Scattering for the quadratic Klein-Gordon equation with inverse-square potential and related problems

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Stephen Deng
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Abstract

We study scattering of the quadratic Klein-Gordon equation with an inverse-square potential

$$\begin{cases} \partial_t^2 u - \Delta u + \frac{a}{|x|^2} u + u = u^2, & (t,x) \in \mathbf{R} \times \mathbf{R}^d \\ u(0,x) = u_0, u_t(0,x) = u_1 \end{cases}$$

under various assumptions on the initial data. The idea is to use the non-resonance structure of the quadratic Klein-Gordon equation. In particular, we study the harmonic analysis adapted to the operator $-\Delta + a/|x|^2$. First, we obtain scattering for the 3D radial small energy problem. Here, the main tools are the refined radial Strichartz estimates which can be obtained in a similar manner to the potential-free case as in Guo-Hani-Nakanishi [Comm. Math. Phys. (2018)], as well as a normal form transform. Next, we obtain a scattering result small energy problem in dimensions $d \geq 3$ (with some restrictions on the coefficient a of the inverse-square potential). Here, the result is obtained using U^p and V^p spaces as studied in [Hadac-Herr-Koch Ann. Inst. H. Poincaré Anal. Non Linéaire (2009)] and for the potential-free QKG(0) in Schottdorf [arXiv:1209.1518 (2012)]. The non-resonance of the QKG(a) is studied using a modulation bound. Furthermore, we obtain a scattering result for the 4D radial large energy problem below the ground state. Here, the usual $L^1 \to L^\infty$ dispersive estimate does not hold for $a < 0$. Nonetheless, a weaker dispersive estimate does hold, as established by Zheng [J. Math. Phys. (2018)]. The main tools for this problem are then weaker frequency-localised dispersive estimates (a combination of the estimates in Guo-Peng-Wang [J. Funct. Anal. (2008)] and those in Zheng [J. Math. Phys. (2018)], Virial-Morawetz estimates as in Dodson-Murphy [Proc. Amer. Math. Soc. (2017)] as well as the reduction of the large energy problem to a small energy problem after large time. We apply these similar methods to also study scattering for the non-linear Schrödinger equation and non-linear Klein-Gordon equation with an exponential-type nonlinearity.

Contents

CHAPTER 1

Introduction and main results

We study the scattering behaviour of the following Cauchy problem of the quadratic Klein-Gordon (QKG(a)) equation with inverse square potential:

$$\begin{cases} \partial_t^2 u - \Delta u + \frac{a}{|x|^2} u + u = u^2, & (t,x) \in \mathbf{R} \times \mathbf{R}^d \\ u(0,x) = u_0, u_t(0,x) = u_1 \end{cases} \tag{1.0.1}$$

where $u : \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}$, $d \geq 3$ and $a > -\left(\frac{d-2}{2}\right)^2 = \mu(0)^2$. We shall denote $-\Delta + \frac{a}{|x|^2}$ by $\mathscr{L}_a$. In fact, it is not immediately clear that properties of the potential-free problem (i.e. QKG(0)) are inherited by the QKG(a), even for a near zero. This is best illustrated by break in translation symmetry for $a \neq 0$. Therefore, we need to recover the tools and arguments used by previous works — namely: Strichartz estimates with improved range of admissible pairs for radial initial data (i.e. radially refined Strichartz estimates) [**17**], bilinear Strichartz estimates [**53**] and the normal form transform [**22**]. The contribution of this thesis is the adaptation of these results to the inverse-square potential setting. The non-resonance structure of the QKG(a) is essential to this study.

The behaviour of the non-linear Schödinger equation (NLS) with inverse-square potential and wave equation (NLW) with inverse-square potential are better understood. Indeed, Strichartz estimates for the NLS with inverse-square potential and NLW with inverse-square potential were obtained by Burq-Planchon-Stalker and Tahvildar-Zadeh [**6**] (also see [**50, 49**]). These results were extended by Miao-Zhang-Zheng [**43**] with an improved range of admissible pairs, at a small loss in angular regularity. Studies related to unique solvability have also been undertaken — see Okazawa-Suzuki-Yokota [**46**] and also Suzuki [**57**], where unique solvability was obtained at the critical coefficient $a = -\left(\frac{d-2}{2}\right)^2$. In addition, the harmonic analysis adapted to the operator $\mathscr{L}_a$ for $d \geq 3$ was studied by Kilip-Miao-Visan-Zhang-Zheng [**33**] via heat kernel bounds of the semigroup $e^{t\mathscr{L}_a}$. Specifically, the multiplier theory (and therefore also the Littlewood-Paley theory) adapted to $\mathscr{L}_a$ is well understood. Thus, for instance, allows the use of frequency decomposition techniques. Studies of the $d = 2$ case have also been considered, for instance, by Burq et al. [**6**]. We also mention some other studies, such as blow-up by Bensouilah-Dinh [**4**] and Csobo-Genoud [**12**]) as well as the stability/instability of standing waves by Bensouilah-Dinh-Zhu [**5**].

Scattering results in the setting of the NLS with inverse-square potential (denoted by NLS(a)) are also well understood. Indeed, the scattering/blow-up dichotomy below the ground state threshold is understood under various assumptions (radial/non-radial, critical/inter-critical, etc.) [**34, 36, 65, 67**]. The global existence/blow-up dichotomy for a class of focusing NLS equations below the ground state threshold has also been studied in both intercritical and critical settings by

Dinh [**13**]. The Virial-Morawetz estimates of Dodson-Murphy [**14**] play a central role in obtaining scattering, and are also central in this thesis. We also mention analogous scattering results for the Hartree equation with inverse-square potential [**9**] and the more generalised setting of the Choquard equation [**41**].

The non-resonance structure of the QKG(a) is central to our study. In the potential-free case, the analogous structure of QKG(0) is studied via the Fourier transform. To make use of the non-resonance structure for $a \neq 0$, we have the following (radial) Hankel transform available to us:

$$(\mathcal{H}_{\nu(a)} f)(r) = \int_0^\infty U_{\nu(a)}(r\rho) f(\rho)\, \rho^{d-1}\, \mathrm{d}\rho,$$

where $U_{\nu(a)}(r\rho) = (r\rho)^{-\frac{d-2}{2}} J_{\nu(a)}(r\rho)$ and $\nu(a) = \sqrt{\left(\frac{d-2}{2}\right)^2 + a}$ — see Chapter 2 below for more details. Using properties of the Bessel function near zero, one can see that in fact the behaviour of $U_{\nu(a)}$ is discontinuous with respect to a. Indeed, we have that $J_{\nu(a)}(z) \sim z^{\nu(a)}$ for z near zero. Hence, we have

$$U_{\nu(a)}(0) = \begin{cases} \infty, & a < 0 \\ \text{const.}, & a = 0 \\ 0, & a > 0 \end{cases}.$$

Thus, it is not immediately clear whether properties of the potential-free case will carry over to solutions of QKG(a) even in the case where a is near 0. As mentioned previously, if $a < 0$, the $L^1 \to L^\infty$ dispersive estimate fails, even though it holds for the classical $a = 0$ case (and also holds for $a > 0$). However, time-decay can be recovered. For the NLS(a), Burq et al. [**6**] recover $L^{p'} \to L^p$ time-decay estimates for $p = 2d/(d-1)$ with time-decay $t^{-1/2}$. Kilip et al. [**33**] instead study certain convergence results that substitute these dispersive estimates. Finally, Zheng [**67**] recovers a weaker time-decay estimates in a weighted L^2 space. This final approach is most relevant to this thesis. Here, the unboundedness of $U(\nu(a))$ at zero for $a < 0$ is the only obstruction. On the other hand, rearrangement breaks down in the range $a > 0$. Indeed, rearrangement only decreases the adapted Sobolev norm

$$\|u\|_{\dot{H}^1_a} := \int_{\mathbf{R}^d} |\nabla u|^2 + \frac{a}{|x|^2}|u|^2\, \mathrm{d}x.$$

in the range $a \leq 0$. The issue lies in the inequality (5.2.2), which implies that for $a > 0$

$$\int_{\mathbf{R}^d} \frac{a}{|x|^2}|u|^2\, \mathrm{d}x \leq \int_{\mathbf{R}^d} \frac{a}{|x|^2}|u^*|^2\, \mathrm{d}x.$$

However, if $a < 0$, then the direction of the inequality is reversed and becomes favourable. This breakdown of rearrangement is an issue when we study the ground state threshold in Chapter 5 as the minimiser of an energy. Rearrangement is used to show that the minimiser must be radial, and thus better compactness embeddings are available to obtain existence.

The QKG(a) has the following conserved energy $E_a(u, u_t)$ defined by

$$\frac{1}{2}\int_{\mathbf{R}^d} |\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2 + \frac{a}{|x|^2}|u(t,x)|^2 + |u(t,x)|^2\, \mathrm{d}x - \frac{1}{3}\int_{\mathbf{R}^d} u(t,x)^3\, \mathrm{d}x.$$

In this thesis, we are especially focused on the QKG(a) in three and four dimensions. In $d = 3$, the model is mass-subcritical and in $d = 4$, the model is mass-critical. Furthermore, the resonance structure of the QKG(a) is essential in our study. This structure is revealed via the Hankel transform, which generalises the analysis of the QKG(a) for $a \neq 0$, and is analogous to the (radial) Fourier transform for the potential-free ($a = 0$) case. The use of the Hankel transform (see [**25, 26**]) allows for the use of the (partial) normal form method by Shatah [**55**] (also see Germain [**16**]) to make use of this resonance structure. We discuss in more detail our main results in the next section.

We also study the related problem of the two-dimensional non-linear Schrödinger equation and non-linear Klein-Gordon equation with exponential nonlinearity and inverse square potential. We shall introduce this study further in Chapter 6.

1.1. Summary of main results

We summarise the results of this thesis. The first result is the following set of refined radial Strichartz estimates adapted to the operator $\mathscr{L}_a$. This result is a generalisation of the estimates in [**17**]. By orthogonality of the spherical harmonics, we can generalise them to a non-radial (and spherically-averaged) setting, though we shall only use the radial version in this thesis.

THEOREM 1.1.1 (Radial refined Strichartz estimates). *Let $d \geq 3$, $k \in \mathbf{Z}$, $2 \leq q, r \leq \infty$ and $u_0 \in L^2_{\mathrm{rad}}(\mathbf{R}^d)$. Let $a > -\left(\frac{d-2}{2}\right)^2$ and furthermore, if $-\left(\frac{d-2}{2}\right)^2 < a < 0$, impose also that $r_0 < r < r_0' = \frac{d}{\sigma}$. Then,*

(a) *(General region) if $q\left(\frac{1}{2} - \frac{1}{r}\right) > \frac{1}{d-1}$ and ω satisfies (3.1.2). Then,*

$$\|e^{it\omega(D_a)} P_k^a u_0\|_{L_t^q L_x^r} \lesssim 2^{k\left(\frac{d}{2} - \frac{d}{r} - \frac{\alpha}{q}\right)} \|u_0\|_{L_x^2(\mathbf{R}^d)}.$$

(b) *(Refined region) if $\frac{2}{2d-1} < q\left(\frac{1}{2} - \frac{1}{r}\right) < \frac{1}{d-1}$ and ω satisfies (3.1.3). Then,*

$$\|e^{it\omega(D_a)} P_k^a u_0\|_{L_t^q L_x^r} \lesssim 2^{k\theta(r,q)} \|u_0\|_{L_x^2(\mathbf{R}^d)}$$

where $\theta(r,q) = \frac{d}{2} - \frac{d}{r} - \frac{\beta}{q} - (\alpha - \beta)\left(\frac{d-1}{2} - \frac{d-1}{r}\right)$.

Furthermore, along the endpoint case $q\left(\frac{1}{2} - \frac{1}{r}\right) = \frac{1}{d-1}$, we have

$$\|e^{it\omega(D_a)} P_k^a u_0\|_{L_t^q L_x^r} \lesssim \langle k(\alpha - \beta)\rangle^{\frac{2}{q}} 2^{k\theta(r,q)} \|u_0\|_{L_x^2(\mathbf{R}^d)}.$$

The above estimates are then used to obtain the following small-energy scattering result in 3D in Chapter 3. The space $\left(\frac{1}{2}, \frac{3}{10} - \kappa, \frac{2}{5} - 3\kappa \,\middle|\, \frac{7}{10} + \kappa\right)$ shall be defined later, and is essential a time-averaged Besov space adapted to $\mathscr{L}_a$ with regularity $\frac{2}{5} - 3\kappa$ at low frequency and $\frac{7}{10} + \kappa$ at high frequency. We follow the argument of Guo-Shen [**22**]. We remark that a similar result can be obtained in 4D, though we do not pursue this in detail. We remark that there is a loss of $1/100$ in the range of a for which the result holds of. This comes from the analysis adapted to $\mathscr{L}_a$, which only holds for a restricted range of L^p when $a < 0$. One is able to increase the range of L^p by restricting the range of a. More precisely, the range of L^p is given by $r_0 < p < r_0' := \frac{d}{\sigma}$, where

$$\sigma = \frac{d-2}{2} - \sqrt{\left(\frac{d-2}{2}\right)^2 + a}.$$

We remark that a similar method has been used to study other models — such as the 3D Gross-Pitaevskii equation [**17**], 3D Zakharov system [**18**] and Klein-Gordon-Zakharov systems [**20**].

THEOREM 1.1.2 (3D radial small energy scattering). *Let $0 < \kappa \ll 1$ be sufficiently small, and suppose that (u_0, u_1) is radial and satisfies $\|(u_0,u_1)\|_{H^1_a \times L^2} \ll 1$. Then, there exists a unique solution $u(t,x)$ to (1.0.1) with $a > \sigma^{-1}(\frac{3}{2}(\frac{3}{10} - \kappa)) \approx -\frac{1}{4} + \frac{1}{100}$ in the space*

$$S = C(\mathbf{R}, H^1_a) \cap \left(\frac{1}{2}, \frac{3}{10} - \kappa, \frac{2}{5} - 3\kappa \,\middle|\, \frac{7}{10} + \kappa\right),$$

that also scatters in the sense that there exists $u_\pm(x) \in H^1_a$ such that

$$\|u - i\langle D_a\rangle^{-1}\partial_t u - e^{it\langle D_a\rangle}u_\pm\|_{H^1_a} \to 0,$$

as $t \to \infty$.

Next, we study the small energy problem in higher dimensions ($d \geq 3$). Since we no longer have access to the refined radial Strichartz estimates, we instead follow the argument of Schottdorf [**53**] to obtain a scattering result. Indeed, the key is to work in U^p and V^p spaces (see Hadac-Herr-Koch [**24**]). For a toy problem, one may refer to the discussion of the non-resonant 2D derivative NLS $i\partial_t u + \Delta u = \partial_{x_1}\bar{u}^2$ by Koch [**38**]. The main problem is that due to duality, the quadratic nonlinearity of the QKG(a) will require trilinear estimates in order to close the scattering argument. Thus, bilinear estimates $L^2 \times L^2 \to L^2$ will allow us to split essentially an L^1 integral into $L^2 \times L^2 \to L^1$ via Hölder's inequality and then split once more into $L^2 \times L^2 \times L^2 \to L^1$. These can then be converted into estimates on U^p and V^p spaces. We have the following result in the range $a > \mathcal{A}_d$ where

$$\mathcal{A}_d = \begin{cases} -\left(\frac{d-2}{2}\right)^2, & d = 3,4 \\ \frac{1}{16}(8d - 3d^2), & d \geq 5. \end{cases} \tag{1.1.1}$$

THEOREM 1.1.3 (Small energy scattering in higher dimensions). *Let $d \geq 3$, $a > \mathcal{A}_d$. Let $(u_0,u_1) \in H^s_a \times H^{s-1}_a$ with $s \geq \frac{d-2}{2}$. Furthermore, assume that u_0 and u_1 are radial. Then the equation QKG(a) (1.0.1) has a global solution in $C(\mathbf{R}, H^s_a) \cap C(\mathbf{R}, H^{s-1}_a)$ that is unique in the space $X^s([0,\infty))$ and scatters as $t \to \pm\infty$.*

Finally, for the quadratic Klein-Gordon equation with inverse-square potential, we study the 4D large energy problem in the radial setting. That is, the dynamics of the QKG(a) below the ground state. This is mainly following the work of Payne-Sattinger [**48**], Ibrahim-Masmoudi-Nakanishi [**30**] and Guo-Shen [**22**] in the potential-free case. The aforementioned breakdown of rearrangement complicates the picture, and in particular, it complicates the existence of the ground state Q_{a^*} (see (5.2.3) below). In this problem, dispersive estimates are essential. However, for negative coefficients a, these are only available in a weaker form — see Zheng [**67**]. The idea here is to convert the large energy problem is a small energy problem after large enough time. The Virial-Morawetz argument of Dodson-Murphy [**14**] is central to this study.

THEOREM 1.1.4 (4D radial large energy scattering). *Let $d = 4$ and $\kappa > 0$ be a sufficiently small constant. Furthermore, suppose that $a > \sigma^{-1}\left(\frac{1}{2}\right) = -1 + \frac{1}{4}$. Suppose that (u_0,u_1) is radial and satisfies*

$$E_a(u_0,u_1) < E(Q_{a^*}, 0).$$

Then, we have the following dichotomy:

(i) *If $\|u_0\|_2 > \|Q_{a^*}\|_{L^2}$, then the solution to (1.0.1) blows up in finite time.*

(ii) *If* $\|u_0\|_2 < \|Q_{a^*}\|_{L^2}$, *then the solution to (1.0.1) satisfies*

$$u(t,x) \in C(\mathbf{R}, H_a^1) \cap \left(\frac{1}{2}, \frac{5}{14} - \kappa, \frac{3}{7} - 4\kappa | \frac{11}{14} + \kappa\right)$$

and scatters in the sense that

$$\left\| u - i\langle D_a\rangle^{-1}\partial_t u - e^{it\langle D_a\rangle} u_\pm \right\|_{H_a^1} \to 0$$

when $t \to \pm\infty$ *and for some* $u_\pm(x) \in H_a^1$.

We also have the following results for the 2D non-linear Schrödinger (NLS) and non-linear Klein-Gordon (NLKG) with inverse-square potential and with exponential-type non-linearity (defined in (6.0.1) and (6.0.2)). We use similar methods to the quadratic Klein-Gordon equation case. In particular, the Virial-Morawetz arguments will be important in this study as well. We also use many ideas from Guo-Shen [**23**] and Ibrahim-Masmoudi-Nakanishi [**30**] to obtain the following result. We shall discuss this result in more detail in Chapter 6.

THEOREM 1.1.5. *Suppose that* $u_0 \in H_a^1(\mathbf{R}^2)$, (α,β) *satisfies conditions (6.3.1),* $m_{\alpha,\beta}$ *is defined by (6.3.7) and* κ_a^* *is a constant defined in Proposition 6.1.8 below. Recall also* $\kappa_0 > 0$ *and* $f(u) := \lambda\left(e^{\kappa_0|u|^2} - 1 - \kappa_0|u|^2\right)u$. *Then,*

(a) *If* $\lambda = -1$, *the solution to (6.0.1) exists globally and scatters provided* $E_S(u_0) < \frac{\kappa_a^*}{2\kappa_0}$.
(b) *If* $\lambda = 1$, *the solution to (6.0.1) exists globally and scatters provided that* $E_S(u_0) + M(u)/2 < m_{\alpha,\beta}$ *and* $K_{\alpha,\beta}(u_0) > 0$, *and* $a > 1$ *or sufficiently close to zero.*
(c) *If* $\lambda = -1$, *the solution to (6.0.2) exists globally and scatters provided* $E_K(u_0) < \frac{\kappa_a^*}{2\kappa_0}$.
(d) *If* $\lambda = 1$, *the solution to (6.0.2) exists globally and scatters provided that* $E_K(u_0) < m_{\alpha,\beta}$ *and* $K_{\alpha,\beta}(u_0) > 0$ *for* $a > 1$ *or sufficiently close to zero.*

1.2. Summary of notation

- $\mathscr{L}_a = -\Delta + \frac{a}{|x|^2}$ and $D_a := \sqrt{\mathscr{L}_a}$
- If $A \leq CB$, then write $A \lesssim B$. If $A \leq CB$ and $B \leq C'A$, then write $A \sim B$. If the constants depend of parameters, for instance $C = C(a)$, then write $A \lesssim_a B$ and $A \sim_a B$ respectively. Also, we write $A \ll B$ when $A < cB$ for some small constant c.
- For $x \in \mathbf{R}^d$, $\langle x \rangle = \sqrt{1 + |x|^2}$.
- Let $L^p(\mathbf{R}^d)$ and $H^s(\mathbf{R}^d)$ denote the standard Lebesgue and Sobolev spaces. Furthermore, let $L^p_{\text{rad}}(\mathbf{R}^d)$ and $H^s_{\text{rad}}(\mathbf{R}^d)$ denote the respective spaces of radial functions. Also, $\mathscr{L}^p_\rho((0,\infty)) = L^p((0,\infty), \rho^{d-1}\mathrm{d}\rho)$
- We shall write $\mathscr{F}u$ to denote the Fourier transform of u.
- The Hankel transform of order ν of a radial function (in the space variable) $u(t,\rho)$ is
$$\mathcal{H}_\nu u(t,\rho) = \int_0^\infty (r\rho)^{-\frac{d-2}{2}} J_\nu(r\rho) u(t,r)\, \mathrm{d}\omega(r)$$
where $\mathrm{d}\omega(r) = r^{d-1}\mathrm{d}r$. We shall also denote the Hankel transform of u (in the space variables) by $\widehat{u}$.
- We always assume that $a > -\left(\frac{d-2}{2}\right)^2$. We define useful choices of orders of the Hankel transform: $\mu(0) = \frac{d-2}{2}$, $\mu(k) = \frac{d-2}{2} + k$, $\nu(a) = \sqrt{\mu(0)^2 + a}$ and $\nu(a,k) = \sqrt{\mu(k)^2 + a}$.
- In Chapter 4, M, N and N' denote dyadic numbers of the form 2^k where $k \in \mathbf{N}$, unless explicitly mentioned to be of the form 2^k where $k \in \mathbf{Z}$. Denote $\sum_N a_N := a_0 + \sum_{n\in\mathbf{N}} a_{2^n}$.

CHAPTER 2

Harmonic analysis associated to the inverse-square potential

2.1. Spherical harmonics decomposition

We shall study the solutions to equations of the form

$$\begin{cases} iu_t(x,t) + \omega\left(\sqrt{-\Delta + \frac{a}{|x|^2}}\right)u(x,t) = 0, & (x,t) \in \mathbf{R}^d \times \mathbf{R} \\ u(0,x) = u_0(x) \end{cases} \tag{2.1.1}$$

by decomposing functions into spherical harmonics. That is, for any $u \in L^2(\mathbf{R}^d)$, we may write

$$u(x) = \sum_{k=0}^{\infty}\sum_{l=1}^{d(k)} a_{k,l}(r)Y_{k,l}(\theta). \tag{2.1.2}$$

where for $k \in \mathbf{N} \cup \{0\}$, the set $\{Y_{k,1}(\theta), ..., Y_{k,d(k)}(\theta)\}$ is the orthogonal basis of the space of spherical harmonics of degree k on $\mathbf{S}^{d-1}$. More specifically, $Y_{k,j}(x) \in L^2(\mathbf{R}^d)$ is a homogeneous polynomial of order k – i.e. $Y(x) = |x|^k Y(x/|x|)$ which is also harmonic (i.e. $\Delta Y = 0$). We first note that if we write u in terms of its spherical harmonic decomposition, then its Fourier transform is given by

$$\mathcal{F}_x u(\xi) = \sum_{k=0}^{\infty}\sum_{l=1}^{d(k)} 2\pi i^k \left(\int_0^\infty (r\rho)^{-\frac{d-2}{2}} J_{\frac{d-2}{2}+k}(r\rho)\cdot a_{k,l}(r) r^{d-1}\,\mathrm{d}r\right) Y_{k,l}(\omega)$$

where $\xi = \rho\omega$ with $\omega \in \mathbf{S}^{d-1}$. We shall denote

$$\mu(k) = \frac{d-2}{2} + k.$$

Spherical harmonics also simplify our study of (2.1.1). In particular, let us consider $u(x) = a(r)Y(\theta)$ where $x = r\theta$ and $Y(\theta)$ is a spherical harmonic of degree k. Then,

$$-\Delta_x\left(a(r)Y(\theta)\right) = -\Delta_x\left(|x|^{-k}a(|x|)Y(x)\right)$$

Evalulating the Laplacian, and using the fact that $\Delta_x Y(x) = 0$ gives

$$-\Delta_x\left(a(r)Y(\theta)\right) = -\left(\partial_r^2 + \frac{d-1}{r}\partial_r\right)(r^{-k}a(r))Y(x) - \frac{2x}{r}\partial_r\left(r^{-k}a(r)\right)\cdot\nabla Y.$$

Now, using the fact that $x \cdot \nabla Y = kY$ and evaluating the derivatives, we find that

$$-\Delta\left(a(r)Y(\theta)\right) = \left(-\partial_r^2 - \frac{d-1}{r}\partial_r + \frac{k(k+d-2)}{r^2}\right)a(r)\cdot Y(\theta).$$

Thus, we also find that

$$\begin{aligned}\left(-\Delta+\frac{a}{|x|^2}\right)(a(r)Y(\theta)) &= \left(-\partial_r^2-\frac{d-1}{r}\partial_r+\frac{k(k+d-2)+a}{r^2}\right)a(r)\cdot Y(\theta)\\ &= \left(-\partial_r^2-\frac{d-1}{r}\partial_r+\frac{\nu(k,a)^2-\mu(0)^2}{r^2}\right)a(r)\cdot Y(\theta)\end{aligned}$$

where $\nu(k,a) := \sqrt{\mu(k)^2+a}$. We may generalise the above argument to a general function in the kth harmonic subspace (which we shall denote by $L^2_{=k}(\mathbf{R}^d)$). Thus, we find that when we restrict to the kth harmonic subspace,

$$\mathscr{L}_a u = A_{\nu(k,a)}u := \left(-\partial_r^2-\frac{d-1}{r}\partial_r+\frac{\nu(k,a)^2-\mu(0)^2}{r^2}\right)u.$$

2.2. The Hankel transform

We shall make use of spherical decomposition via the (generalised) Hankel transform of order ν defined by

$$(\mathcal{H}_\nu f)(\xi) = \int_0^\infty U_\nu(r\rho)f(r\sigma)\,\mathrm{d}\omega(r) \tag{2.2.1}$$

where $\rho=|\xi|, \sigma=\xi/|\xi|$, $\mathrm{d}\omega(r)=r^{d-1}\mathrm{d}r$ and $U_\nu(z)=z^{-\frac{d-2}{2}}J_\nu(z)$. Also, J_ν is the Bessel function of order ν defined as

$$J_\nu(z)=\frac{(z/2)^\nu}{\Gamma(\nu+1/2)\pi^{1/2}}\sum_{m=0}^\infty\frac{(iz)^m}{m!}\int_{-1}^1 t^m(1-t^2)^{\nu-1/2}\,\mathrm{d}t \tag{2.2.2}$$

for $\nu>-1/2$ and $z>0$. We also have the following properties of the Bessel functions, which we shall need in this thesis.

Lemma 2.2.1 (Properties of Bessel functions). *Let $J_\nu(z)$ be the Bessel function of order $\nu>-1/2$ as defined above. Then, for $z>0$*

(i) $|J_\nu(z)|\le Cz^\nu$ *if* $0<z\ll\sqrt{\nu+1}$
(ii) $|J_\nu(z)|\le Cz^{-\frac12}$ *if* $z\gg|\nu^2-\frac14|$.

For radial functions f, we have the following simplification for the Hankel transform:

$$(\mathcal{H}_\nu f)(\rho)=\int_0^\infty U_\nu(r\rho)f(r)\,\mathrm{d}\omega(r).$$

More generally, suppose $f\in L^2_{=k}(\mathbf{R}^d)$. We may use (2.1.2) to write

$$f(x)=\sum_{1\le l\le d(k)}a_l(r)Y_l(\theta).$$

Thus, we have

$$(\mathcal{H}_\nu f)(\xi)=\sum_{1\le l\le d(k)}\left(\int_0^\infty U_\nu(r\rho)a_l(r)\,\mathrm{d}\omega(r)\right)Y_l(\sigma).$$

We have the following properties of the Hankel transform:

Lemma 2.2.2 (see [**6**]). *Let $\mathcal{H}_\nu$ be the Hankel transform of order ν as above. We have*

(i) $\mathcal{H}_\nu$ *is its own inverse:* $\mathcal{H}_\nu=\mathcal{H}_\nu^{-1}$
(ii) $\mathcal{H}_\nu$ *is self-adjoint:* $\mathcal{H}_\nu=\mathcal{H}_\nu^*$

(iii) $\mathcal{H}_\nu$ *is an* L^2*-isometry:* $\|\mathcal{H}_\nu f\|_{L^2} = \|f\|_{L^2}$ *for all* $f \in L^2$*.*
(iv) $\mathcal{H}_\nu(A_\nu\phi)(\xi) = |\xi|^2(\mathcal{H}_\nu\phi)(\xi)$ *for all* $\phi \in L^2$

Let us also record some further properties of the Hankel transform. First, let us discuss the following convolution theorem for the Hankel transform.

LEMMA 2.2.3 (Generalised convolution for the Hankel transform). *Suppose that* $f, g \in L^1(\mathbf{R}^d)$ *are radial. Then, define*

$$f\#_\nu g := \int_0^\infty \tau_x f(y) \cdot g(y)\, \mathrm{d}y$$

where $\tau_x f$ *is the generalised Hankel translation:*

$$\tau_x f(y) := \int_0^\infty f(z) D_\nu(x, y, z)\, \mathrm{d}z,$$

and $D_\nu(x, y, z)$ *is given by*

$$\int_0^\infty U_\nu(x\eta)U_\nu(y\eta)U_\nu(z\eta)\, \mathrm{d}\omega(\eta).$$

Furthermore,

$$\mathcal{H}_\nu(f\#_\nu g) = \mathcal{H}_\nu(f) \cdot \mathcal{H}_\nu(g).$$

Hence, using the fact that $\mathcal{H}_\nu^{-1} = \mathcal{H}_\nu$ *we also obtain that*

$$\mathcal{H}_\nu(fg) = \mathcal{H}_\nu(f)\#_\nu\mathcal{H}_\nu(g).$$

For convenience, we will often omit the subscript ν.

