therefore $B_{00}^K + B_{01}^K = kh = B_{01}^K + B_{11}^K$, hence $B_{00}^K = B_{11}^K$. Similarly $B_{01}^K = B_{10}^K$, thus part 1 of the lemma is shown.

Next we partition each B_{ij} into A_{ij} and C_{ij} such that for all $e = (x, y, z) \in B_{ij}$,

$$\begin{array}{ll} e \in A_{ij} & \text{if } (z \in F \text{ and } i \neq j) \text{ or } (z \in E \text{ and } i = j), \\ e \in C_{ij} & \text{otherwise.} \end{array} \tag{4.4}$$

Consider part 2 of the lemma. Each symbol in E is represented k times in K so $A_{00}^K + A_{11}^K + C_{01}^K + C_{10}^K = k|E| = kh$. Hence, rearranging terms;

$$A_{00}^K + A_{11}^K = kh - \left(C_{01}^K + C_{10}^K\right). \tag{4.5}$$

Similarly, counting symbols of F in K;

$$A_{01}^K + A_{10}^K = kh - \left(C_{00}^K + C_{11}^K\right). \tag{4.6}$$

Now assume that $B_{00}^K - B_{01}^K > 2c$. Then $B_{11}^K - B_{10}^K > 2c$. Summing these two inequalities and substituting $B_{ij}^K = A_{ij}^K + C_{ij}^K$ gives

$$\left(A_{00}^K + A_{11}^K + C_{00}^K + C_{11}^K\right) - \left(A_{01}^K + A_{10}^K + C_{01}^K + C_{10}^K\right) > 4c. \tag{4.7}$$

Substituting (4.5) and (4.6) into (4.7) we obtain that $C_{00}^K + C_{11}^K - C_{01}^K - C_{10}^K > 2c$, which is a contradiction since $C_{00}^K \leqslant c$ and $C_{11}^K \leqslant c$. It follows, by symmetry, that $B_{00}^K - B_{01}^K < -2c$ must also be false. Thus part 2 of the lemma has been shown.

Now if k and h are both odd, then $B_{00}^K - B_{01}^K = B_{00}^K - \left(kh - B_{00}^K\right) = 2B_{00}^K - kh$ is odd, so part 3 of the lemma follows. □

Example 4.7 ([65]). The shading in (4.8) identifies a 7-plex, which we call H, in $\mathcal{Q}_{14}$. Observe that $\Delta_{-2}^H = 0$, $\Delta_2^H = 3$ and $I_{\mathcal{Q}}^H = 1$. Entries marked $_*$ are elements of Δ_*. The circled entries show $I_{\mathcal{Q}} = \bigcup_{i,j \in \{0,1\}} C_{ij}$, where $C_{i,j}$ is as defined in the proof of Lemma 4.6. In proving the next lemma we will show that this 7-partition is indivisible.

	0	2	4	6	8	10	12	1	3	5	7	9	11	13
0	13_*	2	4	6	8	10	12	1	3	5	7	9	11	0_*
2	2	4	6	8	10	12	0	3	5	7	9	11	13	1
4	4	6	8	10	12	0	2	5	7	9	11	13	1	3
6	6	8	10	12	0	2	4	7	9	11	13	1	3	5
8	8	10	12	0	2	4	6	9	11	13	1	3	5	7
10	10	12	0	2	4	6	8	11	13	1	3	5	7	9
12	12	0	2	4	6	8	10	13	1	3	5	7	9	11
1	1	3	5	7	9	11	13	0_*	2_*	4_*	6_*	8_*	10_*	12_*
3	3	5	7	9	11	13	1	4	6	8	10	12	0	2
5	5	7	9	11	13	1	3	6	8	10	12	0	2	4
7	7	9	11	13	1	3	5	8	10	12	0	2	4	6
9	9	11	13	1	3	5	7	10	12	0	2	4	6	8
11	11	13	1	3	5	7	9	12	0	2	4	6	8	10
13	0_*	1	3	5	7	9	11	2_*	4_*	6_*	8_*	10_*	12_*	13_*

(4.8)

The illustrated 7-plex H is defined by:

$$\begin{aligned}
\mathrm{col}(x) &= \begin{cases} E \text{ if } x \in \{1,3,5,6,10,12\}, \\ F \text{ if } x \in \{7,9,11\}, \end{cases} \\
\mathrm{col}(0) &= F \setminus \{13\} \cup \{10\}, \\
\mathrm{col}(2) &= F \setminus \{1\} \cup \{12\}, \\
\mathrm{col}(4) &= F \setminus \{3\} \cup \{2\}, \\
\mathrm{col}(8) &= F \setminus \{11\} \cup \{0\}, \\
\mathrm{col}(13) &= \{4,6,8\} \cup \{1,3,11,13\}.
\end{aligned}$$

Now we are ready to prove Theorem 4.3 in the case that k is odd.

Lemma 4.8 ([65]). *For all odd $h \geqslant 3$, the latin square $\mathcal{Q}_{2h}$ has an indivisible h-partition.*

Proof. First we identify an h-partition. A 7-partition of $\mathcal{Q}_{14}$ is shown in (4.8). For $n = 2h \neq 14$, we show an h-plex H:

$$\mathrm{col}(0) = \{n-2\} \cup F \setminus \{n-1\}, \tag{4.9}$$
$$\mathrm{col}(n-1) = \{y \in E : \ y \equiv 2 \bmod 4\} \cup \{y \in F : \ y \equiv 1 \bmod 4\}, \tag{4.10}$$

For $x \in F \setminus \{n-1\}$,

$$\mathrm{col}(x) = \begin{cases} E & \text{if } x < h, \\ F & \text{if } x \geqslant h. \end{cases} \tag{4.11}$$

For $x \in E \setminus \{0\}$, we specify $\mathrm{col}(x)$ in three cases, as follows.

Case 1: $h \equiv 1 \bmod 4$ $(h \geqslant 5)$.

$$\mathrm{col}(x) = \begin{cases} \{x-4\} \cup F \setminus \{x-3\} & \text{if } x \in E \setminus \{0\},\ x < h \text{ and } x \equiv 0 \bmod 4, \\ \{x-2\} \cup F \setminus \{x-1\} & \text{if } x \in E,\ x > h \text{ and } x \equiv 2 \bmod 4, \\ E & \text{otherwise.} \end{cases}$$

Case 2: $h \equiv 3 \bmod 8$ $(h \geqslant 3)$.
For $h = 3,\ c(2) = \{0, 4, 3\},\ c(4) = \{0, 2, 5\}$.
For $h \geqslant 11$,

$$\mathrm{col}(h-3) = \{0\} \cup F \setminus \{1\},$$

For $a \in \{0, 4, 8, \dots, \frac{1}{2}(h-11)\}$,

$$\begin{aligned} \mathrm{col}(h-7-a) &= \{h-3-a\} \cup F \setminus \{h-2-a\}, \\ \mathrm{col}(h-5-a) &= \{n-6-a\} \cup F \setminus \{n-5-a\}, \\ \mathrm{col}(h+1+a) &= \{4+a\} \cup F \setminus \{5+a\}, \\ \mathrm{col}(h+3+a) &= \{h+1+a\} \cup F \setminus \{h+2+a\}. \end{aligned}$$

Otherwise, $\mathrm{col}(x) = E$.

Case 3: $h \equiv 7 \bmod 8$ $(h \geqslant 15)$
For $a \in \{0, 4, 8, \dots, \frac{1}{2}(h-7)\}$,

$$\begin{aligned} \mathrm{col}(h-7-a) &= \{h-3-a\} \cup F \setminus \{h-2-a\}, \\ \mathrm{col}(h-1-a) &= \{n-6-a\} \cup F \setminus \{n-5-a\}, \\ \mathrm{col}(h+1+a) &= \{a\} \cup F \setminus \{1+a\}, \\ \mathrm{col}(h+3+a) &= \{h+1+a\} \cup F \setminus \{h+2+a\} \quad \text{if } a \leqslant \tfrac{1}{2}(h-15). \end{aligned}$$

Otherwise, $\mathrm{col}(x) = E$.

This completes all cases for h mod 8. It is routine to check that H is an h-plex.

We proceed by contradiction to show that H and $H' = \mathcal{Q}_{2h} \setminus H$ are indivisible.

With $c = 1$, consider the sets A_{ij}, B_{ij} and C_{ij} of $\mathcal{Q}_{2h}$ as described in (4.4). By definition, $\bigcup_{i,j \in \{0,1\}} C_{ij} = I_{\mathcal{Q}}$ where $I_{\mathcal{Q}}$ is given by (2.10).

Note that all cases of H, including that given separately in (4.8), obey (4.11) for their definition of $H \cap (B_{10} \cup B_{11}) \setminus r_{n-1}$. Further, by (4.8), (4.9) and (4.10), we see that $H \cap I_Q = C_{11}$, $H \cap r_{n-1} \cap B_{11} = H \cap \Delta_*$ and $H \cap r_{n-1} \cap A_{11} = H \cap \Delta_2$. We rely on these facts in the following argument.

Assume that H is divisible. Then H contains an odd k-plex $K \subset H$ with $k < h$. It follows from Lemma 2.1 that $K \cap \Delta_* = H \cap \Delta_*$. Therefore $k = r_{n-1}^K \geqslant \Delta_*^K = \frac{1}{2}(h+1)$. Also $I_Q \cap H = C_{11} \subset (K \cap \Delta_*)$, so if $e \in K \setminus C_{11}$, then $e \in A_{ij}$ for some i, j. Consider $K \cap (A_{10} \cup A_{11}) \setminus r_{n-1}$. Following from (4.11) there are $\frac{1}{2}(h-1)$ rows r_x such that $H \cap r_x \subset A_{10}$ and $\frac{1}{2}(h-1)$ rows r_x such that $H \cap r_x \subset A_{11}$. Hence,

$$B_{11}^K = (A_{11} \setminus r_{n-1})^K + \Delta_*^K \geqslant \tfrac{1}{2}k(h-1) + \tfrac{1}{2}(h+1), \tag{4.12}$$

$$B_{10}^K = (A_{10} \setminus r_{n-1})^K + (r_{n-1} \setminus \Delta_*)^K \leqslant \tfrac{1}{2}k(h-1) + \tfrac{1}{2}(h-1). \tag{4.13}$$

Lemma 4.6 requires that $B_{11}^K - B_{10}^K \leqslant 1$ so the inequalities in (4.12) and (4.13) must be equalities. Also $B_{00}^K = B_{11}^K$ so $(B_{00} \cup B_{11})^K = 2B_{11}^K = kh - k + h + 1$ using (4.12). Counting occurrences of the symbols, $k|E| = (B_{00} \cup B_{11} \setminus C_{11})^K$ implies that $kh = kh - k + h$. This contradicts $k < h$ and thus shows that H is indivisible.

Next we consider H'. In this case $I_Q \cap H' = I_Q \setminus C_{11}$. Following similar reasoning as for H we find that if $K \subset H'$ is an odd k-plex, then $(B_{10} \setminus r_{n-1})^K = (A_{10} \setminus r_{n-1})^K = \frac{1}{2}k(h-1)$, and $(B_{11} \setminus r_{n-1})^K = (A_{11} \setminus r_{n-1})^K = \frac{1}{2}k(h-1)$. Hence, by Lemma 4.6,

$$(r_{n-1} \cap B_{10})^K - (r_{n-1} \cap B_{11})^K = \pm 1. \tag{4.14}$$

We also note that $r_{n-1} \cap A_{11} \cap H' = \Delta_2 \cap H'$, with $\Delta_2^{H'} = \frac{1}{2}(h-1)$ and that $\Delta_{-2} = r_1 \cap H'$ so $\Delta_{-2}^{H'} = h$.

Let $J = H' \setminus K$ be a j-plex (with $j > 0$ even). By Lemma 2.1, $\sum_{e \in J \cap I_Q} \Delta(e)$ is even. Thus we consider two cases.

Case 1: $\sum_{e \in J \cap I_Q} \Delta(e) = 0$.
Lemma 2.1 requires that $\Delta_2^J = \Delta_{-2}^J$. Also $j = \Delta_{-2}^J$ so it follows that $(r_{n-1} \cap B_{10})^J = j - \Delta_2^J = 0$. Then $(r_{n-1} \cap B_{10})^K = (r_{n-1} \cap B_{10})^{H'} = \frac{1}{2}(h+1)$. However now $(r_{n-1} \cap B_{10})^K > (r_{n-1} \cap B_{11})^{H'}$ so by (4.14), $(r_{n-1} \cap B_{11})^K = \frac{1}{2}(h-1)$ and $j = \Delta_2^J = 0$, which is a contradiction.

Case 2: $\sum_{e \in J \cap I_Q} \Delta(e) = 2$.
Lemma 2.1 requires that $\Delta_{-2}^J = \Delta_2^J + 1$ so $j = \Delta_2^J + 1$, and $(r_{n-1} \cap B_{10})^K = \frac{1}{2}(h-1)$. Thus, the right hand side of (4.14) is positive, so $\Delta_2^K = \frac{1}{2}(h-3)$ and we have $k = h-2$ and $j = 2$. By the Case 2 assumption, $C_{00} \subset K$, and $(C_{01} \cup C_{10}) \subset J$, hence $B_{11}^J = A_{11}^J = \Delta_2^J + \frac{1}{2}j(h-1) = h$, so by Lemma 4.6, $A_{00}^J = B_{00}^J = B_{11}^J = h$. Thus $A_{00}^J + A_{11}^J + (C_{01} \cup C_{10})^J = 2h + 2$. Since this is not equal to $j|E| = jh$, we have a contradiction.

The contradictions in the two cases show that H' is indivisible. □

4.2.2 Embedding plexes into larger latin squares

The results in this section are jointly due to Bryant, Egan, Maenhaut and Wanless [19].

In this section we will show ways to extend plexes into larger latin squares. By applying our methods to the indivisible plexes of Theorem 4.3 we will prove Theorems 4.9 and 4.10.

Theorem 4.9. *For integers $k \geqslant 2$ and $m \geqslant 1$, there exists a latin square of order $2km$ with an indivisible k-partition.*

We extend on Theorem 4.9 in Section 4.3 where we show indivisible k-partitions in latin squares of odd order.

Theorem 4.10.

$$\kappa(n) \geqslant \max(\tfrac{n}{p}, \lfloor \tfrac{n}{5} \rfloor) \text{ where } p \text{ is the smallest prime divisor of } n. \tag{4.15}$$

We will improve the result of Theorem 4.10 for odd n in Section 4.3.

We begin with some preliminary results on combining (proto)plexes into larger ones. We use definitions and notation associated with partial latin squares from Section 1.12.

Let P and P' be partial latin squares with pairwise disjoint index sets $\mathcal{I}(P)$ and $\mathcal{I}(P')$. We define the *direct sum* $P \oplus P'$ to be the PLS $P \cup P'$ with index set $\mathcal{I}(P) \cup \mathcal{I}(P')$. We define the *product* $P \otimes P'$ to be the PLS

$$P \otimes P' = \big\{\big((x,x'),(y,y'),(z,z')\big) \,:\, (x,y,z) \in P \text{ and } (x',y',z') \in P'\big\}.$$

with index set $\mathcal{I}(P) \times \mathcal{I}(P')$.

The next two results follow immediately from these definitions.

Lemma 4.11. *If $K = K_1 \oplus K_2 \oplus \ldots \oplus K_s$ then K contains a c-protoplex if and only if K_i contains a c-protoplex for each $i \in \{1, 2, \ldots, s\}$.*

Lemma 4.12. *If K is an a-plex of order n and K' is a b-plex of order n', then $K \otimes K'$ is an ab-plex of order nn'. Furthermore, if $K_1, K_2, \ldots, K_s$ is a partition of a latin square L and $K'_1, K'_2, \ldots, K'_t$ is a partition of a latin square L', then $\{K_i \otimes K'_j : i = 1, 2, \ldots, s,\ j = 1, 2, \ldots, t\}$ is a partition of $L \otimes L'$.*

An immediate consequence of Lemma 4.12 is that if $K_1, K_2, \ldots, K_s$ is a k-partition of L and $K'_1, K'_2, \ldots, K'_t$ is a k'-partition of L', then $L \otimes L'$ has a kk'-partition.

Lemma 4.13. *Let K_0 and K_1 be parallel k-plexes in a latin square of order $2k$. Then there is a k-partition of a latin square of order $4k$ all parts of which are isotopic to $K_0 \oplus K_1$.*

Proof. We define a latin square of order $4k$ with index set $\mathcal{I}(K_0) \cup \{i' : i \in \mathcal{I}(K_0)\}$ and a k-partition $\{P_1, P_2, P_3, P_4\}$ where

$$\begin{aligned}
P_1 &= K_0 \cup \big\{(x', y', z') : (x, y, z) \in K_1\big\},\\
P_2 &= \big\{(x', y, z') : (x, y, z) \in K_0\big\} \cup \big\{(x, y', z) : (x, y, z) \in K_1\big\},\\
P_3 &= \big\{(x, y', z') : (x, y, z) \in K_0\big\} \cup \big\{(x', y, z) : (x, y, z) \in K_1\big\},\\
P_4 &= \big\{(x', y', z) : (x, y, z) \in K_0\big\} \cup \big\{(x, y, z') : (x, y, z) \in K_1\big\}.
\end{aligned}$$

It is clear that each P_i is isotopic to $K_0 \oplus K_1$. □

We are now ready to prove Theorem 4.9.