PROOF. The fact that the convolution is zero unless there exists a triangle with side lengths x, y and z follows from [**61**]. Using the definition of the Hankel transform, we see that $f\#g = \mathcal{H}_\nu(\mathcal{H}_\nu(f)\cdot\mathcal{H}_\nu(g))$. Thus, apply the Hankel transform to both sides, we conclude that $\mathcal{H}_\nu(f\#g) = \mathcal{H}_\nu(f) \cdot \mathcal{H}_\nu(g)$. $\square$

REMARK 2.2.4. *In the next section, we discuss an adapted Littlewood-Paley theory. Perhaps one strategy to obtain results such as Bernstein estimates would be to first obtain Young's inequality for the above convolution and then to adapt proofs from the Fourier setting that use convolution arguments. However, we could not obtain Young's inequality. The issue lies in the Hankel translation. Ideally, one could show that*

$$\|\tau_x f(y)\|_{\mathscr{L}^p_y} \lesssim \|f(y)\|_{\mathscr{L}^p_y}.$$

However, if $a < 0$*, then this inequality does not hold even for* $p = 2$*. This is because* $\tau_x f(y) = \mathcal{H}_\nu\left(U(x\rho)\mathcal{H}_\nu f\right)$ *but* $U(z) \notin L^\infty$*, so we cannot apply Hölder's inequality to obtain*

$$\|\tau_x f(y)\|_{\mathscr{L}^2_y} = \|U(x\rho)\mathcal{H}_\nu f\|_{\mathscr{L}^2_\rho} \lesssim \|f(y)\|_{\mathscr{L}^2_y}.$$

2.3. Hankel multipliers and adapted Littlewood-Paley theory

We shall need to consider operators of the form $\omega\left(D_a\right)$. In particular, we shall need both the L^p boundedness of these operators, as well as an explicit representation of these operators as multipliers with respect to the Hankel transform.

Since we shall always study a function u via its spherical decomposition, let us suppose that $u \in L^2_{=k}(\mathbf{R}^d)$. As we saw previously, we have that $\mathscr{L}_a u = A_{\nu(k,a)}u$. Hence, we may appeal to the spectral theory of operator $A_{\nu(k,a)}$ to obtain that

$$\omega\left(D_a\right)u = \omega\left(\sqrt{A_{\nu(k,a)}}\right)u = \mathcal{H}_{\nu(k,a)}\left\{\omega(|\xi|)\mathcal{H}_{\nu(k,a)}u\right\}.$$

More specifically, we obtain from Lemma 2.2.2(1) that

$$u(x) = \mathcal{H}_{\nu(k,a)}\left(\mathcal{H}_{\nu(k,a)}u\right)(x) = \int_0^\infty U_{\nu(k,a)}\left(r|x|\right)\left(\mathcal{H}_{\nu(k,a)}u\right)\left(r\frac{x}{|x|}\right) r^{d-1}\,\mathrm{d}r.$$

Thus, we obtain a resolution of the identity based on the Hankel transform:

$$I = \int_0^\infty E_0^{\nu(k,a)}(r)\,\mathrm{d}r$$

where

$$E_0^{\nu(k,a)}(r)u(x) = U_{\nu(k,a)}\left(r|x|\right)\left(\mathcal{H}_{\nu(k,a)}u\right)\left(r\frac{x}{|x|}\right) r^{d-1}.$$

Now, from Lemma 2.2.2(4) for $u \in L^2_{=k}(\mathbf{R}^d)$, we see that $A_{\nu(k,a)}u = \mathcal{H}_{\nu(k,a)}|\xi|^2\mathcal{H}_{\nu(k,a)}u$ and

$$\mathscr{L}_a = A_{\nu(k,a)} = \int_0^\infty r^2\, E_0^{\nu(k,a)}(r)\mathrm{d}r.$$

Hence, we may define operators in terms of $\sqrt{A_{\nu(k,a)}}$:

$$\omega\left(\sqrt{A_{\nu(k,a)}}\right) = \int_0^\infty \omega(r)\, E_0^{\nu(k,a)}(r)\mathrm{d}r.$$

This may be rewritten as

$$\omega\left(\sqrt{A_{\nu(k,a)}}\right)u(x) = \int_0^\infty U_{\nu(k,a)}\left(r|x|\right)\omega(r)\left(\mathcal{H}_{\nu(k,a)}u\right)\left(r\frac{x}{|x|}\right) r^{d-1}\,\mathrm{d}r.$$

Once again applying Lemma 2.2.2(1), we see that

$$\mathcal{H}_{\nu(k,a)}\left(\omega\left(\sqrt{A_{\nu(k,a)}}\right)u\right) = \omega(|\xi|)\mathcal{H}_{\nu(k,a)}(u) = \omega(\rho)\mathcal{H}_{\nu(k,a)}(u).$$

This gives us an explicit representation for multipliers $\omega(D_a)$.

Next, we discuss L^p boundedness. We have the following Mikhlin-type result from Killip et al. [**33**]:

PROPOSITION 2.3.1 (Mikhlin multipliers of D_a). *Let* $\omega : [0,\infty) \to \mathbf{C}$ *such that* $|\partial^j\omega(\lambda)| \lesssim \lambda^{-j}$ *for all* $j \geq 0$ *and either*

(i) $a \geq 0$ *and* $1 < p < \infty$ *or*

(ii) $-\mu(0)^2 \leq a < 0$ *and* $r_0 < p < r_0' := \frac{d}{\sigma}$ *and* $\sigma = \frac{d-2}{2} - \sqrt{\left(\frac{d-2}{2}\right)^2 + a}$.

Then, $\omega\left(D_a\right)$ *extends to a bounded operator on* $L^p(\mathbf{R}^d)$.

Now, we specialise further and summarise some basic Littlewood-Paley theory adapted to $\mathscr{L}_a$. Let $\phi : [0,\infty) \to [0,1]$ be a smooth function with $\phi(\lambda) = 1$ on $0 \leq \lambda \leq 1$ and $\phi(\lambda) = 0$ for $\lambda \geq 2$. From this, we define $\phi_k(\lambda) = \phi(\lambda/2^k)$ and also $\psi_k(\lambda) = \phi_k(\lambda) - \phi_{k-1}(\lambda)$. Thus, we may define for a radial function u and with $\nu = \nu(a)$ that

$$P^a_{\leq k}u = \mathcal{H}_\nu\phi_k(\rho)\mathcal{H}_\nu u,$$

$$P^a_k u = \mathcal{H}_\nu\psi_k(\rho)\mathcal{H}_\nu u$$

$$P^a_{>k}u = 1 - P^a_{\leq k}u.$$

We may follow the argument of [**59**, Theorem 4.2.2] to conclude that the function space is independent of the choice of ϕ.

The following Bernstein estimates will be important in this work:

Proposition 2.3.2 (Bernstein estimates for P_k^a, [**33**]). *Let $1 < p \le q \le \infty$ for $a \ge 0$ and $r_0 < p \le q < r_0'$ when $-\mu(0)^2 \le a < 0$ and $u \in C_c^\infty(\mathbf{R}^d \backslash \{0\})$. Then,*

(i) *The operators $P_{\le k}^a$, P_k^a are bounded on L^p,*
(ii) *For all $s \in \mathbf{R}$, $\|D_a^s P_k^a u\|_{L^p(\mathbf{R}^d)} \sim 2^{ks} \|P_k^a u\|_{L^p(\mathbf{R}^d)}$.*
(iii) $\|P_k^a u\|_{L^q(\mathbf{R}^d)} \le 2^{kd\left(\frac{1}{p}-\frac{1}{q}\right)} \|P_k^a u\|_{L^p(\mathbf{R}^d)}$
(iv) $\|P_{\le k}^a u\|_{L^q(\mathbf{R}^d)} \le 2^{kd\left(\frac{1}{p}-\frac{1}{q}\right)} \|P_{\le k}^a u\|_{L^p(\mathbf{R}^d)}$

Proof. See [**33**]. □

From this result, we see that we have the commutativity between $\mathcal{L}_a$ and the projector to convert an L^2 estimate to an H_a^s estimate. In addition, we also have the following Littlewood-Paley square function theorem:

Proposition 2.3.3 (Littlewood-Paley square function theorem, [**33**]). *Let $s \ge 0$ and also $1 < p \le q \le \infty$ for $a \ge 0$ and $r_0 < p \le q < r_0'$ when $-\mu(0)^2 \le a < 0$. Then, for any $u \in C_c^\infty(\mathbf{R}^d \backslash \{0\})$. we have that*

$$\|D_a^s u\|_{L^p(\mathbf{R}^d)} \sim \left\| \left(\sum_{k \in \mathbf{Z}} 2^{2ks} |P_k^a u|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbf{R}^d)}.$$

The Mikhlin multiplier theorem above can be used to obtain the following boundedness result for Coifman-Meyer type bilinear multipliers.

Proposition 2.3.4. *Let $\lambda = (\lambda_1, \lambda_2) \in (0, \infty)^2$. Suppose that $m(\lambda)$ is such that for some $s \in \mathbf{N}$, we have*

$$|\partial^\alpha m(\lambda)| \lesssim |\lambda|^{-|\alpha|} \tag{2.3.1}$$

for all partial derivatives with multi-indices $|\alpha| \le s$. Define the operator

$$T_m(f,g)(x) = \int_0^\infty \int_0^\infty m(u,v) U_\nu(ux) U_\nu(vx) \mathcal{H}_\nu f(u) \mathcal{H}_\nu g(v) \, \mathrm{d}\omega(u) \mathrm{d}\omega(v). \tag{2.3.2}$$

Then, for $p, q, r \in (1, \infty)$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ (and also $r_0 < q, r < r_0'$ if $a < 0$), we have

$$\|T_m(f,g)\| \lesssim \|f\|_q \|g\|_r. \tag{2.3.3}$$

Proof. We shall follow the proof in [**64**, Theorem 2.3]. Indeed, we first rewrite T_m as $T_m(f,g)(x) = m(L_1, L_2)(f \otimes g)(x,x)$. Here, $L_1 = \sqrt{\mathcal{L}_a} \otimes I$ and $L_2 = I \otimes \sqrt{\mathcal{L}_a}$, where I is the identity operator. Furthermore, we note that this recharacterisation for T_m can be understood via the joint spectral measure of (L_1, L_2) – see [**64**] for more details.

Let us write $F := f \otimes g$ and let ψ be a smooth function with support in $[1/2, 1]$ such that $\sum_k \psi_k = 1$. Here, we use the notation that $\psi_k(\lambda) := \psi(2^{-k}\lambda)$ for all $\lambda \in [0, \infty)$.

To obtain the result, the idea is the decompose T_m as follows:

$$\begin{aligned} T_m(f,g)(x) &= \sum_{k_1, k_2 \in \mathbf{Z}} (\psi_{k_1}(L_1) \psi_{k_2}(L_2) m(L_1, L_2))(F)(x,x) \\ &= T_1 + T_2 + T_3, \end{aligned}$$

where

$$\begin{aligned}
T_1 &:= \sum_{|k_1-k_2|\le b+2} (\psi_{k_1}(L_1)\psi_{k_2}(L_2)m(L_1,L_2))(F)(x,x),\\
T_2 &:= \sum_{k_1>k_2+b+2} (\psi_{k_1}(L_1)\psi_{k_2}(L_2)m(L_1,L_2))(F)(x,x),\\
T_3 &:= \sum_{k_2>k_1+b+2} (\psi_{k_1}(L_1)\psi_{k_2}(L_2)m(L_1,L_2))(F)(x,x).
\end{aligned}$$

We note that in [**64**], the functions f and g can be taken to belong to a class $\mathcal{A}$ which is dense in L^p for $p \in (1,\infty)$ and such that these sums are finite.

Let us first consider T_1. We further decompose

$$T_1 = \sum_k m_k(\lambda), \tag{2.3.4}$$

where

$$m_k(\lambda) = \psi_k(\lambda_1)\phi_k(\lambda_2)m(\lambda)$$

and $\phi_k(\lambda_2) = \sum_{|k_2-k|\le b+2} \psi_k(\lambda_2)$. By noting the supports of ϕ and ψ, we see that m_k has support in $[2^{k-b-4}, 2^{k+b+4}]^2$. Thus, we may also write

$$m_k(\lambda) = (\tilde{\psi}(\lambda_1)\tilde{\psi}(\lambda_2))\psi_k(\lambda_1)\phi_k(\lambda_2)m(\lambda), \tag{2.3.5}$$

where $\tilde{\psi}$ is a smooth function that equals one on $[2^{-b-3}, 2^{b+3}]$ and vanishes outside of $[2^{-b-4}, 2^{b+4}]$. Consider $M_k(\lambda) := m_k(2^k\lambda)$. Thus, M_k has support in $[-2^{b+4}, 2^{b+4}]$. Define $a = 2^{b+4}$. Then, we may expand M_k via a double Fourier series as

$$M_k(\lambda) = \sum_{n_1,n_2\in\mathbf{Z}} c_{n,k} e^{\frac{\pi i n_1 \lambda_1}{a}} e^{\frac{\pi i n_2 \lambda_2}{a}},$$

where the coefficients are given by

$$c_{n,k} = \frac{1}{4a^2}\int_{-a}^{a}\int_{-a}^{a} (\psi\otimes\phi)(m(2^k\xi)) e^{\frac{\pi i n_1 \xi_1}{a}} e^{\frac{\pi i n_2 \xi_2}{a}}\, \mathrm{d}\xi_1 \mathrm{d}\xi_2.$$

Now, we may apply integration by parts, the assumption (2.3.1) and use the fact that $\psi\otimes\phi$ is compactly supported away from zero to obtain

$$|c_{n,k}| \lesssim (1+|n|)^{-s}$$

for all $n \in \mathbf{Z}^2$ and uniform in k. From this, we use (2.3.5) to obtain

$$m_k(\lambda) = \sum_{n\in\mathbf{Z}^2} c_{n,k} \left(\tilde{\psi}_k(\lambda_1) e^{\frac{2\pi}{a} i n_1 2^{-k}\lambda_1}\right)\left(\tilde{\psi}_k(\lambda_2) e^{\frac{2\pi}{a} i n_2 2^{-k}\lambda_2}\right).$$

Thus, we have that

$$\begin{aligned}
m_k(L_1,L_2) &= \sum_{n\in\mathbf{Z}^2} c_{n,k} \left(\tilde{\psi}_k(L_1) e^{\frac{2\pi}{a} i n_1 2^{-k} L_1}(f)\right)\left(\tilde{\psi}_k(L_2) e^{\frac{2\pi}{a} i n_2 2^{-k} L_2}(g)\right)\\
&=: \sum_{n\in\mathbf{Z}^2} c_{n,k}\, (\psi_k^{n_1}(L_1) f)\, (\psi_k^{n_2}(L_2) g)
\end{aligned}$$

with convergence in L^2. Thus, recalling (2.3.4), we also have that

$$T_1(f,g)(x) = \sum_{n\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} c_{n,k} \psi_k^{n_1}(L)(f)(x) \psi_k^{n_2}(L)(g)(x).$$

Here, we recall that f and g belong to a class $\mathcal{A}$ such that the sum in k is finite. Finally, for T_1, we apply Proposition 2.3.3 and deal with the factors $e^{\frac{2\pi}{a} i n_j 2^{-k}\lambda_j}$ $(j=1,2)$ to obtain for $s > 2\rho + 4$ that

$$\begin{aligned} \|T_1(f,g)\|_p &\lesssim \sum_{n\in\mathbf{Z}^2} (1+|n|)^{-s} \left\| \left(\sum_{k\in\mathbf{Z}} \psi_k^{n_1}(L)(f)(x) \right)^2 \right\|_q \left\| \left(\sum_{k\in\mathbf{Z}} \psi_k^{n_2}(L)(g)(x) \right)^2 \right\|_r \\ &\lesssim \sum_{n\in\mathbf{Z}^2} (1+|n|)^{-s}(1+|n_1|)^{\rho}(1+|n_2|)^{\rho}\|f\|_q\|g\|_r \lesssim \|f\|_q\|g\|_r. \end{aligned}$$

Now, we study the term T_2. The argument for the term T_3 follows by symmetry. Here, we define $\phi_k := \sum_{k_2 < k-b-2} \psi_{k_2}$. Then, we can write

$$\begin{aligned} T_2 &= \sum_{k_1 > k_2+b+2} (\psi_{k_1}(L_1)\psi_{k_2}(L_2)m(L_1,L_2))(F)(x,x) \\ &= \sum_k (\psi_k(L_1)\phi_k(L_2)m(L_1,L_2))(F) \\ &= \sum_k m_k(L_1,L_2)(F), \end{aligned}$$

where, in this case, we set $m_k(\lambda) := \psi_k(\lambda_1)\phi_k(\lambda_2)m(\lambda)$. Thus, m_k is supported in $[2^{k-1}, 2^{k+1}] \times [0, 2^{k-b-1}]$. Hence, we may write

$$m_k(\lambda_1,\lambda_2) = \tilde{\psi}(\lambda_1)\tilde{\phi}(\lambda_2)\psi_k(\lambda_1)\phi_k(\lambda_2)m(\lambda)$$

where $\tilde{\psi}$ is a smooth function that equals to one on $[2^{-1}, 2^1]$ and vanishes outside of $[2^{-2}, 2^2]$, and $\tilde{\phi}$ is a smooth function that equals to one on $[0, 2^{-b-1}]$ and vanishes outside of $[0, 2^{-b}]$. Again, similar to for the T_1 term, we expand $M_k(\lambda) = m_k(2^k\lambda)$ via a double Fourier series. We note that $M_k(\lambda)$ is supported in $[-2,2]^2$. Thus, we obtain

$$M_k(\lambda) = \sum_{n_1,n_2\in\mathbf{Z}} c_{n,k} e^{\frac{\pi i n_1 \lambda_1}{2}} e^{\frac{\pi i n_2 \lambda_2}{2}},$$

where the coefficients are given by

$$c_{n,k} = \frac{1}{16}\int_{-2}^{2}\int_{-2}^{2} (\psi\otimes\phi)(m(2^k\xi)) e^{\frac{\pi i n_1 \xi_1}{2}} e^{\frac{\pi i n_2 \xi_2}{2}}\, \mathrm{d}\xi_1 \mathrm{d}\xi_2.$$

Now, by using integration by parts and the assumption (2.3.1), we obtain

$$|c_{n,k}| \lesssim (1+|n|)^{-s}$$

for all $n \in \mathbf{Z}^2$ and uniform in k.

Thus, similar to above, we have

$$\begin{aligned} T_2(f,g)(x) &= \sum_{n\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} c_{n,k}\psi_k^{n_1}(L)(f)(x)\phi_k^{n_2}(L)(g)(x) \\ &= \sum_{n\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} c_{n,k}\tilde{\psi}_k(L)\left(\psi_k^{n_1}(L)(f)\phi_k^{n_2}(L)(g)\right)(x) \end{aligned}$$

where $\psi_k^{n_1}(\lambda_1) = \tilde{\psi}_k(\lambda_1)e^{\frac{\pi}{2} i n_1 2^{-k}\lambda_1}$, and similarly for $\phi_k^{n_2}(\lambda_2)$. Furthermore, $\tilde{\psi}$ is now taken to be a smooth function which is equal to one on $[2^{-3-b}, 2^{3+b}]$ and vanishes outside of $[2^{-5-b}, 2^{5+b}]$.

Now, we estimate the L^p norm of T_2. Thus, let $h \in L^{p'}$. We observe that

$$\int_0^\infty T_2(f,g)(x)h(x)\, \mathrm{d}\omega(x)$$

is equal to

$$\int_0^\infty \sum_{n\in\mathbf{Z}} \sum_{k\in\mathbf{Z}} c_{n,k}\psi_k^{n_1}(L)(f)(x)\phi_k^{n_2}(L)(g)(x)\tilde{\psi}_k(L)(h)(x)\, \mathrm{d}\omega(x).$$

Thus, using the estimate for $|c_{n,k}|$, we see that

$$\left|\int_0^\infty T_2(f,g)(x)h(x)\, \mathrm{d}\omega(x)\right|$$

is bounded above by

$$\sum_{n\in\mathbf{Z}^2}(1{+}|n|)^{-s}\int_0^\infty \left(\sum_{k\in\mathbf{Z}}|\psi_k^{n_1}(L)(f)|^2\right)^{\frac{1}{2}} \sup_{k\in\mathbf{Z}}\phi_k^{n_2}(L)(g)\left(\sum_{k\in\mathbf{Z}}|\tilde{\psi}_k(L)(h)|^2\right)^{\frac{1}{2}} \mathrm{d}\omega(x).$$

Now, applying Hölder's inequality, this expression is bounded above by

$$\sum_{n\in\mathbf{Z}^2}(1+|n|)^{-s}\left\|\left(\sum_{k\in\mathbf{Z}}|\psi_k^{n_1}(L)(f)|^2\right)^{\frac{1}{2}}\right\|_q \left\|\sup_{k\in\mathbf{Z}}\phi_k^{n_2}(L)(g)\right\|_r.$$

The L^q norm is dealt with the same way as in the T_1 case, while the L^r is bounded above by $(1+|n_2|)^{\rho+2}\|g\|_r$. Indeed, after dealing with the factor as in the T_1 case, the remaining estimate is a corollary of the multiplier theorem (Proposition 2.3.1) – see [**63**]. Thus, we may put together the results for T_1, T_2 and T_3 to obtain the required estimate. □

2.4. Adapted Sobolev and Besov spaces

We also have the following equivalence of Sobolev spaces:

PROPOSITION 2.4.1 (Equivalence of Sobolev norms, [**33**]). *Let $d \geq 3$, $a \geq -\mu(0)^2$ and also $0 < s < 2$. If $1 < p < \infty$ satisfies $\frac{s+\sigma}{d} < \frac{1}{p} < \min\{1, \frac{d-\sigma}{d}\}$ then*

$$\|(-\Delta)^{\frac{s}{2}}u\|_{L^p} \lesssim_{d,p,s} \|D_a^s u\|_{L^p} \text{ for } u \in C_c^\infty(\mathbf{R}^d\backslash\{0\}). \tag{2.4.1}$$

If $\max\{\frac{s}{d}, \frac{\sigma}{d}\} < \frac{1}{p} < \min\{1, \frac{d-\sigma}{d}\}$ then

$$\|D_a^s u\|_{L^p} \lesssim_{d,p,s} \|(-\Delta)^{\frac{s}{2}}u\|_{L^p} \text{ for } u \in C_c^\infty(\mathbf{R}^d\backslash\{0\}).$$

For our purposes, we shall define the following adapted inhomogeneous Besov space. We remark that for $-\mu(0)^2 < a < 0$, we only define these spaces for $r_0 < p < r_0'$, as this is the range where the Bernstein estimates (Proposition 2.3.2) hold. Therefore, we are able to obtain Sobolev embeddings in this range.

$$\|u\|_{B^s_{p,q}} := \|P^a_{\leq 0}u\|_p + \left(\sum_{k\geq 0} 2^{qsk}\|P^a_k u\|_p^q\right)^{\frac{1}{q}}$$

and the following homogeneous Besov space

$$\|u\|_{\dot{B}^s_{p,q}} := \left(\sum_{k\in\mathbf{Z}} 2^{qsk}\|P^a_k u\|_p^q\right)^{\frac{1}{q}}.$$

We write $\dot{B}^s_p := \dot{B}^s_{p,2}$. We also have the following embeddings, which shall be used when we define Besov-type spaces below.

LEMMA 2.4.2 (Embeddings for adapted Besov spaces). *For $a \geq 0$, let $1 \leq p_0 \leq \infty$ and for $-\mu(0)^2 < a < 0$, let $r_0 < p_0 < p_1 < r_0'$. Also, let $1 \leq q \leq \infty$ and $s_1 < s_0$. Then,*

(i) $B^{s_0}_{p_0,q} \hookrightarrow B^{s_1}_{p_1,q}$.
(ii) *If* $s_0 - \frac{d}{p_0} = s_1 - \frac{d}{p_1}$, *then* $\dot{B}^{s_0}_{p_0,q} \hookrightarrow \dot{B}^{s_1}_{p_1,q}$.
(iii) *If* $s_0 - \frac{d}{p_0} \geq s_1 - \frac{d}{p_1}$, *then* $B^{s_0}_{p_0,q} \hookrightarrow B^{s_1}_{p_1,q}$.

PROOF. Property (i) follows from inspection of the definition. Property (iii) follows a similar argument for (ii). Thus, we are left to prove (ii). This follows via the Bernstein estimates (Proposition 2.3.2). Indeed, we have $\|P^a_{\leq 0} u\|_{L^{p_1}} \lesssim \|P^a_{\leq 0} u\|_{L^{p_0}}$. Furthermore, for $k \geq 0$, we use that

$$\|P^a_k u\|_{L^{p_1}} \lesssim 2^{kd\left(\frac{1}{p_0} - \frac{1}{p_1}\right)} \|P^a_k u\|_{L^{p_0}}.$$

Using these facts in the definition gives the result. □

For our purposes, we shall also need the following Besov-type space adapted the operator D_a. Here, we shall use the adapted Littlewood-Paley projections P^a_k in place of the usual Littlewood-Paley projections, as in [**22**] (again note that for $-\mu(0)^2 < a < 0$, we shall only define these Besov-type spaces for $r_0 < p < r_0'$):

(2.4.2)
$$\|u\|_{\dot{B}^{s_L|s_H}_p} := \left(\sum_{k \in \mathbf{Z}, k \leq 0} 2^{2s_L k} \|P^a_k u(x)\|^2_p \right)^{\frac{1}{2}} + \left(\sum_{k \in \mathbf{Z}, k \geq 0} 2^{2s_H k} \|P^a_k u(x)\|^2_p \right)^{\frac{1}{2}}.$$

Notice that this definition is consistent in light of the Bernstein estimates above. Notice also that this definition agrees with the definition for a homogeneous Besov space above for u localised to low or to high frequencies. In this thesis, we shall use these Besov-type spaces in the context of the space-time norm

$$\|u\|_{\left(\frac{1}{q}, \frac{1}{r}, s\right)_I} := \|u(t,x)\|_{L^q_t(I, \dot{B}^s_r)}.$$

and

$$\|u\|_{\left(\frac{1}{q}, \frac{1}{r}, s_L|s_H\right)_I} := \|u(t,x)\|_{L^q_t(I, \dot{B}^{s_L|s_H}_r)}.$$

We have the following embeddings which follow from the definition of the spaces $\dot{B}^{s_L|s_H}_p$. In this thesis, we shall always work using dyadic decomposition, and so it is sufficient to consider the embeddings for low and high frequencies individually.

LEMMA 2.4.3. *For the same range of p as in Lemma 2.4.2, for any $u \in \dot{B}^{s_L|s_H}_p$, and either*

(i) *(low-frequency embedding)* $s^L_1 \leq s^L_0$, *and* $k \leq 0$ *or*
(ii) *(high-frequency embedding)* $s^H_0 \leq s^H_1$, *and* $k \geq 0$ *then*

$$\|P^a_k u\|_{\dot{B}^{s^L_0|s^H_0}_p} \lesssim \|P^a_k u\|_{\dot{B}^{s^L_1|s^H_1}_p} \tag{2.4.3}$$

We shall later denote the low-frequency embedding by

$$\dot{B}^{s^L_1|s^H_1}_p \hookrightarrow_L \dot{B}^{s^L_0|s^H_0}_p,$$

and similarly for the high-frequency embedding.

We also have the following Sobolev embedding.

LEMMA 2.4.4 (Sobolev embedding for Besov-type spaces). *For the same range of p_0, p_1 as in Lemma 2.4.2 and either*

(i) *(low-frequency embedding)* $s_0^L - \frac{d}{p_0} \leq s_1^L - \frac{d}{p_1}$, *and* $k \leq 0$ *or*

(ii) *(high-frequency embedding)* $s_0^H - \frac{d}{p_0} \geq s_1^H - \frac{d}{p_1}$, *and* $k \geq 0$ *then*

$$\|P_k^a u\|_{\dot{B}_{p_1}^{s_1^L | s_1^H}} \lesssim \|P_k^a u\|_{\dot{B}_{p_0}^{s_0^L | s_0^H}}. \tag{2.4.4}$$

PROOF. This follows from localisation, the above Sobolev embedding and Lemma 2.4.3. □

CHAPTER 3

3D small energy scattering: radial case

3.1. Generalised Strichartz estimates for a class of equations

3.1.1. Setup. In this section, we shall obtain generalised Strichartz estimates for the equation

$$\begin{cases} iu_t(x,t) + \omega\left(\sqrt{-\Delta + \frac{a}{|x|^2}}\right) u(x,t) = 0, \quad (x,t) \in \mathbf{R}^d \times \mathbf{R} \\ u(0,x) = P^a_k u_0(x). \end{cases} \tag{3.1.1}$$

Here, $\omega\left(D_a\right) u = \mathcal{H}_{\nu(a)}\left(\omega(\rho)\mathcal{H}_{\nu(a)} u\right)$, $u_0(x) : \mathbf{R}^d \to \mathbf{C}$ is radial and $\omega : \mathbf{R}^+ \to \mathbf{R}$ is a C^3-smooth function. Recall that we only consider $a > -\left(\frac{d-2}{2}\right)^2$ and $d \geq 3$. We need to impose suitable conditions for ω in order to obtain improved estimates. It shall be seen that the following conditions will work. In fact, they are the same as those in [**17**]:

(a(k)) There exists an $\alpha \in \mathbf{R}$ such that for $r \in (2^{k-1}, 2^{k+1})$

$$|\omega'(r)| \gtrsim 2^{k(\alpha-1)} \tag{3.1.2}$$

(b(k)) In addition to (a(k)), there exists $\beta \in \mathbf{R}$ such that $\alpha \geq \beta$ if $k \geq 0$ and otherwise $\alpha \leq \beta$ if $k < 0$ such that for $r \in (2^{k-1}, 2^{k+1})$,

$$|\omega''(r)| \gtrsim 2^{k(\beta-2)} \tag{3.1.3}$$

and $|\omega''(r)/\omega'(r)| \lesssim 2^{-k}$ for $r \in (2^{k-1}, 2^{k+1})$.

For ω satisfying these conditions, we shall obtain the following estimates:

THEOREM 3.1.1 (Radial refined Strichartz estimates). *Let $d \geq 3$, $k \in \mathbf{Z}$, $2 \leq q, r \leq \infty$ and $u_0 \in L^2_{\mathrm{rad}}(\mathbf{R}^d)$ and $a > -\left(\frac{d-2}{2}\right)^2$, and furthermore, if $-\left(\frac{d-2}{2}\right)^2 < a < 0$, impose also that $r_0 < r < r_0' = \frac{d}{\sigma}$. Then,*

(a) *(General region) if $q\left(\frac{1}{2} - \frac{1}{r}\right) > \frac{1}{d-1}$ and ω satisfes (3.1.2). Then,*

$$\|e^{it\omega(D_a)} P^a_k u_0\|_{L^q_t L^r_x} \lesssim 2^{k\left(\frac{d}{2} - \frac{d}{r} - \frac{\alpha}{q}\right)} \|u_0\|_{L^2_x(\mathbf{R}^d)}.$$

(b) *(Refined region) If $\frac{2}{2d-1} < q\left(\frac{1}{2} - \frac{1}{r}\right) < \frac{1}{d-1}$ and ω satisfies (3.1.3). Then,*

$$\|e^{it\omega(D_a)} P^a_k u_0\|_{L^q_t L^r_x} \lesssim 2^{k\theta(r,q)} \|u_0\|_{L^2_x(\mathbf{R}^d)}$$

where $\theta(r,q) = \frac{d}{2} - \frac{d}{r} - \frac{\beta}{q} - (\alpha - \beta)\left(\frac{d-1}{2} - \frac{d-1}{r}\right)$.

Furthermore, along the endpoint case $q\left(\frac{1}{2} - \frac{1}{r}\right) = \frac{1}{d-1}$, we have

$$\|e^{it\omega(D_a)} P^a_k u_0\|_{L^q_t L^r_x} \lesssim \langle k(\alpha - \beta)\rangle^{\frac{2}{q}} 2^{k\theta(r,q)} \|u_0\|_{L^2_x(\mathbf{R}^d)}.$$

PROOF. We shall below prove the following estimates:

$$\|e^{it\omega(D_a)}P_k^a u_0\|_{L_t^2 L_x^r} \lesssim 2^{k\left(\frac{d}{2}-\frac{d}{r}-\frac{\alpha}{2}\right)} \|u_0\|_{L_x^2(\mathbf{R}^d)} \tag{3.1.4}$$

$$\|e^{it\omega(D_a)}P_k^a u_0\|_{L_t^2 L_x^r} \lesssim 2^{k\theta(r,2)} \|u_0\|_{L_x^2(\mathbf{R}^d)} \tag{3.1.5}$$

$$\|e^{it\omega(D_a)}P_k^a u_0\|_{L_t^2 L_x^r} \lesssim \langle k(\alpha-\beta)\rangle 2^{k\theta(r,2)} \|u_0\|_{L_x^2(\mathbf{R}^d)}\,. \tag{3.1.6}$$

We shall interpolate of the above estimates with the estimate $\|e^{it\omega(D_a)}P_k^a u_0\|_{L_t^\infty L_x^2} \lesssim \|u_0\|_{L_x^2(\mathbf{R}^d)}$. In particular, we see that to obtain an estimate on the $L_t^q L_x^r$ norm, we shall need to interpolate between the $L_t^\infty L_x^2$ estimate and the $L_t^q L_x^p$ where $1/p = 1/2 - (q/2)(1/2 - 1/r)$. Doing so, we obtain the above estimates. $\square$

Before we prove these estimates, let us make a few reductions. We shall first use the Hankel transform for radial functions u:

$$(\mathcal{H}_{\nu(a)}f)(s) = \int_0^\infty U_{\nu(a)}(s\rho)f(\rho)\,\mathrm{d}\omega(\rho).$$

First, we apply the Hankel transform to (3.1.1) to obtain a first-order ODE $i\widehat{u}_t + \omega(\rho)\widehat{u} = 0$ with initial condition $\widehat{u}(0,\rho) = \widehat{P_k^a u_0}(\rho)$. We may solve this ODE to obtain that

$$u(t,s) := e^{it\omega(D_a)}P_k^a u_0 = \mathcal{H}_{\nu(a)}\left(e^{it\omega(\rho)}\psi_k(\rho)\mathcal{H}_{\nu(a)}(u_0)\right). \tag{3.1.7}$$

Our goal is to obtain an estimate

$$\|e^{it\omega(D_a)}P_k^a u_0\|_{L_t^q L_x^r} \lesssim C(k)\|u_0\|_{L_x^2}\,.$$

Using the characterisation of u as in (3.1.7), and the fact that u is radial, this is equivalent to the estimate (in the following discussion $C(k)$ is always the same quantity):

$$\left\|\mathcal{H}_{\nu(a)}\left(e^{it\omega(\rho)}\psi_k(\rho)\mathcal{H}_{\nu(a)}(u_0)\right)\right\|_{L_t^q \mathcal{L}_s^r} \lesssim C(k)\|u_0\|_{L_x^2}\,.$$

Thus, replacing u_0 by $\mathcal{H}_{\nu(a)}(u_0)$ and recalling that $\mathcal{H}_\nu^2 = 1$, we reduce to obtaining the estimate

$$\left\|\mathcal{H}_{\nu(a)}\left(e^{it\omega(\rho)}\psi_k(\rho)u_0\right)\right\|_{L_t^q \mathcal{L}_s^r} \lesssim C(k)\|u_0\|_{L_x^2}\,.$$

Using the definition of the Hankel transform, we have

$$\left\|s^{-\frac{d-2}{2}}\int_0^\infty e^{it\omega(\rho)}J_{\nu(a)}(s\rho)\psi_k(\rho)u_0(\rho)\rho^{\frac{d}{2}}\mathrm{d}\rho\right\|_{L_t^q \mathcal{L}_s^r} \lesssim C(k)\|u_0\|_{L_x^2}\,.$$

Now, converting $\mathcal{L}_\rho^s$ to L_x^s, we instead reduce to proving

$$\left\|s^{\frac{d-1}{r}-\frac{d-2}{2}}\int_0^\infty e^{it\omega(\rho)}J_{\nu(a)}(s\rho)\psi_k(\rho)u_0(\rho)\rho^{\frac{d}{2}}\mathrm{d}\rho\right\|_{L_t^q L_s^r} \lesssim C(k)\|u_0\|_{L_x^2}\,.$$

Now, replacing $\psi_k(\rho)$ with $\psi_0(2^{-k}\rho)$, as well as applying a change of variables, we instead reduce to showing

$$\left\|2^{\frac{k}{d}}2^k s^{\frac{d-1}{r}-\frac{d-2}{2}}\int_0^\infty e^{it\omega(2^k\rho)}J_{\nu(a)}(2^k s\rho)\psi_0(\rho)u_0(2^k\rho)\mathrm{d}\rho\right\|_{L_t^q L_s^r} \lesssim C(k)\|u_0\|_{L_x^2}\,.$$

Finally, replacing $u_0(2^k\rho)$ by $u_0(\rho)$, we finally reduce to showing

$$\left\|s^{\frac{d-1}{r}-\frac{d-2}{2}}\int_0^\infty e^{it\omega(2^k\rho)}J_{\nu(a)}(2^k s\rho)\psi_0(\rho)u_0(\rho)\mathrm{d}\rho\right\|_{L_t^q L_s^r} \lesssim 2^{-k}C(k)\|u_0\|_{L_x^2}\,.$$

We shall now denote this integral by

$$T_k^{\nu(a)}\left(u_0\right)(t,s)=\int_0^\infty e^{it\omega(2^k\rho)}J_{\nu(a)}(2^k s\rho)\psi_0(\rho)u_0(\rho)\mathrm{d}\rho.$$

3.1.2. Estimates: part 1. We shall closely follow the method of [**17**] to prove Theorem 3.1.1. In fact the main point of this section and the next shall be to identify differences between the method used in [**17**] and here, and how these differences do not in fact affect the proof. First, we have in the region $|s|<2^{-k}$:

LEMMA 3.1.2. *With the notation as above,*

$$\|\chi_k(s)s^{\frac{d-1}{r}-\frac{d-2}{2}}T_k^{\nu(a)}\left(u_0\right)\|_{L_t^qL_s^r}\lesssim 2^{-k}2^{k\left(\frac{d}{2}-\frac{d}{r}\right)}2^{-\frac{k\alpha}{q}}\|u_0\|_{L_x^2}.$$

Here, we define χ_k as follows. Let $\eta:\mathbf{R}\to[0,1]$ be an even, smooth and radially decreasing function supported in $\{s:|s|\leq 8/5\}$ and such that $\eta\equiv 1$ on $|s|\leq 5/4$. For $k\in\mathbf{Z}$, we define $\chi_k(s):=\eta(s/2^k)-\eta(s/2^{k-1})$ and $\chi_{\leq k}(s):=\eta(s/2^k)$.

PROOF. See [**17**, p. 11]. Since we have that $\nu(a)=\sqrt{\left(\frac{d-2}{2}\right)^2+a}>0$, the proof can be adapted immediately to this context. $\square$

To deal with the region $|s|\geq 2^{-k}$, we decompose

$$T_k^{\nu(a)}\left(u_0\right)=\sum_{j\geq -k}T_{j,k}^{\nu(a)}(u_0)$$

where

$$T_{j,k}^{\nu(a)}(u_0)=\chi_j(s)\int_0^\infty e^{it\omega(2^k\rho)}J_{\nu(a)}(2^k s\rho)\psi_0(\rho)u_0(\rho)\mathrm{d}\rho.$$

We have the following estimates:

LEMMA 3.1.3. *Suppose that $k\in\mathbf{Z}$, ω satisfies condition (a(k)), $j\geq -k$ and $2\leq q\leq r\leq\infty$. Then, $\|T_{j,k}^{\nu}u_0\|_{L_t^qL_s^r}\lesssim 2^{-(j+k)\left(\frac{1}{2}-\frac{1}{q}\right)}2^{-\frac{k}{r}}2^{-\frac{k\alpha}{q}}\|u_0\|_{L_x^2}$.*

PROOF. See [**17**, p. 12] $\square$

Thus, we obtain that for $\frac{1}{q}<(d-1)\left(\frac{1}{2}-\frac{1}{r}\right)$

$$\begin{aligned}\|\chi_{\geq -k}(s)s^{\frac{d-1}{r}-\frac{d-2}{2}}T_k^{\nu(a)}\left(u_0\right)\|_{L_t^qL_s^r} &\lesssim \sum_{j\geq -k}2^{j\left(\frac{d-1}{r}-\frac{d-2}{2}\right)}\|T_{j,k}^{\nu(a)}u_0\|_{L_t^qL_s^r}\\ &\lesssim 2^{k\left(\frac{d-1}{r}-\frac{d-2}{2}\right)}2^{-(j+k)\left(\frac{1}{2}-\frac{1}{q}\right)}2^{-\frac{k}{r}}2^{-\frac{k\alpha}{q}}\|u_0\|_{L_x^2}\\ &\lesssim 2^{-k}2^{k\left(\frac{d}{2}-\frac{d}{r}\right)}2^{-\frac{k\alpha}{q}}\|u_0\|_{L_x^2}.\end{aligned}$$

Thus, combining this with Lemma 3.1.2, we obtain Theorem 3.1.1(a).

3.1.3. Estimates: part 2. Now, we move onto the refined estimates (part (b) of Theorem 3.1.1). We make a slight adjustment as compared to [**17**] in that we have for general ν (as opposed to $(d-2)/2$) that we may write the Bessel function $J_\nu(r)$ as

(3.1.8)

$$J_\nu(r)=\frac{e^{i\left(r-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}+e^{-i\left(r-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}}{2r^{1/2}}+C_dr^{\frac{d-2}{2}}e^{-ir}E_+(r)-C_d'r^{\frac{d-2}{2}}e^{ir}E_-(r),$$

as in [**56**]. Interestingly, we notice that changing ν amounts to a phase translation and so in fact, will not affect our estimates. More precisely, we may apply Van der Corput's lemma to obtain the same results. We recall Van der Corput's lemma:

LEMMA 3.1.4. *Suppose ϕ is a real-valued smooth function on (a,b) and that $|\phi^{(k)}(x)| \geq 1$ for all $x \in (a,b)$. Then, if $k \geq 2$ or $k=1$ and ϕ' is monotonic:*

$$\left| \int_a^b e^{i\lambda\phi(x)}\psi(x)\,\mathrm{d}x \right| \leq \frac{c_k}{\lambda^k}\left(|\psi(b)| + \int_a^b |\psi'(x)|\,\mathrm{d}x \right).$$

Thus, we may still decompose $T^{\nu}_{j,k}u_0$ into the same two components and write $T^{\nu}_{j,k}u_0 = M^{\nu}_{j,k}u_0 + E^{\nu}_{j,k}u_0$ where

$$M^{\nu}_{j,k}u_0(s) = \chi_{j+k}(s)\int e^{i(t\omega(2^k\rho)+s\rho)}\psi_0(\rho)u_0(\rho)(s\rho)^{-\frac{1}{2}}\,\mathrm{d}\rho + \text{c.c.},$$

and

$$E^{\nu}_{j,k}u_0(s) = \chi_{j+k}(s)\int e^{i(t\omega(2^k\rho)+s\rho)}\psi_0(\rho)u_0(r)(s\rho)^{\frac{d-2}{2}}E_+(s\rho)\,\mathrm{d}\rho + \text{c.c.}.$$

Here c.c. denotes the complex conjugate of the first term. With the same decomposition we may proceed as in [**17**] to obtain Theorem 3.1.1(b).

REMARK 3.1.5. *Finally, let us remark that we can obtain spherically averaged estimates for non-radial initial data using the orthogonality of the spherical harmonics. In particular, given some well-behaved initial data u_0, we may decompose it using spherical harmonics:*

$$u_0(x) := \sum_{k=0}^{\infty}\sum_{l=1}^{d(k)} a_{k,l}(r)Y_{k,l}(\theta).$$

Thus, we solve (3.1.1) on each harmonic subspace to obtain a solution

$$e^{it\omega(D_a)}u_0 := \sum_{k=0}^{\infty}\sum_{l=1}^{d(k)} b_{k,l}(r)Y_{k,l}(\theta).$$

Using the representation of $\mathcal{L}_a$ on the kth spherical harmonic $L^2_{=k}(\mathbf{R}^d)$, we reduce to solving the PDE (with b denoting $b_{k,l}$):

$$\begin{cases} ib_t(r,t) + \omega\left(A_{\nu(a,k)}\right)b(x,t) = 0, & (r,t) \in (0,\infty)\times\mathbf{R} \\ v(0,r) = a_{k,l}(r) \end{cases} \tag{3.1.9}$$

Taking the Hankel transform of order $\nu(a,k)$, we find that

$$b_{k,l} = \mathcal{H}_{\nu(a,k)}\left(e^{it\omega(\rho)}\mathcal{H}_{\nu(a,k)}(a_{k,l})\right).$$

Now, in order to obtain the estimate $\|e^{it\omega(D_a)}u_0\|_{L^q_t\mathcal{L}^r_sL^2(\mathbf{S}^{d-1})} \lesssim C(k)\|u_0\|_{L^2_x}$, it is equivalent to obtain the estimate

$$\left\| \sum_{k,l}\mathcal{H}_{\nu(a,k)}\left(e^{it\omega(\rho)}\mathcal{H}_{\nu(a,k)}(a_{k,l})\right)\right\|_{L^q_t\mathcal{L}^r_sL^2(\mathbf{S}^{d-1})} \lesssim C(k)\left\|u_0\right\|_{L^2_x}.$$

Again, using the fact that $\mathcal{H}^2_\nu = \mathrm{id}$, we reduce to showing

$$\left\| \sum_{k,l}\mathcal{H}_{\nu(a,k)}\left(e^{it\omega(\rho)}a_{k,l}\right)\right\|_{L^q_t\mathcal{L}^r_sL^2(\mathbf{S}^{d-1})} \lesssim C(k)\left\|u_0\right\|_{L^2_x}.$$

Now, using the L^2 orthogonality of the spherical harmonics on both sides, we may further reduce to showing

$$\left\| \mathcal{H}_{\nu(a,k)} \left(e^{it\omega(\rho)} a_{k,l} \right) \right\|_{L^q_t \mathcal{L}^r_s \ell^2_{k,l}} \lesssim C(k) \left\| a_{k,l} \right\|_{\ell^2_{k,l}}.$$

Thus, it suffices to prove the above estimate for each pair (k, l) and then take the $\ell^2_{k,l}$ sum of these estimates. This then gives spherically-averaged Strichartz estimates.

3.2. 3D radial small-energy scattering

In this section, our goal shall be to use our estimates from Section 3.1 to study scattering for the following equation in three dimensions:

$$\begin{cases} \partial_t^2 u - \Delta u + \frac{a}{|x|^2} u + u = u^2, & (t, x) \in \mathbf{R} \times \mathbf{R}^3 \\ u(0, x) = u_0, u_t(0, x) = u_1. \end{cases} \tag{3.2.1}$$

Recall as before that we shall denote $D_a = \sqrt{\mathcal{L}_a}$. We first obtain the following estimates. The further restriction when $-\mu(0)^2 < a < 0$ is needed in order to perform Sobolev embeddings in the proof of scattering later.

Proposition 3.2.1. *Let $u_0 \in L^2$ be radial. Let $d \geq 3$, and $2 \leq q, r \leq \infty$ (and also if $-\mu(0)^2 < a < 0$, then further restrict $r_0 < r < r_0'$) with $\frac{1}{q} + \frac{d-1}{r} < \frac{d-1}{2}$ (which we shall refer to as admissible pairs in the general region) Then,*

$$\| e^{it\langle D_a \rangle} P_k^a \phi \|_{(\frac{1}{q}, \frac{1}{r}, \frac{2}{q} + \frac{d}{r} - \frac{d}{2} | \frac{1}{q} + \frac{d}{r} - \frac{d}{2})} \lesssim \| P_k^a \phi \|_2.$$

Similarly, we have the dual estimate with admissible pairs (q, r) and $(\tilde{q}, \tilde{r})$ in the general region:

$$\left\| \int_0^t e^{i(t-s)\langle D_a \rangle} F(s) \, \mathrm{d}s \right\|_{(\frac{1}{q}, \frac{1}{r}, \frac{2}{q} + \frac{d}{r} - \frac{d}{2} | \frac{1}{q} + \frac{d}{r} - \frac{d}{2})} \lesssim \| F \|_{(\frac{1}{\tilde{q}'}, \frac{1}{\tilde{r}'}, -(\frac{2}{\tilde{q}'} + \frac{d}{\tilde{r}'} - \frac{d}{2}) | -(\frac{1}{\tilde{q}'} + \frac{d}{\tilde{r}'} - \frac{d}{2}))}$$

Furthermore, for $\frac{1}{q} + \frac{d-1}{r} > \frac{d-1}{2}$ and $\frac{2}{q} + \frac{2d-1}{r} < \frac{2d-1}{2}$ (which we refer to as admissible pairs in the refined region), then

$$\| e^{it\langle D_a \rangle} P_k^a \phi \|_{(\frac{1}{q}, \frac{1}{r}, \frac{2}{q} + \frac{d}{r} - \frac{d}{2} | \frac{d}{2} - 1 - \frac{1}{q} - \frac{d-2}{r})} \lesssim \| P_k^a \phi \|_2.$$

Also, we have the dual estimate with admissible pairs (q, r) and $(\tilde{q}, \tilde{r})$ in the refined region:

$$\left\| \int_0^t e^{i(t-s)\langle D_a \rangle} F(s) \, \mathrm{d}s \right\|_{(\frac{1}{q}, \frac{1}{r}, \frac{2}{q} + \frac{d}{r} - \frac{d}{2} | \frac{d}{2} - 1 - \frac{1}{q} - \frac{d-2}{r})}$$
$$\lesssim \| F \|_{(\frac{1}{\tilde{q}'}, \frac{1}{\tilde{r}'}, -(\frac{2}{\tilde{q}'} + \frac{d}{\tilde{r}'} - \frac{d}{2}) | -(\frac{d}{2} - 1 - \frac{1}{\tilde{q}} - \frac{d-2}{\tilde{r}}))}.$$

Proof. Let $\omega(\xi) = \langle \xi \rangle = (1 + \xi^2)^{1/2}$. Following the notation in Theorem 3.1.1, we may estimate ω' and ω'' to obtain that $\alpha = 2$ and $\beta = 2$ if $k \leq 0$, $\alpha = 1$ and $\beta = -1$ if $k > 0$ satisfy the assumptions. The idea here is to substitute these values into the exponents as in Theorem 3.1.1, and to combine this with the adapted Littlewood-Paley square function theorem, we obtain our estimates. For instance, let us consider the first set of estimates. In this case, for low frequencies $j \leq 0$, we have

$$\| e^{it\langle D_a \rangle} P_j^a u_0 \|_{L^q_t L^r_x} \lesssim 2^{j s_L} \| P_j^a u_0 \|_{L^2_x}$$

with $s_L = \frac{d}{2} - \frac{2}{q} - \frac{d}{r}$. Meanwhile for high frequencies $j \geq 0$ we have

$$\|e^{it\langle D_a\rangle} P_j^a u_0\|_{L_t^q L_x^r} \lesssim 2^{js_H} \|P_j^a u_0\|_{L_x^2}$$

where $s_H = \frac{d-2}{r} + \frac{1}{q} + 1 - \frac{d}{2}$. Therefore, we may use the Littlewood-Paley square function theorem for P_k^a (Proposition 2.3.3) to obtain the above estimates. More specifically, we have

$$\begin{aligned}
\|e^{it\langle D_a\rangle} P_j^a \phi\|_{(\frac{1}{q},\frac{1}{r},-s_L|-s_H)} &= \Bigg\| \left(\sum_{k\in\mathbf{Z},k\leq 0} 2^{2(-s_L)k} \|P_k^a e^{it\langle D_a\rangle} P_j^a \phi(x)\|_r^2 \right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{k\in\mathbf{Z},k\geq 0} 2^{2(-s_H)k} \|P_k^a e^{it\langle D_a\rangle} P_j^a \phi(x)\|_r^2 \right)^{\frac{1}{2}} \Bigg\|_{L_t^q} \\
&\lesssim \left(\sum_{k\in\mathbf{Z},k\leq 0} 2^{2(-s_L)k} \|e^{it\langle D_a\rangle} P_k^a P_j^a \phi(x)\|_{L_t^q L_x^r}^2 \right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{k\in\mathbf{Z},k\geq 0} 2^{2(-s_H)k} \|e^{it\langle D_a\rangle} P_k^a P_j^a \phi(x)\|_{L_t^q L_x^r}^2 \right)^{\frac{1}{2}} \\
&\lesssim \left(\sum_{k\in\mathbf{Z},k\leq 0} \|P_k^a P_j^a \phi(x)\|_2^2 \right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{k\in\mathbf{Z},k\geq 0} \|P_k^a P_j^a \phi(x)\|_2^2 \right)^{\frac{1}{2}} \\
&\lesssim \|P_j^a \phi\|_2.
\end{aligned}$$

□

The main result of this chapter is

THEOREM 3.2.2. *Let $0 < \kappa \ll 1$ be sufficiently small, and suppose that (u_0, u_1) is radial and satisfies $\|(u_0, u_1)\|_{H_a^1 \times L^2} \ll 1$, then there exists a unique solution $u(t,x)$ to (3.2.1) with $a > \sigma^{-1}(\frac{3}{2}(\frac{3}{10} - \kappa))$ in the space*

$$S(I) = C(\mathbf{R}, H_a^1) \cap \left(\frac{1}{2}, \frac{3}{10} - \kappa, \frac{2}{5} - 3\kappa \,\middle|\, \frac{7}{10} + \kappa \right)_{\mathbf{R}},$$

that also scatters: there exists $u_\pm(x) \in H_a^1$ such that

$$\|u - i\langle D_a\rangle \partial_t u - e^{it\langle D_a\rangle} u_\pm\|_{H_a^1} \to 0,$$

as $t \to \pm\infty$.

We also recall the definition $\sigma := \frac{d-2}{2} - \sqrt{\left(\frac{d-2}{2}\right)^2 + a} = \frac{d-2}{2} - \sqrt{\mu(0)^2 + a}$

3.2.1. Motivation: low frequency interactions. Now we have the language to motivate the following application of the above estimates. First, we shall perform a change of variables $U(t,x) = u(t,x) - i\langle D_a\rangle^{-1}u_t(t,x)$. From this, we get a first-order equation (see the next section for more details):

$$i\partial_t U + \langle D_a\rangle U = \frac{1}{4}\langle D_a\rangle^{-1}\left(U^2 + 2U\bar{U} + \bar{U}^2\right), \quad U(0,x) = \phi(x).$$

In the rest of this subsection, we shall discuss the simplified equation

$$i\partial_t U + \langle D_a\rangle U = \langle D_a\rangle^{-1}U^2, \quad U(0,x) = \phi(x). \tag{3.2.2}$$

We now want to establish well-posedness for this equation. To do this, we shall proceed with a contraction-mapping argument. Firstly, by Duhamel's principle, we have

$$\Phi_\phi U = e^{it\langle D_a\rangle}\phi - i\int_0^t e^{i(t-s)\langle D_a\rangle}\langle D_a\rangle^{-1}U^2 \,\mathrm{d}s.$$

Let $S_I = \left(0, \frac{1}{2}, 0\,|1\right)_I \cap \left(\frac{1}{2}, \frac{3}{10}, \frac{2}{5}|\frac{7}{10}\right)_I$, $X_M = \{U \in S_I : \|U\|_{S_I} \le M\}$ for some M and $I = [0,T]$ which we shall choose later so that $(\Phi_\phi|_{X_M}, d)$ with $\Phi_\phi : X_M \to X_M$ is a contraction with respect to with $d(U,V) = \|U - V\|_{S_I}$. Note that (X_M, d) is complete. In this case, we have

$$\|\Phi_\phi U\|_{\left(0,\frac{1}{2},0|1\right)_I} \lesssim \left\|e^{it\langle D_a\rangle}\phi\right\|_{\left(0,\frac{1}{2},0|1\right)_I} + \left\|\int_0^t e^{i(t-s)\langle D_a\rangle}\langle D_a\rangle^{-1}U^2 \,\mathrm{d}s\right\|_{\left(0,\frac{1}{2},0|1\right)_I}.$$

We handle the first term by frequency localisation and the second term by an inhomogeneous Strichartz estimate with norm $L^2_{t\in I}H^1_x$ to obtain

$$\|\Phi_\phi U\|_{\left(0,\frac{1}{2},0|1\right)_I} \lesssim \|\phi\|_{H^1_a} + \left\|\langle D_a\rangle^{-1}U^2\right\|_{L^1_t H^1_x} \lesssim \|\phi\|_{H^1_x} + \left\|U^2\right\|_{L^1_t L^2_x}.$$

Now, we apply Hölder's inequality for the finite time interval and also Hölder's inequality in x to obtain

$$\|\Phi_\phi U\|_{\left(0,\frac{1}{2},0|1\right)_I} \lesssim \|\phi\|_{H^1_a} + T\,\|U\|^2_{L^\infty_t L^4_x}.$$

Now, we use the embedding $H^1_a \hookrightarrow L^4$ to obtain that

$$\|\phi\|_{H^1_a} + T\,\|U\|^2_{L^\infty_t L^4_x} \lesssim \|\phi\|_{H^1_a} + T\,\|U\|^2_{L^\infty_t H^1_a} \lesssim \|\phi\|_{H^1_a} + T\|U\|^2_{S_I}.$$

We may do the same for the second norm in $S(I)$ to obtain that

$$\|\Phi_\phi U\|_{S_I} \lesssim \|\phi\|_{H^1_a} + T\|U\|^2_{S_I}.$$

Thus, if we set $M = 2\|\phi\|_{H^1_a}$ and T sufficiently small, then we find $\|\Phi_\phi U\|_{S_I} \le M$ so that indeed $\Phi_\phi : X_M \to X_M$. Now, we also verify that the contraction condition is satisfied. In this case, we find that after using an inhomogeneous Strichartz estimate:

$$\begin{aligned}
\|\Phi_\phi U - \Phi_\phi V\|_{S_I} &\lesssim \|\langle D_a\rangle^{-1}(U^2 - V^2)\|_{L^2_t W^{4/3,1}_x} \\
&\lesssim T\|U - V\|_{S_I}\|U + V\|_{S_I} \\
&\lesssim MT\|U - V\|_{S_I}.
\end{aligned}$$

Now, if T is chosen sufficiently small, then indeed, (Φ_ϕ, d) is a contraction, meaning that we have a unique solution in X_M. To extend this uniqueness to all of S_I, let us consider an interval $[0,t^*] \subseteq [0,T]$. Then, applying the above estimate, we obtain

$$\begin{aligned} \|U-V\|_{S_I} &\lesssim \|\langle D_a\rangle^{-1}(U^2-V^2)\|_{L_t^2 W_x^{4/3,1}} \\ &\lesssim t^*\|U-V\|_{S_I}\|U+V\|_{S_I} \\ &\lesssim Mt^*\|U-V\|_{S_I}. \end{aligned}$$

Thus, if we set t^* to be so that $MCt^* < 1$, then in fact, we see that $\|U-V\|_{S_I} < (1/2)\|U_V\|_{X_1}$, so that indeed $U = V$. We may use a similar argument to obtain continuous dependence on data.