Proof. (of Theorem 4.9) Let L be the latin square of order $2k$ with an indivisible k-partition guaranteed by Theorem 4.3. For $m \notin \{2, 6\}$, we apply Lemma 4.12 to L and L' where L' is a latin square of order m possessing a 1-partition. For $m = 2$ we apply Lemma 4.13 to L. For $m = 6$ we apply Lemma 4.12 to L and a latin square of order 3, followed by Lemma 4.13. In all cases, Lemma 4.11 guarantees that the resulting k-partition is indivisible. □

We will soon construct latin squares of various additional orders which contain k-plexes that are direct sums of two k-protoplexes. By taking one of these to be a k-protoplex from Theorem 4.3, the resulting k-plexes will be indivisible. First we prove restrictions on the order of a latin square that contains a k-plex that is a direct sum of two k-protoplexes.

Lemma 4.14. *If $K = K_0 \oplus K_1$ is a k-plex, where K_0 is a k-protoplex of order m and K_1 is a k-protoplex of order $n - m$, then $n^2 - (3m + k)n + 3m^2 \geqslant 0$.*

Proof. Let L be the latin square containing K and define

$$X = \big\{(x, y, z) \in L : x \in \mathcal{I}(K_0),\ y \in \mathcal{I}(K_1)\big\}.$$

Consider the m symbols of $\mathcal{I}(K_0)$. The number of occurrences of each of these in the rows indexed by $\mathcal{I}(K_0)$ is m, and the number of occurrences in K_0 is k. Hence the number of occurrences of each of these symbols in X is at most $m - k$. Similarly, the number of occurrences of each of the $n - m$ symbols of $\mathcal{I}(K_1)$ in X is at most $n - m - k$. Since the total number of occurrences of symbols in X is $m(n - m)$, it follows that $m(n - m) \leqslant m(m - k) + (n - m)(n - m - k)$. Rearranging, we obtain the required inequality. □

Putting $m = 2k$ in Lemma 4.14, we obtain $(n - 3k)(n - 4k) \geqslant 0$ and thus $n \leqslant 3k$ or $n \geqslant 4k$. However, $n < 3k$ implies that K_1 is a k-protoplex of order less than k, which is impossible. Thus, we have the following corollary which tells us the orders of latin squares in which it may be possible to embed our indivisible k-protoplexes of order $2k$.

Corollary 4.15. *If $K = K_0 \oplus K_1$ is a k-plex, where K_0 is a k-protoplex of order $2k$ and K_1 is a k-protoplex of order $n-2k$, then either $n = 3k$ or $n \geqslant 4k$.*

In what follows, we will show that there exists a latin square of order n with an indivisible k-plex for all n allowed (given that our k-plex is a direct sum involving a k-protoplex of order $2k$) by Corollary 4.15, except for n in the range $4k < n < 5k$.

Lemma 4.16. *A k-protoplex K of order $m \geqslant 2k$ can be extended to a k-plex of order $2m-k$ that is a direct sum involving K.*

Proof. The result is trivial in the case $m = 3$ and $k = 1$ so assume otherwise. Let K_0 be a k-protoplex of order m and let L_1 be a latin square of order $m-k$ containing a k-plex K_1. Since $m \geqslant 2k$ (and $(m,k) \neq (3,1)$), L_1 exists by Corollary 1.17. We now show that $K_0 \oplus L_1$ is completable, thus showing that $K_0 \oplus K_1$ is the required k-plex. To complete $K_0 \oplus L_1$ we need to

(1) fill the rectangle $R_{01} = \big\{(x,y) : x \in \mathcal{I}(K_0),\ y \in \mathcal{I}(K_1)\big\}$,

(2) fill the rectangle $R_{10} = \big\{(x,y) : x \in \mathcal{I}(K_1),\ y \in \mathcal{I}(K_0)\big\}$, and

(3) fill the unfilled cells with row and column indices in $\mathcal{I}(K_0)$.

Each of (1)-(3) may be done independently, and each is equivalent to finding a 1-factorisation of an $(m-k)$-regular bipartite graph with m vertices in each part. The existence of such factorisations is well known, and is an easy consequence of Hall's Theorem [96]. □

Lemma 4.17. *Let $m \geqslant k \geqslant 1$ and $n \geqslant 2m+k$. For any k-plex K of order m, there exists a k-plex $K \oplus K'$ of order n.*

Proof. Let L be a latin square of order m containing a k-plex K. Let S be a set of $n-m$ symbols distinct from $\mathcal{I}(L)$ and let γ be a permutation applying a single $n-m$ cycle to S. By [152], and as mentioned in Section 1.11, there exists a (Parker) latin square $L' \supset L$ of order n, indexed by $\mathcal{I}(L) \cup S$ and such that L' has an automorphism α that applies γ simultaneously to the row, column and symbol indices. We obtain the required k-plex by taking the union of K with k orbits of α on entries whose row, column and symbol indices are in S. The condition $n \geqslant 2m+k$ ensures that $|S| - m \geqslant k$, so there are enough suitable orbits for this construction. □

Note that applying Lemma 4.16 or Lemma 4.17 to the indivisible k-plex of order $2k$ from Theorem 4.3, we obtain an indivisible k-plex of order n for $n = 3k$ and all $n \geqslant 5k$, respectively. Also, applying Theorem 4.9 with $m = 2$ gives us, for each $k \geqslant 2$, a latin square of order $4k$ with an indivisible k-plex. We state these facts in the following corollary.

Corollary 4.18. *For all $k \geqslant 2$, there exists an indivisible k-plex of order n for $n = 3k$, $n = 4k$ and all $n \geqslant 5k$.*

We are now ready to prove Theorem 4.10 which states that $\kappa(n) \geqslant \max(\frac{n}{p},\ \lfloor\frac{n}{5}\rfloor)$ where p is the smallest prime divisor of n.

Proof. (of Theorem 4.10) Let p be the smallest prime divisor of n. If $p = 2$ or $p = 3$, then the existence of a latin square of order n with an indivisible $(\frac{n}{p})$-plex is guaranteed by Theorem 4.3 or the $n = 3k$ result in Corollary 4.18, respectively. If $p \geqslant 5$, then the existence of a latin square of order n with an indivisible $\lfloor\frac{n}{5}\rfloor$-plex follows from the $n \geqslant 5k$ result in Corollary 4.18. □

4.3 Latin squares of odd order

We now investigate, by direct construction, indivisible plexes in latin squares of odd order.

4.3.1 Indivisible partitions of $\mathcal{O}_{m,k}$

We use the latin family $\mathcal{O}_{m,k}$ which is defined for all odd integers $m, k \geqslant 3$.

Lemma 4.19. *For odd integers $m \geqslant 3$ and $k \geqslant 3$, the latin square $\mathcal{O}_{m,k}$ of order mk has an indivisible k-partition.*

Proof. For each $i \in \mathbb{Z}_m$, a k-plex K_i is given by

$$\operatorname{col}\big((a,b)\big) = \big\{(\ a+i,\ d\ ) :\ d \in \mathbb{Z}_k \text{ for each } (a,b) \in \mathcal{I}(\mathcal{O}_{m,k})\big\}. \tag{4.16}$$

It is routine to verify that (4.16) identifies a k-partition of $\mathcal{O}_{m,k}$. We need to show that each K_i is indivisible.

Assume that $C \subset K_i$ is a c-plex for some $0 < c < k$ and some $i \in \mathbb{Z}_m$. Consider the projection $\pi : \mathcal{O}_{m,k} \to \mathbb{Z}_k$ of $\Delta(\mathcal{O}_{m,k})$ onto the second coordinate. Let $\pi_* = \big(\pi^{-1}(1) \cup \pi^{-1}(-1)\big) \cap C$. The construction of $\mathcal{O}_{m,k}$ and K_i imply that π is zero on $C \setminus \pi_*$ and that $|\pi_*| \in \{c, 2c\}$. Specifically, if $i \in \{0, m-1\}$, then $\pi_* \subseteq \pi^{-1}(1)$ and $|\pi_*| = c$. If $i \in \{1, 3, 5, \dots, m-2\}$ then $\pi_* \subseteq \pi^{-1}(-1)$ and $|\pi_*| = 2c$. Otherwise $i \in \{2, 4, 6, \dots, m-3\}$, so $\pi_* \subseteq \pi^{-1}(1)$ and $|\pi_*| = 2c$. In each case, Lemma 2.1 implies that k divides c which contradicts $0 < c < k$. Thus K_i is indivisible. □

Lemma 4.19 yields no improvement to the lower bound on $\kappa(n)$ stated by (4.15). However, for any odd n divisible by 3 or 5, we match the bound of (4.15) with a simpler construction than was used to prove Theorem 4.10. Using $\mathcal{O}_{m,k}$, we can apply the technique in the proof of Theorem 4.9 to obtain a latin square of order dmk with a neat indivisible k-partition, except in cases $d \in \{2, 6\}$.

We can now prove Theorem 4.1.

Proof. (of Theorem 4.1) Assume that $n = mk$ where m, k are positive integers such that $k \neq n > 2$ and $(m, k) \neq (6, 1)$. For $k = 1$ we rely on Theorem 1.16. For all odd integers $k \geqslant 3$ and $m \geqslant 3$, Lemma 4.19 shows the required k-partition. For all even m and $k \geqslant 2$, we use an indivisible k-partition given by Theorem 4.9. $\square$

4.3.2 Indivisible partitions of $\mathcal{H}_n$

We now use the family $\mathcal{H}_n$ which is defined for all odd $n \geqslant 5$.

Lemma 4.20. *If P is $(k, k+1)$-partition of $\mathcal{H}_{2k+1}$ such that Δ_2 is in one part and Δ_{-2} is in the other, then P consists of an indivisible k-plex and a $(k+1)$-plex that contains no c-plex for $1 < c < k$.*

Proof. Let $K \subset \mathcal{H}_{2k+1}$ be a k-plex such that K satisfies the hypothesis. Now $|\Delta_2| = |\Delta_{-2}| = k$, so respecting the rows of K and the sum required by Lemma 2.1, K has exactly one element of $\Delta_1 \cup \Delta_{-1}$ and it must be in r_1. Any c-plex $C \subset K$ must intersect Δ_*, but no proper subset of $\Delta_* \cap K$ will satisfy Lemma 2.1. Thus K is indivisible. Similarly, if $C \subset K'$ is a c-plex for some $c > 1$, then C contains some element of Δ_a for $a = \pm 2$. By Lemma 2.1, C must then contain k such elements and it follows that C is an indivisible k-plex. $\square$

Lemma 4.21. *The latin square $\mathcal{H}_n$ has an indivisible $(1, k^2)$-partition.*

Proof. We show two parts of the partition.

A transversal T is given by

$$\operatorname{col}(x) = \begin{cases} \{n - \frac{1}{2}x\} & \text{if } x \text{ is even}, \\ \{\frac{1}{2}(n - x)\} & \text{otherwise.} \end{cases} \tag{4.17}$$

For each $(x, y, z) \in T$ let $t(x) = y$. We define a k-plex K_1, parallel to T, as follows:

$$\operatorname{col}(x) = \begin{cases} \{1, 3, 5, \ldots, n-2\} & \text{if } x = 0, \\ \{2, 3, 4, \ldots, t(x) - 1\} \cup \{n - 1\} & \text{if } x = 1, \\ \{0, 2, 4, \ldots, n-3\} & \text{if } x = 2, \\ \{t(x) + 1,\ t(x) + 2,\ t(x) + 3, \ldots,\ t(x) + k\} & \text{otherwise.} \end{cases}$$

It is routine to show that K_1 is a k-plex and that it is parallel to T. Thus, $K_2 = \mathcal{H}_n \setminus (K_1 \cup T)$ is a k-plex and we have shown a $(1, k^2)$-partition. By Lemma 4.20, this partition is indivisible. $\square$

In (4.18) we illustrate the indivisible partition of Lemma 4.21 in $\mathcal{H}_9$. Entries marked $_*$ identify elements of Δ_*.

	0	1	2	3	4	5	6	7	8
0	1_*	3_*	2	5_*	4	7_*	6	0_*	8
1	0_*	2	3	4	5	6	7	8	1_*
2	2	1_*	4	3_*	6	5_*	8	7_*	0_*
3	3	4	5	6	7	8	0	1	2
4	4	5	6	7	8	0	1	2	3
5	5	6	7	8	0	1	2	3	4
6	6	7	8	0	1	2	3	4	5
7	7	8	0	1	2	3	4	5	6
8	8	0	1	2	3	4	5	6	7

(4.18)

We now prove Theorem 4.2 and so conclude the main results of this chapter.

Proof. (of Theorem 4.2) The case for all even $n > 2$ is shown by Theorem 4.3. For $n = 3$, the species of $\mathbb{Z}_3$ possesses a 1-partition. The case for all odd $n > 3$ is shown by Lemma 4.21. $\square$

Although an argument using Lemma 2.1 for an indivisible $(k+1)$-plex in $\mathcal{H}_{2k+1}$ is not obvious, we wonder if this family does admit larger indivisible plexes. Computations verify that $\mathcal{H}_n$ has an indivisible $(k, k+1)$-partition for all $n \leqslant 11$. Also, for $n \leqslant 9$, we confirmed that $\mathcal{H}_n$ has no larger indivisible plex. In (4.19) we give an example of an indivisible $(k, k+1)$-partition of $\mathcal{H}_9$ (left) and $\mathcal{H}_{11}$ (right). In each case, other than the absence of transversals in the larger part, Lemma 4.20 explains the indivisibility. The entries marked $_*$ show elements of Δ_*.

	0	1	2	3	4	5	6	7	8
0	1_*	3_*	2	5_*	4	7_*	6	0_*	8
1	0_*	2	3	4	5	6	7	8	1_*
2	2	1_*	4	3_*	6	5_*	8	7_*	0_*
3	3	4	5	6	7	8	0	1	2
4	4	5	6	7	8	0	1	2	3
5	5	6	7	8	0	1	2	3	4
6	6	7	8	0	1	2	3	4	5
7	7	8	0	1	2	3	4	5	6
8	8	0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7	8	9	10
0	1_*	3_*	2	5_*	4	7_*	6	9_*	8	0_*	10
1	0_*	2	3	4	5	6	7	8	9	10	1_*
2	2	1_*	4	3_*	6	5_*	8	7_*	10	9_*	0
3	3	4	5	6	7	8	9	10	0	1	2
4	4	5	6	7	8	9	10	0	1	2	3
5	5	6	7	8	9	10	0	1	2	3	4
6	6	7	8	9	10	0	1	2	3	4	5
7	7	8	9	10	0	1	2	3	4	5	6
8	8	9	10	0	1	2	3	4	5	6	7
9	9	10	0	1	2	3	4	5	6	7	8
10	10	0	1	2	3	4	5	6	7	8	9

(4.19)

Curiously, computation shows that in $\mathcal{H}_9$ every indivisible 5-plex occurs in a $(4, 5)$-partition obeying the hypothesis of Lemma 4.20. Thus, every indivisible 5-plex in

$\mathcal{H}_9$ occurs in an indivisible $(4,5)$-partition. In smaller members of $\mathcal{H}_n$ alternative partitions of Δ_* can yield an indivisible $(k,k+1)$-partition. In $\mathcal{H}_7$ the complement of an indivisible 4-plex in $\mathcal{H}_7$ is not necessarily indivisible. The smallest family member $\mathcal{H}_5$ does exhibit every one of its indivisible 3-plexes in an indivisible $(2,3)$-partition. However, every 2-plex in $\mathcal{H}_5$ is indivisible since it does not have two parallel transversals. It would be interesting to know if the observed condition in $\mathcal{H}_9$ occurs in larger latin squares.

Problem 4.22. For $n > 9$, does there exist a latin square L of order n that possesses an indivisible k-plex and every indivisible k-plex in L is in an indivisible $(k, n-k)$-partition?

As an aside to indivisible plexes, we next point out that $\mathcal{H}_n$ is a confirmed bachelor. We thus offer, next, a very simple alternative proof of Theorem 1.23 for the case of all odd $n > 3$.

Theorem 4.23. *The elements* $(1,0,0)$ *and* $(1,n-1,1)$ *of* $\mathcal{H}_n$ *are transversal-free. Hence, for all odd* $n > 3$*, the latin square* $\mathcal{H}_n$ *is a confirmed bachelor latin square of order* n*.*

Proof. Assume that $T \subset \mathcal{H}_n$ is a transversal and that $(1,0,0) \in T$. No other element of $\Delta_1 \cup \Delta_{-1}$ is possible in T as each of them agrees with $(1,0,0)$ in at least one coordinate. Considering r_0 and r_2, T has at most one element from each of Δ_2 and Δ_{-2}, but then $\sum_{e \in T} \Delta(e) \equiv 0 \bmod n$ is impossible. Thus, by Lemma 2.1, $(1,0,0)$ is not in a transversal. Similarly, no transversal in $\mathcal{H}_n$ contains $(1,n-1,1)$. $\square$

4.3.3 Indivisible partitions of $\mathbb{Z}_n$

The existence of indivisible 2-partitions in $\mathbb{Z}_n$ for n even is shown by Theorem 1.27. The next result is first evidence, in general, of indivisible plexes in the Cayley table of $\mathbb{Z}_n$ when it does possess transversals.

Lemma 4.24. *For all odd* n *and integers* $t \equiv n \bmod 4$*, the Cayley table of* $\mathbb{Z}_n$ *has an indivisible* $(1^t, 2^{(n-t)/2})$*-partition.*

Proof. The existence of a 1-partition is well known. For $a \in \{0,1,\ldots,n-1\}$, a transversal T_a is given by $\mathrm{col}(x) = \{x+a\}$.

For each $d \in D = \{0,1,2,\ldots,2\lfloor \frac{n}{4} \rfloor - 1\}$, a 2-plex J_d is next shown.

If d is even,

$$\mathrm{col}(x) = \begin{cases} \{x+2d+1,\ x+2d+4\} & \text{if } x = \left\lfloor \frac{n}{2} \right\rfloor - 2, \\ \{x+2d+1,\ x+2d+2\} & \text{if } (x = \left\lfloor \frac{n}{2} \right\rfloor - 1) \text{ or } (x = n-2), \\ \{x+2d+2,\ x+2d+3\} & \text{if } x = n-1, \\ \{x+2d+1,\ x+2d+3\} & \text{otherwise,} \end{cases}$$

and if d is odd,

$$\mathrm{col}(x) = \begin{cases} \{x+2d,\ x+2d+1\} & \text{if } x = \left\lfloor \frac{n}{2} \right\rfloor - 2, \\ \{x+2d+1,\ x+2d+2\} & \text{if } (x = \left\lfloor \frac{n}{2} \right\rfloor - 1) \text{ or } (x = n-2), \\ \{x+2d-1,\ x+2d+2\} & \text{if } x = n-1, \\ \{x+2d,\ x+2d+2\} & \text{otherwise.} \end{cases}$$

If d is even, then J_d contains the 3 elements $(\,n-3,\ 2d,\ 2d-3\,)$, $(\,n-2,\ 2d,\ 2d-2\,)$ and $(\,n-2,\ 2d-1,\ 2d-3\,)$. Since each pair of these triples agree in some coordinate, J_d cannot be divided into parallel transversals. Similarly, if d is odd, using elements in rows $\left\lfloor \frac{n}{2} \right\rfloor - 2$ and $\left\lfloor \frac{n}{2} \right\rfloor - 3$ shows that J_d is indivisible.

It is routine to show that, for a fixed even $d \in D$, the union $J_d \cup J_{d+1}$ is equal to the 4-plex given by

$$\mathrm{col}(x) = \{x+2d+1,\ x+2d+2,\ x+2d+3,\ x+2d+4\}.$$

Thus it follows that for a fixed even $d \in D$,

$$J_d \cup J_{d+1} = T_{2d+1} \cup T_{2d+2} \cup T_{2d+3} \cup T_{2d+4}.$$

An example of the required partition is given by

$$\big\{T_a \ :\ a \in \{0, n-1, n-2, \ldots, n-t-2\}\big\} \cup \big\{J_d \ :\ d \in \{0, 1, 2, \ldots, 2\left\lfloor \tfrac{n-t}{4} \right\rfloor - 1\}\big\}.$$

$\square$

For $n \equiv 3 \bmod 4$, Lemma 4.24 falls short of an indivisible $(1, 2^{(n-1)/2})$-partition which we expect by Conjecture 4.34 in Section 4.6.

In Section 4.5 we will table data on the indivisible partitions of every type occurring in $\mathbb{Z}_n$ for $n \leqslant 9$.

4.4 Subsquares and restrictions on odd plexes

In this section we investigate ways in which plexes and partitions may fail to contain odd plexes.

Assume that L is a latin square of order $n = a + b$ and that $A \subseteq L$ is a subsquare of order a. A well known necessary condition is that $a \leqslant b$. By Theorem 1.35, the condition that $a \leqslant b$ is sufficient for embedding of a latin square A' of order a into some latin square of order $n = a + b$. Existence of the subsquare A in L implies that, up to permutation of the rows and columns, L has the form:

$$L = \begin{pmatrix} A & S \\ R & B \end{pmatrix}, \tag{4.20}$$

where B, R and S are $b \times b$, $b \times a$, $a \times b$ submatrices respectively. Let C denote the subset of B whose symbols are in A and let $D = B \setminus C$. Suppose that K_A is a $(b-a)$-plex in A. Simple counting shows that $K_A \cup D$ is a $(b-a)$-plex of L.

Let K be a k-plex in L and assume that $|A \cap K| = x$. Counting cells of K in the rows of A implies that $|K \cap C| = ak - x$. Similarly, $|K \cap R| = ak - x$. It follows that

$$|K \cap D| = bk - 2(ak - x). \tag{4.21}$$

In particular, $|K \cap D|$ is odd if and only if b and k are both odd. In 1944, Mann [117] used this argument to prove that a latin square of order $n \equiv 1 \bmod 4$, with a subsquare of order $(n-1)/2$, has no orthogonal mate. As Mann showed, the number of disjoint transversals of such a latin square L is at most $|D| = b < n$. By a similar argument using (4.21), if b is odd and $D \subseteq P \subseteq L$ for some plex P, then $L \setminus P$ contains no odd plexes. For example, shading in (4.22) identifies a 2-plex P for which the complementary 6-plex contains no odd plex.

$$\left(\begin{array}{ccc|ccccc} 2 & 1 & 0 & 5 & 6 & 3 & 4 & 7 \\ 1 & 0 & 2 & 6 & 7 & 5 & 3 & 4 \\ 0 & 2 & 1 & 7 & 5 & 4 & 6 & 3 \\ \hline 5 & 6 & 7 & 3 & 4 & 0 & 1 & 2 \\ 6 & 7 & 5 & 4 & 3 & 2 & 0 & 1 \\ 3 & 5 & 4 & 0 & 2 & 1 & 7 & 6 \\ 4 & 3 & 6 & 1 & 0 & 7 & 2 & 5 \\ 7 & 4 & 3 & 2 & 1 & 6 & 5 & 0 \end{array}\right) \tag{4.22}$$

More generally:

Lemma 4.25. *Suppose that n and j are positive integers such that $\frac{2}{3}n \leqslant j \leqslant n$ and $j \equiv 2(n-1) \bmod 4$. Then there exists a latin square of order n with a j-plex that contains no odd plex.*

Proof. First consider small n. If $n \in \{1,3,4,7\}$, then the lemma is true vacuously. If $n \in \{2,5\}$, then $(n,j) = (2,2)$ or $(5,4)$. Case $(2,2)$ is trivial, and $(5,4)$ is shown by a latin square of order 5 whose 3 transversals coincide on a single entry. The complement of any transversal is therefore a 4-plex containing no odd plex.

To prove all remaining cases we will show existence of an appropriate L of the form (4.20), such that b is odd. We shall identify a j-plex $J \subseteq L$ such that $D \cap J = \emptyset$. The lemma will then immediately follow from (4.21).

Set $a = j/2$. The hypothesis $\frac{2}{3}n \leqslant j \leqslant n$ implies that $\frac{1}{3}n \leqslant a \leqslant \frac{1}{2}n$. Now $a \leqslant \frac{1}{2}n$ is necessary and sufficient to obtain a latin square of the form (4.20). Also $a \geqslant \frac{1}{3}n$ ensures that $a \geqslant n - 2a = b - a$, so the subsquare A is big enough to contain a $(b-a)$-plex. Corollary 1.17 confirms that $(b-a) \leqslant a$ is sufficient for us to choose a latin square A, of order $a > 2$, that possesses a $(b-a)$-plex K_A.

For odd $n \geqslant 9$, $j = 2a \equiv 0 \bmod 4$ implies that a is even. Also, $a \geqslant \frac{1}{3}n \geqslant 3$. On the other hand, for even $n \geqslant 6$, $j = 2a \equiv 2 \bmod 4$ implies that a is odd, and $4 \leqslant \frac{2}{3}n \leqslant 2a \equiv 2 \bmod 4$ implies that $a \geqslant 3$. In either case, $b = n - 3a$ is an odd integer and $a \geqslant 3$.

Now $K = K_A \cup D$ is a $(b-a)$-plex in L. Thus, $J = L \setminus K$ is the required even j-plex. □

For $j = n \equiv 2 \bmod 4$, Lemma 4.25 says no more than a well known fact; a latin square of order $n = 2a$ with a subsquare of odd order a possesses no odd plex. (e.g. Theorem 1.25.) There is no guarantee in Lemma 4.25 that L possesses any odd plexes, other than L itself when n is odd. We next show that certain odd plexes exist when extra structural constraints are imposed on L. We need the following result which is an immediate consequence of the existence of Parker squares [152, Thm 6]. Parker squares were mentioned in Section 1.11.

Lemma 4.26. *There exists a Latin square L of the form* (4.20) *which has a cyclic automorphism α of order b. The subsquare A of L occurs in the rows and columns indexed by the fixed points of α.*

Theorem 4.27. *Assume that n, j, k are positive integers such that $\frac{2}{3}n \leqslant j \leqslant n$, $j \equiv 2(n-1) \bmod 4$ and $k = n - j$. Then there exists a latin square of order n with a $(1^k, j)$-partition in which the j-plex contains no odd plex.*

Proof. We follow the proof of Lemma 4.25, but use the extra structure from Lemma 4.26. The cases where $n \in \{1,2,3,4,5,7\}$ are exactly as in the previous proof. For all a except 2 and 6, we use as the subsquare A, a latin square of order a with a 1-partition guaranteed by Theorem 1.16. In the proof of Lemma 4.25 we have $a > 2$, thus our only concern is if $a = 6$. If $a = 6$, then there exists a latin square with a $(1^4, 2)$-partition (see Section 4.5, [76] or [150]) which satisfies all but one case. The exception is when $n = 17$, $a = 6$, $b = 11$ and $k = 5$. However, example (4.23) then shows the required $(1^5, 12)$-partition. Five parallel transversals are marked by subscripts a, b, c, d and e. Together they cover D so the complementary 12-plex contains no odd plex, by (4.21).

$$\left(\begin{array}{cccccccc|ccccccccc}
0 & 1_b & 2 & 3_d & 4_e & 5 & 6 & 7 & 8 & 9 & 10 & 11_c & 12 & 13 & 14 & 15_a & 16 \\
1_a & 2_e & 3 & 4 & 5 & 6 & 7 & 0 & 9 & 10 & 11 & 12 & 13 & 14_c & 15_b & 16 & 8_d \\
2_b & 3_c & 4_a & 5 & 6 & 7 & 0 & 1 & 10 & 11_e & 12 & 13 & 14 & 15_d & 16 & 8 & 9 \\
3 & 4_d & 5 & 6_a & 7 & 0_e & 1 & 2 & 11 & 12 & 13 & 14_b & 15_c & 16 & 8 & 9 & 10 \\
4_c & 5_a & 6_b & 7 & 0 & 1 & 2 & 3 & 12_d & 13 & 14 & 15 & 16 & 8 & 9_e & 10 & 11 \\
5 & 6 & 7_e & 0 & 1 & 2 & 3_a & 4 & 13 & 14 & 15 & 16 & 8 & 9 & 10_d & 11_b & 12_c \\
6 & 7 & 0_c & 1 & 2_a & 3 & 4 & 5_d & 14 & 15 & 16 & 8 & 9 & 10_e & 11 & 12 & 13_b \\
7 & 0 & 1_d & 2_c & 3_b & 4 & 5 & 6 & 15 & 16 & 8 & 9 & 10 & 11 & 12 & 13_e & 14_a \\
\hline
8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16_a & 0_b & 1_c & 2_d & 3_e & 4 & 5 & 6 & 7 \\
9 & 10 & 11 & 12 & 13 & 14 & 15 & 16_c & 7_b & 8_a & 0_d & 1_e & 2 & 3 & 4 & 5 & 6 \\
10 & 11 & 12 & 13 & 14 & 15 & 16_b & 8_e & 6_c & 7_d & 9_a & 0 & 1 & 2 & 3 & 4 & 5 \\
11 & 12 & 13 & 14 & 15 & 16_d & 8_c & 9_b & 5_e & 6 & 7 & 10_a & 0 & 1 & 2 & 3 & 4 \\
12 & 13 & 14 & 15 & 16 & 8_b & 9_d & 10 & 4 & 5_c & 6_e & 7 & 11_a & 0 & 1 & 2 & 3 \\
13_d & 14 & 15 & 16_e & 8 & 9_c & 10 & 11 & 3 & 4 & 5 & 6 & 7 & 12_b & 0_a & 1 & 2 \\
14_e & 15 & 16 & 8 & 9 & 10 & 11 & 12 & 2 & 3 & 4_b & 5 & 6_d & 7_a & 13_c & 0 & 1 \\
15 & 16 & 8 & 9 & 10_c & 11 & 12_e & 13_a & 1 & 2 & 3 & 4 & 5_b & 6 & 7 & 14_d & 0 \\
16 & 8 & 9 & 10_b & 11_d & 12_a & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7_c & 15_e
\end{array}\right) \tag{4.23}$$

In all remaining cases, each of the required k disjoint transversals can be found by taking the union of a transversal of A with an appropriate orbit of α in B. $\square$

Corollary 4.28. *For $n \geqslant 6$ and even $j \leqslant \frac{1}{3}n$, there exists a latin square of order n with a j-plex J that can be refined into parallel 2-plexes but J contains no odd plex.*

Proof. We choose L according to Lemma 4.26 and ensure that $A = \mathbb{Z}_j$. The condition $j \leqslant \frac{1}{3}n$ implies $j = a \leqslant (b - a) = n - 2j$. So D contains at least j orbits of α that include the b symbols of L that do not occur in A. Let K_B be the union of j such orbits. Then $J = A \cup K_B$ is a j-plex of L. Now $A = \mathbb{Z}_j$ has a 2-partition but has no odd plex, by Theorem 1.27. By combining a 2-plex of A with two orbits of α from K_B we obtain a 2-plex of L. Thus J can easily be refined into parallel 2-plexes. However it contains no odd plex, since A contains no odd plex. $\square$

In the proof of the next lemma, we construct, among other things, an $(n-2)$-plex with no odd plex in a latin square of order $n \equiv 2 \bmod 4$ that also possesses transversals.

Lemma 4.29. *For even integers $n \geqslant 6$ and $j \leqslant n$ such that $j \equiv 0 \bmod 4$, there exists a latin square L of order n such that*

1. *Any partition of L has at most j odd parts.*
2. *If $j' \equiv 0 \bmod 4$ and $0 \leqslant j' \leqslant j$, then L has an indivisible $(1^{j'}, 2^{(n-j')/2})$-partition.*

3. If j'' is even and $j'' \leqslant n - j$, then L has a j''-plex that contains no odd plex.

Proof. We prove the statement for members of the family $\mathcal{V}_{n,j}$, as defined in Section 2.4.2. We recall that, as specified for $\mathcal{V}_{n,j}$, $u = j/4$, $h = n/2$ and $U = \{0, 1, 2, \ldots, u-1\}$. Also, $|\Delta_*| = j$ and every element e in Δ_* has $\Delta(e) = h$. Thus, Part 1 of the lemma is shown by Lemma 2.1.

For each $v \in U$ and each $i \in \{0, 1, 2, 3\}$, we a transversal $T_{v,i}$ is given by

$$\mathrm{col}(x) = \begin{cases} \{x + 2v\} & \text{if } (i = 2 \text{ and } x = 0) \text{ or } (i = 3 \text{ and } x = h), \\ & \text{or } (i = 0 \text{ and } 0 < x < h) \text{ or } (i = 1 \text{ and } x > h) \\ \{x + 2v + 1\} & \text{if } (i = 0 \text{ and } x \geqslant h) \text{ or } (i = 1 \text{ and } x < h), \\ \{x + h + 2v\} & \text{if } (i = 0 \text{ and } x = 0) \text{ or } (i = 1 \text{ and } x = h) \\ & \text{or } (i = 2 \text{ and } 0 < x < h) \text{ or } (i = 3 \text{ and } x > h), \\ \{x + h + 2v + 1\} & \text{if } (i = 2 \text{ and } x \geqslant h) \text{ or } (i = 3 \text{ and } x < h). \end{cases}$$

It is routine to show that the collection $\{T_{v,i}\}$ for $v \in U$ and $i \in \{0, 1, 2, 3\}$ defines a $(1^j, n-j)$-partition. It should be clear that for a fixed $v \in U$, the union, $\bigcup_{i \in \{0,1,2,3\}} T_{v,i}$ is equivalent to the 4-plex given by

$$\mathrm{col}(x) = \{x + 2v,\ x + 2v + 1,\ x + h + 2v,\ x + h + 2v + 1\}. \tag{4.24}$$

In what follows, we will define 2-plexes which will either refine this 4-plex, or are parallel to it.