We shall now discuss the global well-posedness and scattering of this problem for small data. Here, we shall identify the main obstructions: non-radial data and low frequency terms. We shall then devote the remainder of this chapter to dealing with this obstruction via a normal form transform in the flavour of [**22**]. The normal form transform in the mentioned paper admits an integration by parts for low frequencies which essentially converts the quadratic term into a cubic term, and thus fixes this issue. In the rest of this section, we shall first discuss why the quadratic term is an issue, and why this issue does not arise for a cubic term. The idea is that we need the radial assumption in order to make valid choices for admissible pairs to close the arguments that follow.

In order to obtain global well-posedness of the problem, we want an estimate for the solution U which is independent of the length of the time interval I. Due to the quadratic term U^2 in (3.2.2), we want an estimate of the form

$$\|\Phi_\phi U\|_{S_I} \lesssim \|\phi\|_{H_a^1} + C\|U\|_{S_I}^2$$

where $C > 0$ independent of the length of the interval I. If we are able to obtain such a bound, then for $\|\phi\|_{H_a^1} < \epsilon$ for some ϵ sufficiently small (i.e. for small data), we obtain that

$$\|\Phi_\phi U\|_{S_I} < \frac{M}{2} + M^2 < M,$$

which would give a global-in-time bound for small data as we may set $I = \mathbf{R}$. As we will see later, the key point is that the pair $(q,r) = (2,4)$ is admissible in Proposition 3.2.1 – that is, for radial initial data, which allows our argument to proceed. In the non-radial case, we do not have Strichartz estimates for this pair.

With this, let us now see another problem: low frequencies. To see why low-frequency components are an issue, let us first decompose the solution U into high and low frequencies: $U = P^a_{\geq 1}U + P^a_{\leq 0}U := U_H + U_L$. Then, $U^2 = U_H^2 + 2U_HU_L + U_L^2$. Firstly, we see that the high-high interactions can be controlled by S_I via the $(q',r') = (1,2)$ estimate:

$$\left\| \int_0^t e^{i(t-s)\langle D_a\rangle}\langle D_a\rangle^{-1}U_H^2 \,\mathrm{d}s \right\|_{S_I} \lesssim \|U_H^2\|_{\left(1,\frac{1}{2},0|0\right)} \lesssim \|U_H\|^2_{\left(\frac{1}{2},\frac{1}{4},0|0\right)}.$$

Now, using Sobolev embeddings and the fact that we have are studying the high-frequency component, we have

$$\|U_H\|_{\left(\frac{1}{2},\frac{1}{4},0|0\right)} \lesssim \|U_H\|_{\left(\frac{1}{2},\frac{3}{10},0|\frac{3}{20}\right)} \lesssim \|U_H\|_{\left(\frac{1}{2},\frac{3}{10},0|\frac{7}{10}\right)}$$

so that

$$\left\| \int_0^t e^{i(t-s)\langle D_a\rangle}\langle D_a\rangle^{-1}U_H^2 \,\mathrm{d}s \right\|_{S_I} \lesssim \|U_H^2\|_{\left(\frac{1}{2},\frac{1}{4},0|0\right)} \lesssim \|U_H\|^2_{\left(\frac{1}{2},\frac{3}{10},0|\frac{7}{10}\right)} \lesssim \|U_H\|^2_{S_I}.$$

Let us first look at the low-low interactions. Similar to before, we have, for instance that

$$\begin{aligned} \left\| \int_0^t e^{i(t-s)\langle D_a \rangle} \langle D_a \rangle^{-1} U_L^2 \, \mathrm{d}s \right\|_{S_I} &\lesssim \|U_L^2\|_{\left(1, \frac{1}{2}, 0|0\right)} \\ &\lesssim \|U_L\|^2_{\left(\frac{1}{2}, \frac{1}{4}, 0|0\right)} \end{aligned}$$

We now note that have the embedding

$$\|U_L\|_{\left(\frac{1}{2}, \frac{1}{4}, 0|0\right)} \lesssim \|U_L\|_{\left(\frac{1}{2}, \frac{3}{10}, \frac{3}{20}|0\right)},$$

and since we are working the low-frequency component, we cannot close the argument because we cannot control this norm by $\left(\frac{1}{2}, \frac{3}{10}, \frac{2}{5}|0\right)$. Thus, we cannot control the low-low interactions using Sobolev embeddings. A similar argument works for other choices of admissible pairs (q, r) and also for the low-high interactions. Thus, we shall now remove these low-frequency terms via a normal-type transform, and in doing so, we shall be able to obtain uniform estimates and obtain scattering for (3.2.1).

3.2.2. Normal-type transform. In this section, we shall follow the arguments in [**22**] to obtain a normal-type transform in order to eliminate certain interactions (for instance low-low interactions) which will allow us to obtain some uniform estimates and scattering results. In particular, we take the Hankel transform of order $\nu = \nu(a)$ to obtain the ordinary differential equation $\partial_t^2 \widehat{u} + \left(\rho^2 + 1\right) \widehat{u} = \mathcal{H}_\nu(u^2)$. Factorising and letting

$$U(t, x) = u(t, x) - i \langle D_a \rangle^{-1} u_t(t, x)$$

(corresponding to $\widehat{u}(t, \rho) = \widehat{u}(t, \rho) - i(1 + \rho^2)^{1/2} \widehat{u}_t(t, x)$ in frequency space), we obtain the first order ODE

$$(i\partial_t + \langle D_a \rangle) U = \langle D_a \rangle^{-1} u^2 = \frac{1}{4} \langle D_a \rangle^{-1} \left(U^2 + 2U\overline{U} + \overline{U}^2\right). \tag{3.2.3}$$

As we shall see later, it shall suffice that we consider the equation

$$(i\partial_t + \langle D_a \rangle) U = \langle D_a \rangle^{-1} U^2. \tag{3.2.4}$$

We shall define a normal-form transform $\Omega(U, U)$ for (3.2.4) shortly, and discuss how it can be modified for the other quadratic terms in (3.2.3). The idea is we want to consider $\left(U + i \langle D_a \rangle^{-1} \Omega(U, U)\right)(t, x)$. This normal-form transform satisfies the following equation in analogy to (3.2.4):

$$\begin{aligned} (i\partial_t + \langle D_a \rangle) \left(U + i \langle D_a \rangle^{-1} \Omega(U, U)\right) &= \langle D_a \rangle^{-1} U^2 + i\Omega(U, U) \\ &\quad + i \langle D_a \rangle^{-1} \Omega(i\partial_t U, U) \\ &\quad + i \langle D_a \rangle^{-1} \Omega(U, i\partial_t U). \end{aligned}$$

Next, making use of the fact that $i\partial_t U = -\langle D_a\rangle U + \langle D_a\rangle^{-1} U^2$, we obtain that

$$\begin{aligned}(i\partial_t + \langle D_a\rangle)\Big(U + i\langle D_a\rangle^{-1}\Omega(U,U)\Big) &= \langle D_a\rangle^{-1}U^2 + i\Omega(U,U)\\ &\quad +i\langle D_a\rangle^{-1}\Omega(-\langle D_a\rangle U,U)\\ &\quad +i\langle D_a\rangle^{-1}\Omega(U,-\langle D_a\rangle U)\\ &\quad +i\langle D_a\rangle^{-1}\Omega(\langle D_a\rangle^{-1}U^2,U)\\ &\quad +i\langle D_a\rangle^{-1}\Omega(U,\langle D_a\rangle^{-1}U^2).\end{aligned}$$

Now, we shall derive a suitable $\Omega(U,U)$ for our purposes. Consider the quadratic term $\langle D_a\rangle^{-1}U^2 + i\Omega(U,U) + i\langle D_a\rangle^{-1}\Omega(-\langle D_a\rangle U,U) + i\langle D_a\rangle^{-1}\Omega(U,-\langle D_a\rangle U)$ (i.e. the first three lines of the right-hand side of (3.2.5)). First, using the convolution formula for the Hankel transform (Lemma 2.2.3), we note that

$$\begin{aligned}\langle D_a\rangle^{-1}U^2 &= \mathcal{H}_\nu\langle\rho\rangle^{-1}\mathcal{H}_\nu U^2\\ &= \mathcal{H}_\nu\langle\rho\rangle^{-1}(\mathcal{H}_\nu U \# \mathcal{H}_\nu U)\\ &= \mathcal{H}_\nu\langle\rho\rangle^{-1}\int_0^\infty\int_0^\infty D_\nu(x,y,\rho)\cdot\mathcal{H}_\nu(U)(x)\mathcal{H}_\nu(U)(y)\,\mathrm{d}\omega(x)\mathrm{d}\omega(y).\end{aligned}$$

Thus, if we define

$$\mathcal{H}_\nu\Omega(U,U) = \int_0^\infty\int_0^\infty \frac{D_\nu(x,y,\rho)}{i(\langle x\rangle + \langle y\rangle - \langle\rho\rangle)} m(x,y)\mathcal{H}_\nu(U)(x)\mathcal{H}_\nu(U)(y)\,\mathrm{d}\omega(x)\mathrm{d}\omega(y),$$

we see that $i\Omega(U,U) + i\langle D_a\rangle^{-1}\Omega(-\langle D_a\rangle U,U) + i\langle D_a\rangle^{-1}\Omega(U,-\langle D_a\rangle U)$ is equal to

$$\int_0^\infty\int_0^\infty \frac{D_\nu(x,y,\rho)}{i(\langle x\rangle + \langle y\rangle - \langle\rho\rangle)}\left(1 - \frac{\langle x\rangle}{\langle\rho\rangle} - \frac{\langle y\rangle}{\langle\rho\rangle}\right) m(x,y)\mathcal{H}_\nu(U)(x)\mathcal{H}_\nu(U)(y)\,\mathrm{d}\omega(x)\mathrm{d}\omega(y).$$

Thus, the resonance term $\mathcal{H}_\nu T_{\mathrm{Res}}(U,U)$ is equal to

$$\int_0^\infty\int_0^\infty (1 - m(x,y))D_\nu(x,y,\rho)\mathcal{H}_\nu(U)(x)\mathcal{H}_\nu(U)(y)\,\mathrm{d}\omega(x)\mathrm{d}\omega(y). \tag{3.2.5}$$

We claim that since the convolution is zero unless one can form a triangle with sides of length x, y and ρ. In particular, we have particular, $\rho \le x + y$ and from the modulation bound (e.g. see [**53**]), we find that

$$\langle x\rangle + \langle y\rangle - \langle\rho\rangle \ge \langle x\rangle + \langle y\rangle - \langle x+y\rangle \ge \frac{1}{\langle\min\{|x|,|y|,|x+y|\}\rangle} \gtrsim_\beta 1 \tag{3.2.6}$$

if we assume that $\min\{|x|,|y|\} \lesssim 2^\beta$ for some large constant $\beta > 0$. Similarly, we may obtain that $|\langle x\rangle \pm \langle y\rangle \pm \langle\rho\rangle| \gtrsim_\beta 1$. Now, to verify the claim, we use Fubini's theorem (see [**60**] for details) to obtain the following weighted version of the convolution:

$$\begin{aligned}&\mathcal{H}_\nu(\rho^\sigma\mathcal{H}_\nu(f)\mathcal{H}_\nu(g))(r)\\ &= \int_0^\infty\int_0^\infty (rxy)^{-\frac{d-2}{2}} f(x)g(y)\int_0^\infty J_\nu(x\rho)J_\nu(y\rho)J_\nu(r\rho)\rho^{1-\nu}\,\mathrm{d}\rho\,\mathrm{d}\omega(x)\mathrm{d}\omega(y)\end{aligned}$$

Now, we may use the identity (see [**61**, p. 411(3)])

$$\begin{aligned}&\int_0^\infty J_\nu(x\rho)J_\nu(y\rho)J_\nu(r\rho)\rho^{1-\nu}\,\mathrm{d}\rho\\ =\quad & C_\nu(rxy)^{-\nu}\Delta_{x,y,r},\end{aligned}$$

where $\Delta_{x,y,r}$ is the area of the triangle with sides of length x, y and r and zero otherwise. Thus, we define the weighted convolution $\#_{\mathrm{w}}$ via

$$f\#_{\mathrm{w}}g := \int_0^\infty\int_0^\infty (ruv)^{-\frac{d-2}{2}-\nu}\Delta^{2\nu-1}_{x,y,r}f(x)g(y)\,\mathrm{d}\omega(x)\mathrm{d}\omega(y)$$

such that

$$\mathcal{H}_\nu(\rho^\sigma\mathcal{H}_\nu(f)\mathcal{H}_\nu(g)) = f\#_{\mathrm{w}}g. \tag{3.2.7}$$

For the purposes of this thesis, $1-m(x,y)$ is a sum of terms of the form $\phi_j(x)\phi_k(y)$ where ϕ_j, ϕ_k are Paley-Littlewood multipliers. Thus, (3.2.5) is a sum of terms of the form $\mathcal{H}_\nu(\mathcal{H}_\nu(P_jU)\mathcal{H}_\nu(P_kU))$. Thus, we may localise the factor of ρ^σ in (3.2.7) via $\mathcal{H}_\nu(P_jU)$ and $\mathcal{H}_\nu(P_kU)$. Now, applying the above identity, we obtain the claim.

Returning to (3.2.5) we see that

$$\begin{aligned}(i\partial_t+\langle D_a\rangle)\Big(U+i\langle D_a\rangle^{-1}\Omega(U,U)\Big) \quad = \quad & \langle D_a\rangle^{-1}\mathcal{H}_\nu T_{\mathrm{Res}}(U,U)\\ & +i\langle D_a\rangle^{-1}\Omega(\langle D_a\rangle^{-1}U^2,U)\\ & +i\langle D_a\rangle^{-1}\Omega(U,\langle D_a\rangle^{-1}U^2).\end{aligned}$$

Therefore,

$$U(t,x) \quad = \quad e^{it\langle D_a\rangle}\Big(U_0+i\langle D_a\rangle^{-1}\Omega(U,U)(0)\Big)-i\langle D_a\rangle^{-1}\Omega(U,U) \tag{3.2.8}$$

$$-i\int_0^t e^{i(t-s)\langle D_a\rangle}\langle D_a\rangle^{-1}\mathcal{H}_\nu T_{\mathrm{Res}}(U,U)\,\mathrm{d}s \tag{3.2.9}$$

$$+\int_0^t e^{i(t-s)\langle D_a\rangle}\langle D_a\rangle^{-1}\Omega(\langle D_a\rangle^{-1}U^2,U)\,\mathrm{d}s \tag{3.2.10}$$

$$+\int_0^t e^{i(t-s)\langle D_a\rangle}\langle D_a\rangle^{-1}\Omega(U,\langle D_a\rangle^{-1}U^2)\,\mathrm{d}s. \tag{3.2.11}$$

Finally, let us remark on frequency decomposition. Let $\beta>0$ be a large parameter and

$$\begin{aligned}LL &:= \{(j,k)\in\mathbf{Z}^2 : j,k\le -\beta+10\}\\ HL &:= \{(j.k)\in\mathbf{Z}^2 : j\ge -\beta+10, k\le -\beta+10\}\\ LH &:= \{(j,k)\in\mathbf{Z}^2 : (k,j)\in HL\}\\ HH &:= \{(j,k)\in\mathbf{Z}^2 : j,k\ge -\beta+10\}\end{aligned}$$

so that we may write

$$U(x)U'(x) = \left(\sum_{(j,k)\in HH}+\sum_{(j,k)\in HL}+\sum_{(j,k)\in LH}+\sum_{(j,k)\in LL}\right)(P^a_jUP^a_kU').$$

Define for $S\subseteq\mathbf{Z}^2$

$$m_S(x,y) = \sum_{(j,k)\in S}\phi_j(x)\phi_k(y)$$

and also

$$(UU')_S := \mathcal{H}_\nu \int_0^\infty \int_0^\infty D_\nu(x,y,\rho) m_S(x,y) \mathcal{H}_\nu(U)(x)\mathcal{H}_\nu(U)(y) \, \mathrm{d}\omega(x)\mathrm{d}\omega(y).$$

Using the convolution formula for the Hankel transform, this is equal to

$$(UU')_S = \sum_{(j,k)\in S} P_j^a U P_k^a U'.$$

3.2.3. Small energy scattering in $\mathbf{R}^3$. In this section, we shall establish some uniform estimates for small initial data to establish some scattering of the 3D quadratic Klein-Gordon equation with inverse-square potential. Here, we shall use the normal form transform with $m = m_{LL}$ so that $T_{\mathrm{Res}}(U,U) = (UU')_{HH+HL+LH}$. From the above discussion, we also have heuristically that $\Omega(U,U') \sim (UU')_{LL}$. Let $\epsilon, \kappa > 0$ be sufficiently small (this extra space in the exponents is used for large energy problems). We shall define two spaces

$$S_a(I) = \left(0, \frac{1}{2}, 0\,|1\right)_I \cap \left(\frac{1}{2}, \frac{3}{10} - \kappa, \frac{2}{5} - 3\kappa \,\middle|\, \frac{7}{10} + \kappa\right)_I,$$

and also the space

$$\tilde{S}_a(I) = \left(\frac{1}{2} - \epsilon, \frac{1}{4} + \epsilon, 5\epsilon\right) \cap \left(\frac{1}{3}, \frac{1}{6}, -\epsilon|\epsilon\right).$$

We shall see below that $\tilde{S}(I)$ can be controlled by $S(I)$. We also note that $S(I)$ is chosen to obtain H_a^1 estimates in Proposition 3.2.1.

In this section, we shall obtain a uniform estimate for $\|U(t,x)\|_{S(I)}$ in terms of the norms of (3.2.8 – 3.2.11). Thus, we will need to estimate the resonance term, the boundary term and the trilinear term.

LEMMA 3.2.3 (Resonance term). *Let U and U' be radial. Then,*

$$\left\| \int_0^t e^{i(t-s)\langle D_a\rangle} \langle D_a\rangle^{-1} (UU')_{HH+HL+LH} \, \mathrm{d}s \right\|_{S(I)} \lesssim_\beta \|U\|_{\tilde{S}(I)} \|U'\|_{\tilde{S}(I)}. \tag{3.2.12}$$

PROOF. For both norms in $S(I)$, we apply the inhomogeneous Strichartz estimate with dual pair $(\frac{1}{\tilde{q}'}, \frac{1}{\tilde{r}'})$ chosen to be (after calculating the corresponding regularities) $(1-2\epsilon, \frac{1}{2}+2\epsilon, 2\epsilon|4\epsilon)$. We start with the HH case. First, note that by frequency decomposition and Hölder inequality, we have for $(j,k)\in HH$ that

$$\|P_j^a U P_k^a U'\|_{(1-2\epsilon, \frac{1}{2}+2\epsilon, 2\epsilon|4\epsilon)} \lesssim_\beta 2^{-\epsilon(j+k)} \|P_j^a U\|_{(\frac{1}{2}-\epsilon, \frac{1}{4}+\epsilon, 6\epsilon)} |P_k^a U'\|_{(\frac{1}{2}-\epsilon, \frac{1}{4}+\epsilon, 6\epsilon)}.$$

Thus, we have

$$\begin{aligned} \left\| \int_0^t e^{i(t-s)\langle D_a\rangle} \langle D_a\rangle^{-1} (UU')_{HH} \, \mathrm{d}s \right\|_{S(I)} &\lesssim_\beta \sum_{(j,k)\in HH} \|P_j^a U P_k^a U'\|_{(1-2\epsilon, \frac{1}{2}+2\epsilon, 2\epsilon|4\epsilon)} \\ &\lesssim_\beta \sum_{(j,k)\in HH} 2^{-\epsilon j} \|P_j^a U\|_{(\frac{1}{2}-\epsilon, \frac{1}{4}+\epsilon, 6\epsilon)} \\ &\qquad \cdot 2^{-\epsilon k} \|P_k^a U'\|_{(\frac{1}{2}-\epsilon, \frac{1}{4}+\epsilon, 6\epsilon)}. \\ &\lesssim_\beta \|P_{\geq 0}^a U\|_{(\frac{1}{2}-\epsilon, \frac{1}{4}+\epsilon, 6\epsilon)} \\ &\qquad \cdot \|P_{\geq 0}^a U'\|_{(\frac{1}{2}-\epsilon, \frac{1}{4}+\epsilon, 6\epsilon)}. \end{aligned}$$

This is what we needed. Let us now verify that $\left(\frac{1}{2}-\epsilon, \frac{1}{4}+\epsilon, 6\epsilon\right)$ is controlled by $S(I)$. By the above Sobolev embedding, we see that for $j \geq -\beta + 10$

$$\left(\frac{1}{2}-\epsilon, (1-2\epsilon)\left(\frac{3}{10}-\kappa\right)+2\epsilon\cdot\frac{1}{2}, s\right) \hookrightarrow_H \left(\frac{1}{2}-\epsilon, \frac{1}{4}+\epsilon, 6\epsilon\right)$$

with $s = 6\epsilon + 3\left((1-2\epsilon)\left(\frac{3}{10}-\kappa\right) - 2\epsilon\cdot\frac{1}{2} - \frac{1}{4} - \epsilon\right)$. Since $s < (1-2\epsilon)\left(\frac{7}{10}+\kappa\right)+2\epsilon$ we also obtain that

$$\left(\frac{1}{2}-\epsilon, (1-2\epsilon)\left(\frac{3}{10}-\kappa\right), (1-2\epsilon)\left(\frac{7}{10}+\kappa\right)+2\epsilon\right) \hookrightarrow_H \left(\frac{1}{2}-\epsilon, (1-2\epsilon)\left(\frac{3}{10}-\kappa\right), s\right).$$

From this, we may conclude that

$$(1-2\epsilon)\left(\frac{1}{2}, \frac{3}{10}-\kappa, \frac{7}{10}+\kappa\right)+2\epsilon\left(0, \frac{1}{2}, 1\right) \hookrightarrow_H \left(\frac{1}{2}-\epsilon, \frac{1}{4}+\epsilon, 6\epsilon\right),$$

and indeed $\left(\frac{1}{2}-\epsilon, \frac{1}{4}+\epsilon, 6\epsilon\right)$ is controlled by $S(I)$. The HL and LH cases can be handled similarly. We shall deal with the HL case, as the LH case is identical. Here, we choose the dual pair $(\frac{1}{\tilde{q}'}, \frac{1}{\tilde{r}'})$ to be $(\frac{1}{2}, \frac{3}{4}-\epsilon, \frac{1}{4}+\epsilon)$. Again, using frequency decomposition and Hölder inequality, we first obtain for $(j,k) \in HL$ that

$$\|P_j^a U P_k^a U'\|_{(\frac{1}{2}, \frac{3}{4}-\epsilon, \frac{1}{4}+\epsilon)} \quad \lesssim \quad 2^{\epsilon(-j+k)}\|P_j^a U\|_{(\frac{1}{2}-\epsilon, \frac{1}{4}+\epsilon, \frac{1}{4}+2\epsilon)}\|P_k^a U'\|_{(\epsilon, \frac{1}{2}-2\epsilon, -\epsilon)}.$$

Thus, we have

$$\begin{aligned}
\left\|\int_0^t e^{i(t-s)\langle D_a\rangle}\langle D_a\rangle^{-1}(UU')_{HL}\,\mathrm{d}s\right\|_{S(I)} &\lesssim_\beta \sum_{(j,k)\in HL}\|P_j^a U P_k^a U'\|_{(\frac{1}{2}, \frac{3}{4}-\epsilon, \frac{1}{4}+\epsilon)} \\
&\lesssim_\beta \sum_{(j,k)\in HL} 2^{-\epsilon j}\|P_j^a U\|_{(\frac{1}{2}-\epsilon, \frac{1}{4}+\epsilon, \frac{1}{4}+2\epsilon)} \\
&\qquad\qquad \cdot 2^{\epsilon k}\|P_k^a U'\|_{(\epsilon, \frac{1}{2}-2\epsilon, -\epsilon)} \\
&\lesssim_\beta \|P_{\geq 0}^a U\|_{(\frac{1}{2}-\epsilon, \frac{1}{4}+\epsilon, \frac{1}{4}+2\epsilon)} \\
&\qquad \cdot\|P_{\leq 0}^a U'\|_{(\epsilon, \frac{1}{2}-2\epsilon, -\epsilon)}.
\end{aligned}$$

Similar to in the HH case above, we have $S(I) \hookrightarrow_H (\frac{1}{2}-\epsilon, \frac{1}{4}+\epsilon, \frac{1}{4}+2\epsilon)$. For the low-frequency part, we first use Sobolev embedding to obtain

$$\left(\epsilon, (1-2\epsilon)\cdot\frac{1}{2}+2\epsilon\cdot\left(\frac{3}{10}-\kappa\right), s\right) \hookrightarrow_L \left(\epsilon, \frac{1}{2}-2\epsilon, -\epsilon\right)$$

where $s = 3\left((1-2\epsilon)\cdot\frac{1}{2}+2\epsilon\cdot\left(\frac{3}{10}-\kappa\right)-\frac{1}{2}+2\epsilon)-\epsilon\right) > (1-2\epsilon)(\frac{2}{5}-3\kappa)$. Thus, we have $S(I) \hookrightarrow_L (\epsilon, \frac{1}{2}-2\epsilon, -\epsilon)$. $\square$

We move on to the boundary term. We shall need the following result regarding the boundedness of $\Omega(f,g)$.

Lemma 3.2.4 (Boundary term). *Let U and U' be radial. Then, for $a > \sigma^{-1}(\frac{3}{2}(\frac{3}{10}-\kappa))$,*

$$\left\|\langle D_a\rangle^{-1}\,\Omega(U,U')\right\|_{S(I)} \lesssim_\beta \|U\|_{S(I)}|U'\|_{S(I)}. \tag{3.2.13}$$

PROOF. Again, we convert our H_a^1 estimates to L^2 estimates to compensate for the $\langle D_a \rangle^{-1}$ term. Thus, we need to consider

$$\|\Omega(U,U')\|_{\left(0,\frac{1}{2},0\right)}$$

and

$$\|\Omega(U,U')\|_{\left(\frac{1}{2},\frac{3}{10}-\kappa,\frac{2}{5}-3\kappa\middle|\frac{7}{10}+\kappa\right)}.$$

First, we deal with the $\left(0,\frac{1}{2},0\right)$ norm. We apply Sobolev-Besov embedding as well as Proposition 2.3.4 to obtain that

$$\begin{aligned}
\|\Omega(U,U')\|_{\left(0,\frac{1}{2},0\right)} &\lesssim \sum_{(j,k)\in LL} \|P_j^a U\|_{(0,\frac{1}{4},0)} \|P_k^a U'\|_{(0,\frac{1}{4},0)} \\
&\lesssim_\beta \sum_{(j,k)\in LL} \|P_j^a U\|_{(0,\frac{1}{2},\frac{3}{4})} \|P_k^a U'\|_{(0,\frac{1}{2},\frac{3}{4})} \\
&\lesssim_\beta \sum_{(j,k)\in LL} 2^{\frac{3}{4}j} \|P_j^a U\|_{(0,\frac{1}{2},0)} 2^{\frac{3}{4}k} \|P_k^a U'\|_{(0,\frac{1}{2},0)} \\
&\lesssim \|P_{\leq 0} U\|_{S(I)} |P_{\leq 0} U'\|_{S(I)}.
\end{aligned}$$

It remains now for us to control the other norm in $S(I)$. We have

$$\begin{aligned}
&\|\Omega(U,U')\|_{\left(\frac{1}{2},\frac{3}{10}-\kappa,\frac{2}{5}-3\kappa\right)} \\
&\lesssim \sum_{(j,k)\in LL} \|P_j^a U\|_{\left(\frac{1}{4},\frac{1}{2}\left(\frac{3}{10}-\kappa\right),0\right)} \|P_k^a U'\|_{\left(\frac{1}{4},\frac{1}{2}\left(\frac{3}{10}-\kappa\right),0\right)} \\
&\lesssim \sum_{(j,k)\in LL} 2^{\frac{1}{2}j} \|P_j^a U\|_{\left(\frac{1}{4},\frac{1}{2}\left(\frac{3}{10}-\kappa\right),-\frac{1}{2}\right)} 2^{\frac{1}{2}k} \|P_k^a U'\|_{\left(\frac{1}{4},\frac{1}{2}\left(\frac{3}{10}-\kappa\right),-\frac{1}{2}\right)} \\
&\lesssim \|P_{\leq 0}^a U\|_{\left(\frac{1}{4},\frac{1}{2}\left(\frac{3}{10}-\kappa\right),-\frac{1}{2}\right)} \|P_{\leq 0}^a U'\|_{\left(\frac{1}{4},\frac{1}{2}\left(\frac{3}{10}-\kappa\right),-\frac{1}{2}\right)}.
\end{aligned}$$

Finally, we verify that $S(I)$ controls $\left(\frac{1}{4},\frac{1}{2}\left(\frac{3}{10}-\kappa\right),-\frac{1}{2}\right)$ for low frequencies. We have

$$\left(\frac{1}{4},\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\left(\frac{3}{10}-\kappa\right),s\right) \hookrightarrow \left(\frac{1}{4},\frac{1}{2}\left(\frac{3}{10}-\kappa\right),-\frac{1}{2}\right)$$

where $s=\frac{3}{4}-\frac{1}{2}>\frac{1}{2}\cdot(\frac{2}{5}-3\kappa)$ for a sufficiently small choice of κ. Thus, we indeed have the low frequency embedding.

Notice also that $\frac{1}{2}(\frac{3}{10}-\kappa)\notin(r_0,r_0')$ unless we add a restriction that $a>\sigma^{-1}(\frac{3}{2}(\frac{3}{10}-\kappa))$ – i.e. the range for r for the case $d=3$ is

$$\frac{1}{3}\sigma<\frac{1}{r}<1-\frac{1}{3}\sigma.$$

Thus, we need to set $\sigma<\frac{3}{2}(\frac{3}{10}-\kappa)$ to obtain the required restriction. □

LEMMA 3.2.5 (Trilinear term). *Let U and U' be radial. Then,*

(3.2.14)
$$\left\|\int_0^t e^{i(t-s)\langle D_a\rangle} \langle D_a\rangle^{-1} \Omega(\langle D_a\rangle^{-1} UU'',U')\,\mathrm{d}s)\right\|_{S(I)} \lesssim_\beta \|U\|_{\tilde{S}(I)} \|U'\|_{\tilde{S}(I)} \|U''\|_{\tilde{S}(I)}.$$

PROOF. Again, we shall begin by applying the inhomogeneous Strichartz estimate. In this case, we shall choose the dual pair $(\tilde{q}',\tilde{r}')=(1,2)$. Thus, we find that using (3.2.6) along with Hölder's inequality:

$$\begin{aligned}\|\Omega(\langle D_a\rangle^{-1}UU'',U')\|_{(1,\frac{1}{2},0|0)} &\lesssim \|UU''\|_{(\frac{2}{3},\frac{1}{3},0)}\left\|\|P^a_{j_3}U'\|_{(\frac{1}{3},\frac{1}{6},0)}\right\|_{\ell^1_{j_3}} \\ &= \|UU''\|_{(\frac{2}{3},\frac{1}{3},0)}\left\|\|P^a_{j_3}U'\|_{(\frac{1}{3},\frac{1}{6},-\epsilon|\epsilon)}\right\|_{\ell^2_{j_3}}.\end{aligned}$$

Also, for $(j_1,j_2)\in\mathbf{Z}^2$,

$$\begin{aligned}\|P^a_{j_1}UP^a_{j_2}U''\|_{(\frac{2}{3},\frac{1}{3},0)} &\lesssim 2^{\epsilon j_1-2\epsilon j_1^+}\|P^a_{j_1}U\|_{(\frac{1}{3},\frac{1}{6},-\epsilon|\epsilon)} \\ &\quad\cdot 2^{\epsilon j_2-2\epsilon j_2^+}\|P^a_{j_2}U''\|_{(\frac{1}{3},\frac{1}{6},-\epsilon|\epsilon)}.\end{aligned}$$

Now, note that

$$\begin{aligned}\|P^a_{j_1}UP^a_{j_2}U''P^a_kU'\|_{L^1_tL^2_x} &\lesssim \left\|\|P^a_{j_1}U\|_{(\frac{1}{3},\frac{1}{6},0|0)}\right\|_{\ell^1_{j_1}}\cdots\left\|\|P^a_kU'\|_{(\frac{1}{3},\frac{1}{6},0|0)}\right\|_{\ell^1_k} \\ &\lesssim \left\|\|P^a_{j_1}U\|_{(\frac{1}{3},\frac{1}{6},-\epsilon|\epsilon)}\right\|_{\ell^2_{j_1}}\cdots\left\|\|P^a_kU'\|_{(\frac{1}{3},\frac{1}{6},-\epsilon|\epsilon)}\right\|_{\ell^2_{j_3}}.\end{aligned}$$

Now, we note that $\tilde{S}(I)\hookrightarrow\left(\frac{1}{3},\frac{1}{6},-\epsilon|\epsilon\right)$ to obtain the required result. □

We should also check that this component of $\tilde{S}(I)$ can also be controlled by $S(I)$. In this case, we need to interpolate between $\frac{2}{3}$ of $\left(\frac{1}{2},\frac{3}{10}-\kappa,\frac{2}{5}-3\kappa\,\middle|\,\frac{7}{10}+\kappa\right)$ and $\frac{1}{3}$ of $(0,\frac{1}{2},0|1)$. Let us first check the following embedding for high and low frequencies:

$$\left(\frac{1}{3},\frac{2}{3}\left(\frac{3}{10}-\kappa\right)+\frac{1}{3}\cdot\frac{1}{2},s_L\,|s_H\right)\hookrightarrow\left(\frac{1}{3},\frac{1}{6},-\epsilon|\epsilon\right).$$

For low-frequency we see that we need

$$s_L\le-\epsilon-\frac{3}{6}+2\left(\frac{3}{10}-\kappa\right)+\frac{1}{3}\cdot\frac{3}{2},$$

and for high-frequency we see that we need

$$\epsilon-\frac{3}{6}+2\left(\frac{3}{10}-\kappa\right)+\frac{1}{3}\le s_H.$$

We see that we may choose such values of s_L and s_H so that

$$\frac{2}{3}\left(\frac{2}{5}-3\kappa\right)\le s_L$$

and we can choose s_H so that

$$s_H\le\frac{2}{3}\left(\frac{7}{10}+\kappa\right)+\frac{1}{3},$$

so we indeed obtain the required embedding to control $\tilde{S}_I$. Thus, we obtain the following perturbed Strichartz estimate:

Proposition 3.2.6 (Perturbed Strichartz estimates). *Suppose $d=3$. Let $0<\epsilon<\kappa\ll1$ and U is a solution to (3.2.8 - 3.2.11) where U_0 is radial and $a>\sigma^{-1}(\frac{3}{2}(\frac{3}{10}-\kappa))$. Then, we have*

$$\|U\|_{S(I)}\lesssim\|U_0\|_{H^1_a}+\|U\|^2_{S(I)}+\|U\|^2_{S(I)}+\|U\|^3_{S(I)}.$$

Thus, we also have small data scattering for (3.2.1).

CHAPTER 4

Small energy scattering in higher dimensions

4.1. Function spaces

In this section, we shall review the definition and basic properties of U^p and V^p spaces (see [**24**]). The U^p spaces were used earlier by Koch and Tataru [**39, 40**], while the V^p spaces were introduced much earlier by Wiener [**62**]. Firstly, we shall denote the set of finite partitions of the form $-\infty = t_0 < t_1 < \cdots < t_K = \infty$ by $\mathcal{Z}$. The set of finite partitions of the form $-\infty < t_0 < t_1 < \cdots < t_K < \infty$ shall be denoted by $\mathcal{Z}_0$. Then, for $1 \leq p < \infty$ and $\{t_k\}_{0 \leq k \leq K} \in \mathcal{Z}$ and $\{\phi_k\}_{0 \leq k \leq K} \subset L^2(\mathbf{R}^d, \mathbf{C})$ such that $\sum_{k=0}^{K} \|\phi_k\|_{L^2}^p = 1$ and $\phi_0 = 0$, a function of the form

$$a = \sum_{k=1}^{K} \mathbf{1}_{[t_{k-1}, t_k)} \phi_{k-1}$$

a U^p**-atom**. We then define the atomic space U^p as

$$U^p := \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j \in L^\infty(\mathbf{R}, L^2) : a_j \text{ are } U^p \text{ atoms and } \{\lambda_j\}_{1 \leq j \leq \infty} \in \ell^1 \right\}$$

endowed with the norm

$$\|u\|_{U^p} := \inf \left\{ \|\lambda_j\|_{\ell^1} : u = \sum_{j=1}^{\infty} \lambda_j a_j, \lambda_j \in \mathbf{C}, a_j \text{ are } U^p \text{ atoms} \right\}.$$

Also, we define for $1 \leq p < \infty$ the space V^p of **bounded** p**-variation** as the space of all functions $v : \mathbf{R} \to L^2$ such that $\lim_{t \to \infty} v(t) = 0$ and such that $\lim_{t \to -\infty} v(t)$ exists and also satisfies

$$\|v\|_{V^p} := \sup_{\{t_k\}_{0 \leq k \leq K} \in \mathcal{Z}} \left(\sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{L^2}^p \right)^{\frac{1}{p}} < \infty.$$

We define $V_*^p \subset V^p$ as the subspace of all functions $v \in V^p$ such that $\lim_{t \to -\infty} v(t) = 0$, $\lim_{t \to \infty} v(t)$ exists and $||v||_{V^p} < \infty$. Also, denote by V_{rc} the space of all right-continuous $v \in V^p$ and $V_{*,\mathrm{rc}}$ the space of all right-continuous $v \in V_*^p$.

Lemma 4.1.1 (Basic properties of U^p and V^p spaces). *Let $1 \leq p < q < \infty$.*

(i) *U^p, V^p, V_*^p and $V_{*,\mathrm{rc}}^p$ are Banach spaces.*
(ii) *Let $u \in U^p$. Then $u : \mathbf{R} \to L^2$ is a right-continuous.*
(iii) *Let $u \in U^p$. Then, $\lim_{t \to -\infty} u(t) = 0$ and $\lim_{t \to \infty} u(t)$ exists in L^2.*
(iv) *Let $v \in V^p$. Then, $\lim_{t \to \pm\infty} v(t)$ exist in L^2.*
(v) *The embeddings $V^p \hookrightarrow V^q$ and $V_*^p \hookrightarrow V_*^q$ are continuous.*
(vi) *The embeddings $U^p \hookrightarrow V_{*,\mathrm{rc}}^p \hookrightarrow U^q \hookrightarrow L_t^\infty(\mathbf{R}, L_x^2(\mathbf{R}^d))$ are continuous.*

LEMMA 4.1.2 (Duality). *Let $1 < p < \infty$. Then, $(U^p)^* = V^{p'}$ in the sense that there exists an isometric isomorphism $T : V^{p'} \to (U^p)^*$ such that $T(v) = B(\cdot, v)$. In particular, suppose that $u \in V^1_*$ is absolutely continuous on compact intervals and $v \in V^{p'}$. Then,*

$$B(u,v) = -\int_{-\infty}^{\infty} (u'(t), v(t))_{L^2}\ \mathrm{d}t.$$

Furthermore, we have the following dual characterisation of $\|\cdot\|_{U^p}$ as

$$\|u\|_{U^p} = \sup_{v \in V^{p'},\ \|v\|_{V^{p'}}=1} \left| \int_{-\infty}^{\infty} (u'(t), v(t))_{L^2}\ \mathrm{d}t \right|.$$

The above spaces U^p and V^p provide a framework in which we may define function spaces adapted to the operators $e^{\pm it\langle D_a\rangle}$. We now build adapted function spaces with have desirable properties to close the contraction mapping argument:

$$U^p_\pm = \left\{ u : \mathbf{R} \to L^2 : e^{\mp it\langle D_a\rangle} u \in U^p \right\}$$

with norm

$$\|u\|_{U^p_\pm} := \left\| e^{\mp it\langle D_a\rangle} u \right\|_{U^p}.$$

We also define the space

$$V^p_\pm = \left\{ v : \mathbf{R} \to L^2 : e^{\mp it\langle D_a\rangle} u \in V^p \right\}$$

with norm

$$\|v\|_{V^p_\pm} := \left\| e^{\mp it\langle D_a\rangle} u \right\|_{V^p}.$$

We remark that $U^p_\pm$ is again an atomic space with atoms of the form $e^{\pm it\langle D_a\rangle}a$ where a is an U^p-atom. We define some relevant projections. Let $M \in 2^{\mathbf{Z}}$ be a dyadic number. Note that in this chapter, dyadic numbers shall be denoted usually by M, N and N'. Here, their relative size to each other is more important, and so this notation is more convenient. Also, let ϕ be a smooth, even, non-negative function such that $\phi(t) = 1$ on $|t| < 1$ and $\phi(t) = 0$ on $|t| > 2$. Set also $\psi(t) = \phi(t) - \phi(2t)$ and $\psi_N(t) = \psi(t/N)$. Next we define the time frequency projection

$$\mathcal{F}_t(Q_N u) = \psi_N \mathcal{F}_t u.$$

Recall the space frequency projection is defined as $P^a_N u := \psi_N(D_a)u$ so that for $u \in L^2_{=k}(\mathbf{R}^d)$, we have

$$P^a_N u = \mathcal{H}_{\nu(k)} \psi_N(\rho) \mathcal{H}_{\nu(k)} u.$$

Define the modulation projection

$$Q^\pm_M u := e^{\pm t\langle D_a\rangle} Q_M e^{\mp t\langle D_a\rangle} u \tag{4.1.1}$$

such that for $u \in L^2_{=k}(\mathbf{R}^d)$ and recalling the notation $\widetilde{u} := \mathscr{F}_t \mathcal{H}_\nu u$, where in this case $\nu = \sqrt{\mu(k)^2 + a}$, we have

$$\widetilde{Q^\pm_M u} := \psi_M(\tau \mp \langle\rho\rangle)\widetilde{u}.$$

We define $Q^\pm_{\le M} u$ in a similar manner so that on each harmonic subspace, we have

$$\widetilde{Q^\pm_{\le M} u} = \phi_M(\tau \mp \langle\rho\rangle)\widetilde{u}.$$

We have the following estimates related to the above projections. The proofs are the same as in [**24**] as the estimates may be reduced to estimates on U^p and V^p via the definition (4.1.1).

LEMMA 4.1.3. *Let M be a dyadic number. Let $1 \le p < \infty$ (and if $-\left(\frac{d-2}{2}\right)^2 < a < 0$, then also $r_0 < p < r_0'$). Then,*

(i) $\|Q_M^{\pm} u\|_{L^2(\mathbf{R}^d)} \lesssim M^{-\frac{1}{2}} \|u\|_{V_{\pm}^2}$
(ii) $\|Q_{\geq M}^{\pm} u\|_{L^2(\mathbf{R}^d)} \lesssim M^{-\frac{1}{2}} \|u\|_{V_{\pm}^2}$
(iii) $\|Q_{<M}^{\pm} u\|_{U_{\pm}^p} \lesssim \|u\|_{U_{\pm}^p}$, $\|Q_{\geq M}^{\pm} u\|_{U_{\pm}^p} \lesssim \|u\|_{U_{\pm}^p}$
(iv) $\|Q_{<M}^{\pm} u\|_{V_{\pm}^p} \lesssim \|u\|_{V_{\pm}^p}$, $\|Q_{\geq M}^{\pm} u\|_{V_{\pm}^p} \lesssim \|u\|_{V_{\pm}^p}$

The U^p spaces are able to inherit L^2-based multilinear estimates related to free solutions via the following transfer principle. This shall be especially relevant later when we obtain bilinear Strichartz estimates which are of the form $L^2 \times L^2 \to L^2$. This transfer principle allows us to convert this to an estimate, say, $U_{\pm_1}^2 \times U_{\pm_2}^2 \to L^2$, which we shall need for the trilinear estimates, as mentioned in the discussion at the start of the chapter.

LEMMA 4.1.4 (Transfer principle). *Let $T_0 : L^2 \times \cdots \times L^2 \to L^1_{\mathrm{loc}}(\mathbf{R}^d, \mathbf{C})$ be a m-linear operator. Suppose for some $1 \le p, q \le \infty$ that*

$$\left\| T_0(e^{\pm_1 \cdot \langle D_a \rangle} \phi_1, ..., e^{\pm_m \cdot \langle D_a \rangle} \phi_m) \right\|_{L_t^p(\mathbf{R}, L_x^q(\mathbf{R}^d)} \lesssim \|\phi_1\|_{L^2} \cdots \|\phi_m\|_{L^2}.$$

There exists an operator $T : U_{\pm_1}^p \times \cdots \times U_{\pm_m}^p \to L_t^p(\mathbf{R}, L_x^q(\mathbf{R}^d))$ satisfying

$$\|T(u_1, ..., u_m)\|_{L_t^p(\mathbf{R}, L_x^q(\mathbf{R}^d))} \lesssim \|u_1\|_{U_{\pm_1}^p} \cdots \|u_m\|_{U_{\pm_m}^p}$$

such that $T(u_1, ..., u_m)(t)(x, y) = T_0(u_1(t), ..., u_m(t))(x, y)$ a.e.

The above transfer principle allows us to bring L^2-based multilinear estimates within the framework of the U^p and V^p spaces, but at the moment, only for U^p spaces. We have the following result which allows us to form V^p estimates as well.

LEMMA 4.1.5 (Interpolation). *Let $q > 1$, E a Banach space, $T : U_{\pm}^q \to E$ a bounded linear operator such that $\|Tu\|_E \le C_q \|U\|_{U_{\pm}^q}$ for $u \in U_{\pm}^q$. Suppose also that for some $1 \le p < q$, it holds that $\|Tu\|_E \le C_p \|u\|_{U_{\pm}^p}$ where $0 < C_p \le C_q$ for all $u \in U_{\pm}^p$. Then for all $u \in V_{-,\mathrm{rc},\pm}^p$, we have*

$$\|Tu\|_E \le \frac{4C_p}{2\alpha_{p,q}} \left(\ln \left(\frac{C_q}{C_p} + 2\alpha_{p,q} + 1 \right) \right) \|u\|_{V_{\pm}^p}, \tag{4.1.2}$$

where $\alpha_{p,q} = (1 - p/q)\ln(2)$.

Finally, we define the function spaces in which we shall perform the contraction mapping argument

$$\begin{aligned} \|u\|_{X_{\pm}^s} &= \left(\sum_N N^{2s} \|P_N^a u^{\pm}\|_{U_{\pm}^2}^2 \right)^{\frac{1}{2}} \\ \|u\|_{Y_{\pm}^s} &= \left(\sum_N N^{2s} \|P_N^a u^{\pm}\|_{V_{\pm}^2}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

We put these spaces together and define $X_s := X_+^s \times X_-^s$ and $Y_s := Y_+^s \times Y_-^s$.

4.2. Bilinear Strichartz estimates

4.2.1. Bilinear estimates for radial initial data. Let us begin our study of the Klein-Gordon equation with inverse-square potential. As discussed above, we shall do via the equivalent first-order system:

$$\left\{ \begin{array}{l} \pm i u_t(x,t) + \langle D_a \rangle u(x,t) = 0, \quad (t,x) \in \mathbf{R} \times \mathbf{R}^d \\ u(0,x) = u_0(x). \end{array} \right. \tag{4.2.1}$$

In particular, we shall obtain the following bilinear Strichartz estimate initially for radial initial data. In this chapter, we shall consider the range $a > \mathcal{A}_d$ where

$$\mathcal{A}_d = \left\{ \begin{array}{ll} -\left(\frac{d-2}{2}\right)^2, & d = 3,4 \\ \frac{1}{16}(8d - 3d^2), & d \geq 5. \end{array} \right. \tag{4.2.2}$$

This is the range for which the L^4 norm is enough to control the Hankel transform at the origin when $a < 0$. We note that $\sigma^{-1}\left(\frac{d}{4}\right) = \frac{1}{16}(8d - 3d^2)$.

Theorem 4.2.1 (Bilinear Strichartz estimates for $M \ll N$ (radial case)). *Suppose that $d \geq 3$, $a > \mathcal{A}_d$. Suppose that $M, N \in 2^{\mathbf{N}}$ and $u_M, u_N \in L^2$ are radial with spatial frequency supported at frequencies M and N respectively (i.e. $P_k^a u_M = u_M$ and $P_k^a u_N = u_N$). Also, denote $\pm_i \in \{+,-\}$ for $i = 1,2$. We have the following bilinear Strichartz estimates:*

$$\left\| \left(e^{\pm_1 it\langle D_a \rangle} u_M\right) \left(e^{\pm_2 it\langle D_a \rangle} u_N\right) \right\|_{L_t^2 L_x^2} \lesssim_a M^{\frac{d}{4}} N^{\frac{d}{4} - \frac{1}{2}} \|u_M\|_{L^2} \|u_N\|_{L^2}. \tag{4.2.3}$$

Remark 4.2.2. *The coefficient can still be improved, as in [**53**]. However, it is sufficient for our purposes. The issue lies in the estimate (4.2.5) below. Furthermore, we use the notation $\mathrm{d}\omega(x,y)$ to denote $\mathrm{d}\omega(x)\mathrm{d}\omega(y) = x^{d-1}\mathrm{d}x \cdot y^{d-1}\mathrm{d}y$. This use will be justified via Fubini's theorem.*

Proof. Let us discuss the proof in the case where $\pm_1 = \pm_2 = +$. The other cases are similar, as will be noted below in the proof. We shall use the fact that u_M and u_N are radial to reduce the L^2 norm to the radial $\mathscr{L}^2$ norm. Note that by a slight abuse of notation, we shall write $u_M(z) = u_M(|z|)$. Next, by duality,

$$\text{LHS of (4.2.3)} = \sup_{\|G\|_{L_t^2 \mathscr{L}_x^2} = 1} \left| \left\langle G, \left(e^{it\langle D_a \rangle} u_M\right) \left(e^{it\langle D_a \rangle} u_N\right) \right\rangle_{L_t^2 \mathscr{L}_x^2} \right|.$$

Thus, the fact that $\widehat{f} \# \widehat{g} = \widehat{fg}$, the inner product becomes

$$\begin{aligned} &\left\langle \widehat{G}, \left(e^{it\langle \rho \rangle} \widehat{u_M}(\rho)\right) \# \left(e^{it\langle \rho \rangle} \widehat{u_N}(\rho)\right) \right\rangle_{L_t^2 \mathscr{L}_\rho^2} \\ &= \int_{\mathbf{R}} \left\langle \widehat{G}, \left(e^{it\langle \rho \rangle} \widehat{u_M}(\rho)\right) \# \left(e^{it\langle \rho \rangle} \widehat{u_N}(\rho)\right) \right\rangle_{\mathscr{L}_\rho^2} \mathrm{d}t. \end{aligned}$$

Now, we expand the Hankel convolution[1]:

$$\left(e^{it\langle \rho \rangle} \widehat{u_M}(\rho)\right) \# \left(e^{it\langle \rho \rangle} \widehat{u_N}(\rho)\right) = \int_0^\infty e^{it(\langle x \rangle + \langle y \rangle)} \widehat{u_M}(x) \widehat{u_N}(y) D(x,y,\rho) \, \mathrm{d}\omega(x,y).$$

[1]For ease of notation, we shall omit the extra integral signs. Also, recall the norm $\|u\|_{\mathscr{L}^p} := \int_0^\infty u(x) \, \mathrm{d}\omega(x) = \int_0^\infty u(x) \, x^{d-1} \mathrm{d}x$.

Thus, by Fubini's theorem, we see that it is suitable to apply the Fourier transform in the time variable, and obtain the following simplification for the inner product:

$$\int_0^\infty \left\langle \widetilde{G}(\langle x\rangle + \langle y\rangle, \rho), \widehat{u_M}(x)\widehat{u_N}(y)D(x,y,\rho)\right\rangle_{\mathscr{L}^2_\rho} \, \mathrm{d}\omega(x,y),$$

which is equal to

$$\int_0^\infty \overline{G^*(\langle x\rangle + \langle y\rangle, \rho)}, \widehat{u_M}(x)\widehat{u_N}(y)D(x,y,\rho)\, \mathrm{d}\omega(x,y,\rho).$$

Thus, in order to prove (4.2.3), it is equivalent to prove

$$\begin{aligned} &\left|\int_0^\infty G(\langle x\rangle + \langle y\rangle, \rho), \widehat{u_M}(x)\widehat{u_N}(y)D(x,y,\rho)\, \mathrm{d}\omega(x,y,\rho)\right| \\ &\qquad \lesssim_a \mathbf{KG}_d \|G(u,\rho)\|_{L^2_u\mathscr{L}^2_\rho}\|u_M\|_{\mathscr{L}^2_\rho}\|u_N\|_{\mathscr{L}^2_\rho}. \end{aligned} \tag{4.2.4}$$

First, we shall perform a change of variables: $u = \langle x\rangle + \langle y\rangle \sim \ \langle y\rangle$ and $v = x$ with $\mathrm{d}u\mathrm{d}v \sim \mathrm{d}x\mathrm{d}y$. (Here, the other cases for $\pm_1, \pm_2$ can be handled similarly as we have assumed that $1 \leq M \ll N$, so we always have $\langle x\rangle \ll \langle y\rangle$.) Thus, we obtain also with the Cauchy-Schwarz inequality that the left-hand side of (4.2.4) is bounded by

$$\begin{aligned} & \|G\|_{L^2_u\mathscr{L}^2_\rho} \int_0^\infty \left(\int_0^\infty \left(\widehat{u_M}(x)\widehat{u_N}(y)D(x,y,\rho)\right)^2 y^{d-1}\, \mathrm{d}\omega(y)\mathrm{d}\omega(\rho)\right)^{\frac12} \, \mathrm{d}\omega(x) \\ \lesssim\ & \|G\|_{L^2_u\mathscr{L}^2_\rho} M^{\frac d2}\left(\int_0^\infty \left(\widehat{u_M}(x)\widehat{u_N}(y)D(x,y,\rho)\right)^2 y^{d-1}\, \mathrm{d}\omega(x,y,\rho)\right)\right)^{\frac12} \\ \lesssim\ & \|G\|_{L^2_u\mathscr{L}^2_\rho} M^{\frac d2} N^{\frac{d-1}{2}}\left(\int_0^\infty \left(\widehat{u_M}(x)\widehat{u_N}(y)D(x,y,\rho)\right)^2 \, \mathrm{d}\omega(x,y,\rho)\right)\right)^{\frac12}. \end{aligned}$$

Thus, it remains to deal with the integral. For this, we use Lemma 4.2.3 which we shall prove shortly. Combined with the fact that $x \sim M$ and $y \sim N$, we obtain the required result. □

We now prove the integral estimate required above, as well as a more general estimate. We again note the slight abuse of notation where we have identified a radial function $f(x)$ as a function in $|x|$.

LEMMA 4.2.3. *Let u and v be radial. For $a > \mathcal{A}_d$, $\nu := \nu(a,k)$ and $d \geq 3$, we have*

$$\int_0^\infty \left(u(x)v(y)D_\nu(x,y,\rho)\right)^2 \, \mathrm{d}\omega(x,y,\rho) \lesssim \|U_\nu\|^2_{\mathscr{L}^4} \left\|\frac{u(x)}{x^{\frac d4}}\right\|^2_{\mathscr{L}^2_x}\left\|\frac{v(y)}{y^{\frac d4}}\right\|^2_{\mathscr{L}^2_y}. \tag{4.2.5}$$

PROOF. Using the definition of $D_\nu(x,y,\rho)$ (we shall from now on omit the subscript ν) and Fubini's theorem, we find that the left-hand side of (4.2.5) is

$$\int_0^\infty u(x)^2v(y)^2U(x\eta)U(y\eta)U(\rho\eta)U(x\tilde\eta)U(y\tilde\eta)U(\rho\tilde\eta)\, \mathrm{d}\omega(x,y,\rho,\eta,\tilde\eta).$$

The idea is to use the orthogonality of Bessel functions in the ρ variable. Thus, we write the above integral as

$$\int_0^\infty u(x)^2v(y)^2U(x\eta)U(y\eta)U(x\tilde\eta)U(y\tilde\eta)\left[\int_0^\infty U(\rho\eta)U(\rho\tilde\eta)\, \mathrm{d}\omega(\rho)\right] \mathrm{d}\omega(x,y,\eta,\tilde\eta).$$

We may deal with the inner integral using the following Bessel function identity

$$\int_0^\infty J_\nu(\rho\eta)J_\nu(\rho\tilde{\eta})\rho \, \mathrm{d}\rho = \frac{1}{\eta}\delta(\eta - \tilde{\eta}).$$

Then, the whole integral becomes

$$\begin{aligned} &\int_0^\infty (u(x)U(x\eta))^2 \, (v(y)U(y\eta))^2 \, \mathrm{d}\omega(x,y,\eta) \\ &= \left\| \|u(x)U(x\eta)\|_{\mathscr{L}^2_x} \|u(y)U(y\eta)\|_{\mathscr{L}^2_y} \right\|^2_{L^2_\eta}. \end{aligned} \tag{4.2.6}$$

Now, we use Hölder's inequality and interchange of norms to obtain

$$\begin{aligned} (4.2.6) \quad &\leq \quad \|u(x)U(x\eta)\|^2_{L^4_\eta \mathscr{L}^2_x} \|u(y)U(y\eta)\|^2_{\mathscr{L}^4_\eta \mathscr{L}^2_\eta} \\ &\leq \quad \|u(x)U(x\eta)\|^2_{\mathscr{L}^2_x \mathscr{L}^4_\eta} \|u(y)U(y\eta)\|^2_{\mathscr{L}^2_y \mathscr{L}^4_\eta}. \end{aligned}$$

By scaling, we also have

$$\begin{aligned} \|u(x)U(x\eta)\|^2_{\mathscr{L}^2_x \mathscr{L}^4_\eta} \quad &= \quad \left\| u(x)\|U(x\eta)\|_{\mathscr{L}^4_\eta} \right\|^2_{\mathscr{L}^2_x} \\ &= \quad \left\| \frac{u(x)}{x^{\frac{d}{4}}} \|U(\eta)\|_{\mathscr{L}^4_\eta} \right\|^2_{\mathscr{L}^2_x}. \end{aligned}$$

Now, we note that this $\mathscr{L}^4$ norm is finite only for $a > \mathcal{A}_d$. Therefore, we obtain the required result. $\square$

REMARK 4.2.4. *We can try to improve the estimate (4.2.5). We can try the following estimate:*

$$\begin{aligned} &\int_0^\infty (u(x)U(x\eta))^2 \, (v(y)U(y\eta))^2 \, \mathrm{d}\omega(x,y,\eta) \\ \lesssim \quad &\int_0^\infty u(x)^2 v(y)^2 \frac{1}{y^d} \int_0^\infty U\left(\frac{x\eta}{y}\right)^2 U(\eta)^2 \, \mathrm{d}\omega(\eta) \, \mathrm{d}\omega(x,y) \end{aligned}$$

Now, we split the inner integral into regions $x\eta/y > C$ and $x\eta/y < C$ to estimate the Bessel function. Suppose that $U(z) = U_\nu(z)$ is bounded near 0 – i.e. $\nu > \frac{d-2}{2}$. In this case, the $x\eta/y < C$ integral is bounded and for the $x\eta/y > C$ integral we have

$$\int_{\eta > Cy/x}^\infty \left(\frac{x\eta}{y}\right)^{2(-\frac{d-2}{2}-\frac{1}{2})} \eta^{2(-\frac{d-2}{2}-\frac{1}{2})} \, \mathrm{d}\eta.$$

If we have $y \leq x$, then $y/x \leq 1$, and so we obtain a bound on the inner integral that is independent of x and y. Putting everything together, we have the estimate

$$\int_0^\infty (u(x)v(y)D_\nu(x,y,\rho))^2 \, \mathrm{d}\omega(x,y,\rho) \lesssim \|U_\nu\|_{\mathscr{L}^4} \|u(x)\|^2_{\mathscr{L}^2_x} \left\| \frac{v(y)}{y^{\frac{d}{2}}} \right\|^2_{\mathscr{L}^2_y}.$$

However, because we imposed that $y \leq x$, this estimate gives weaker coefficients for the bilinear Strichartz estimates (Theorem 4.2.3) compared to (4.2.5).