Let $H = \{0, 1, 2, \ldots, h-1\}$. For each $d \in H$, we show a 2-plex J_d, as follows. For $n \equiv 2 \bmod 4$,

$$\mathrm{col}(x) = \{x + d,\ x + h + d\}, \tag{4.25}$$

and for $n \equiv 0 \bmod 4$,

$$\mathrm{col}(x) = \begin{cases} \{x + d,\ x + h + d\} & \text{if } x < h - 1 \text{ or } x = n - 1, \\ \{x + d + (-1)^d,\ x + h + d + (-1)^d\} & \text{otherwise.} \end{cases} \tag{4.26}$$

We next show that J_d is indivisible. If $n \equiv 2 \bmod 4$, then J_d is the union of h intercalates and hence is indivisible. So suppose that $n \equiv 0 \bmod 4$ and let $q = n/4$. In the case when d is even, consider J_d in rows $n-2$, $n-1$, $q-1$ and $3q-2$. These entries lie outside of Δ_*. They are

$$\begin{array}{ll} (\,n-2,\ d-1,\ d-3\,), & (\,n-2,\ h+d-1,\ h+d-3\,), \\ (\,n-1,\ d-1,\ d-2\,), & (\,n-1,\ h+d-1,\ h+d-2\,), \\ (\,q-1,\ q+d-1,\ h+d-2\,), & (\,q-1,\ 3q+d-1,\ d-2\,), \\ (\,3q-2,\ q+d-1,\ d-3\,), & (\,3q-2,\ 3q+d-1,\ h+d-3\,). \end{array}$$

In any partition of these 8 triples into two sets, one of the sets will contain a pair of triples that agree in some coordinate. Hence J_d cannot be divided into parallel transversals. The case for odd d is similarly shown by using the triples in rows $h-2$, $h-1$, $q-2$ and $3q-1$. This proves that J_d is indivisible.

Comparing (4.24) with (4.25) and (4.26), shows that for a fixed $v \in U$,

$$J_{2v} \cup J_{2v+1} = \bigcup_{i \in \{0,1,2,3\}} T_{v,i}. \tag{4.27}$$

Proof of Part 2 and Part 3: For $j' \equiv 0 \bmod 4$ and $0 \leqslant j' \leqslant j$, an indivisible $(1^{j'}, 2^{(n-j')/2})$-partition is given by

$$\big\{T_{v,i} : v \in \{0,1,2,\dots,\tfrac{1}{4}j'-1\},\, i \in \{0,1,2,3\}\big\} \cup \big\{J_d : d \in \{\tfrac{1}{2}j', \tfrac{1}{2}j'+1, \dots, h-1\}\big\}.$$

The equivalence (4.27) implies that, as required, the plexes in the partition are parallel. Thus we have shown Part 2. For Part 3, set $j' = j$ in the $(1^{j'}, 2^{(n-j')/2})$-partition of Part 2. Since, by Part 1, this partition has at most j odd parts we form, by union of an appropriate number of 2-plexes from this partition, a j''-plex for each even $j'' \leqslant n-j$. This completes our proof of the lemma. □

Permitting $n = 4$ for the family $\mathcal{V}_{n,j}$ in the proof of Lemma 4.29 yields $\mathcal{V}_{4,0}$ and $\mathcal{V}_{4,4}$, the species of $\mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, respectively. The plexes and partitions given in (4.24), (4.26) and (4.27) work for $n = 4$ but Parts 2 and 3 of Lemma 4.29 fail for $\mathcal{V}_{4,4}$. As recorded in [76], and confirmed in the next section, every 2-plex in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is divisible.

Corollary 4.30. *For all $n \geqslant 5$ and*

$$k = \begin{cases} 1 & \text{if } n \equiv 1 \bmod 4, \\ 3 & \text{if } n \equiv 3 \bmod 4, \\ 2 & \text{if } n \text{ is even,} \end{cases}$$

there exists a latin square L of order n such that L has at least k disjoint transversals and L possesses an $(n-k)$-plex that contains no odd plex. If $n \not\equiv 2 \bmod 4$ then the $(n-k)$-plex occurs in a $(1^k, n-k)$-partition.

Proof. The case $n \equiv 2 \bmod 4$ is shown by Lemma 4.29. Theorem 4.27 shows every other case, except $(n,k) = (7,3)$. For the exception we rely on an example. Table 4.6 in the next section confirms the existence of a latin square of order 7 with an indivisible $(1^3, 4)$-partition. □

We leave open the possibility of an $(n-1)$-plex with no odd plex among the latin squares of order $n \equiv 3 \bmod 4$. If an example exists it must be of order at least 11 since, by computation, among the latin squares of order 7 every 6-plex contains an odd plex.

4.5 Latin squares of small order

The computational results in this section have been independently confirmed by Ian Wanless.

As mentioned in Section 1.14.1, several earlier reports in the literature consider plexes in small latin squares but the value of $\kappa(n)$ for small n has not been explicitly computed. We compute $\kappa(n)$ exactly for $n \leqslant 8$ and find that $\kappa(9) \in \{6, 7\}$. In the process we show that Conjecture 1.2 is true for the latin squares of order 9.

We summarise, in Section 4.5.2, the results of a complete enumeration of the plexes, indivisible plexes and indivisible partitions in all species representatives of order $n \leqslant 8$. Our computations for order 9 are later explained in Section 4.5.3. These computations were expensive. We did not keep a precise account of the computation time for this project but estimate that in total it was several decades.

To establish that a given k-plex was indivisible we checked, by exhaustive search, that it contained no smaller c-plex, for $1 \leqslant c \leqslant \frac{1}{2}k$. The author and Wanless independently wrote programs and ran them on a representative of each species for orders $n \in \{4, 5, 6, 7, 8\}$. We thus obtained two independent sets of raw data from which the tables are drawn. Information on the paratopy group size of each species representative was used in the final stage of calculations.

4.5.1 Definitions and explanation of tables

A *distinctive type* of indivisible partition is one that occurs in some, but not all species of order n. A *category* is a set of species that possess the same (distinctive) types of indivisible partitions. Tables in this section contain data on categories for each order $n \in \{6, 7, 8\}$. The marks in these tables are ✓ or ✗ depending on whether or not the specified indivisible partition type occurs. For each order $n \in \{6, 7, 8\}$ we will also present a table summarising enumerative data. The data under each of the column headings of these tables means as follows:

Column heading	Explanation
Plex/indiv. partition type	A short description of the object counted. "k-plex" means all k-plexes including any indivisible. "indiv k-plex" means only indivisible k-plexes. A type is always an indivisible partition type.
Exist num. of species	The number of species in which the object exists.
Maximum	The maximum, over all of the species of order n, of the count of the object in a species.
Minimum	Similarly, the minimum count over all species.
Non-zero minimum	Similarly, the minimum count over all species when restricted to species in which the object exists.
Mean species to 3 s.f.	Arithmetic mean to 3 significant figures where each species representative has equal weight.
Weighted mean to 3 s.f.	Similarly, where each species has weight proportional to the number of latin squares in the species. Equivalently, the expected value for a latin square of order n that is chosen uniformly at random.

Within all tables, notation for partition types is abbreviated by omitting both parentheses and separating commas. This is unambiguous since the order is less than 10.

For each order $n \in \{6, 7, 8\}$ we will table enumerative data for species representatives of all group tables. In these tables the entry in the group column may contain a mark '$*$'. This mark indicates that the specified group achieves the maximum value, over all species of order n, for the corresponding "Plex/indiv. partition type".

4.5.2 Species of order less than 9

We vary slightly from the general table layout for very small orders $n = 3, 4, 5$ to give a complete count for each species. In the trivial cases $n \in \{1, 2\}$, clearly $\kappa(n) = n$.

Data notes: 1 species of order $n = 3$. The unique species is of the group table of $\mathbb{Z}_3$. It contains exactly three 1-plexes and has a (unique) 1-partition. There are no indivisible 2-plexes so $\kappa(3) = 1$.

Data notes: 2 species of order $n = 4$. The two species are the groups $\mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Their data is in Table 4.1. The species of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_3$ are the only known species with no set of $\lfloor n/2 \rfloor$ parallel indivisible 2-plexes. Neither species of order 4 has an indivisible partition of type $(1^2, 2)$. The largest indivisible plex is a 2-plex, thus $\kappa(4) = 2$.

Data notes: 2 species of order $n = 5$. In Table 4.2 we present data on the unique group $\mathbb{Z}_5$ and the single other non-group species. The largest indivisible plex is a 3-plex found in the non-group table, hence $\kappa(5) = 3$. Neither species has an indivisible partition of type $(1^2, 3)$ or $(1^3, 2)$.

Table 4.1: Plexes and indivisible partitions in species of order $n = 4$.

Plex/indivisible partition type	Species $\mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
1-plex	0	8
2-plex	12	12
indiv 2-plex	12	0
2^2	6	0
1^4	0	2

Table 4.2: Plexes and indivisible partitions in species of order $n = 5$.

Plex/indivisible partition type	Species $\mathbb{Z}_5$	Non-group
1-plex	15	3
2-plex	130	30
indiv 2-plex	100	30
indiv 3-plex	0	12
23	0	12
12^2	150	9
1^5	3	0

Table 4.3: Categories of order $n = 6$.

Category	Distinctive indivisible partition type 24	3^2	123	$1^3 3$	$1^2 2^2$	$1^4 2$	Number of species
1	✓	✓	✓	✗	✓	✓	1
2	✓	✓	✓	✗	✓	✗	1
3	✓	✓	✗	✗	✗	✗	1
4	✗	✓	✓	✗	✓	✓	1
5	✗	✓	✓	✗	✓	✗	1
6	✗	✓	✗	✗	✗	✗	1
7	✗	✗	✓	✓	✓	✓	1
8	✗	✗	✓	✓	✗	✓	1
9	✗	✗	✗	✗	✗	✗	4

Data notes: 12 species of order $n = 6$. Integer partitions of 6 identify 11 types of indivisible partitions to consider. No species possesses a 1-partition (see Theorem 1.16) but every species does possess an indivisible 2-partition. No species of order 6 possesses an indivisible 5-plex and there are no indivisible $(1^2, 4)$-partitions. Thus 4 types of indivisible partition, $(6), (1, 5), (1^2, 4)$ and (1^6) never occur and 1 type is common to all species. We found that 6 types occur in some species but not all, so these are the distinctive types. In particular, we record that $\kappa(6) = 4$.

Considering the six distinctive types of indivisible partitions, we found as shown in Table 4.3, that the latin squares of order 6 fall into 9 different categories. The four species of category 9, including the two group tables; $\mathbb{Z}_6$ and D_6, possess no odd plexes and do not have an indivisible 4-plex. They are the only known species of latin squares of order $n > 2$ that possess only one type of indivisible partition. Three species, categories 1, 2 and 3, possess an indivisible 4-plex. Each of these three species has an indivisible $(2, 4)$-partition. The example (4.28) is in category 2.

Table 4.4: Plexes and indivisible partitions in species of order $n = 6$.

Plex/ indivisible partition type	Exist num. of species	Maximum	Minimum	Non-zero minimum	Mean species to 3 s.f.	Weighted mean to 3 s.f.
1-plex	6	32	0	8	7.33	6.86
2-plex	12	1539	239	239	490	303
3-plex	8	1792	0	512	523	615
indiv 2-plex	12	1539	195	195	459	289
indiv 3-plex	8	640	0	200	310	422
indiv 4-plex	3	72	0	12	8.67	12.4
24	3	72	0	12	8.67	12.4
3^2	6	320	0	88	90.7	141
123	6	696	0	144	147	158
2^3	12	5949	131	131	917	283
$1^3 3$	2	72	0	40	9.33	1.84
$1^2 2^2$	5	144	0	4	16.3	15.2
$1^4 2$	4	56	0	2	7.50	1.99

Table 4.5: Plexes and indivisible partitions in groups of order $n = 6$.

Plex/indivisible partition type	Species $\mathbb{Z}_6$	D_6
indiv 2-plex	1539∗	567
2^3	5949∗	945

Every species with a 3-plex has an indivisible 3-plex, however two of these species, categories 7 and 8, have no indivisible 3-partition. Indivisible partitions of type $(1^4, 2)$ and $(1, 2, 3)$ were found to occur if and only if a species has enough disjoint transversals. Enumerative data for the species of order $n = 6$ is summarised in Table 4.4. Enumerative data for the two species of group tables of order $n = 6$ is in Table 4.5.

Example order $n = 6$: an indivisible 4-plex. Shading identifies an indivisible $(2, 4)$-partition in (4.28). This latin square, from category 2, is the smallest member of the latin family $\mathcal{Q}_n$. Lemma 2.1 can be used to explain that the 4-plex is indivisible.

Table 4.6: Categories of order $n = 7$.

Category	25	34	124	1^34	1^223	1^43	1^32^2	1^52	1^7	Number of species
	Distinctive indivisible partition type									
1	✓	✓	✓	✗	✓	✗	✓	✗	✗	1
2	✗	✓	✓	✓	✓	✓	✓	✓	✓	2
3	✗	✓	✓	✓	✓	✓	✓	✓	✗	33
4	✗	✓	✓	✓	✓	✓	✓	✗	✗	62
5	✗	✓	✓	✓	✓	✗	✓	✗	✗	9
6	✗	✓	✓	✓	✓	✗	✗	✗	✗	2
7	✗	✓	✓	✗	✓	✓	✓	✓	✓	1
8	✗	✓	✓	✗	✓	✓	✓	✓	✗	10
9	✗	✓	✓	✗	✓	✓	✓	✗	✓	1
10	✗	✓	✓	✗	✓	✓	✓	✗	✗	6
11	✗	✓	✓	✗	✓	✗	✓	✗	✗	12
12	✗	✓	✓	✗	✓	✗	✗	✗	✗	5
13	✗	✓	✓	✗	✗	✗	✗	✗	✗	1
14	✗	✗	✓	✓	✓	✓	✓	✗	✓	1
15	✗	✗	✗	✗	✓	✓	✓	✓	✓	1

To see that the 2-plex is indivisible consider the entries marked $_*$.

$$\begin{pmatrix} 5 & 2 & 4 & 1 & 3 & 0 \\ 2 & 4 & 0 & 3_* & 5_* & 1 \\ 4 & 0 & 2 & 5_* & 1 & 3 \\ 1 & 3 & 5 & 0 & 2 & 4 \\ 3 & 5 & 1 & 4 & 0 & 2 \\ 0 & 1 & 3 & 2 & 4 & 5 \end{pmatrix} \tag{4.28}$$

Data notes: 147 species of order $n = 7$. There are 15 partition types to be checked. We found that no species has an indivisible partition of type $(7), (1, 6)$ or $(1^2, 5)$. Every species possesses an indivisible partition of type $(1, 3^2)$, $(2^2, 3)$ and $(1, 2^3)$. There are 9 distinctive types of indivisible partitions and these lead to the 15 different categories shown in Table 4.6. Precisely one species, category 1, has an indivisible $(2, 5)$-partition and it is illustrated below in (4.29). Hence, $\kappa(7) = 5$.

Evidently, an indivisible 5-plex of order 7 is rare, however all but one species possesses an indivisible 4-plex. The exception is the species of the unique group $\mathbb{Z}_7$ (category 15). All but two species possess an indivisible $(3, 4)$-partition. The exceptions are the Cayley table of $\mathbb{Z}_7$ and the Steiner quasigroup (see [40, p24]) of order 7 (category 14). In Table 4.7 we summarise enumerative data for order 7. In Table 4.8 we give data for the group $\mathbb{Z}_7$. Wanless [150] observed that $\mathbb{Z}_7$ achieves the maximum number of k-plexes for each k. We see by Table 4.8 that it achieves the maximum value for

Table 4.7: Plexes and indivisible partitions in species of order $n = 7$.

Plex/ indivisible partition type	Exist num. of species	Maximum	Minimum	Non-zero minimum	Mean species to 3 s.f.	Weighted mean to 3 s.f.
1-plex	147	133	3	3	23.0	20.4
2-plex	147	23184	2676	2676	3500	3330
3-plex	147	310198	42731	42731	47200	45100
indiv 2-plex	147	19404	2676	2676	3370	3240
indiv 3-plex	147	54880	17360	17360	35900	36400
indiv 4-plex	146	10284	0	328	3020	3160
indiv 5-plex	1	4	0	4	0.0272	0.0125
25	1	4	0	4	0.0272	0.0125
34	145	10212	0	136	2460	2600
124	146	2292	0	72	596	591
13^2	147	29351	5996	5996	13500	12900
2^23	147	566538	28728	28728	64200	62500
1^34	109	672	0	1	9.74	4.17
1^223	146	120540	0	192	5010	3290
12^3	147	976864	1442	1442	13100	5580
1^43	117	7546	0	1	74.1	14.7
1^32^2	139	189042	0	3	1630	205
1^52	47	9702	0	1	70.2	2.14
1^7	6	635	0	1	4.43	0.0202

Table 4.8: Plexes and indivisible partitions in group of order $n = 7$.

Plex/indivisible partition type	Species $\mathbb{Z}_7$
1-plex	133∗
2-plex	23184∗
3-plex	310198∗
indiv 2-plex	19404∗
indiv 3-plex	54880∗
13^2	29351∗
$2^2 3$	566538∗
$1^2 23$	120540∗
12^3	976864∗
$1^4 3$	7546∗
$1^3 2^2$	189042∗
$1^5 2$	9702∗
1^7	635∗

every type of indivisible plex and partition that is has.

Example order $n = 7$: an indivisible 5-plex. Shaded entries in the left hand example in (4.29) identify an indivisible $(2, 5)$-partition of the unique species of order 7 possessing an indivisible 5-plex. This species is in category 1. It is symmetric and has automorphism (34)(56). Together, these generate a paratopism group of order 4. It has exactly four indivisible $(2, 5)$-partitions, which comprise a single orbit of the paratopism group. It has 23 transversals, 1 subsquare of order 3 and 14 intercalates. On the right, the shading identifies an indivisible $(3, 4)$-partition in the same species.

$$\begin{pmatrix} 0 & \boxed{1} & 2 & \boxed{3} & 4 & 5 & 6 \\ \boxed{1} & 2 & \boxed{0} & 5 & 6 & 4 & 3 \\ 2 & 0 & 1 & 6 & \boxed{5} & \boxed{3} & 4 \\ 3 & 5 & \boxed{6} & \boxed{4} & 2 & 1 & 0 \\ \boxed{4} & \boxed{6} & 5 & 2 & 3 & 0 & 1 \\ 5 & 4 & 3 & 1 & \boxed{0} & 6 & \boxed{2} \\ 6 & 3 & 4 & 0 & 1 & \boxed{2} & \boxed{5} \end{pmatrix} \qquad \begin{pmatrix} \boxed{0} & \boxed{1} & \boxed{2} & 3 & 4 & 5 & 6 \\ \boxed{1} & 2 & 0 & \boxed{5} & \boxed{6} & 4 & 3 \\ 2 & 0 & 1 & \boxed{6} & \boxed{5} & \boxed{3} & 4 \\ 3 & \boxed{5} & 6 & \boxed{4} & 2 & 1 & \boxed{0} \\ 4 & 6 & 5 & 2 & \boxed{3} & \boxed{0} & \boxed{1} \\ 5 & \boxed{4} & \boxed{3} & 1 & 0 & 6 & \boxed{2} \\ \boxed{6} & 3 & \boxed{4} & 0 & 1 & \boxed{2} & 5 \end{pmatrix} \tag{4.29}$$

Data notes: 283657 species of order $n = 8$. There are 22 types to be considered. No species has an indivisible partition of type (8), $(1, 7)$, $(1^2, 6)$ or $(2, 6)$. The other 18 types are distinctive partition types. We found that 4600 species possess an indivisible 5-plex. Hence, $\kappa(8) = 5$.

The categories of order 8 are tabled in two parts according to whether or not the squares contain an indivisible 5-plex. Table 4.9 collates the 4600 species with an

Table 4.9: Categories of order $n = 8$ for species possessing an indivisible 5-plex.

Cat.	Distinctive indivisible partition type														Num. of
	35	125	134	1^35	1^224	1^23^2	12^23	1^44	1^323	1^22^3	1^53	1^42^2	1^62	1^8	species
1	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	2
2	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗	2
3	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	3
4	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	295
5	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗	216
6	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✗	✓	✗	✗	29
7	✓	✓	✓	✗	✓	✓	✓	✗	✓	✓	✓	✓	✗	✗	1
8	✓	✓	✓	✗	✓	✓	✓	✗	✓	✓	✗	✓	✓	✓	1
9	✓	✓	✓	✗	✓	✓	✓	✗	✓	✓	✗	✓	✗	✗	24
10	✓	✓	✓	✗	✓	✓	✓	✗	✓	✓	✗	✗	✗	✗	11
11	✓	✓	✓	✗	✓	✓	✓	✗	✗	✓	✗	✗	✗	✗	1
12	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	2
13	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗	1
14	✓	✗	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	26
15	✓	✗	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	2221
16	✓	✗	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗	1245
17	✓	✗	✓	✗	✓	✓	✓	✓	✓	✓	✗	✓	✗	✗	50
18	✓	✗	✓	✗	✓	✓	✓	✓	✓	✓	✗	✗	✗	✗	2
19	✓	✗	✓	✗	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	1
20	✓	✗	✓	✗	✓	✓	✓	✗	✓	✓	✓	✓	✗	✓	1
21	✓	✗	✓	✗	✓	✓	✓	✗	✓	✓	✓	✓	✗	✗	1
22	✓	✗	✓	✗	✓	✓	✓	✗	✓	✓	✗	✓	✓	✓	1
23	✓	✗	✓	✗	✓	✓	✓	✗	✓	✓	✗	✓	✗	✓	4
24	✓	✗	✓	✗	✓	✓	✓	✗	✓	✓	✗	✓	✗	✗	35
25	✓	✗	✓	✗	✓	✓	✓	✗	✓	✓	✗	✗	✗	✗	9
26	✓	✗	✓	✗	✓	✓	✓	✗	✓	✗	✗	✗	✗	✗	1
27	✓	✗	✓	✗	✓	✓	✓	✗	✗	✓	✗	✗	✗	✗	3
28	✓	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	6
29	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	2
30	✗	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	5
31	✗	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	250
32	✗	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗	142
33	✗	✓	✓	✗	✓	✓	✓	✓	✓	✓	✗	✓	✗	✗	2
34	✗	✓	✓	✗	✓	✓	✓	✗	✓	✓	✗	✓	✗	✗	3
35	✗	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	2

Table 4.10: Categories of order $n = 8$ for species with no indivisible 5-plex.

Cat.	Distinctive indivisible partition type														Num. of
	4^2	134	2^24	23^2	1^224	1^23^2	12^23	1^44	1^323	1^22^3	1^53	1^42^2	1^62	1^8	species
36	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	1866
37	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	206706
38	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	31
39	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗	69708
40	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	1
41	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✗	6
42	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✗	✗	408
43	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	41
44	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✗	15
45	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✗	✓	4
46	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✗	✗	14
47	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✗	✓	✓	✓	15
48	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✗	✓	✓	✗	4
49	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✗	✓	✗	✓	19
50	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✗	✓	✗	✗	159
51	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✗	✗	✗	✗	25
52	✓	✓	✓	✓	✓	✓	✓	✗	✗	✓	✗	✗	✗	✗	3
53	✓	✗	✓	✓	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	26
54	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	1
55	✗	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	2
56	✗	✗	✓	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	1
57	✗	✗	✗	✓	✗	✓	✓	✗	✓	✓	✓	✓	✓	✓	2

Table 4.11: Plexes and indivisible partitions in species of order $n = 8$.

Plex/ indivisible partition type	Exist num. of species	Maximum	Minimum	Non-zero minimum	Mean species to 3 s.f.	Weighted mean to 3 s.f.
1-plex	283624	384	0	8	61.1	61.0
2-plex	283657	465976	23592	23592	57600	57500
3-plex	283656	28979840	0	2211280	3630000	3630000
4-plex	283657	114209676	10890348	10890348	14300000	14300000
indiv 2-plex	283657	460096	23457	23457	56800	56800
indiv 3-plex	283656	14388224	0	1908032	3170000	3170000
indiv 4-plex	283655	10098036	0	5376	1410000	1410000
indiv 5-plex	4600	19168	0	1	0.265	0.137
35	4194	18356	0	1	0.240	0.123
4^2	283651	2679882	0	48	67100	66800
125	989	828	0	1	0.0261	0.0149
134	283622	4809618	0	4608	812000	812000
2^24	283655	9249292	0	25344	2380000	2380000
23^2	283656	1901046272	0	34296746	89000000	88800000
1^35	11	2	0	1	0.0000458	0.0000356
1^224	283622	500568	0	1600	61100	61100
1^23^2	283624	70983936	0	36232	1150000	1140000
12^23	283624	2101675776	0	1728990	13200000	13100000
2^4	283657	3402475552	372045	372045	6450000	6350000
1^44	283224	8646	0	1	130	130
1^323	283617	91225856	0	768	116000	113000
1^22^3	283623	736603392	0	7710	339000	326000
1^53	282808	609792	0	1	165	148
1^42^2	283569	36365952	0	2	2740	2150
1^62	211470	634368	0	1	14.2	4.12
1^8	2024	70272	0	1	0.930	0.0147

indivisible 5-plex. Each of these species possesses an indivisible partition of types (4^2), $(2^2, 4)$, $(2, 3^2)$ and (2^4).

In Table 4.10 we report on the majority; that is the 279057 species with no indivisible 5-plex. All of these species possess an indivisible (2^4)-partition. Category 37 shows that most species do not have an indivisible 5-plex or a 1-partition but do have an indivisible partition for every type not ruled out by the aforementioned constraints. All but two species possess an indivisible 4-plex. The exceptions are the species of the group tables of $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ which form category 57. All but one species possesses an indivisible 3-plex. The exception, category 56, is the species of $\mathbb{Z}_8$ which has no odd plexes by Theorem 1.27. It is unique with respect to this property. In Table 4.11 we summarise the enumerative data for order $n = 8$.

Example order $n = 8$: Species with most indivisible 5-plexes. The mean number of indivisible 5-plexes in Table 4.11 reflects that, at this order, indivisible 5-plexes are uncommon. Indeed, some 2783 of the 4600 species possessing an indivisible 5-plex have only one of them. In this sense, the latin square in (4.30), of category 9, is extreme. It uniquely has the maximum number, that is 19168 indivisible 5-plexes. On the left, one indivisible $(3, 5)$-partition is marked. On the right, the 4 intercalates marked do not intersect any of the 16 transversals in the square.

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ 2 & 3 & 1 & 4 & 6 & 7 & 5 & 0 \\ 3 & 2 & 4 & 1 & 7 & 6 & 0 & 5 \\ 4 & 5 & 6 & 7 & 2 & 0 & 1 & 3 \\ 5 & 4 & 7 & 6 & 0 & 2 & 3 & 1 \\ 6 & 7 & 5 & 0 & 1 & 3 & 2 & 4 \\ 7 & 6 & 0 & 5 & 3 & 1 & 4 & 2 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ 2 & 3 & 1 & 4 & 6 & 7 & 5 & 0 \\ 3 & 2 & 4 & 1 & 7 & 6 & 0 & 5 \\ 4 & 5 & 6 & 7 & 2 & 0 & 1 & 3 \\ 5 & 4 & 7 & 6 & 0 & 2 & 3 & 1 \\ 6 & 7 & 5 & 0 & 1 & 3 & 2 & 4 \\ 7 & 6 & 0 & 5 & 3 & 1 & 4 & 2 \end{pmatrix} \tag{4.30}$$

Group tables of order $n = 8$. In Table 4.12, we present data on the number of plexes, indivisible plexes and indivisible partitions for the five group tables of order $n = 8$. None of the five species possesses an indivisible 5-plex or an indivisible $(1^4, 4)$-partition. The Abelian groups in Table 4.12 account for many of the maximum values recorded in Table 4.11. The absence of odd plexes in $\mathbb{Z}_8$ is explained by Theorem 1.27. Each of the non-cyclic groups of order 8 has 384 transversals, as explained by Bedford and Whitaker [11]. Theorem 1.59, by Balasubramanian, states that a latin square of even order has an even number of transversals. However, as mentioned in Section 1.14.2, Balasubramanian's statement does not generalise in an obvious way to k-plexes. Nevertheless, it is striking that every value in Table 4.12 is an even number. This is explained by the next result, due to Ian Wanless.

Lemma 4.31. *For integers $n \geqslant k \geqslant 1$, let m be the greatest common divisor of n and k. Let L be the Cayley table of a group $(G, \star)$ of order n. The number of k plexes in L is a multiple of n/m. Likewise, the number of indivisible k-plexes in L is a multiple of n/m.*

Table 4.12: Plexes and indivisible partitions in groups of order $n = 8$.

Plex/indivisible partition type	Group table $\mathbb{Z}_8$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	D_8	Q_8
1-plex	0	384*	384*	384*	384*
2-plex	460096	462120	465976*	240008	236744
3-plex	0	28979840*	28907648	14590848	14625792
4-plex	113925996	114021324	114209676*	57461196	57353676
indiv 2-plex	460096*	432744	435736	210344	207272
indiv 3-plex	0	11326720	11203584	5707008	5831424
indiv 4-plex	28416	0	0	6144	6912
4^2	0	0	0	768	0
134	0	0	0	6144	9216
2^24	420096	0	0	26112	32256
23^2	0	1135096064	1143668736	278490880	283778304
1^224	0	0	0	4608	4608
1^23^2	0	70983936*	66060288	36537600	39034368
12^23	0	2101675776*	2063308800	499349760	506898432
2^4	3402475552*	2717863186	2906655710	333162978	303664290
1^323	0	91225856*	83091456	41943040	42633216
1^22^3	0	719491968	736603392*	169825152	157148160
1^53	0	569856	344064	555264	609024
1^42^2	0	28231296	36365952*	15490560	13766784
1^62	0	272384	634368*	373376	233472
1^8	0	23040	70272*	33408	32256

Proof. Let Γ be the group of right translations acting on the k-plexes of L by $K \mapsto \{(r, c \star g, s \star g) : (r, c, s) \in K\}$ for each $g \in G$. Let K be a k-plex of L and Γ_K the stabiliser of K under the action of Γ. Then Γ_K acts semi-regularly on the k entries in the first row of K, which means $|\Gamma_K|$ divides k. The orbit of K under Γ has cardinality $n/|\Gamma_K|$, so $|\Gamma_K|$ divides n and hence also divides m. As this is true for arbitrary K, the total number of k-plexes in L must be a multiple of n/m. The same is true for indivisible plexes since Γ preserves indivisibility of plexes. $\square$

4.5.3 Species of order 9

Data: 19270853541 species of order $n = 9$. As mentioned in Section 1.14.1, we used a program described in [119] to generate isotopy class representatives of order 9. This program was also used in [118] in confirming Conjecture 1.1 for order 9, thereby showing that $\kappa(9) < 9$. We found, in the following manner, that $\kappa(9) < 8$. First, we obtained a list of isotopy class representatives which possess a transversal that has no parallel transversal. In other words, we identified all isotopy classes with a $(1, 8)$-partition in which the 8-plex contains no transversal. This list reduced to a list of 36007 species representatives, which we shall refer to as Ω. Subsequently, we established that for each species in Ω, a 2-plex is contained in those 8-plexes with no transversal, thereby confirming that $\kappa(9) < 8$. As we have just mentioned, each species of order 9 has either 2 parallel transversals or possesses a 2-plex. Hence, Conjecture 1.2 is true for order 9.

Unfortunately, we were unable to perform an exhaustive search for an indivisible 7-plex. In a small sample, we found that many species have well over a million 2-plexes so it is hard to test a lot of species. Although finding all 2-plexes by our programs is not viable, we did investigate the possibility of an indivisible $(1^2, 7)$-partition, a much easier computation. The computation was made a little more efficient by adjusting the latin square generator to output species representatives, rather than isotopy representatives. The adjustment was made by Brendan McKay who also contributed to a program designed to record species representatives with a $(1^2, 7)$-partition in which the 7-plex contains no transversal. We thus recorded, in a list which we call Ψ, precisely 10270 such species representatives. Exactly 2093 species representatives appear in both Ψ and Ω.

Among the species in Ω and Ψ we found that there are no indivisible 7-plexes. We did not search all of them for an indivisible 6-plex. However, we found one species in Ω that has two indivisible 6-plexes. Thus, $\kappa(9) \in \{6, 7\}$. We also conclude that if $\kappa(9) \neq 6$, then there exists an indivisible $(2, 7)$-partition.

Example order $n = 9$: a species of Ω with indivisible 6-plexes. We illustrate in (4.31) the two, slightly different, indivisible 6-plexes which prove that $\kappa(9) \geqslant 6$. In each case, the 3-plex marked divides in two different ways to give indivisible $(1, 2, 6)$-partitions. The latin square has a trivial paratopy group so the plexes are in different orbits.

In both cases, equation (4.21) shows that the 6-plex has no odd plexes. (The bold entries comprise the set D, as defined in Section 4.4.) We leave open explanation of the absence of even plexes in these 6-plexes.

$$\left(\begin{array}{cccc|ccccc}
0 & 1 & 2 & 3 & 7 & 6 & 5 & 8 & 4\\
1 & 0 & 3 & 2 & 6 & 5 & 4 & 7 & 8\\
2 & 3 & 0 & 1 & 4 & 8 & 7 & 5 & 6\\
3 & 2 & 1 & 0 & 8 & 7 & 6 & 4 & 5\\
\hline
4 & 6 & 7 & 8 & \mathbf{5} & 3 & 0 & 2 & 1\\
5 & 8 & 6 & 7 & 2 & \mathbf{4} & 1 & 3 & 0\\
6 & 7 & 5 & 4 & 3 & 1 & \mathbf{8} & 0 & 2\\
7 & 4 & 8 & 5 & 1 & 0 & 2 & \mathbf{6} & 3\\
8 & 5 & 4 & 6 & 0 & 2 & 3 & 1 & \mathbf{7}
\end{array}\right)
\qquad
\left(\begin{array}{cccc|ccccc}
0 & 1 & 2 & 3 & 7 & 6 & 5 & 8 & 4\\
1 & 0 & 3 & 2 & 6 & 5 & 4 & 7 & 8\\
2 & 3 & 0 & 1 & 4 & 8 & 7 & 5 & 6\\
3 & 2 & 1 & 0 & 8 & 7 & 6 & 4 & 5\\
\hline
4 & 6 & 7 & 8 & \mathbf{5} & 3 & 0 & 2 & 1\\
5 & 8 & 6 & 7 & 2 & \mathbf{4} & 1 & 3 & 0\\
6 & 7 & 5 & 4 & 3 & 1 & \mathbf{8} & 0 & 2\\
7 & 4 & 8 & 5 & 1 & 0 & 2 & \mathbf{6} & 3\\
8 & 5 & 4 & 6 & 0 & 2 & 3 & 1 & \mathbf{7}
\end{array}\right)
\tag{4.31}$$

The subsquare of order 4 in L of (4.31) is the species of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ so it has 8 distinct transversals. We can form a transversal, $T = T_A \cup D$, where T_A is a transversal in $A = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and D is the set marked by the bold entries of L. Then, $L \setminus T$ contains no odd plex by (4.21). We discovered that each species of Ω has similar structure.