4.2.2. Weighted Hankel convolution estimates. In this section, we shall first prove the following lemma, which we shall use to prove the bilinear Strichartz estimate for comparable frequencies. We shall focus on the radial case. We will also need the spacetime Hankel transform which we define as $\mathcal{H}^{\nu}_{tx}u := \mathscr{F}_t\mathcal{H}_{\nu,x}u$. As we have seen previously, it is useful to take the Fourier transform of the time variable. We denote the corresponding convolution by $\#_{tx}$. From the multiplicative structure of the usual convolution on t and the Hankel convolution of x, we can readily deduce the multiplicative structure of $\#_{tx}$. Indeed, we find that for well-behaved f and g, we have

$$\begin{aligned}
\mathscr{F}_t\mathcal{H}_x(fg)(\rho) &= \mathscr{F}_t\int_0^\infty \tau_\rho\widehat{f}(t,x)\widehat{g}(t,x)\,\mathrm{d}\omega(x)\\
&= \mathscr{F}_t\int_0^\infty \widehat{f}(t,y)\widehat{g}(t,x)D(x,y,\rho)\,\mathrm{d}\omega(x,y)\\
&= \int_0^\infty \mathscr{F}_t(\widehat{f}(t,y)\widehat{g}(t,x))D(x,y,\rho)\,\mathrm{d}\omega(x,y)\\
&= \int_{\mathbf{R}}\int_0^\infty \widetilde{f}(\tau-s,y)\widetilde{g}(s,x)D(x,y,\rho)\,\mathrm{d}\omega(x,y)\mathrm{d}s,
\end{aligned}$$

where $\widetilde{u} := \mathcal{H}_{tx}(u)$. As usual, we shall suppress the order of the Hankel transform unless it is unclear.

LEMMA 4.2.5. *For $\nu(k,a)$ with $a > \mathcal{A}_d$, suppose that u and v are radial and their spacetime Hankel transforms of order $\nu(k,a)$ are supported on sets A and B – i.e.* $\operatorname{supp}(\mathcal{H}_{tx}u) \subseteq A$ *and* $\operatorname{supp}(\mathcal{H}_{tx}v) \subseteq B$. *Then,*

$$\|uv\|_{L^2_t\mathscr{L}^2_x} \leq_a \left(\sup_{\tau\in\mathbf{R}}|I(\tau)|\right)^{\frac12}\left\|\frac{\widetilde{u}}{x^{\frac{d}{4}}}\right\|_{L^2_t\mathscr{L}^2_x}\left\|\frac{\widetilde{v}}{x^{\frac{d}{4}}}\right\|_{L^2_t\mathscr{L}^2_x},$$

and for $A^* = \{(-\tau,x) : (\tau,x)\in A\}$,

$$I(\tau) = \int_{\mathbf{R}}\int_0^\infty \mathbf{1}_{(\tau,0)+A^*}(s,x)\mathbf{1}_B(s,y)\,\mathrm{d}\omega(x,y)\mathrm{d}s.$$

PROOF. We first note that

$$\begin{aligned}
\|uv\|^2_{L^2_t\mathscr{L}^2_x} &= \|\widetilde{u}\#_{tx}\widetilde{v}\|^2_{L^2_\tau\mathscr{L}^2_\rho}\\
&= \|\mathbf{1}_A\widetilde{u}\#_{tx}\mathbf{1}_B\widetilde{v}\|^2_{L^2_\tau\mathscr{L}^2_\rho}.
\end{aligned}$$

We may write this final norm as

$$\int_{\mathbf{R}}\int_0^\infty\left(\int_{\mathbf{R}}\int_0^\infty \mathbf{1}_A\widetilde{u}(\tau-s,x)\mathbf{1}_B\widetilde{v}(s,y)D(x,y,\rho)\,\mathrm{d}\omega(x,y)\mathrm{d}s\right)^2\,\mathrm{d}\omega(\rho)\mathrm{d}\tau. \tag{4.2.7}$$

We shall apply the Cauchy-Schwarz to the inner integrals to obtain:

$$\begin{aligned}
&\int_{\mathbf{R}}\int_0^\infty \mathbf{1}_A\widetilde{u}(\tau-s,x)\mathbf{1}_B\widetilde{v}(s,y)D(x,y,\rho)\,\mathrm{d}\omega(x,y)\mathrm{d}s\\
&\qquad\leq\left(\int_{\mathbf{R}}\int_0^\infty \mathbf{1}_A(\tau-s,x)\mathbf{1}_B(s,y)\,\mathrm{d}\omega(x,y)\mathrm{d}s\right)^{\frac12}\\
&\qquad\quad\cdot\left(\int_{\mathbf{R}}\int_0^\infty (\widetilde{u}(\tau-s,x)\widetilde{v}(s,y)D(x,y,\rho))^2\,\mathrm{d}\omega(x,y)\mathrm{d}s\right)^{\frac12}\\
&\qquad:= I^{\frac12}J^{\frac12}.
\end{aligned}$$

Thus, (4.2.7) becomes $\int\int IJ\,\mathrm{d}\omega(\rho)\mathrm{d}s$. Let us first study the integral $I = I(\tau)$:

$$\begin{aligned} I(\tau) &= \int_{\mathbf{R}}\int_0^\infty \mathbf{1}_A(\tau - s, x)\mathbf{1}_B(s,y)\ \mathrm{d}\omega(x,y)\mathrm{d}s \\ &= \int_{\mathbf{R}}\int_0^\infty \mathbf{1}_{(\tau,0)+A^*}(s,x)\mathbf{1}_B(s,y)\ \mathrm{d}\omega(x,y)\mathrm{d}s. \end{aligned}$$

Here, $A^* = \{(-\tau, x) : (\tau, x) \in A\}$. Now, we notice that for a fixed $s \in \mathbf{R}$, the product $\mathbf{1}_{(\tau,0)+A^*}(s,x)\mathbf{1}_B(s,y)$ is nonzero if $x \in E_{(\tau,0)+A^*}(s)$ and $y \in E_B(s)$ where for a set $C \subseteq \mathbf{R}\times(0,\infty)$, $E_C(s) = \{x : (s,x) \in C\}$.

Thus, going back to (4.2.7), where we see that we will again encounter the integral from above:

$$\begin{aligned} (4.2.7) &\leq \int_{\mathbf{R}}\int_0^\infty I(\tau)J\ \mathrm{d}\omega(\rho)\mathrm{d}\tau \\ &\leq \sup_\tau |I(\tau)| \int_{\mathbf{R}}\int_{\mathbf{R}}\int_0^\infty \left(\widetilde{u}(\tau - s, x)\widetilde{v}(s,y)D(x,y,\rho)\right)^2\ \mathrm{d}\omega(x,y,\rho)\mathrm{d}s\mathrm{d}\tau \\ &= \sup_\tau |I(\tau)| \int_{\mathbf{R}}\int_{\mathbf{R}}\int_0^\infty \left(\widetilde{u}(\tau, x)\widetilde{v}(s,y)D(x,y,\rho)\right)^2\ \mathrm{d}\omega(x,y,\rho)\mathrm{d}s\mathrm{d}\tau. \end{aligned}$$

Now, we use the same convolution estimate as above (Lemma 4.2.3) to obtain the required result. $\square$

4.2.3. Bilinear Strichartz estimates for comparable frequencies. Let us now use the above to obtain the bilinear Strichartz estimates for comparable frequencies. This means we want an estimate with $M \sim N$ for the term

$$\left\|\left(e^{\pm_1 it\langle D_a\rangle}u_M\right)\left(e^{\pm_2 it\langle D_a\rangle}u_N\right)\right\|_{L_t^2L_x^2}.$$

We consider the case where $\pm_1 = \pm_2 = +$. The other cases are handled in a similar manner, as shall be seen below.

We shall follow the method of thickened spheres used by Selberg [**54**] (see also Schottdorf [**53**]). Thus, we shall approximate the spacetime supports $\delta(\tau \pm \langle\rho\rangle)$ by $\epsilon^{-1}\mathbf{1}_{|\tau\pm\langle\rho\rangle|<\epsilon}$. In particular, we thicken the support of $e^{\pm it\langle D_a\rangle}u_M$ to $A = \{(t,\rho) : \rho \sim M, |\tau \mp \langle\rho\rangle| < \epsilon\}$ and likewise we thicken the support of $e^{\pm it\langle D_a\rangle}u_N$ to $B = \{(t,\rho) : \rho \sim N, |\tau \mp \langle\rho\rangle| < \epsilon\}$. The goal now is to obtain a bound on $I_1(\tau)$ above using these choices for A and B and then finally take the limit as $\epsilon \to 0$ to get the bilinear estimates.

First, we notice that the supremum is attained when $(\tau, 0) + A^*$ and B are situated in the below diagram – i.e. $(\tau,0)+A^*$ completely overlaps with B. We also notice that the other cases for $\pm_1, \pm_2$ can be handled in a similar way to the $\pm_1 = \pm_2 = +$ case as the only difference amounts to a reflection, which does not affect the geometry of the problem.

Recall the integral for $I(\tau)$:

$$I(\tau) = \int_{\mathbf{R}}\int_0^\infty \mathbf{1}_{(\tau,0)+A^*}(s,x)\mathbf{1}_B(s,y)\ \mathrm{d}\omega(x,y)\mathrm{d}s.$$

We first have that for $M > 1$ and $\epsilon \ll 1$

$$\int_0^\infty \mathbf{1}_{(\tau,0)+A^*}(s,x)\ \mathrm{d}\omega(x) \leq \int_\rho^{\rho+2\epsilon} x^{d-1}\mathrm{d}x \lesssim \rho^{d-1}\epsilon \sim M^{d-1}\epsilon,$$

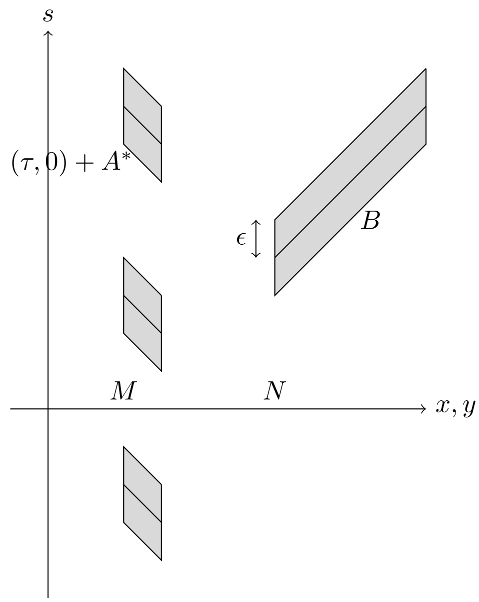

FIGURE 1. The supremum $\sup_\tau |I_1(\tau)|$ is attained when τ is chosen so that $(\tau, 0) + A^*$ and B are located as in the above diagram. Here, A and B are precisely the thickened supports mentioned previously. In particular, $A = \{(t, \rho) : \rho \sim M, |\tau - \langle\rho\rangle| < \epsilon\}$ and $B = \{(t, \rho) : \rho \sim N, |\tau - \langle\rho\rangle| < \epsilon\}$.

and similarly,

$$\int_0^\infty \mathbf{1}_B(s, y)\, \mathrm{d}\omega(y) \lesssim N^{d-1}\epsilon.$$

Finally, we observe that s is non-zero only on an interval of length comparable to M. Therefore

$$I(\tau) \lesssim M M^{d-1} N^{d-1} \epsilon^2$$

so that after taking $\epsilon \to 0$ in the approximation above we then obtain the bilinear estimates for $M \sim N$:

$$\begin{aligned} \left\| \left(e^{it\langle D_a\rangle} u_M\right) \left(e^{it\langle D_a\rangle} u_N\right) \right\|_{L_t^2 L_x^2} &\lesssim M^{\frac{1}{2}} (MN)^{\frac{d-1}{2}} N^{-\frac{d}{2}} \|u_M\|_{L^2} \|u_N\|_{L^2} \\ &\sim M^{\frac{d-2}{2}} N^{\frac{1}{2}} \|u_M\|_{L^2} \|u_N\|_{L^2}. \end{aligned} \tag{4.2.8}$$

We may apply a slightly modified argument to obtain the estimate in the $M \ll N$ case.

4.3. Small data scattering in higher dimensions

4.3.1. U^p and V^p-based estimates. We shall now use the transfer principle (Lemma 4.1.4) to convert the above bilinear Strichartz estimates into the form which we will need for the proof of scattering. Our first application of the transfer principle shall be to the following L^4 estimate. As above, we assume that all functions here are radial.

PROPOSITION 4.3.1. *Let $d \geq 3$, $a > \mathcal{A}_d$ and suppose that u_M has support at frequency M. Then,*

$$\left\| e^{\pm it\langle D_a\rangle} u_M \right\|_{L^4} \lesssim M^{\frac{d-1}{4}} \|u_M\|_{L^2} \tag{4.3.1}$$

and therefore

$$\|u_M\|_{L^4} \lesssim M^{\frac{d-1}{4}} \|u_M\|_{U^4_\pm}. \tag{4.3.2}$$

PROOF. The estimate (4.3.2) follows from an application of the transfer principle (c) to (4.3.1). Thus, it remains to verify (4.3.1). We shall rewrite it as a bilinear estimate:

$$\left\| e^{it\langle D_a\rangle} u_M \right\|^2_{L^4} = \left\| e^{it\langle D_a\rangle} u_M \overline{e^{it\langle D_a\rangle} u_M} \right\|_{L^2}.$$

Now, this is a bilinear Strichartz estimate, and so the result follows immediately from (4.2.8). $\square$

We now discuss the main application of the transfer principle to the bilinear Strichartz estimates. We shall use the above L^4 estimates. Indeed, we have

PROPOSITION 4.3.2. *Let $d \geq 3$, $a > \mathcal{A}_d$, $M, N \in 2^{\mathbf{N}}$. Furthermore, suppose that u_M, u_N, v_M and v_N are radial. We have*

(a) *For $u_M \in U^2_{\pm_1}$, $u_N \in U^2_{\pm_2}$*

$$\|u_M u_N\|_{L^2} \lesssim \begin{cases} M^{\frac{d}{4}} N^{\frac{d-2}{4}} \|u_M\|_{U^2_{\pm_1}} \|u_N\|_{U^2_{\pm_2}}, & M \ll N \\ M^{\frac{d-2}{2}} N^{\frac{1}{2}} \|u_M\|_{U^4_{\pm_1}} \|u_N\|_{U^4_{\pm_2}}, & M \sim N. \end{cases}$$

(b) *For $v_M \in V^2_{\pm_1}$, $v_N \in V^2_{\pm_2}$*

$$\|v_M v_N\|_{L^2} \lesssim \begin{cases} M^{\frac{d}{4}} N^{\frac{d-2}{4}} \left(\log\left(\frac{N}{M}\right)\right)^2 \|v_M\|_{V^2_{\pm_1}} \|v_N\|_{V^2_{\pm_2}}, & M \ll N \\ M^{\frac{d-2}{2}} N^{\frac{1}{2}} \|v_M\|_{V^2_{\pm_1}} \|v_N\|_{V^2_{\pm_2}}, & M \sim N. \end{cases}$$

PROOF. Firstly, we obtain the $U^2_{\pm_1} \times U^2_{\pm_2} \to L^2$ estimate via the transfer principle in the $M \ll N$ case. Thus, we move on to verify the $U^4_{\pm_1} \times U^4_{\pm_2} \to L^2$ estimates in the $M \sim N$ case. We first have $\|u_M u_N\|_{L^2} \leq \|u_M\|_{L^4}\|u_N\|_{L^4}$. Thus, we may apply the $U^4_\pm \to L^4$ estimate (4.3.2) to obtain

$$\|u_M u_N\|_{L^2} \leq (MN)^{\frac{d-1}{4}} \|u_M\|_{U^4_{\pm_1}} \|u_N\|_{U^4_{\pm_2}}. \tag{4.3.3}$$

Note that for $M \sim N$, we have $(MN)^{(d-1)/4} \sim M^{(d-2)/2} N^{1/2}$. This completes the proof of (a).

We obtain the $V^2_{\pm_1} \times V^2_{\pm_1} \to L^2$ estimates in the $M \sim N$ case from (4.3.3) and the embedding $V^2_\pm \hookrightarrow U^4_\pm$. Thus, it remains to verify the $V^2_{\pm_1} \times V^2_{\pm_1} \to L^2$ estimates in the $M \ll N$ case. We first define an operator $Tv := u_M P^a_N v$. In this case, we use (4.3.3) and the embedding $U^2_\pm \hookrightarrow U^4_\pm$ to obtain

$$\|T\|_{U^4_{\pm_2} \to L^2} \lesssim (MN)^{\frac{d-1}{4}} \|u_M\|_{U^2_{\pm_1}}.$$

Furthermore, from the bilinear estimates, we also have

$$\|T\|_{U^2_{\pm_2} \to L^2} \lesssim M^{\frac{d}{4}} N^{\frac{d-2}{4}} \|u_M\|_{U^2_{\pm_1}}.$$

Thus, we use Proposition 4.1.5 to obtain

$$\|T\|_{V^2_{\pm_2}\to L^2} \lesssim M^{\frac{d}{4}} N^{\frac{d-2}{4}} \log\left(\frac{N}{M}\right) \|u_M\|_{U^2_{\pm_1}}.$$

From this we have a $U^2_{\pm_1} \times V^2_{\pm_2} \to L^2$ estimate, so it remains to repeat the argument to finally obtain a $V^2_{\pm_1} \times V^2_{\pm_2} \to L^2$. Now, define an operator $Su := v_N P^a_M u$. We have the estimates

$$\|S\|_{U^4_{\pm_1}\to L^2} \lesssim (MN)^{\frac{d-1}{4}} \|v_N\|_{V^2_{\pm_2}}$$

and also

$$\|S\|_{U^2_{\pm_1}\to L^2} \lesssim M^{\frac{d}{4}} N^{\frac{d-2}{4}} \log\left(\frac{N}{M}\right) \|v_N\|_{V^2_{\pm_2}}.$$

Thus, again applying Proposition 4.1.5 we obtain

$$\|S\|_{V^2_{\pm_1}\to L^2} \lesssim M^{\frac{d}{4}} N^{\frac{d-2}{4}} \left(\log\left(\frac{N}{M}\right)\right)^2 \|u_M\|_{V^2_{\pm_2}}.$$

This completes the proof of (b). □

4.3.2. Trilinear estimates. For the remainder of this section, we shall work towards proving Theorem 1.1.3, which is analogous to the work of Schottdorf [**53**] for the potential-free case. In this section, we shall first use the bilinear Strichartz estimates obtained above in order to prove the following trilinear estimates. These estimates shall then be used to prove the scattering result.

PROPOSITION 4.3.3 (Trilinear estimates). *Let $d \geq 3$, $s \geq \frac{d-2}{2}$, $N \sim N'$. Furthermore, suppose that u_M, v_N and $w_{N'}$ are radial. Then,*

$$\frac{1}{N}\left|\sum_{M\ll N}\int_{\mathbf{R}}\int_{\mathbf{R}^d} u_M v_N w_{N'}\,\mathrm{d}x\mathrm{d}t\right| \lesssim \left(\sum_{M\ll N} L^{2s}\|u_M\|^2_{V^2_{\pm_1}}\right)^{\frac{1}{2}} \|v_N\|_{V^2_{\pm_2}} \|w_{N'}\|_{V^2_{\pm_3}} \tag{4.3.4}$$

and also

$$\left(\sum_{M\lesssim N} M^{-2}M^{2s} \sup_{\|w_M\|_{V^2_{\pm_3}}=1}\left|\int_{\mathbf{R}}\int_{\mathbf{R}^d} u_N v_{N'} w_M\,\mathrm{d}x\mathrm{d}t\right|^2\right)^{\frac{1}{2}} \lesssim N^s\|u_N\|_{V^2_{\pm_1}} (N')^s \|v_{N'}\|_{V^2_{\pm_2}}. \tag{4.3.5}$$

PROOF. The first step is to show that the following low modulation integral is zero. The consequence of this is that we assume at least one of $u_M = u^h_M$, $v_N = v^h_{H'}$ or $w_{N'} = w^h_{N'}$ in the above integral. That is, we may place one of these terms in high modulation. Here, we decompose u_M, v_N and $w_{N'}$ into high and low modulation components. For instance, $u_M = u^l_M + u^h_M$ where $u^h_M = Q^{\pm}_{>L}(u_M)$, where L shall be chosen later.

LEMMA 4.3.4 (Low modulation integral is zero). *The integral*

$$\int_{\mathbf{R}}\int_{\mathbf{R}^d} u^l_M v^l_N w^l_{N'}\,\mathrm{d}x\mathrm{d}t \tag{4.3.6}$$

vanishes.

Proof of Lemma 4.3.4. We consider the following convolution of order $\nu = \nu(0) = \frac{d-2}{2}$:

$$\left(\mathcal{H}_{tx}^{\nu(a)}(u_M)\#_\nu \mathcal{H}_{tx}^{\nu(a)}(v_N)\#_\nu \mathcal{H}_{tx}^{\nu(a)}(w_{N'})\right)(0,\epsilon)$$

for some small $\epsilon > 0$ to be chosen later. In particular, by using the definition for $Q^{\pm}_{<L}$, this expression is equal to

$$\int_{\mathbf{R}}\int \mathbf{1}_{\{|\tau_1\pm\langle\rho_1\rangle|<L\}}\widetilde{u_M}(\tau_1,\rho_1)\mathbf{1}_{\{|\tau_2\pm\langle\rho_2\rangle|<L\}}\widetilde{v_N}(\tau_2,\rho_2) \cdot \mathbf{1}_{\{|\tau_3\pm\langle\rho_3\rangle|<L\}}\widetilde{w_{N'}}(\tau_3,\rho_3)\cdot D_\nu(x,z,\rho_2)D_\nu(\rho_1,\rho_3,\epsilon)\,\mathrm{d}\omega(\rho_1,\rho_2,\rho_3,z)\mathrm{d}t.$$

In particular, for the time variable, we obtain the relation

$$\tau_1+\tau_2+\tau_3 = 0.$$

Furthermore, by noting that the integral is zero unless there is a triangle with sides of length x, z and ρ_2, as well as a triangle with sides of length ρ_1, ρ_3 and ϵ, we also obtain the relation

$$\rho_1 < \rho_2+\rho_3+\epsilon.$$

From the modulations, we obtain that

$$6L \geq 2\left(\tau_1 \pm \langle\rho_1\rangle + \tau_2 \pm \langle\rho_2\rangle + \tau_3 \pm \langle\rho_3\rangle\right) = |\pm\langle\rho_1\rangle \pm \langle\rho_2\rangle \pm \langle\rho_3\rangle|.$$

Now, choose ϵ so that $2\langle\rho_2\rangle \geq \langle\rho_2+\epsilon\rangle$. Then, for instance, in the case $(\pm_1,\pm_2,\pm_3) = (-,+,+)$, we have that

$$\begin{aligned} 2(\langle\rho_2\rangle+\langle\rho_3\rangle-\langle\rho_1\rangle) &\geq \langle\rho_2+\epsilon\rangle+\langle\rho_3\rangle-\langle\rho_1\rangle+(\langle\rho_3\rangle-\langle\rho_1\rangle)\\ &\geq \langle\rho_2+\epsilon\rangle+\langle\rho_3\rangle-\langle\rho_1\rangle\\ &\geq M^{-1}. \end{aligned}$$

In the second line, we used that ρ_1 is localised to M and ρ_3 is localised to $N' > M$. In the third line, we combined the condition $\rho_1 < \rho_2+\rho_3+\epsilon$ with the following modulation bound (see Schottdorf [**53**]) above with $\rho_2+\epsilon$ instead of ρ_2.

Lemma 4.3.5. *Let $\rho_1+\rho_2 = \rho_3$. Then, $\langle\rho_1\rangle+\langle\rho_2\rangle-\langle\rho_3\rangle \gtrsim \langle\rho_{\min}\rangle^{-1}$ where we write $\rho_{\min} = \min\{\rho_1,\rho_2,\rho_3\}$.*

In particular, we obtain that $6L \geq M^{-1}$. So, choosing $L = (cM)^{-1}$ with $c > 0$ sufficiently large, we may conclude that the above integral is zero, and therefore at least one of u^l_M, v^l_N or $w^l_{N'}$ is zero. From this, we may conclude that the integral vanishes. □

Now, we prove the trilinear estimates. Let us begin with (4.3.4). First, assume that $u_M = u^h_M$. Then, by using the modulation estimate Lemma 4.1.3(i) and the

$N \sim N'$ bilinear Strichartz estimate, we bound the left-hand side of (4.3.4) by

$$\begin{aligned}
&\frac{1}{N}\sum_{M\ll N}\|u_M\|_{L^2}\|v_N w_{N'}\|_{L^2}\\
&\quad\lesssim \frac{1}{N}\sum_{M\ll N} M^{\frac{1}{2}}\|u_M\|_{V^2_{\pm_1}}\|v_N w_{N'}\|_{L^2}\\
&\quad\lesssim \frac{1}{N}\left(\sum_{M\ll N} M^{2s}\|u_M\|^2_{V^2_{\pm_1}}\right)^{\frac{1}{2}}\left(\sum_{M\ll N} M^{1-2s}\|v_N w_{N'}\|_{L^2}\right)^{\frac{1}{2}}\\
&\quad\lesssim \frac{1}{N}\left(\sum_{M\ll N} M^{2s}\|u_M\|^2_{V^2_{\pm_1}}\right)^{\frac{1}{2}}\left(\sum_{M\ll N} M^{1-2s}N^{d-1}\|v_N\|^2_{L^2}\|w_{N'}\|^2_{L^2}\right)^{\frac{1}{2}}.
\end{aligned}$$

Let us study the second summation. We notice that for $s \geq \frac{d-2}{2}$

$$\frac{1}{N}\left(\sum_{M\ll N} M^{1-2s}N^{d-1}\right)^{\frac{1}{2}} \lesssim N^{-1}N^{\frac{d-2s}{2}} \lesssim 1.$$

Thus, we obtain the required result for this case. Next, we study the case where $w_{N'} = w^h_{N'}$. The case where $v_N = v^h_N$ is handled in the same way. We have

$$\begin{aligned}
\text{LHS of (4.3.4)} \quad &\lesssim \quad \frac{1}{N}\sum_{M\ll N}\|w_{N'}\|_{L^2}\|u_M v_N\|_{L^2}\\
&\lesssim \quad \frac{1}{N}\sum_{M\ll N} M^{\frac{1}{2}}\|w_{N'}\|_{V^2_{\pm_1}}\|u_M v_N\|_{L^2}\\
&\lesssim \quad \frac{1}{N}\sum_{M\ll N} M^{\frac{1}{2}}M^{\frac{d}{4}}N^{\frac{d-2}{4}}\left(\log\left(\frac{N}{M}\right)\right)^2\|u_M\|_{V^2_{\pm_1}}\|v_N\|_{V^2_{\pm_2}}\|w_{N'}\|_{V^2_{\pm_3}}.
\end{aligned}$$

We apply Cauchy-Schwarz to obtain the $\sum M^{2s}\|u_M\|_{V^2_{\pm_1}}$ term, and thus we are left with

$$(N)^{-2}\sum_{M\ll N} MM^{\frac{d}{2}}M^{\frac{d-2}{2}}M^{-2s}\left(\log\left(\frac{N}{M}\right)\right)^4 \lesssim N^{-1}\sum_{M\ll N} M^{d-1-2s} \lesssim 1.$$

Here we have used that $\left(\log\left(\frac{N}{M}\right)\right)^4 \lesssim \frac{N}{M}$. This verifies (4.3.4) and now we verify (4.3.5). Again, there are two cases to study since the cases of similar frequencies are handled in the same manner. Firstly, consider the case where $w_M = w^h_M$. In this case, we have

$$\begin{aligned}
\left(\text{LHS of (4.3.5)}\right)^2 \quad &\lesssim \quad \sum_{M\lesssim N} M^{-1}M^{2s}\|u_N v_{N'}\|^2_{L^2}\\
&\lesssim \quad N^{d-2}N'\sum_{M\lesssim N} M^{2s-1}\|u_N\|^2_{V^2_{\pm_1}}\|v_{N'}\|^2_{V^2_{\pm_2}}\\
&\lesssim \quad N^{d+2s-1}N\|u_N\|^2_{V^2_{\pm_1}}\|v_{N'}\|^2_{V^2_{\pm_2}}\\
&\lesssim \quad N^{2s}\|u_N\|^2_{V^2_{\pm_1}}(N')^{2s}\|v_{N'}\|^2_{V^2_{\pm_2}}.
\end{aligned}$$

In the last line, we have used the fact that $s \geq \frac{d-2}{2}$. In the other case where $v_{N'} = v^h_{N'}$, we have

$$\begin{aligned}
(\text{LHS of } (4.3.5))^2 &\lesssim \sup_{\|w_M\|_{V^2_{\pm_3}}=1} \sum_{M \lesssim N} M^{2s-1} \|v_{N'}\|^2_{V^2_{\pm_2}} \|u_N w_M\|^2_{L^2} \\
&\lesssim \sum_{M \lesssim N} M^{2s-1} M^{\frac{d}{2}} N^{\frac{d-2}{2}} \left(\log\left(\frac{N}{M}\right)\right)^4 \|u_N\|^2_{V^2_{\pm_1}} \|v_{N'}\|^2_{V^2_{\pm_2}}. \\
&\lesssim N^{2s} \|u_N\|^2_{V^2_{\pm_1}} (N')^{2s} \|v_{N'}\|^2_{V^2_{\pm_2}}.
\end{aligned}$$