Lemma 4.32. *If L is a latin square of order* 9*, then the following conditions are equivalent.*

1. *L has a transversal that has no parallel transversal;*
2. *L has a subsquare that is paratopic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$;*
3. *L has exactly 8 transversals that have no parallel transversal and each of them is the union of a transversal of a subsquare A, of order* 4*, and its corresponding set D, as defined in Section* 4.4*;*
4. *L has an* 8*-plex that contains no odd plex;*
5. *L belongs to one of the* 36007 *species in Ω.*

Proof. Direct computation showed that $5 \implies 3$. There is only one species of order four that has transversals, so $3 \implies 2$. The implication $2 \implies 4$ follows by (4.21). The fact that $4 \implies 1$ is immediate. Likewise $1 \implies 5$, by the definition of Ω. $\square$

As an independent check of our computations, we built all species of order 9 with a subsquare paratopic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and confirmed that there are 36007 such species. It would be interesting to know whether the behaviour seen in Lemma 4.32 is a one-off, or whether analogous statements hold for some larger $n \equiv 1 \bmod 4$.

Data notes: 6 species of interest of order $n = 9$. For a small selection of species of order 9, as next listed, we present in Table 4.13 the number of plexes and indivisible plexes, and note the existence or otherwise of indivisible partitions. The indivisible partitions with two parts are counted.

Species	**Source: interest**
$\mathbb{Z}_9$	: species of group table.
$\mathbb{Z}_3 \oplus \mathbb{Z}_3$	: species of group table.
A	(4.31) : the single species thus far identified with an indivisible 6-plex.
B	[118, p281] : unique species with the minimum number of transversals.
C	[118, p281] : unique species with the maximum number of intercalates.
D	[155] : of an infinite family $\mathcal{W}_n$ which has at most $n-3$ disjoint transversals.

None of the six species possess an indivisible 7-plex or an indivisible partition of type $(3,6)$, $(1^3,6)$ or $(1^4,5)$. The types (9), $(1,8)$ and $(1^2,7)$ were ruled out by the earlier mentioned search of all species of order 9. All of the six species possess an indivisible partition of type $(2,3,4)$, (3^3), $(1^2,3,4)$, $(1,2^2,4)$, $(1,2,3^2)$, $(2^3,3)$, $(1^3,2,4)$, $(1^3,3^2)$ $(1^2,2^2,3)$, $(1,2^4)$, $(1^4,2,3)$, $(1^3,2^3)$ and $(1^5,2^2)$. We omit from Table 4.13 such partition types for which data is common to all six species.

Table 4.13: Plexes and indivisible partitions in six species of order $n = 9$.

Plex/indivisible partition type	Species $\mathbb{Z}_9$	$\mathbb{Z}_3 \oplus \mathbb{Z}_3$	A	B	C	D
1-plex	2025	2241	176	68	801	415
2-plex	11312217	11299068	1361837	1266808	1328760	983984
3-plex	3805235823	3805353948	416827284	422663812	419501388	383230707
4-plex	67338677862	67339012014	7490644777	7482295830	7492688982	7920057672
indiv 2-plex	10517040	10331712	1356971	1266052	1212120	959909
indiv 3-plex	1572440364	1439897904	383502764	411021388	300228264	336935606
indiv 4-plex	12636	23328	922080208	1320329954	314944938	1944551389
indiv 5-plex	✗	✗	1862	✗	✗	14829
indiv 6-plex	✗	✗	2	✗	✗	✗
45	✗	✗	69	✗	✗	3558
126	✗	✗	4	✗	✗	✗
135	✗	✗	✓	✗	✗	✓
14^2	✗	✗	✓	✓	✓	✓
$2^2 5$	✗	✗	✓	✗	✗	✓
$1^2 25$	✗	✗	✓	✗	✗	✓
$1^5 4$	✗	✗	✓	✓	✓	✓
$1^6 3$	✓	✓	✗	✗	✓	✓
$1^7 2$	✓	✓	✗	✗	✓	✗
1^9	✓	✓	✗	✗	✓	✗

Table 4.13 shows that the six species are of different categories and identifies 10 types of distinctive partitions. In Section 5.4 of the next chapter we identify species of order 9 possessing at most 3 disjoint transversals. Thus, partition types $(1^5, 2^2)$ and $(1^4, 2, 3)$ common to the species of Table 4.13, are distinctive. Type $(1^4, 5)$ is also distinctive as we found examples among the species in Ω. In summary, of the 30 types of indivisible partitions to consider, at least 24 types are known to occur and we confirm that at least 13 of those are distinctive types. At least 3 types never occur: (9), $(1, 8)$ and $(1^2, 7)$.

We leave open the possibility of a latin square of order 9 with any one of the following properties:

1. has an indivisible 7-plex, thus an indivisible $(2, 7)$-partition,
2. has an indivisible partition of type $(3, 6)$ or $(1^3, 6)$,
3. has no indivisible 4-plex,
4. has no indivisible partition of any of the following 11 types:
   $(2, 3, 4)$, (3^3), $(1^2, 3, 4)$, $(1, 2^2, 4)$, $(1, 2, 3^2)$, $(2^3, 3)$,
   $(1^3, 2, 4)$, $(1^3, 3^2)$, $(1^2, 2^2, 3)$, $(1, 2^4)$ or $(1^3, 2^3)$.

In (4.32) we show an example of the two species of Table 4.13 that possess an indivisible (4, 5)-partition. On the left is species A and on the right is species D.

$$
\left(\begin{array}{cccc|ccccc}
0 & 1 & 2 & 3 & 7 & 6 & 5 & 8 & 4 \\
1 & 0 & 3 & 2 & 6 & 5 & 4 & 7 & 8 \\
2 & 3 & 0 & 1 & 4 & 8 & 7 & 5 & 6 \\
3 & 2 & 1 & 0 & 8 & 7 & 6 & 4 & 5 \\
\hline
4 & 6 & 7 & 8 & 5 & 3 & 0 & 2 & 1 \\
5 & 8 & 6 & 7 & 2 & 4 & 1 & 3 & 0 \\
6 & 7 & 5 & 4 & 3 & 1 & 8 & 0 & 2 \\
7 & 4 & 8 & 5 & 1 & 0 & 2 & 6 & 3 \\
8 & 5 & 4 & 6 & 0 & 2 & 3 & 1 & 7
\end{array}\right)
\qquad
\left(\begin{array}{ccccccccc}
1 & 0 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
3 & 2 & 1 & 4 & 5 & 6 & 7 & 8 & 0 \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 \\
0 & 4 & 3 & 6 & 7 & 8 & 5 & 1 & 2 \\
4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 & 3 \\
5 & 6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 \\
6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 \\
7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
8 & 1 & 5 & 2 & 3 & 4 & 0 & 6 & 7
\end{array}\right)
\tag{4.32}
$$

In Table 4.14 we summarise our results for $\kappa(n)$ for the small orders $n \leqslant 11$. The values $\kappa(10)$ and $\kappa(11)$ are justified by Theorem 4.2 and (4.19), respectively.

Table 4.14: $\kappa(n)$ for $n \leqslant 11$

Order n	1	2	3	4	5	6	7	8	9	10	11
$\kappa(n)$	1	2	1	2	3	4	5	5	6 or 7	$\geqslant 5$	$\geqslant 6$

4.6 Concluding remarks

The data of the last section offers a solid resource for testing and further development of general results about existence of plexes and partitions. As mentioned in Section 1.14, very little theory exists on the number of plexes in latin squares. We mentioned in that section the known general results for $k = 1$. For $k > 1$, we know of no general results other than Lemma 4.31 about the number of k-plexes; indivisible or otherwise.

A frustrating open problem is whether or not every latin square of order $n > 2$ contains a non-trivial plex. We emphasise this important open problem, alternatively expressed:

Problem 4.33. Is $\kappa(n) < n$, for all $n > 2$?

A proof of Conjecture 1.2 would resolve Problem 4.33.

In the next chapter we will report on computations proving that Conjecture 1.3 is true for order $n \leqslant 9$. The data on indivisible partitions favours the next, slightly stronger, statement.

Conjecture 4.34. *Every latin square of order $n > 3$ has $\lfloor n/2 \rfloor$ parallel* indivisible *2-plexes.*

Our computations show that few species of order $n \leqslant 8$ fail to possess an indivisible k-plex or an indivisible $(k, n-k)$-partition for $k \in \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$. However, for any strictly larger k the incidence of indivisible k-plexes is much rarer. For order $n = 9$, we have not tested enough species to address the likelihood of an indivisible $\lceil \frac{n}{2} \rceil$-plex. We leave open whether or not the family $\mathcal{H}_n$, of Section 4.3.2, possesses an indivisible $\lceil \frac{n}{2} \rceil$-plex for all odd $n > 3$. We conjecture on a problem first raised in [65] and which remains open for odd $n \geqslant 13$. Theorem 4.3 answers the case for even $n \geqslant 4$.

Conjecture 4.35. *For each $n \geqslant 4$, there exists a latin square of order n with an indivisible $(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$-partition.*

The next statement is similar in expression to Problem 3.24.

Problem 4.36. For integers $a < b < c$, does there exist a latin square L with an indivisible a-plex, an indivisible c-plex and a divisible b-plex, but L has no indivisible b-plex?

In contrast to Problem 3.24, Conjecture 1.3 does not immediately imply a parity condition, on either the order of L or on the integers $\{a, b, c\}$, of an example meeting the conditions of Problem 4.36. Our data shows that no example exists among latin squares of order $n \leqslant 8$.

The techniques of Section 4.2.2 yield many different types of indivisible partitions in larger latin squares. The results of that section leave open the following special case of the problem of Conjecture 1.39.

Problem 4.37. For $4k < n < 5k$, can a k-protoplex of order $2k$ always be extended to a k-plex of order n?

Avoidance of odd plexes in large parts was explained in Section 4.4. Thus, we understand the absence of odd plexes in the indivisible 6-plexes (4.31). We leave open explanation of why they contain no even plex.

Further study of indivisible plexes in $\mathbb{Z}_n$ is due. Our small theoretic contribution in Section 4.3.3 does not explain larger indivisible plexes found in $\mathbb{Z}_n$ for $n \leqslant 9$. Our data confirms an observation in [150] that the commutative groups tend to have many plexes. Therefore, the absence of indivisible k-plexes, for $k > n/2$, in those Cayley tables is not so surprising. As a benchmark, if indivisible k-plexes occur in $\mathbb{Z}_n$ then we might reasonably expect other latin squares of order n to also contain indivisible k-plexes.

Whenever k properly divides n, Theorem 4.1 confirms the existence of an indivisible k-partition in some latin square of order n. Few latin squares of small composite order fail to contain such a partition so it would be of interest to identify conditions under which such failure occurs, particularly for odd orders. By Theorem 1.25, examples exist if k is odd and n is even. Also, in Chapter 3 we saw how the number of odd plexes in a partition can be constrained. A search, of a small sample (100000) of randomly generated latin squares of order 9, found none that fail to contain an indivisible 3-partition. It would be useful to survey the types of indivisible partitions in a larger sample of order 9 and in latin squares of larger order.

In the next chapter, on latin squares with restricted transversals, we will see further behaviour that is not evident among latin squares of small order and their partitions with parts of size 1.

For a given latin square L, define $K(L)$ to be the maximum k such that L has an indivisible k-plex. For large n, can we expect that almost all latin squares of order n will possess almost all types of indivisible partitions other than those types with part sizes near to, or exceeding, $K(L)$? Among the species of small order, we have not seen a huge variation in $K(L)$. This suggests the following problem.

Problem 4.38. Consider, for fixed but arbitrary n,

$$\min\big\{K(L)\ :\ L \text{ is a latin square of order } n\big\}.$$

How does this minimum value of $K(L)$ behave for larger n?

Chapter 5

Latin squares with restricted transversals

5.1 Introduction

In this chapter we consider several ways in which transversals in latin squares can be restricted.

For a given latin square L of order n, we define:

$$\lambda(L) = \max\big\{\, m : \ L \text{ possesses a } (1^m, n-m)\text{-partition}\big\}.$$

Lemma 4.29 showed the following.

Lemma 5.1. *For each even $n \geqslant 6$ and each integer $j \equiv 0 \bmod 4$, there exists a latin square $L = \mathcal{V}_{n,j}$ such that $\lambda(L) = j$.*

In Section 5.3 we prove our main result which is Theorem 5.2. It proves the existence of arbitrarily large latin squares with $\lambda(L) = 1$.

Theorem 5.2. *For all even $n \geqslant 10$, except perhaps if n is a power of 2, there exists a latin square of order n that has a transversal but every transversal coincides on a single entry.*

The possibility of an infinite family of latin squares of odd order with a constant λ value remains an open question.

As mentioned in Section 1.8, a confirmed bachelor square is a latin square with at least one transversal-free entry. A corollary of Theorem 5.2 is a new construction of confirmed bachelor latin squares for orders $n \equiv 3 \bmod 4$ when $n \geqslant 11$. Our construction of latin squares with at least one transversal-free entry is a general method. It is described in Section 5.3.

We define for all positive integers n:

$$\mu(n) = \min\big\{ \lambda(L) : L \text{ is a latin square of order } n \big\}.$$

Clearly $\mu(n) = 0$ for all even n, thus we are concerned with the case that n is odd. If Conjecture 1.1 is true, then $\mu(n) \geqslant 1$ for all odd n. A trivial upper bound, $\mu(n) \leqslant n-2$, follows immediately from the existence of a bachelor latin square of order n. However, a better bound can be derived by considering the number of disjoint transversals in two such constructions in the literature.

Similar to earlier discussion in Section 4.4, the next theorem follows from the reasoning of Mann [117]. Let t be a positive integer.

Theorem 5.3. *Assume that L is a latin square of order $4t+1$ such that L contains a subsquare of order $2t$. Then $\lambda(L) \leqslant 2t+1$.*

Theorem 5.3 is useful in Section 5.4 when we consider our data for order 9. For orders $n = 4t+3$, the reasoning of Evans [73] also gives the result that $\lambda \leqslant 2t+1$ for his construction. Together, these results by Mann and Evans justify the following.

Theorem 5.4. *If n is odd and $n > 3$, then $\mu(n) \leqslant \frac{1}{2}(n+1)$.*

We know of no general argument that improves on the bound given by Theorem 5.4, although it is not tight in the cases where μ is known. In Section 5.4 we report on computations proving that $\mu(9) = 3$. We found that almost all species of order 9 possess a λ value of at least 5. These computations, as we will explain, also prove Conjecture 1.3 for the latin squares of order $n \leqslant 9$. For order $n \in \{5, 7\}$ there is a latin square whose transversals coincide on one entry, hence $\mu(5) = \mu(7) = 1$. For $n \in \{1, 3\}$ we have $\mu(n) = n$.

In Section 5.4 we also report on computations concerning the number of transversal-free entries in the species of order $n \leqslant 9$. Our data supports the view that almost all large latin squares have no transversal-free entries.

For odd $n > 3$, the existence of a confirmed bachelor latin square of order n as stated in Theorem 1.23 was proved in [155] using the family $\mathcal{W}_n$. They did not name the family; it is a convenience introduced here. The definition of $\mathcal{W}_n$ is included in Section 2.5.2. As observed in [155], the 3 elements of $\Delta_* \cap r_{n-1} \subset \mathcal{W}_n$ are transversal-free.

We note that for $n \geqslant 9$, as displayed in (2.28), there are 7 elements in $\Delta_* \backslash (\Delta_2 \cup \Delta_{-2}) \subset \mathcal{W}_n$. It follows from Lemma 2.1 that, for $n \geqslant 9$, all 7 of these elements are transversal-free. This proves the following.

Theorem 5.5. *For all odd $n \geqslant 9$, there exists a latin square of order n with at least 7 transversal-free entries.*

We recorded in Theorem 4.23 that the family $\mathcal{H}_n$ has at least two transversal-free entries.

In Section 5.2 we prove the following theorem using an infinite family in which the proportion of transversal-free entries is asymptotic to one ninth. Theorem 5.6 is due to Ian Wanless.

Theorem 5.6. *For all odd* $m \geqslant 3$*, there exists a latin square of order* $3m$ *that contains an* $(m-1) \times m$ *latin subrectangle consisting of transversal-free entries.*

5.2 Transversals in $\mathcal{D}_{3m}$

In this section we prove Theorem 5.6 using the family $\mathcal{D}_{3m}$ which is defined in Chapter 2 for all odd $m \geqslant 3$.

All results in this section, including the construction of the family $\mathcal{D}_{3m}$, are the work of Ian Wanless. The family $\mathcal{D}_{3m}$ is a generalisation of an interesting example of order 9 which was obtained by the data project reported in the next section.

Suppose that T is a transversal of $\mathcal{D}_{3m}$. We define x_{ij} to be the number of elements of T of the form $\big((i,b),(j,d),(e,f)\big)$ where b, d and f are arbitrary and $e = i + j$ in $\mathbb{Z}_3$. We define y_{ij} to be the number of elements of T of the same form, but where $e \neq i + j$ in $\mathbb{Z}_3$. Finally, we let z be the number of elements of T of the form $\big((0,1),(0,d),(0,f)\big)$ where d and f are arbitrary. A number of constraints are immediate from the definition of $\mathcal{D}_{3,m}$. We will make repeated implicit use of the bounds $0 \leqslant x_{ij} \leqslant m$, $0 \leqslant y_{ij} \leqslant 1$, $0 \leqslant z \leqslant 1$ and the fact that $y_{02} = y_{11} = y_{20} = 0$. Moreover, the construction of $\mathcal{D}_{3m}$ forces

$$y_{00} + y_{01} \leqslant 1 \qquad y_{10} + y_{12} \leqslant 1 \qquad y_{21} + y_{22} \leqslant 1, \tag{5.1}$$

$$0 \leqslant x_{00} - z \leqslant m - 2. \tag{5.2}$$

Also, the need for T to include one representative from each row, column and symbol of $\mathcal{D}_{3m}$ implies

$$x_{00} + x_{01} + x_{02} + y_{00} + y_{01} = m, \tag{5.3}$$

$$x_{20} + x_{21} + x_{22} + y_{21} + y_{22} = m, \tag{5.4}$$

$$x_{00} + x_{10} + x_{20} + y_{00} + y_{10} = m, \tag{5.5}$$

$$x_{02} + x_{12} + x_{22} + y_{12} + y_{22} = m, \tag{5.6}$$

$$x_{00} + x_{12} + x_{21} + y_{01} + y_{10} + y_{22} = m, \tag{5.7}$$

$$x_{01} + x_{10} + x_{22} + y_{00} + y_{12} + y_{21} = m. \tag{5.8}$$

Adding (5.3), (5.5) and (5.7), then subtracting (5.4), (5.6) and (5.8), gives

$$3x_{00} - 3x_{22} + y_{00} + 2y_{01} + 2y_{10} - 2y_{12} - 2y_{21} - y_{22} = 0. \tag{5.9}$$

Moreover, Lemma 2.1 necessitates that

$$3 \mid y_{00} + 2y_{01} + 2y_{10} + y_{12} + y_{21} + 2y_{22}, \tag{5.10}$$
$$m \mid x_{22} + y_{10} + y_{12} - z. \tag{5.11}$$

The above restrictions are enough to show that certain entries in $\mathcal{D}_{3m}$ are not in any transversal.

Lemma 5.7. *No transversal of* $\mathcal{D}_{3m}$ *includes an entry in row* $(1, 0)$ *and column* $(0, d)$, *where* d *is arbitrary.*

Proof. We are required to show $y_{10} = 0$, so assume for the sake of contradiction that $y_{10} = 1$. By (5.1) it follows that $y_{12} = 0$.

First suppose that $z = 1$.

From (5.11), $x_{22} = 0$. To satisfy (5.10) and (5.1) the only possibilities are

(i) $y_{01} = y_{22} = 1$, $y_{00} = y_{21} = 0$,

(ii) $y_{00} = 1$, $y_{01} = y_{21} = y_{22} = 0$,

(iii) $y_{21} = 1$, $y_{00} = y_{01} = y_{22} = 0$.

In all three cases x_{00} may be calculated from (5.9) but its value violates (5.2).

We conclude that $z = 0$, and hence $x_{22} = m - 1$ from (5.11). Again to satisfy (5.10) and (5.1) we must have (i), (ii) or (iii). However, (i) and (5.9) imply $x_{00} = m - 2$, which together with (5.7) gives the contradiction $x_{12} + x_{21} < 0$.

Similarly, (ii) and (5.9) imply $x_{00} = m - 2$, which with (5.5) leads to $x_{20} = 0$. However, this is impossible since having $x_{20} = y_{21} = y_{22} = 0$ prevents the transversal from including any entry in row $(2, 0)$.

Finally, (iii) and (5.9) imply $x_{00} = m - 1$ which breaches (5.2). We have exhausted all possibilities, and are forced to conclude that $y_{10} = 0$ as required. □

Lemma 5.8. *No transversal of* $\mathcal{D}_{3m}$ *includes an entry in row* $(0, b)$ *and column* $(0, d)$, *where* $b \in \{2, 3, \ldots, m - 1\}$ *and* d *is arbitrary.*

Proof. We are required to show that $x_{00} - z = 0$ and may assume, given Lemma 5.7, that $y_{10} = 0$. From (5.9) we have

$$0 = 3(x_{00} - z) - 3(x_{22} + y_{12} - z) + y_{00} + 2y_{01} + y_{12} - 2y_{21} - y_{22}. \tag{5.12}$$

As $m > 1$, we see from (5.11) that the only possible values for $x_{22} + y_{12} - z$ are 0 and m. In the latter case, (5.12) and (5.2) yield the immediate contradiction

$$0 \leqslant 3(m - 2) - 3m + y_{00} + 2y_{01} + y_{12} \leqslant -2.$$

That leaves the case when $x_{22} + y_{12} - z = 0$. Here, (5.12) and (5.1) yield

$$0 \geqslant 3(x_{00} - z) - y_{21} - (y_{21} + y_{22}) \geqslant 3(x_{00} - z) - 2.$$

Given that $x_{00} - z$ is a non-negative integer, it must be zero and we are done. $\square$

Lemmas 5.7 and 5.8 combine to prove Theorem 5.6. Note that in both lemmas the symbols occurring in the transversal-free entries have the form $(0, f)$ for arbitrary f. Thus the transversal-free entries do form a latin subrectangle as claimed.

Computational results summarised in Section 5.4 show that there is no latin square of order 9 whose set of transversal-free entries contains a subrectangle of larger dimensions than 2×3, although three species of order 9 have more than 6 transversal-free entries. We confirmed that Lemmas 5.7 and 5.8 identify the only transversal-free entries in $\mathcal{D}_9$. Also by computer, $\lambda(\mathcal{D}_9) = 5$ which is slightly less than the value implied by the lemmas.

Certainly, $\mathcal{D}_{3m}$ can never provide a counterexample to Conjecture 1.1. For example a transversal of $\mathcal{D}_{3m}$ is given by

$$\mathrm{col}\big((a,b)\big) = \begin{cases} \{(\,a,\ 0\,)\} & \text{if } b = 0, \\ \{(\,a-1,\ b\,)\} & \text{otherwise.} \end{cases}$$

5.3 Transversals in $\mathcal{U}_n$ and $\mathcal{B}_n$

In this section we prove Theorem 5.2. We use the latin squares $\mathcal{U}_n$ and $\mathcal{B}_n$ defined in Chapter 2.

We remind the reader that $\mathcal{B}_n$ is defined for $n = 2h$, where $h \geqslant 5$ and h is odd.

Lemma 5.9. *The latin square $\mathcal{B}_n$ has a transversal. Every transversal in $\mathcal{B}_n$ coincides on a single element e, where $e = (4, 4, 4)$ if $n = 10$ and otherwise $e = (1, n-3, 1)$.*

Proof. Assume that $T \subset \mathcal{B}_n$ is a transversal. First we prove the second statement of the lemma. Lemma 2.1 requires that $\sum_{e \in T} \Delta(e) \equiv h \bmod n$, an odd value. The possible choices for odd Δ values are displayed (2.22). Respecting that T has at most one element in each row and each column, the odd sums obtained by using odd Δ values alone are ± 1, ± 3. Hence a sum of $\pm(h-1)$ or $\pm(h-3)$ must be met by elements of Δ_a where a is even. These sets Δ_a are listed in (2.23).

Considering the rows, T has at most one element in each of the sets $(\Delta_{-4} \cap r_4) \cup \Delta_{-3}$, $(\Delta_4 \cap r_0) \cup \Delta_1$, $\Delta_{h-7} \cap r_6$ and $\Delta_{h+7} \cap r_{h-1}$. It follows that, if $n = 10$, T contains the element $(4, 4, 4) \in \Delta_{-4}$ and an element of Δ_{-1}. If $n > 10$, then T must contain the element $(1, n-3, 1) \in \Delta_3$, some element from $\Delta_4 \cap r_0$ and, for $n > 14$, we also need an element of $\Delta_{h-7} \cap r_6$.

Next we show that a transversal T exists in $\mathcal{B}_n$.

If $n = 10$, then T is given by

$$\big\{(0,7,7),\ (1,9,9),\ (2,0,2),\ (3,2,5),\ (4,4,4),\\ (5,3,8),\ (6,5,1),\ (7,6,3),\ (8,8,6),\ (9,1,0)\big\}.$$

If $n = 14$, then T is given by;

$$\big\{(0,0,4),\ (1,11,1),\ (2,12,0),\ (3,13,2),\ (4,2,6),\ (5,3,8),\ (6,5,11),\\ (7,6,13),\ (8,4,12),\ (9,1,10),\ (10,7,3),\ (11,8,5),\ (12,9,7),\ (13,10,9)\big\}.$$

For $n > 14$, we specify T in two cases.

Case 1: $n \equiv 2 \bmod 8$

$$\mathrm{col}(x) = \begin{cases} \{0\} & \text{if } x = 0, \\ \{n-3\} & \text{if } x = 1, \\ \{h+3\} & \text{if } x = 6, \\ \{h-1\} & \text{if } x = h-3 \text{ and } n \equiv 10 \bmod 16, \\ \{h-5\} & \text{if } x = h+1 \text{ and } n \equiv 2 \bmod 16, \\ \{1\} & \text{if } x = n-1, \\ \{x+1\} & \text{if } x > 0 \text{ and } x \equiv 0 \bmod 4, \\ \{x-2\} & \text{if } 1 < x < n-1 \text{ and } x \equiv 1 \bmod 4, \\ \{x-1\} & \text{if } x \equiv 3 \bmod 4, \\ \{x-10\} & \text{if } 14 \leqslant x < h+5 \text{ and } x \equiv 6 \bmod 8, \\ \{x+2\} & \text{if } x \geqslant h+5 \text{ and } x \equiv 2 \bmod 4, \\ \{x+6\} & \text{otherwise.} \end{cases}$$

Case 2: $n \equiv 6 \bmod 8$

$$\mathrm{col}(x) = \begin{cases} \{h-3\} & \text{if } x = 6, \\ \{h-4\} & \text{if } x = h+2, \\ \{x-1\} & \text{if } x \in \{h, h-1\}, \\ \{x-2\} & \text{if } 4 \leqslant x \leqslant 8 \text{ and } x \notin \{6, h, h+1\}, \\ \{x-4\} & \text{if } \big(1 \leqslant x \leqslant 3\big) \text{ or } \big(11 \leqslant x \leqslant h-2 \text{ and } x \in F\big), \\ \{x-8\} & \text{if } x = 9 \text{ or } \big(12 \leqslant x \leqslant h+1 \text{ and } x \equiv 0 \bmod 4\big), \\ \{x\} & \text{if } x = 0 \text{ or } \big(10 \leqslant x \leqslant h-5 \text{ and } x \equiv 2 \bmod 4\big), \\ \{x-3\} & \text{otherwise.} \end{cases}$$

□

We remind the reader that $\mathcal{U}_n$ is defined for $n = 2mq \geqslant 6$, where $q \geqslant 3$ and q is odd, and $m = 2^t$ for $t \geqslant 1$. In other words, for all $n \geqslant 12$ such that $n \equiv 0 \bmod 4$ and n is not a power of 2.

Lemma 5.10. *The latin square $\mathcal{U}_n$ has a transversal. Every transversal in $\mathcal{U}_n$ contains the element $(0, n-m, 0)$.*

Proof. Assume that $T \subset \mathcal{U}_n$ is a transversal. Lemma 2.1 requires that $\sum_{e \in T} \Delta(e) = qm \equiv 0 \mod m$. Hence, T must contain an even number of elements from $\Delta_{-m/2} \cup \Delta_{m/2}$. However, these elements form an intercalate so T contains at most one, hence zero, of them. The only way to satisfy Lemma 2.1 is for T to include one element from each of Δ_m and Δ_{2m} and, if $q \neq 3$, also one element from $\Delta_{(q-3)m}$. Thus, T includes two elements of $\Delta_m \cup \Delta_{2m}$ and one of those must be from outside r_{n-m}. Hence, the element $(0, n-m, 0)$ is required in T.

We now identify a transversal $T \subset \mathcal{U}_n$.

Case 1: $n \equiv 4 \bmod 8$
We have $n = 2mq$ with $m = 2$. Then T is given by

$$\operatorname{col}(x) = \begin{cases} \{n-m\} & \text{if } x = 0, \\ \{mq-2m\} & \text{if } x = m \text{ and } q > 3, \\ \{m\} & \text{if } x = n-m, \\ \{0\} & \text{if } x = n-2m, \\ \{x+2m+1\} & \text{if } x \in E \text{ and } mq-2m \leqslant x < n-2m, \\ \{x-2m+1\} & \text{if } x \in F \text{ and } x > mq, \\ \{x\} & \text{otherwise.} \end{cases}$$

Case 2: $n \equiv 0 \bmod 8$
We have $n = 2mq$ with $m \geqslant 4$. Then T is given by

$$\operatorname{col}(x) = \begin{cases} \{n-m\} & \text{if } x = 0, \\ \{mq-2m\} & \text{if } x = m \text{ and } q > 3, \\ \{m\} & \text{if } x = n-m, \\ \{mq+1\} & \text{if } x = mq-2m, \\ \{x+3\} & \text{if } x \in E \text{ and } mq \leqslant x < n-m, \\ \{x-1\} & \text{if } x \in F \text{ and } mq < x < n-m, \\ \{x+1\} & \text{if } n-m < x < n, \\ \{x\} & \text{otherwise.} \end{cases}$$

□

Together, Lemmas 5.9 and 5.10 cover all cases to prove Theorem 5.2.

The proof of Lemma 5.10 depends on the existence of a proper odd divisor q. It is possible to modify the definition for $\mathcal{U}_n$ to permit $q = 1$ but in that case each of the 3 elements of Δ_m alone satisfy Lemma 2.1, so the most we can say is that $\lambda(\mathcal{U}_n) \leqslant 3$. For the case that n is a power of 2, a promising stronger restriction occurs in the family $\mathcal{A}_n$.

The family $\mathcal{A}_n$ is defined for $n = 16d$, where $d \geqslant 1$.

Lemma 5.11. *A transversal in $\mathcal{A}_{16d}$ intersects the elements $(10d, 0, 4d)$ and $(4d, 13d, 14d)$.*

Proof. Assume that T is a transversal in $\mathcal{A}_n$. Lemma 2.1 requires that $\sum_{e\in T} \Delta(e) \equiv 8d \bmod n$. The possible choices are shown in (2.20). Ensuring at most one element from each row and column means that T contains elements $(10d, 0, 4d) \in \Delta_{-6d}$, $(4d, 13d, 14d) \in \Delta_{-3d}$ and one element of Δ_d. □

By computer we found that for $n \leqslant 64$, $\mathcal{A}_n$ does possess transversals. A general pattern for finding one is not known.

Conjecture 5.12. *The latin square $\mathcal{A}_n$ has a transversal. Hence, Theorem 5.2 holds for all even orders $n \geqslant 10$.*

We conclude this section by noting that, by the next statement, our latin squares of even order n with $\lambda = 1$ yield a confirmed bachelor latin square of order $(n + 1)$.

Theorem 5.13. *Suppose that L is a latin square of order n and that T is a transversal in L such that $L \setminus T$ contains no transversal. Then there exists a latin square L' of order $(n + 1)$ such that L' has at least one transversal-free entry.*

Proof. Let $T \subset L$ be a transversal and let $\mathcal{I}(L) = \mathbb{Z}_n$. We construct L' by prolongation of the transversal T. That is, the latin square L' of order $(n + 1)$ is given by

$$L'_{n+1}[x, y] = \begin{cases} n & \text{if } x = y = n \text{ or } (x, y, z) \in T, \\ z & \text{if } x = n \text{ and } (x', y, z) \in T, \\ z & \text{if } y = n \text{ and } (x, y', z) \in T, \\ L[x, y] & \text{if otherwise } (x, y, z) \in L. \end{cases} \tag{5.13}$$

Consider the element $(n, n, n) \in L'$. This element must be transversal-free as otherwise there is a transversal contained in $L \setminus T$. □

Applying Theorem 5.13 with $L = \mathcal{B}_n$ justifies our claim in Section 5.1 of a new construction for a confirmed bachelor latin square for all orders $n \equiv 3 \bmod 4$, when $n \geqslant 11$.

5.4 Latin squares of small order

The computational results in this section have been independently confirmed by Ian Wanless.

For order 9, we used a program to generate isotopy class representatives. The program, which we also used in Section 4.5.3, is described in [119]. We report on two projects involving restricted transversals.