In the last line, we again used the fact that $s \geq \frac{d-2}{2}$. □

4.3.3. Proof of small-energy scattering (Theorem 1.1.3). We may now rigorously prove the small-energy scattering. Indeed, we first have the following result which will allow for a contraction mapping

THEOREM 4.3.6. *Let $s \geq \max\left(\frac{1}{2}, \frac{d-2}{2}\right)$ and $a > \mathcal{A}_d$. Define*

$$I^{\pm}(f,g) := \int_0^t e^{\pm i(t-s)\langle D_a \rangle} \frac{fg}{2\langle D_a \rangle} \, \mathrm{d}s.$$

We have $I_{\pm_1,\pm_2} : Y^s \times Y^s \to X^s$ where

$$I_{\pm_1,\pm_2}\left((u^+,u^-),(v^+,v^-)\right) := \left(I^+(u^{\pm_1}, v^{\pm_2}), I^-(u^{\pm_1}, v^{\pm_2})\right).$$

That is, there exists a constant $C = C(a,d)$ such that

$$\|I(u,v)\|_{X^s} \leq C \|u\|_{Y^s} \|v\|_{Y^s}.$$

Furthermore, from the fact that $X^s \hookrightarrow Y^s$, we have

$$I : X^s \times X^s \to X^s$$

and

$$I : Y^s \times Y^s \to Y^s.$$

PROOF. We shall consider the I^+ component as I^- may be handled in a similar manner. First, we use frequency decomposition, so that after invoking symmetry, we need to study the following two terms

$$S_1 := \left\| \sum_N \sum_{M \ll N} I(u_M, v_N) \right\|_{X^s}, \qquad S_2 := \left\| \sum_N \sum_{N \sim N'} I(u_N, v_{N'}) \right\|_{X^s}.$$

We shall start with S_1. The idea is to unpack the definition of X^s so that we are left to study the U^2_+ pieces

$$\left\| P^a_{N'} \sum_M \sum_{M \ll N} I(u_M, v_N) \right\|_{U^2_+}.$$

Now, we may use duality to obtain

$$\begin{aligned}\left\| P^a_{N'} \sum_{M \ll N} I^+(u_M, v_N) \right\|_{U^2_+} &= \left\| P^a_{N'} \sum_{M \ll N} \int_0^t e^{i(t-s)\langle D_a \rangle} \frac{u_M v_N}{2\langle D_a \rangle} \, \mathrm{d}s \right\|_{U^2_+} \\ &\lesssim \frac{1}{N'} \left\| \sum_{M \ll N} \int_0^t e^{-is\langle D_a \rangle} u_M v_N \right\|_{U^2} \\ &= \frac{1}{N'} \sup_{\|w\|_{V^2}=1} \left| \sum_{M \ll N} B\left(e^{-it\langle D_a \rangle} u_M v_N, w \right) \right| \\ &= \frac{1}{N'} \sup_{\|w_{N'}\|_{V^2_+}=1} \left| \sum_{M \ll N} \int \int u_M v_N w_{N'} \, \mathrm{d}x \, \mathrm{d}t \right|.\end{aligned}$$

We note here that we may now take $N \sim N'$ due to the convolution structure, as discussed above. We may use the trilinear estimate (4.3.4) to obtain

$$\left\| P^a_{N'} \sum_{M \ll N} I^+(u_M, v_N) \right\|_{U^2_+} \lesssim \left(\sum_{M \ll N} N^{2s} \|u_M\|^2_{V_{\pm_1}} \right)^{\frac{1}{2}} \|v_N\|_{V^2_{\pm_2}}.$$

Therefore, putting all the U^2_+ pieces back together,

$$\sum_{N \sim N'} (N')^{2s} \left\| P^a_M \sum_{M \ll N} I(u_M, v_N) \right\|^2_{U^2_+} \lesssim \|u\|^2_{Y^s} \|v\|^2_{Y^s}.$$

Thus, it remains to study S_2. We have

$$S_2 \le \sum_N \sum_{N \sim N'} \|I^+(u_N, v_{N'})\|_{X^s} \lesssim \sum_N \sum_{N \sim N'} \left(\sum_{M \lesssim N} M^{2s} \left\| P^a_M I^+(u_N, v_{N'}) \right\|_{U^2_+} \right)^{\frac{1}{2}}.$$

Here, we have again used the previously discussed convolution structure to restrict $M \lesssim N$. By using duality, we may apply our trilinear estimate (4.3.5) to obtain

$$S_2 \lesssim \sum_N \sum_{N \sim N'} N^{2s} \|u_N\|_{V^2_{\pm_1}} (N')^{2s} \|v_{N'}\|_{V^2_{\pm_2}} \lesssim \|u\|_{Y^s} \|v\|_{Y^s}.$$

Thus, we have shown the required result. $\square$

THEOREM 4.3.7. *Let $d \ge 3$, $a > \mathcal{A}_d$, $(u_0, u_1) \in H^s_{a,\mathrm{rad}} \times H^{s-1}_{a,\mathrm{rad}}$ with $s \ge \frac{d-2}{2}$. Then the equation QKG(a) (1.0.1) has a global solution in $C(\mathbf{R}, H^s_a) \cap C(\mathbf{R}, H^{s-1}_a)$ that is unique in the space $X^s([0,\infty))$ and scatters as $t \to \pm\infty$.*

PROOF. We shall work in the following restricted space:

$$X^s([0,\infty)) := \{u \in C([0,\infty), H^s) : \exists v \in X^s \text{ s.t. } v(t) = u(t), t \in [0,\infty)\}$$

We have $T^\pm u^\pm := e^{\pm it\langle D_a \rangle} u_0^\pm \mp i I^\pm(u)$, where

$$I^\pm(u) = \int_0^t e^{\pm i(t-s)\langle D_a \rangle} \frac{u^2}{2\langle D_a \rangle} \, \mathrm{d}s.$$

Recall that $u = u^+ + u^-$. Define the set $C_M = \{u \in X^s([0,\infty)) : \|u\|_{X^s([0,\infty))} \leq M\}$. We see that for M sufficiently small and initial data u_0 also sufficiently small $\|e^{it\langle D_a\rangle} u^{\pm}(0)\|_{H^s_a} < \epsilon$, we have

$$\|e^{\pm it\langle D_a\rangle} u_0^{\pm} \mp iI^{\pm}(u)\|_{X^s_{\pm}([0,\infty))} \lesssim \epsilon + M^2 \leq M.$$

Next, we check the contraction condition:

$$\|I^{\pm}(f) - I^{\pm}(g)\|_{X^s_{\pm}([0,\infty))} \lesssim \left(\|f\|_{X^s([0,\infty))} + \|g\|_{X^s([0,\infty))}\right) \|f-g\|_{X^s([0,\infty))}.$$

Thus, for $\delta \ll 1$ sufficiently small, we obtain the existence of a unique solution in C_M. Now, we prove scattering. From Theorem 4.3.6, we see that

$$e^{\mp it\langle D_a\rangle} P_N^a I^{\pm}(u) \in V^2_{*,\mathrm{rc}}$$

From the properties of $V^2_{*,\mathrm{rc}}$, this means that $\lim_{t\to\infty} e^{\mp it\langle D_a\rangle} P_N^a I^{\pm}(u)$ exists. Thus,

$$\sum_N N^{2s} \|P_N^a I^{\pm}(u)\|^2_{V^2_{\pm}} \lesssim 1.$$

Therefore, $\lim_{t\to\infty} e^{\mp it\langle D_a\rangle} I^{\pm}(u) \in H^s_a$. Thus, we have

$$e^{\mp} it\langle D_a\rangle u^{\pm} \to u^{\pm}(0) \mp i \lim_{t\to\infty} e^{\mp it\langle D_a\rangle} I^{\pm}(u) \in H^s_a.$$

$\square$

CHAPTER 5

4D dichotomy of dynamics below the ground state

5.1. Time-decay estimates

5.1.1. Time-decay L^p estimates. In this section, we shall establish time-decay frequency-localised estimates for radial initial data in. In particular, we find that for $a > 0$, the situation is much the same as in the $a = 0$ potential-free case. Whereas for the $a < 0$ case, we shall need to combine the methods of Guo-Wang-Peng [**21**] and Zheng [**67**] in order to obtain time-decay estimates with a weight.

In particular, Zheng obtains the following L^p time-decay estimates for the propagator $e^{it\mathscr{L}_a}$:

PROPOSITION 5.1.1 (L^p time-decay estimates for the propagator $e^{it\mathscr{L}_a}$). *Let u be radial, $2 \le p \le \infty$, $\delta = 1 - \frac{2}{p}$ and $\sigma = \frac{d-2}{2} - \nu(a)$. Then,*

$$(5.1.1) \qquad \begin{cases} \|e^{it\mathscr{L}_a}u\|_{L^p(\mathbf{R}^d)} \lesssim |t|^{-\frac{d}{2}\delta}\|u\|_{L^{p'}(\mathbf{R}^d)}, & a \ge 0 \\ \left\|\frac{1}{(1+|x|^{-\sigma})^\delta}e^{it\mathscr{L}_a}u\right\|_{L^p(\mathbf{R}^d)} & \\ \qquad\lesssim |t|^{(-\frac{d}{2}+\sigma)\delta}\left\|(1+|x|^{-\sigma})^\delta u\right\|_{L^{p'}(\mathbf{R}^d)}, & -\left(\frac{d-2}{2}\right)^2 < a < 0. \end{cases}$$

We shall obtain frequency-localised estimates for a class of dispersive semigroups $e^{it\omega(D_a)}$ where we localise with Littlewood-Paley projectors P_k^a adapted to $\mathscr{L}_a$ (see Section 2.3).

First, we impose some assumptions on ω. In particular, because we are mainly interested in the semigroup with $\omega(r) = (1+r^2)^{1/2}$, we shall apply the assumptions in [**21**] and assume that $\omega : \mathbf{R}^+ \to \mathbf{R}$ is smooth and satisfies both (H) and (L) below:

(H) There exists $m_1 > 0$ such that for $k = 2, 3, ...$ we have $|\omega'(r)| \sim r^{m_1-1}$ and $|\omega^{(k)}(r)| \lesssim r^{m_1-k}$ for $r \ge 1$. Furthermore, there exists α_1 such that $|\omega''(r)| \sim r^{\alpha_1-2}$ for $r \ge 1$.

(L) There exists $m_2 > 0$ such that for $k = 2, 3, ...$ we have $|\omega'(r)| \sim r^{m_2-1}$ and $|\omega^{(k)}(r)| \lesssim r^{m_2-k}$ for $0 < r < 1$. Furthermore, there exists α_2 such that $|\omega''(r)| \sim r^{\alpha_2-2}$ for $0 < r < 1$.

PROPOSITION 5.1.2 (Frequency-localised L^p time-decay estimates). *Let u be radial, $d \ge 3$ and recall that $\sigma = \frac{d-2}{2} - \nu(a)$. Furthermore, assume $\omega : \mathbf{R}^+ \to \mathbf{R}$ is smooth away from the origin and satisfies (H) and (L) above. Let $2 \le p \le \infty$ and $\delta := 1 - \frac{2}{p}$.*

(i) *If $k \ge 0$ and $a \ge 0$ then*

$$(5.1.2) \qquad \left\|e^{it\omega(D_a)}P_k^a u\right\|_{L^p} \lesssim |t|^{-\frac{d-1+\theta}{2}\delta}2^{k(d-\frac{m_1(d-1+\theta)}{2}-\frac{\theta(\alpha_1-m_1)}{2})\delta}\|u_0\|_{L^{p'}}.$$

(ii) *If* $k \geq 0$ *and* $-\left(\frac{d-2}{2}\right)^2 < a < 0$, *then*

$$\left\| \frac{1}{(1+|x|^{-\sigma})^{\delta}} e^{it\omega(D_a)} P_k^a u \right\|_{L^p} \lesssim |t|^{-\frac{d-1+\theta}{2}\delta} 2^{k(d-\sigma-\frac{m_1(d-1+\theta)}{2}-\frac{\theta(\alpha_1-m_1)}{2})\delta} \|(1+|x|^{-\sigma})^{\delta} u\|_{L^{p'}}. \tag{5.1.3}$$

(iii) *If* $k < 0$ *and* $a \geq 0$ *then*

$$\left\| e^{it\omega(D_a)} P_k^a u \right\|_{L^p} \lesssim |t|^{-\frac{d-1+\theta}{2}\delta} 2^{k(d-\frac{m_2(d-1+\theta)}{2}-\frac{\theta(\alpha_2-m_2)}{2})\delta} \|u_0\|_{L^{p'}}. \tag{5.1.4}$$

(iv) *If* $k < 0$ *and* $-\left(\frac{d-2}{2}\right)^2 < a < 0$, *then*

$$\left\| \frac{1}{(1+|x|^{-\sigma})^{\delta}} e^{it\omega(D_a)} P_k^a u \right\|_{L^p} \lesssim |t|^{-\frac{d-1+\theta}{2}\delta} 2^{k(d-\sigma-\frac{m_2(d-1+\theta)}{2}-\frac{\theta(\alpha_2-m_2)}{2})\delta} \|(1+|x|^{-\sigma})^{\delta} u\|_{L^{p'}}. \tag{5.1.5}$$

Proof. We shall first prove the $L^\infty - L^1$ decay estimate, and then interpolate with the L^2 estimate to obtain the above result. Furthermore, we shall focus on the high-frequency case, with the proof for the low-frequency case being similar. We may use the definition of the Hankel transform of order $\nu = \nu(a)$ to write

$$\begin{aligned}
(e^{it\omega(D_a)} P_k^a u)(t,r) &= \int_0^\infty (r\rho)^{-\frac{d-2}{2}} J_\nu(r\rho) e^{it\omega(\rho)} \psi(2^{-k}\rho) \widehat{u}(\rho) \rho^{d-1}\, \mathrm{d}\rho \\
&= 2^{kd} \int_0^\infty (r2^k\rho)^{-\frac{d-2}{2}} J_\nu(r2^k\rho) e^{it\omega(2^k\rho)} \psi(\rho) \widehat{u}(2^k\rho) \rho^{d-1}\, \mathrm{d}\rho \\
&= 2^{kd} \int_0^\infty (r2^k\rho)^{-\frac{d-2}{2}} J_\nu(r2^k\rho) e^{it\omega(2^k\rho)} \psi(\rho) \\
&\qquad \cdot \int_0^\infty (s2^k\rho)^{-\frac{d-2}{2}} J_\nu(s2^k\rho) u(s) s^{d-1}\, \mathrm{d}s \rho^{d-1}\, \mathrm{d}\rho \\
&= \int_0^\infty (r2^k\rho)^{-\frac{d-2}{2}} J_\nu(r2^k\rho) e^{it\omega(2^k\rho)} \psi(\rho) \\
&\qquad \cdot \int_0^\infty (s\rho)^{-\frac{d-2}{2}} J_\nu(s\rho) u(2^{-k}s) s^{d-1}\, \mathrm{d}s\ \rho^{d-1}\, \mathrm{d}\rho.
\end{aligned}$$

Since our goal currently is to estimate this term in L^∞, we may replace $2^k r$ by r, so that we shall focus on the integral

$$\begin{aligned}
I(t,r) &= \int_0^\infty (r\rho)^{-\frac{d-2}{2}} J_\nu(r\rho) e^{it\omega(2^k\rho)} \psi(\rho) \\
&\qquad \cdot \int_0^\infty (s\rho)^{-\frac{d-2}{2}} J_\nu(s\rho) u(2^{-k}s) s^{d-1}\, \mathrm{d}s\ \rho^{d-1}\, \mathrm{d}\rho \\
&= \int_0^\infty u(2^{-k}s) s^{d-1} K(t,s,r)\, \mathrm{d}s \\
&= \left(\int_0^1 + \int_1^\infty\right) u(2^{-k}s) s^{d-1} K(t,s,r)\, \mathrm{d}s =: \mathcal{I}_1(t,r) + \mathcal{I}_2(t,r)
\end{aligned}$$

where

$$K(t,r,s) = (rs)^{-\frac{d-2}{2}} \int_0^\infty J_\nu(r\rho) J_\nu(s\rho) e^{it\omega(2^k\rho)} \psi(\rho) \rho\, \mathrm{d}\rho. \tag{5.1.6}$$

We shall first study the case when $r < 1$, where we can use the behaviour of the Bessel function near 0. Recall that $\sigma := \frac{d-2}{2} - \nu$. Then, we have

$$\begin{aligned}
\mathcal{I}_1(t,r) &= \int_0^1 u(2^{-k}s)s^{d-1}K(t,s,r)\,\mathrm{d}s \\
&\sim \int_0^1 u(2^{-k}s)s^{d-1}(rs)^{-\sigma}\,\mathrm{d}s \int_0^\infty e^{it\omega(2^k\rho)}\psi(\rho)\rho^{2\nu+1}\,\mathrm{d}\rho.
\end{aligned}$$

By bounding the inside integral and after using a change of variables $2^{-k}s \mapsto s$, we obtain the estimates

$$\begin{cases}
\|\mathcal{I}_1(t,x)\|_{L^\infty_x} \lesssim 2^{kd}\|u\|_{L^1_x}, & a \geq 0 \\[2ex]
\left\|\frac{1}{1+|x|^{-\sigma}}\mathcal{I}_1(t,x)\right\|_{L^\infty_x} \lesssim 2^{k(d-\sigma)}\|(1+|x|^{-\sigma})u\|_{L^1_x}, & -\left(\frac{d-2}{2}\right)^2 < a < 0
\end{cases} \tag{5.1.7}$$

Next, we observe that

$$\frac{1}{it\omega'(2^k\rho)2^k}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(e^{it\omega(2^k\rho)}\right) = e^{it\omega(2^k\rho)}.$$

Thus, for the inside integral, we find that by repeated use of integration by parts, for any $q \in \mathbf{N}$ is equal to

$$\frac{1}{(it2^k)^q}\sum_{m=0}^{q}\sum_{l_1+\cdots+l_q\in\Lambda^q_m} C_{q,m}\int_0^\infty e^{it\omega(2^k\rho)}\prod_{j=1}^{q}\partial_\rho^j\left(\frac{1}{\omega'(2^k\rho)}\right)\partial_\rho^{q-m}(\psi(\rho)\rho^{2\nu+1})\,\mathrm{d}\rho$$

where

$$\Lambda^q_m = \{l_1,...,l_q \in \mathbf{Z}^+ : 0 \leq l_1 < ... < l_q \leq q,\ l_1 + ... + l_q = m\}.$$

Now, using the facts that

$$\left|\frac{\mathrm{d}^n}{\mathrm{d}\rho^n}\psi(\rho)\rho^{2\nu+1}\,\mathrm{d}\rho\right| \leq C_{n,\nu}$$

for some $C_{n,\nu} > 0$ and

$$\frac{\mathrm{d}^n}{\mathrm{d}\rho^n}\left(\frac{1}{\omega'(2^k\rho)}\right) \leq c_n 2^{-k(m_1-1)} \tag{5.1.8}$$

we obtain that

$$\begin{cases}
\|\mathcal{I}_1(t,x)\|_{L^\infty_x} \lesssim |t|^{-q}2^{k(d-m_1q)}\|u\|_{L^1_x}, & a \geq 0 \\[2ex]
\left\|\frac{1}{1+|x|^{-\sigma}}\mathcal{I}_1(t,x)\right\|_{L^\infty_x} \lesssim |t|^{-q}2^{k(d-\sigma-m_1q)}\|(1+|x|^{-\sigma})u\|_{L^1_x}, & -\left(\frac{d-2}{2}\right)^2 < a < 0
\end{cases}.$$

Finally, by interpolation with (5.1.7) we may conclude that for $\theta \geq 0$

$$\begin{cases}
\|\mathcal{I}_1(t,x)\|_{L^\infty_x} \lesssim |t|^{-\theta}2^{k(d-m_1\theta)}\|u\|_{L^1_x}, & a \geq 0 \\[2ex]
\left\|\frac{1}{1+|x|^{-\sigma}}\mathcal{I}_1(t,x)\right\|_{L^\infty_x} \lesssim |t|^{-\theta}2^{k(d-\sigma-m_1\theta)}\|(1+|x|^{-\sigma})u\|_{L^1_x}, & -\left(\frac{d-2}{2}\right)^2 < a < 0
\end{cases}.$$

Now, let us estimate $\mathcal{I}_2(t,r)$ for $r \leq 1$. In this case, we use the $s \to \infty$ asymptotic behaviour of $J_\nu(s\rho)$. In particular, because (at least up to phase translations) the asymptotic behaviour is identical to the $\nu = \frac{d-2}{2}$ case studied in [**21**], we should expect to obtain the same results. For completeness, let us include these details.

Since in this case, r is small, we use the same approximation for $J_\nu(r\rho)$. We use that

$$(s\rho)^{-\frac{d-2}{2}}J_\nu(s\rho) \sim \mathrm{Re}(e^{is\rho}h(s\rho))$$

where $|\partial_r^k h(r)| \lesssim_k r^{-\frac{d-1}{2}-k}$ (see (3.1.8)). From this, we obtain the estimate for $s \geq 1$ and $k \geq 0$ by induction:

$$\left|\frac{\partial^n}{\partial\rho^n}\left(h(s\rho)\psi(\rho)\rho^{d-1-\sigma}\right)\right| \lesssim_n s^{-\frac{d-1}{2}}. \tag{5.1.9}$$

In this case, we find that $\mathcal{I}_2(t,r)$ is comparable to

$$\begin{aligned}
&\int_1^\infty u(2^{-k}s)s^{d-1}\int_0^\infty (r\rho)^{-\sigma}\left[e^{is\rho}h(s\rho)+e^{-is\rho}\bar{h}(s\rho)\right]\psi(\rho)e^{it\omega(\rho)}\rho^{d-1}\,\mathrm{d}\rho\,\mathrm{d}s\\
=\;& r^{-\sigma}\int_1^\infty u(2^{-k}s)s^{d-1}\int_0^\infty \left[e^{is\rho}h(s\rho)+e^{-is\rho}\bar{h}(s\rho)\right]\psi(\rho)e^{it\omega(\rho)}\rho^{d-1-\sigma}\,\mathrm{d}\rho\,\mathrm{d}s\\
=\;& r^{-\sigma}\int_1^\infty u(2^{-k}s)s^{d-1}\int_0^\infty e^{i(t\omega(\rho)+s\rho)}h(s\rho)\psi(\rho)\rho^{d-1-\sigma}\,\mathrm{d}\rho\,\mathrm{d}s\\
&\quad + r^{-\sigma}\int_1^\infty u(2^{-k}s)s^{d-1}\int_0^\infty e^{i(t\omega(\rho)-s\rho)}\bar{h}(s\rho)\psi(\rho)\rho^{d-1-\sigma}\,\mathrm{d}\rho\,\mathrm{d}s\\
:=\;& B_1+B_2
\end{aligned}$$

We are now reduced to studying B_i for $i=1,2$. For B_1, we notice that $\omega_1(r) := t\psi'(2^k\rho)+s\rho$ satisfies $\omega_1'(\rho) \geq ct2^{km_1}$ so that (5.1.8) holds if ω replaced by ω_1. Hence, we may apply the same method as above and find that for any $\theta \geq 0$,

$$|B_1| \lesssim |t|^{-\theta}2^{k(d-m_1\theta)}\|u\|_{L_x^\infty}.$$

Likewise, for the integral B_2, we notice that (5.1.8) holds also for ω replaced by $\omega_2(r) := t\psi'(2^k\rho)-s\rho$ as long as

$$s > 2\sup_{\rho\in[1/2,2]} t2^k\psi'(2^k\rho)$$

or

$$s < \frac{1}{2}\inf_{\rho\in[1/2,2]} t2^k\psi'(2^k\rho).$$

Hence, using (5.1.9), we also have for all $\theta \geq 0$ that

$$|B_2| \lesssim |t|^{-\theta}2^{k(d-m_1\theta)}\|u\|_{L_x^\infty}.$$

Now, in the remaining case where

$$\frac{1}{2}\inf_{\rho\in[1/2,2]} t2^k\psi'(2^k\rho) \leq s \leq 2\sup_{\rho\in[1/2,2]} t2^k\psi'(2^k\rho),$$

we use (5.1.9) to obtain

$$|B_2| \lesssim 2^{kd}s^{-\frac{d-1}{2}}\|u\|_{L_x^\infty} \lesssim t^{-\frac{d-1}{2}}2^{k(d-\frac{(d-1)m_1}{2})}\|u\|_{L_x^\infty}. \tag{5.1.10}$$

Furthermore, by assumption $|\omega''(r)| \geq t2^{k\alpha_1}$. Therefore, $|\psi_2''(r)| \geq t2^{k\alpha_1}$. Thus, applying van der Corput's lemma, we obtain

$$|B_2| \lesssim (t2^{k\alpha_1})^{-\frac{1}{2}}\int_0^\infty \left|\frac{\mathrm{d}}{\mathrm{d}\rho}\left(h(s\rho)\psi(\rho)\rho^{d-1-\sigma}\right)\right|\,\mathrm{d}\rho \lesssim 2^{k(d-\frac{d}{2}(m_1+\frac{\alpha-m_1}{d}))}\|u\|_{L_x^\infty}$$

By interpolating this with (5.1.10), we obtain

$$|B_2| \lesssim |t|^{-\frac{d-1+\theta}{2}} 2^{k(d-\frac{m_1(d-1+\theta}{2}-\frac{\theta(\alpha_1-m_1)}{2})} \|u\|_{L^\infty_x}$$

We notice the absence of the $2^{-k\sigma}$ term in the $s > 1$ case. Since we are dealing with the high-frequency ($k \geq 0$) case, we may remove this factor from the estimate on B_1 in order to combine with the estimate for B_2.

Thus, it remains for us to estimate $I(t,r)$ when $r \geq 1$. This case is similar to the $r \leq 1$ case except for the fact we no longer to deal with the weight $r^{-\sigma}$. Thus, adding everything together, we obtain (i). The $k \geq 0$ case may be done in a similar manner, except now we must insert the factor of $2^{-k\sigma}$ to the estimate for B_2.

Finally, to obtain the other estimates for $p \geq 2$, we interpolate with the L^2 estimate

(5.1.11)

$$\begin{cases} \|e^{it\omega(D_a)}P_k^a u\|_{L^2(\mathbf{R}^d)} = \|u_0\|_{L^2(\mathbf{R}^d)}, & a \geq 0 \\ \left\|\frac{1}{1+|x|^{-\sigma}} e^{it\omega(D_a)}P_k^a u\right\|_{L^2(\mathbf{R}^d)} \leq \|(1+|x|^{-\sigma})P_k^a u\|_{L^2(\mathbf{R}^d)} & -\left(\frac{d-2}{2}\right)^2 < a < 0 \end{cases}$$

This gives us the above time-decay estimates. □

Using $\omega(z) = (1+|z|^2)^{1/2}$, we also obtain time-decay estimates for the Klein-Gordon propagator with inverse square potential:

PROPOSITION 5.1.3 (Frequency-localised L^p time-decay estimates for Klein–Gordon). *Let u be radial, $d \geq 3$ and recall that $\sigma = \frac{d-2}{2} - \nu(a)$. For $k \in \mathbf{Z}$ and $2 \leq p \leq \infty$ we have*

(i) *For $k \geq 0$*

$$\begin{cases} \left\|e^{it\omega(D_a)}P_k^a u\right\|_{L^p(\mathbf{R}^d)} \lesssim |t|^{-\frac{d-1+\theta}{2}\delta} 2^{k\left(\frac{d+1+\theta}{2}\right)\delta} \|u_0\|_{L^{p'}}, \quad a \geq 0 \\ \left\|\frac{1}{(1+|x|^{-\sigma})^\delta} e^{it\omega(D_a)}P_k^a u\right\|_{L^p(\mathbf{R}^d)} \\ \qquad \lesssim |t|^{-\frac{d-1+\theta}{2}\delta} 2^{k\left(\frac{d+1+\theta}{2}\right)\delta} \|(1+|x|^{-\sigma})^\delta u_0\|_{L^{p'}}, \quad -\left(\frac{d-2}{2}\right)^2 < a < 0 \end{cases}$$

(ii) *For $k < 0$*

$$\begin{cases} \left\|e^{it\omega(D_a)}P_k^a u\right\|_{L^p(\mathbf{R}^d)} \lesssim |t|^{-\frac{d-1+\theta}{2}\delta} 2^{k\left(\frac{d+1+\theta}{2}\right)\delta} \|u_0\|_{L^{p'}}, \quad a \geq 0 \\ \left\|\frac{1}{(1+|x|^{-\sigma}})^\delta e^{it\omega(D_a)}P_k^a u\right\|_{L^p(\mathbf{R}^d)} \\ \qquad \lesssim |t|^{-\frac{d-1+\theta}{2}\delta} 2^{k(1-\theta-\sigma)\delta} \|(1+r^{-\sigma})^\delta u_0\|_{L^{p'}}, \quad -\left(\frac{d-2}{2}\right)^2 < a < 0 \end{cases}$$

5.2. Blow-up/global well-posedness dichotomy

In this section, we shall obtain a blow-up/global well-posedness dichotomy for QKG(a) (1.0.1). First, let us list some notation and terminology (see [**22, 30, 48**]) that we will need for this section as well as the next. While we are only interested in the case where $d = 4$, we shall first discuss the general framework as studied in the above papers. Consider the equation

$$\partial_t^2 u + \mathscr{L}_a u + u = f(u)$$

where $f(u) = u^{p+1}$ for $p \in \{1, 2, ...\}$. Write $F(u) = \int f(u)\ \mathrm{d}u = \frac{1}{p+2}u^{p+2}$ and $G(u) = uf(u) - 2F(u) = \frac{p}{p+2}u^{p+2}$. Also, define the energy $E_a(u, u_t)$ by

$$\frac{1}{2}\int_{\mathbf{R}^d} |\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2 + \frac{a}{|x|^2}|u(t,x)|^2 + |u(t,x)|^2\ \mathrm{d}x - \frac{1}{p+2}\int_{\mathbf{R}^d} u(t,x)^3\ \mathrm{d}x.$$

For $\phi \in H_a^1$, the stationary energy is given by

$$\begin{aligned} J_a(\phi) &= \frac{1}{2}\int_{\mathbf{R}^d} |\nabla\phi(t,x)|^2 + \frac{a}{|x|^2}|\phi(t,x)|^2 + |\phi(t,x)|^2\ \mathrm{d}x - \frac{1}{p+2}\int_{\mathbf{R}^d} u(t,x)^3\ \mathrm{d}x \\ &= \frac{1}{2}\left(\|\phi\|_{\dot{H}_a^1} + \|\phi\|_2^2\right) - \frac{1}{3}\int_{\mathbf{R}^d} u(t,x)^3\ \mathrm{d}x. \end{aligned}$$

The potential well is $j_a(\lambda) = \mathscr{L}_{a,b} J_a(\phi) := J_a(e^{\alpha\lambda}\phi(e^{-\beta\lambda}x))$. This may be written as

$$j_a(\lambda) = \frac{1}{2}e^{(2\alpha+(d-2)\beta)\lambda}\|\phi\|_{\dot{H}_a^1}^2 + \frac{1}{2}e^{(2\alpha+d\beta)\lambda}\|\phi\|_2^2 - \frac{1}{p+2}e^{(p+2)\alpha+d\beta)\lambda}\int_{\mathbf{R}^d} \phi^{p+2}\ \mathrm{d}x.$$

Define also the sign functional $K_{\alpha,\beta;a}(\phi) = K_{\alpha,\beta;a}(\phi) := \partial_\lambda|_{\lambda=0} j_a(\lambda)$, which is equal to

$$\frac{1}{2}(2\alpha + (d-2)\beta)\|\phi\|_{\dot{H}_a^1}^2 + \frac{1}{2}(2\alpha + d\beta)\|\phi\|_2^2 - \frac{1}{p+2}((p+2)\alpha + d\beta)\int_{\mathbf{R}^d} \phi^{p+2}\ \mathrm{d}x.$$

Finally, define the minimal energy with respect to $K_{\alpha,\beta;a}$ by

$$m_{\alpha,\beta}(a) := \begin{cases} \inf\left\{J_a(\phi) : \phi \in H_a^1\backslash\{0\}, K_{\alpha,\beta;a}(\phi) = 0\right\}, & \mu(0)^2 < a < 0 \\ \inf\left\{J_a(\phi) : \phi \in H_a^1\backslash\{0\} \text{ radial}, K_{\alpha,\beta;a}(\phi) = 0\right\}, & a > 0. \end{cases} \tag{5.2.1}$$

We shall need the extra assumption that the test functions are radial in the case where $a > 0$. This is because in this case we do not have access to rearrangement inequalities and therefore automatically reduce to the radial case. Instead, it is an assumption we need to add. Indeed, let u^* be, say, the Schwarz rearrangement of u. Then, we have $\|u^*\|_2 = \|u\|_2$ and $\|\nabla u^*\|_2 \leq \|\nabla u\|_2$. We rewrite this last inequality as $\|u^*\|_{\dot{H}^1} \leq \|u\|_{\dot{H}^1}$ However, we also have the inequality

$$\int fg\ \mathrm{d}x \leq \int f^*g^*\ \mathrm{d}x, \tag{5.2.2}$$

for any suitable f, g such that the right-hand side is finite. This inequality means that we only have $\|u^*\|_{\dot{H}_a^1} \leq \|u\|_{\dot{H}_a^1}$ if $a \leq 0$.