Search 1: for transversal-free entries. This search of latin squares of order 9 produced a list of isotopy classes which possess transversal-free entries. The list was then reduced to species representatives. We recorded that 20011 species of order 9 possess at least one transversal-free entry. For orders $n \leqslant 8$ we ran our programs directly on a list of species representatives. In Table 5.1 we summarise the latin squares of order $4 \leqslant n \leqslant 9$ according to the number of transversal-free entries they contain. The single species for each order 1, 2 and 3 have, respectively, 0,4 and 0 transversal-free entries.

Search 2: for partitions with transversals. In this search of the latin squares of order 9 we kept a list of each isotopy class representative of order 9 which failed to possess a $(1^5, 2^2)$-partition. Reducing this list to species representatives we obtained a list Γ of 182 species representatives. A further search of the species in Γ showed that each of them has a $(1^3, 2^3)$-partition. Thus, all species of order 9 possess a $(1, 2^4)$-partition. Combined with known results for $n \leqslant 8$, first shown by Wanless [150], and confirmed by our data in Section 4.5.2, we proved the following.

Theorem 5.14. *Conjecture* 1.3 *is true for the latin squares of order* $n \leqslant 9$.

By further tests on species in Γ we found that the species of order 9 satisfy the following.

1. Exactly 156 species fail to possess a $(1^5, 2^2)$-partition or a $(1^6, 3)$-partition, but do have a $(1^5, 4)$-partition.
2. Hence, in total 19270853515 species have $\lambda \geqslant 5$.
3. Exactly 26 species fail to possess a $(1^5, 4)$-partition. Of these, 23 species have a $(1^4, 5)$-partition, hence have $\lambda = 4$. The remaining 3 species each have $\lambda = 3$.
4. Exactly 7 species in Γ have an order 4 subsquare, and that subsquare is paratopic to the Cayley table of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Hence, Lemma 4.32 applies to these 7 species. All 7 of these species have $\lambda = 4$.
5. It follows, from 4. and Theorem 5.3, that the other 36000 species in Ω (the species of Lemma 4.32) must have $\lambda = 5$. Similarly, any latin square of order 9 containing a subsquare isotopic to the Cayley table of $\mathbb{Z}_4$ must have $\lambda = 5$, however we did not calculate the number of species that this involves.

To the extent that we achieved for $n = 9$, we summarise in Table 5.2 the species of order $n \leqslant 9$ according to λ. Data for $n \leqslant 8$ in Table 5.2 was first summarised in [150] building upon the studies for $n \leqslant 7$ mentioned in Section 1.14.1. The mark ✗ indicates that zero species possess the λ value, and ✓ means that at least one species has the stated λ value.

We next illustrate a few examples of interest from the two projects.

The unique species of order 9 achieving the maximum number of transversal-free entries is one well known to us. The latin square L_1, shown below in semi-symmetric form, is isotopic to $\mathcal{W}_9$. It has 415 transversals. Shading shows its 10 transversal-free entries which correspond to all of the elements $\Delta_* \subset \mathcal{W}_9$. Other data on the indivisible plexes and partitions of $\mathcal{W}_9$ is recorded in Table 4.13 (Species D). Also, by computation, we note that $\lambda(\mathcal{W}_9) = 6$.

$$L_1 = \begin{pmatrix} 0 & 3 & 1 & 8 & 7 & 2 & 6 & 5 & 4 \\ 2 & 6 & 4 & 0 & 8 & 7 & 1 & 3 & 5 \\ 5 & 0 & 3 & 2 & 1 & 6 & 4 & 8 & 7 \\ 1 & 7 & 2 & 4 & 3 & 5 & 8 & 6 & 0 \\ 8 & 2 & 6 & 3 & 5 & 4 & 7 & 0 & 1 \\ 7 & 8 & 0 & 5 & 4 & 3 & 2 & 1 & 6 \\ 6 & 1 & 5 & 7 & 2 & 8 & 0 & 4 & 3 \\ 4 & 5 & 8 & 1 & 6 & 0 & 3 & 7 & 2 \\ 3 & 4 & 7 & 6 & 0 & 1 & 5 & 2 & 8 \end{pmatrix}$$

The next two latin squares, L_2 and L_3 shown below, represent the only other species of order 9 with more than 6 transversal-free entries. All three squares L_1, L_2 and L_3 have at least one subsquare of order 3.

L_2 has 287 transversals, 7 transversal-free entries (as shaded) and $\lambda(L_2) = 6$.

$$L_2 = \left(\begin{array}{ccc|ccc|ccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 0 & 4 & 5 & 3 & 7 & 8 & 6 \\ 2 & 0 & 1 & 5 & 3 & 4 & 8 & 6 & 7 \\ \hline 3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 & 0 \\ 4 & 7 & 3 & 2 & 8 & 6 & 0 & 5 & 1 \\ 7 & 3 & 4 & 8 & 6 & 0 & 5 & 1 & 2 \\ \hline 6 & 5 & 8 & 1 & 2 & 7 & 4 & 0 & 3 \\ 5 & 8 & 6 & 7 & 0 & 1 & 2 & 3 & 4 \\ 8 & 6 & 7 & 0 & 1 & 2 & 3 & 4 & 5 \end{array}\right) \qquad L_3 = \left(\begin{array}{ccc|ccc|ccc} 0 & 1 & 2 & 4 & 5 & 3 & 6 & 7 & 8 \\ 1 & 2 & 0 & 5 & 3 & 4 & 7 & 8 & 6 \\ 4 & 0 & 1 & 3 & 2 & 5 & 8 & 6 & 7 \\ \hline 2 & 8 & 6 & 0 & 1 & 7 & 4 & 5 & 3 \\ 8 & 6 & 4 & 1 & 7 & 0 & 5 & 3 & 2 \\ 6 & 4 & 8 & 7 & 0 & 1 & 3 & 2 & 5 \\ \hline 3 & 7 & 5 & 2 & 8 & 6 & 0 & 1 & 4 \\ 7 & 5 & 3 & 8 & 6 & 2 & 1 & 4 & 0 \\ 5 & 3 & 7 & 6 & 4 & 8 & 2 & 0 & 1 \end{array}\right)$$

L_3 is isotopic to its transpose by the symbol permutation (38)(57). It has 8 transversal-free entries (shaded) and 92 transversals in total. This latin square exhibits both types of restrictions on its transversals. It is also one of three species recording the least λ value for order 9, that is $\lambda(L_3) = 3$.

The other two species which achieve $\mu(9) = 3$ are illustrated below by L_4 and L_5. Note that L_3 differs from L_4 in only 6 entries, a 3-cycle between symbols 2 and 4.

L_4 has 84 transversals and zero transversal-free entries. The square is nearly symmetric, just transpose, and then apply the symbol permutation (36)(58). A case analysis using Lemma 2.1 can be used to show that a transversal in L_4 must include 2 entries from the centre subsquare (with symbols $\{0, 1, 7\}$) and it follows from this that it has at most 3 disjoint transversals.

$$L_4 = \left(\begin{array}{ccc|ccc|ccc}
0 & 1 & 2 & 4 & 5 & 3 & 6 & 7 & 8 \\
1 & 2 & 0 & 5 & 3 & 4 & 7 & 8 & 6 \\
2 & 0 & 1 & 3 & 4 & 5 & 8 & 6 & 7 \\
\hline
4 & 8 & 6 & 0 & 1 & 7 & 2 & 5 & 3 \\
8 & 6 & 4 & 1 & 7 & 0 & 5 & 3 & 2 \\
6 & 4 & 8 & 7 & 0 & 1 & 3 & 2 & 5 \\
\hline
3 & 7 & 5 & 2 & 8 & 6 & 0 & 1 & 4 \\
7 & 5 & 3 & 8 & 6 & 2 & 1 & 4 & 0 \\
5 & 3 & 7 & 6 & 2 & 8 & 4 & 0 & 1
\end{array}\right) \qquad
L_5 = \left(\begin{array}{ccc|cccccc}
\mathbf{2} & 1 & \mathbf{0} & 4 & 5 & 3 & 7 & 8 & 6 \\
1 & 0 & 2 & 7 & 8 & 6 & 4 & 5 & 3 \\
\mathbf{0} & 2 & 1 & 8 & 6 & 7 & 5 & 3 & 4 \\
\hline
5 & 8 & 7 & 3 & 0 & 4 & 6 & 1 & 2 \\
3 & 6 & 8 & 5 & 4 & 0 & 2 & 7 & 1 \\
4 & 7 & 6 & 0 & 3 & 5 & 1 & 2 & 8 \\
8 & 5 & 4 & 6 & 1 & 2 & 3 & 0 & 7 \\
6 & 3 & 5 & 2 & 7 & 1 & 8 & 4 & 0 \\
7 & 4 & 3 & 1 & 2 & 8 & 0 & 6 & 5
\end{array}\right)$$

The latin square L_5 is isotopic to its transpose. It has 168 transversals, including at least one through every entry, one subsquare of order 3 and 48 intercalates. A parity argument shows that any odd plex must intersect the order 3 subsquare. In fact, every one of the transversals in L_3 contains exactly one of the three elements marked by the bold entries of the subsquare but an explanation of this remains open.

For $n = 9$, members of our other latin families of odd order show no special significance in terms of the types of restrictions of transversals we have described. The family $\mathcal{H}_n$ yielded large indivisible plexes in Chapter 4. By computation, $\lambda(\mathcal{H}_9) = 7$ and its only transversal-free entries are the two specified by Theorem 4.23. Also $\mathcal{O}_{3,3}$, from another family featured in Chapter 4, has zero transversal-free entries and $\lambda(\mathcal{O}_{3,3}) = 9$. Species A of Table 4.13, the latin square that we identified with an indivisible 6-plex (and a member of Ω) has 176 transversals, $\lambda = 5$ and zero transversal-free entries.

The number of transversals in latin squares is studied in [118]. The authors found there that the species of order n exhibiting the least number of transversals for order $n \in \{5, 7, 9\}$ were all examples of semi-symmetric latin squares. For $n = 9$, the minimum number of transversals is 68 (Species B of Table 4.13), and the average number is 214 [118]. Species B has $\lambda = 5$ and zero transversal-free entries. Species C of Table 4.13 has, for order 9, the maximum number of intercalates [118]. It has 801 transversals, $\lambda = 9$ and zero transversal-free entries.

Table 5.1: Species of order $4 \leqslant n \leqslant 9$ according to number of transversal-free entries.

Number of transversal free entries	Order n					
	4	5	6	7	8	9
0	1	1	2	54	267932	19270833530
1				11	13165	18066
2				26	1427	1853
3				12	253	54
4				12	508	21
5				6	89	7
6			1	8	65	7
7				3	33	1
8				4	48	1
9					25	
10				1	27	1
11				1	9	
12		1	2	6	9	
13				1	2	
14					2	
16	1		1	1	27	
18					1	
20					1	
28					1	
36			6	1		
64					33	

Table 5.2: Species of order $n \leqslant 9$ according to λ.

	Order n					
λ	4	5	6	7	8	9
0	1	✗	6	✗	33	✗
1	✗	1	✗	1	✗	✗
2	✗	✗	2	5	7	✗
3	✗	✗	✗	24	46	3
4	1	✗	4	68	712	23
5		1	✗	43	71330	✓
6			✗	✗	209505	✓
7				6	✗	✓
8					2024	✗
9						✓

Another type of restriction on the transversals of a latin square is the following, weaker, restriction within a partition. For a given latin square L, we define:

$$\alpha(L) = \min\big\{\, a : \; L \text{ has a } (1^a, n-a)\text{-partition with no 1-plex in the } (n-a)\text{-plex}\big\}.$$

By definition, for a given L, $\alpha(L) \leqslant \lambda(L)$. Our study of $\alpha(L)$ is quite limited as our computational projects targeted different questions. However for order 9, we found all of the species with $\alpha = 1$ and $\alpha = 2$. These are, respectively, the 36007 species of Lemma 4.32, and 8177 of the species in Ψ that we identified in Section 4.5.3 during the process of ruling out the existence of an indivisible $(1^2, 7)$-partition.

For all positive integers n we define:

$$\beta(n) = \min\big\{\, \alpha(L) : \; L \text{ is a latin square of order } n\big\}.$$

Conjecture 1.1 implies $\beta(n) \geqslant 1$ for all odd n, and we know by $\mathbb{Z}_n$ that $\beta(n) = 0$ for all even n. In contrast to what we know of $\mu(n)$, Corollary 4.30 showed that for $n \geqslant 5$, $\beta(n) \leqslant 1$ if $n \equiv 1 \bmod 4$, and $\beta(n) \leqslant 3$ if $n \equiv 3 \bmod 4$. Some improvement, if indeed possible, of the upper bound of $\beta(n)$ remains open for $n \equiv 3 \bmod 4$, where $n \geqslant 11$. The values, $\beta(5) = \beta(7) = 1$ and $\beta(3) = 3$, are not specifically reported in our data tables.

Unlike the restriction $\lambda(L) < n$, a small value $\alpha(L)$ does not in general immediately imply that the given latin square L has no 1-partition. As we have reported, if L is of order 9 then $\alpha(L) = 1$ implies that $\lambda(L) \leqslant 5$. We record as follows, $\alpha(L)$ for the examples illustrated in this section: $\alpha(\mathcal{W}_9) = \alpha(L_2) = 3$, $\alpha(L_3) = \alpha(L_4) = 2$, $\alpha(L_5) = \lambda(L_5) = 3$. For our family members of order 9 we found that: $\alpha(\mathcal{D}_9) = 2$, $\alpha(\mathcal{H}_9) = 4$ and $\alpha(\mathcal{O}_{3,3}) = 3$.

5.5 Concluding remarks

The computational results of this chapter suggest that almost all large latin squares have no transversal-free entries. The various types of restrictions on the transversals that we have considered show that the existence of one type of restriction does not, in general, imply another. For example, L_4 has a small value of λ but no transversal-free entries.

Our data suggests that Theorems 5.2 and 5.6 identify what may be considered exotic behaviour for larger n. Indeed, all of the bachelor families mentioned here and in other chapters may be considered rare for larger n if the more recent views regarding Conjecture 1.24 hold true. Those more recent views [118, 155] suggested that, asymptotically, the proportion of bachelor latin squares is equal to zero.

Theorem 5.6 shows that the family $\mathcal{D}_{3m}$ has quadratically many transversal-free entries and is thus much stronger than Theorem 5.5. It would be interesting to know if a result similar to Theorem 5.6 applies to orders that are not divisible by 3.

An extension of Theorem 5.2 to all even $n \geqslant 10$ remains open. This would be resolved by a proof of Conjecture 5.12 that $\mathcal{A}_n$ has a transversal. Data in Table 5.2 shows that if L is of even order $n \leqslant 8$ then $\lambda(L) \neq 1$. Thus, Theorem 5.2 cannot be extended to smaller even n.

Collectively, Theorem 5.2, Lemma 5.1, the data in Table 5.2 and that on indivisible partitions in Section 4.5.2 support the following.

Conjecture 5.15. *For all even $n \geqslant 10$ and each $m \in \{0, 1, 2 \ldots, n-2\} \cup \{n\}$, there exists a latin square of order n such that $\lambda(L) = m$.*

Far less is known about λ for the latin squares of odd order. Our finding that $\mu(9) = 3$ offers more support for Conjecture 1.1. There appears room to improve on Theorem 5.4 which was drawn from work in [73, 117]. In fact, for odd orders n and $\lambda < n$, it remains to be shown in general that any particular value of λ is actually achieved. We note that for order $n = 11$ the construction by Evans [73] has an indivisible $(1^6, 5)$-partition. Indivisibility of the 5-plex follows from the same argument stated in [73] and which justifies Theorem 5.4. If it is true that $\lambda = (n+1)/2$ for the construction in [73], then it would follow that there is an indivisible $(1^{(n+1)/2}, (n-1)/2)$-partition for $n \geqslant 7$ when $n \equiv 3 \bmod 4$.

It would be interesting to determine whether or not $\mu(n) < n/2$ for all odd $n > 3$. In particular:

Problem 5.16. Is $\mu(n)$ unbounded $n \to \infty$?

Afterword

Two articles based on results from this thesis are [19] and [67]. Results of Chapter 3 are published in [67]. The results in Section 4.2 are in [19]. These two articles were already mentioned.

Shortly after submission of this thesis, we received the news that a further article based on results from Chapter 4 has been accepted for publication. The article, [68], contains Theorem 4.1 and the results in Sections 4.3.1 and 4.4. Also reported in [68] are the computations and data on indivisible partitions in Section 4.5, and the computational proof of Theorem 5.14.

Also, just after submission of the thesis, the author proved two of the stated conjectures.

Conjecture 4.35 was proved by showing an indivisible $(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$-partition in $\mathcal{H}_n$. This gives a slightly improved lower bound for $\kappa(n)$. The new result, along with Theorems 4.2 and 4.23, and related work in Section 4.3.2, is submitted for publication [66].

Conjecture 5.12 was proved by showing a transversal in $\mathcal{A}_n$. This gives the anticipated extension of Theorem 5.2 to all even orders $n \geqslant 10$. The extended result, along with Theorem 5.6, Table 5.1 and related work in Chapter 5, is submitted for publication [69].

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