We now study a dichotomy with respect to the sets

$$\begin{aligned} \mathcal{K}_{\alpha,\beta;a}^+ &:= \{(u_0, u_1) \in H_a^1 \times L^2 : E_a(u_0, u_1) < m_{\alpha,\beta;a}, K_{\alpha,\beta;a}(u_0) \geq 0\} \\ \mathcal{K}_{\alpha,\beta;a}^- &:= \{(u_0, u_1) \in H_a^1 \times L^2 : E_a(u_0, u_1) < m_{\alpha,\beta;a}, K_{\alpha,\beta;a}(u_0) < 0\} \end{aligned}$$

Indeed, we have the following blow-up/global well-posedness dichotomy with respect to $\mathcal{K}_{1,0;a}^\pm$.

THEOREM 5.2.1. *Suppose $u \in C(I, H_a^1)$ is the solution to (1.0.1) with initial data $u(0,\cdot) = u_0$ and $u_t(0,\cdot) = u_1$ where I is the maximal lifespan interval. Furthermore, if $a > 0$, assume that the initial data is radial.*

- *If $(u_0, u_1) \in \mathcal{K}_{1,0;a}^+$, then u is global, and*
- *if $(u_0, u_1) \in \mathcal{K}_{1,0;a}^-$, then u blows up in finite time.*

Proof. The proof is the same as in Payne-Sattinger [**48**] (see also [**22, 30**]). First, by obtaining a contradiction, we shall prove that the optimiser ϕ of $m_{1,0;a}$ does not change sign, so that

$$m_{1,0;a} = \inf\left\{J_a(\phi) : \phi \in H^1_a\backslash\{0\}, K_{1,0;a}(\phi) = 0,\ \phi > 0\right\}.$$

Indeed, suppose that the optimiser ϕ does change sign. Then, from the definition of $K_{1,0;a}$, we would have $K_{1,0;a}(|\phi|) < K_{1,0;a}(\phi) = 0$. Define $j_a(\lambda) := J_a(e^\lambda|\phi|)$ so that

$$j_a'(\lambda) = K_{1,0;a}(e^\lambda|\phi|) = e^{2\lambda}\|u\|^2_{H^1_a} - e^{3\lambda}\|u\|_3^3$$

and also

$$j_a''(\lambda) = 2e^{2\lambda}\|u\|^2_{H^1_a} - 3e^{3\lambda}\|u\|_3^3.$$

We first observe that

- $\lim_{\lambda\to\infty} j_a'(\lambda) = -\infty$,
- $\lim_{\lambda\to-\infty} j_a'(\lambda) = 0$ and
- $j_a''(\lambda) > 0$ for λ sufficiently small (i.e. e^λ near 0).

From these observations, we find that $\lambda \mapsto j_a(\lambda)$ is convex for sufficiently small λ. From this and the fact $j_a'(0) < 0$, we conclude that there exists $\bar\lambda < 0$ such that $j_a'(\bar\lambda) = K_{1,0;a}(e^{\bar\lambda}) = 0$. Thus, $K_{1,0;a}(e^{\bar\lambda}\phi) > 0$ and $e^{\bar\lambda}\phi$ is an admissible in (5.2.1). From the definition of $K_{1,0;a}$, we also have that $K_{1,0;a}(e^\lambda\phi) > 0$ for $\lambda < 0$. Indeed, in this case, $e^{3\lambda} < e^{2\lambda}$, so that by assumption

$$K_{1,0;a}(e^\lambda\phi) > e^{3\lambda}K_{1,0;a}(\phi) = 0.$$

Thus, noticing that $\frac{\mathrm{d}}{\mathrm{d}\lambda}J_a(e^\lambda u) = K_{1,0;a}(e^\lambda u)$ we may integrate over $\lambda \in [\bar\lambda, 0]$ to obtain

$$J_a(e^{\bar\lambda}|\phi|) < J_a(e^{\bar\lambda}\phi) < J_a(\phi) = m_{1,0;a}.$$

Thus we see that $e^{\bar\lambda}|\phi|$ is admissible but attains a smaller value $J_a(e^{\bar\lambda}|\phi|)$ than the infimum $m_{1,0;a}$. Since for any admissible ϕ, we may find $\bar\lambda \le 0$ so that $J_a(e^{\bar\lambda}|\phi|) < J_a(u)$, we may simply assume u is positive.

We can also show that

$$m_{1,0;a} = \inf\left\{\frac{1}{6}\|\phi\|^2_{H^1_a} : \phi \in H^1_a\backslash\{0\}, K_{1,0;a}(\phi) \le 0, \phi > 0\right\}.$$

Note that $G_0(\phi) := J_a(u) - \frac{1}{3}K_{1,0;a}(u) = \frac{1}{6}\|u\|_{H^1_a}$. For $K_{1,0;a}(\phi) = 0$, the two functionals J and G_0 already coincide. Thus, it remains for us to verify that for all $K_{1,0;a}(\phi) < 0$ we have $G_0(\phi) > m_{1,0;a}$. We first observe that this means $g(\lambda) = J_a(e^\lambda\phi)$ satisfies $g'(\lambda) = K_{1,0;a}(e^\lambda\phi)$ and $g'(0) < 0$. Therefore, using a similar argument to above, there exists a $\lambda^* < 0$ such that $g'(\lambda^*) = K_{1,0;a}(e^{\lambda^*}\phi) = 0$. Again, similar to above, we obtain

$$m_{1,0;a} \le J_a(e^{\lambda^*}\phi) = G_0(e^{\lambda^*}\phi) < G_0(\phi),$$

as required.

Using this recharacterisation, we now establish existence of the optimiser. Let $\phi_n > 0$ be a minimising sequence so that $G_{1,0;a}(\phi_n) := \frac{1}{6}\|\phi_n\|_{H^1_a} \to m_{1,0;a}$ as $n \to \infty$. Note that $K_{1,0;a}(\phi_n) \le 0$. For $-\left(\frac{d-2}{2}\right)^2 < a \le 0$, we may apply Schwarz rearrangement to allow the ϕ_n to be radial. Otherwise, we need an extra assumption in the admissible class for (5.2.1) that the test functions must be radial. In either case, we may now use the compact embedding $H^1_{\mathrm{rad},a} \subseteq L^3$ to obtain strong convergence to a limit ϕ^a_∞ (which we shall write as ϕ_∞) in L^3 up to a subsequence. This limit satisfies $K_{1,0;a}(\phi_\infty) \leq 0$, $J_a(\phi_\infty) \leq m_{1,0;a}$ and $G_0(\phi_\infty) \leq m_{1,0;a}$.

Furthermore, we can show that $\phi_\infty \neq 0$. Indeed, suppose that $\phi_n \to 0$ in L^3. From $K_{1,0;a}(\phi_n) \leq 0$ and $\int \phi_n^3 = \int |\phi_n|^3$ we have $\|\phi_n - 0\|_{H^1_a} \leq \|\phi_n - 0\|_{L^3}$. Thus, ϕ_n will also converge strongly to zero in H^a_1 and using the fact that $\|u\|^3_{L^3} \lesssim \|u\|^3_{H^1_a}$, we must have $K_{1,0;a}(\phi_n) > 0$ for large n, which is a contradiction.

Suppose that $K_{1,0;a}(\phi_\infty) < 0$. Then, there exists a $\lambda^* < 0$ such that

$$K_{1,0;a}\left(e^{\lambda^*}\phi_\infty\right) = 0.$$

Therefore

$$m_{1,0;a} \leq G_0\left(e^{\lambda^*}\phi_\infty\right) = e^{2\lambda^*} G_0\left(\phi_\infty\right) < m_{1,0;a}.$$

Thus, we must have $K_{1,0;a}(\phi_\infty) = 0$, whence ϕ_n converges strongly to ϕ_∞ in H^1 and $J_a(\phi_\infty) = m_{1,0;a}$. We shall leave the explicit characterisation of this optimiser until after this proof.

Next, we show that $\mathcal{K}^\pm_{1,0;a}$ is invariant under the flow of (1.0.1). Let $u(t,x)$ be the solution to (1.0.1), and let I denote its maximal lifespan. We notice first that $E(u,u_t) = E(u_0,u_1) < m_{1,0;a}$. Furthermore, let $(u_0,u_1) \in \mathcal{K}^+_{1,0;a}$. If $K_{1,0;a}(u(t^*)) = 0$ then it follows that $u(t^*) = 0$. Thus, using that $\|u\|^3_{L^3} \lesssim \|u\|^3_{H^1_a}$, we obtain $\|u(t)\|^3_3 = o(\|u\|^2_{H^1_a})$, so that $K_{1,0;a}(u(t)) \geq 0$. This shows that $u(t,x) \in K^+_{1,0;a}$ for all $t \in I$. This implies the invariance of $\mathcal{K}^-_{1,0;a}$ as well.

Finally, we shall show the dichotomy, starting with global well-posedness in $\mathcal{K}^+_{1,0;a}$. Thus, assume that the initial data (u_0,u_1) belongs in $\mathcal{K}^+_{1,0;a}$. Let u be the solution to (1.0.1) with this inital data. Furthermore, let I be the maximal lifespan. By invariance of $\mathcal{K}^+_{1,0;a}$ under the flow of (1.0.1), we have $K_{1,0;a}(u(t)) \geq 0$ for all $t \in I$ and

$$E(u,u_t) \geq \frac{1}{6}\int_{\mathbf{R}^d} |\nabla u|^2 + u^2 + \frac{a}{|x|^2}u^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbf{R}^d} u_t^2 \,\mathrm{d}x.$$

Thus, $E(u_0,u_1) \sim \|(u,u_t)\|^2_{H^1\times L^2}$. Thus, the solution may be extended to $\mathbf{R}$ by using the local theory.

Thus, it remains to establish blow-up in $\mathcal{K}^-_{1,0;a}$. We note that $K_{1,0;a}(\phi)$ has an upper bound in $\mathcal{K}^-_{1,0;a}$. Indeed, first note that for $\phi \in \mathcal{K}^-_{1,0;a}$, we have $\int_{\mathbf{R}^d} \phi^3 \,\mathrm{d}x > 0$. Consider $g(\lambda) = J_a(e^\lambda \phi)$ as above. By assumption, $g'(0) = K_{1,0;a}(\phi) < 0$. Therefore, there exists $\lambda^* < 0$ such that $g'(\lambda^*) = 0$. We also have $g''(\lambda) \leq 2g'(\lambda)$. If we integrate this identity over $\lambda \in [\lambda^*, 0]$, we obtain $g'(0) \leq 2(g(0) - g(\lambda^*))$. Thus, $K_{1,0;a}(\phi) \leq -2(m_{1,0;a} - J_a(\phi))$. Thus, $-K_{1,0;a}(\phi) > \delta$ for some $\delta > 0$.

Let us show blow-up for $t > 0$ (the $t < 0$ case is similar). First, assume that the solution exists for all $t > 0$ and consider $y(t) = \|u(t)\|^2_{L^2}$. Since u solves the QKG(a),

$$\partial_t^2 y(t) = 2\left(\|u_t\|_2^2 - K_{1,0;a}(u(t))\right) = 5\|u_t\|^2_2 - 6E(u_0,u_t) + \|u\|^2_{H^1_a}.$$

Using the upper bound as above, we conclude that $\partial_t^2 y(t) > 2\delta > 0$, and therefore $y(t) \to \infty$ as $t \to \infty$. Also, since for large t, we have $\|u(t)\|^2_{H^1_a} > 6E(u_0,u_1)$. We may use Cauchy-Schwarz to obtain

$$\partial_t^2 y(t) > 5\|u_t\|^2_2 \geq \frac{5}{4}\frac{(\partial_t y(t))^2}{y(t)}.$$

Thus,

$$\partial_t^2 y(t)^{-\frac{1}{4}} = -\frac{1}{4} y^{-\frac{9}{4}} \left(yy'' - \frac{5}{4}(y')^2 \right) < 0,$$

which is a contradiction to the fact that $y \to \infty$. Therefore, we must have blow-up in finite time. □

LEMMA 5.2.2. *The optimiser ϕ_∞^a in the above Theorem 5.2.1 satisfies the equation*

$$-\Delta Q_a + Q_a + \frac{a}{|x|^2} Q_a = Q_a^2.$$

Furthermore, $m_{1,0;a}^a = E(Q_a, 0)$.

PROOF. We shall derive the Euler-Lagrange corresponding $J_a(\phi)$ subject to the constraint $K_{1,0;a}(\phi) = 0$. Thus, consider a perturbation $\phi + w(\tau)$ such that $J_a(\phi + \epsilon(\tau))$ attains a minimum at $w = 0$. We need to ensure that this perturbation satisfies the constraint. This is the same procedure as Evans [**15**]. Indeed, first consider a test function $\phi + \tau v + \sigma w$. We know that $K_{1,0;a}(\phi + \tau v + \sigma w)$ is equal to

$$\begin{aligned} & \int_{\mathbf{R}^d} (\nabla(\phi + \tau v + \sigma w))^2 \, \mathrm{d}x + \int_{\mathbf{R}^d} (\phi + \tau v + \sigma w)^2 \, \mathrm{d}x \\ + \quad & \int_{\mathbf{R}^d} \frac{a}{|x|^2} (\phi + \tau v + \sigma w)^2 \, \mathrm{d}x - \int_{\mathbf{R}^d} (\phi + \tau v + \sigma w)^3 \, \mathrm{d}x. \end{aligned}$$

Then,

$$\begin{aligned} \partial_\tau K_{1,0;a}(\tau, \sigma) \quad = \quad & 2 \int_{\mathbf{R}^d} \nabla(\phi + \tau v + \sigma w) \cdot \nabla v \, \mathrm{d}x + 2 \int_{\mathbf{R}^d} (\phi + \tau v + \sigma w) v \, \mathrm{d}x \\ + \quad & \int_{\mathbf{R}^d} \frac{2a}{|x|^2} (\phi + \tau v + \sigma w) v \, \mathrm{d}x - \int_{\mathbf{R}^d} 3(\phi + \tau v + \sigma w)^2 v \, \mathrm{d}x, \end{aligned}$$

and

$$\begin{aligned} \partial_\sigma K_{1,0;a}(\tau, \sigma) \quad = \quad & 2 \int_{\mathbf{R}^d} \nabla(\phi + \tau v + \sigma w) \cdot \nabla w \, \mathrm{d}x + 2 \int_{\mathbf{R}^d} (\phi + \tau v + \sigma w) w \, \mathrm{d}x \\ + \quad & \int_{\mathbf{R}^d} \frac{2a}{|x|^2} (\phi + \tau v + \sigma w) w \, \mathrm{d}x - \int_{\mathbf{R}^d} 3(\phi + \tau v + \sigma w)^2 w \, \mathrm{d}x. \end{aligned}$$

We choose w so that $\partial_\sigma K_{1,0;a}(0,0) \neq 0$. Now, since $K_{1,0;a}(0,0)$ is zero and C^1, we may apply the implicit function theorem to obtain $g : \mathbf{R} \to \mathbf{R}$ such that $g(0) = 0$ and $K_{1,0;a}(\tau, g(\tau)) = 0$ for τ sufficiently small. If we differentiate this last identity, we obtain $\partial_\tau K_{1,0;a}(\tau, g(\tau)) + g'(\tau)\partial_\sigma K_{1,0;a}(\tau, g(\tau)) = 0$. Now, let $\tau = 0$ to obtain

$$g'(0) = -\frac{\partial_\tau K_{1,0;a}(0,0)}{\partial_\sigma K_{1,0;a}(0,0)}.$$

Thus, if we let $k(\tau) = \tau v + g(\tau) w$, we obtain that $\phi + k(\tau)$ is admissible in (5.2.1), and we may use it to derive the Euler-Lagrange. Consider $h(\tau) := J_a(\phi + k(\tau))$. Then from $h'(0) = 0$, we obtain

$$\begin{aligned} h'(0) \quad = \quad & \int_{\mathbf{R}^d} \nabla\phi \cdot (\nabla v + g'(0)\nabla w) \, \mathrm{d}x + \int_{\mathbf{R}^d} \frac{a}{|x|^2} \phi (v + g'(0) w) \, \mathrm{d}x \\ & + \int_{\mathbf{R}^d} \phi (v + g'(0) w) \, \mathrm{d}x - \int_{\mathbf{R}^d} \phi^2 (v + g'(0) w) \, \mathrm{d}x = 0. \end{aligned}$$

Now, substituting the value of $g'(0)$ from (5.2.3), we obtain with

$$\mu := \frac{\int_{\mathbf{R}^d} \nabla u \cdot \nabla w + \frac{a}{|x|^2} uw + uw - u^2 w \, \mathrm{d}x}{\int_{\mathbf{R}^d} 2\nabla u \cdot \nabla w + 2\frac{a}{|x|^2} uw + 2uw - 3u^2 w \, \mathrm{d}x}$$

that

$$\int_{\mathbf{R}^d} \left(-\Delta u + \frac{a}{|x|^2} u + u - u^2 \right) v \, \mathrm{d}x = \mu \int_{\mathbf{R}^d} \left(-2\Delta u + 2\frac{a}{|x|^2} u + 2u - 3u^2 \right) v \, \mathrm{d}x.$$

If we let $v = u$, we notice that the numerator of μ is precisely $K_{1,0;a}(u)$, which is zero by assumption. Therefore, the right-hand side of the above identity vanishes, and we have

$$\int_{\mathbf{R}^d} \left(-\Delta u + \frac{a}{|x|^2} u + u - u^2 \right) v \, \mathrm{d}x = 0.$$

Since this holds for all $v \in H^1$, we arrive at the Euler-Lagrange as in the statement of the lemma. □

We shall write

$$Q_{a^*} = \begin{cases} Q_a, & -\left(\frac{d-2}{2}\right)^2 < a < 0 \\ Q_a^{\mathrm{rad}} & a > 0. \end{cases} \tag{5.2.3}$$

We also have the following global well-posedness/blow-up dichotomy with respect to $\mathcal{K}^{\pm}(a)$ defined as

$$\begin{aligned} \mathcal{K}^+(a) &:= \{(u_0, u_1) \in H_a^1 \times L^2 : E(u_0, u_1) < E(Q, 0), \|u_0\|_2 < \|Q_{a^*}\|_2\} \\ \mathcal{K}^-(a) &:= \{(u_0, u_1) \in H_a^1 \times L^2 : E(u_0, u_1) < E(Q, 0), \|u_0\|_2 > \|Q_{a^*}\|_2\} \end{aligned}$$

THEOREM 5.2.3. *Let $d = 4$. Suppose $u \in C(I, H_a^1)$ is the solution to (1.0.1) with initial data $u(0, \cdot) = u_0$ and $u_t(0, \cdot) = u_1$ where I is the maximal lifespan interval. If $(u_0, u_1) \in \mathcal{K}^+(a)$, then u is global, and if $(u_0, u_1) \in \mathcal{K}^-(a)$, then u blows up in finite time. Furthermore, $m_{1,0;a} = E(Q, 0)$.*

5.3. 4D radial large-energy scattering

5.3.1. Gagliardo-Nirenberg inequality. We now shift our attention to the behaviour of the QKG(a) with initial data $\|u_0\|_2 < \|Q_{a^*}\|_2$, where we shall study scattering. The first ingredient in this endeavour is the following sharp Gagliardo-Nirenberg inequality with mass critical exponent.

PROPOSITION 5.3.1 (Gagliardo-Nirenberg inequality [**13**]). *Let $a > -\left(\frac{d-2}{2}\right)^2$ and denote $a \wedge 0 := \min\{a, 0\}$. For any $g \in H_a^1$ we have*

$$\|g\|_{\frac{2(d+2)}{d}}^{\frac{2(d+2)}{d}} \leq \frac{d+2}{d} \left(\frac{\|g\|_2}{\|Q_{a \wedge 0}\|_2} \right)^{\frac{4}{d}} \|g\|_{\dot{H}_a^1}^2. \tag{5.3.1}$$

If we further assume that g is radial then for $a > 0$ we have

$$\|g\|_{\frac{2(d+2)}{d}}^{\frac{2(d+2)}{d}} \leq \frac{d+2}{d} \left(\frac{\|g\|_2}{\|Q_a^{\mathrm{rad}}\|_2} \right)^{\frac{4}{d}} \|g\|_{\dot{H}_a^1}^2. \tag{5.3.2}$$

Recall the notation

$$Q_{a^*} = \begin{cases} Q_a, & -\left(\frac{d-2}{2}\right)^2 < a < 0 \\ Q_a^{\mathrm{rad}} & a > 0. \end{cases}$$

Suppose that there exists $A < 1$ such that $\|g\|_2 < A\|Q_{a^*}\|_2$. In other words, suppose there is a gap between $\|g\|_2$ and $\|Q_{a^*}\|_2$. In this case, we have the following result which follows from the above Proposition 5.3.1:

$$\|g\|_{\dot{H}_a^1}^2 - \frac{d}{d+2}\|g\|_{\frac{2(d+2)}{d}}^{\frac{2(d+2)}{d}} \geq \left(\frac{1}{A} - 1\right)\|g\|_2^2. \tag{5.3.3}$$

We shall need the above result with dependence on t. Indeed,

Proposition 5.3.2. *Let $d = 4$ and $p = 1$. Suppose $\|u_0\|_2 < \|Q_{a^*}\|_2$ and $E(u_0, u_1) < E(Q_{a^*}, 0)$. Let $u(t, x) \in C(I, H_a^1)$ be a solution to (1.0.1) with initial data (u_0, u_1). Then, for some $A = A(E(u_0, u_1)) < 1$ we have*

$$\|u(t)\|_2 < A\|Q_{a^*}\|_2.$$

Proof. The proof is the same as in potential-free case [**22**]. Using the energy identity

$$\|Q_{a^*}\|_{H_a^1}^2 = \|Q_{a^*}\|_3^3 \tag{5.3.4}$$

and Pohozaev identity (see e.g. [**13**, p. 287]))

$$\frac{d-2}{2}\|Q_{a^*}\|_{\dot{H}_a^1} + \frac{d}{2}\|Q_{a^*}\|_2^2 = \frac{d}{p+2}\|Q_{a^*}\|_3^3, \tag{5.3.5}$$

we shall show that

$$E(Q_{a^*}, 0) = \frac{1}{2}\|Q_{a^*}\|_2^2. \tag{5.3.6}$$

Indeed, suppose for some $t \in I$, we have $\|u(t)\|_2 = \|Q_{a^*}\|_2$. By (5.3.1), we have $K(u(t)) \geq 0$. Thus,

$$E(u(t), u_t(t)) \geq \frac{1}{2}\|u(t)\|_2^2 = \frac{1}{2}\|Q_{a^*}\|_2^2.$$

This contradicts the fact that $E(u_0, u_1) < E(Q_{a^*}, 0)$. Thus, we have $\|u(t, \cdot)\|_2 < \|Q\|_2$.

Next, from the fact that $E(u_0, u_1) < E(Q_{a^*}, 0)$, there exists a constant $A < 1$ such that

$$E(u(t), u_t(t)) < \frac{A^2}{2}\|Q_{a^*}\|_2^2.$$

From this, we obtain that $\|u(t)\|_2 < A\|Q_{a^*}\|_2$ for all $t \in I$. ☐

5.3.2. Virial-Morawetz estimates. We shall now obtain Virial/Morawetz estimates. These estimates will be used to obtain L^3 decay after large time.

Proposition 5.3.3. *Let $d = 4, p = 1$ and $u(t, x) \in C(\mathbf{R}, H^1)$ be a solution to (1.0.1) the QKG(a) with initial data $(u_0, u_1) \in H_{a,\mathrm{rad}}^1 \times L_{\mathrm{rad}}^2$. Suppose that $E_a := E_a(u_0, u_1) > 0$ and $\|u\|_{H_a^1}^2 + \|u_t\|_2^2 \sim E_a$. If there exists $A < 1$ such that $\|u(t)\|_2 \leq A\|Q_{a^*}\|_2$ for all $t \in \mathbf{R}$, then*

$$\int_T^{2T}\int_{|x|\leq R} |u|^3 \,\mathrm{d}x\mathrm{d}t \leq C(E, A)\left(R + TR^{-\frac{3}{2}}\right).$$

Proof. Suppose $h : \mathbf{R}^d \to \mathbf{R}^d$ and $q(x) : \mathbf{R}^d \to \mathbf{R}$. Denote by h_j the jth coordinate of the function $h(x)$. We first consider the following Morawetz identity with

$$M(t) := -\int u_t(h \cdot \nabla u + qu)\,\mathrm{d}x.$$

Differentiating in time and using integration by parts, we obtain that $\partial_t M(t)$ is equal to

$$
\begin{aligned}
&-\int u_{tt}(h\cdot\nabla u+qu)\ \mathrm{d}x-\int u_t(h\cdot\nabla u_t+qu_t)\ \mathrm{d}x\\
&=-\int\left(\Delta u-\frac{a}{|x|^2}u-u+u^2\right)(h\cdot\nabla u+qu)\ \mathrm{d}x-\int u_t(h\cdot\nabla u_t+qu_t)\ \mathrm{d}x.
\end{aligned}
$$

To deal with the second integral, we have

$$
-\int u_t(h\cdot u_t+q\nabla u_t)\ \mathrm{d}x=-\int\frac{1}{2}\nabla u_t^2+qu_t\ \mathrm{d}x=\int\left(-\frac{1}{2}\mathrm{div}h+q\right)u_t\ \mathrm{d}x
$$

We may do the same thing for the u term in the first integral. Now, to deal with the u^2 term, we recall $F(u):=\int f(u)\ \mathrm{d}u$ and $G(u):=uf(u)-2F(u)$. Then, we may write

$$
\begin{aligned}
-\int u^2h\cdot\nabla u+qu\cdot u^2\ \mathrm{d}x &= -\int h\cdot\nabla F(u)+quf(u)\ \mathrm{d}x\\
&= \int\left(-\frac{1}{2}\mathrm{div}h+q\right)(-2F(u))+\int q\left(uf(u)-2F(u)\right)\ \mathrm{d}x\\
&= \int\left(-\frac{1}{2}\mathrm{div}h+q\right)(-2F(u))+\int qG(u)\ \mathrm{d}x.
\end{aligned}
$$

To deal with the Δu term, we may use the product rule. All together, we find that $\partial_t M(t)$ is now equal to

$$
\begin{aligned}
&\sum_{j,k=1}^{d}\int\partial_k u\partial_k h_j\partial_j u\ \mathrm{d}x+\frac{1}{2}\int|u|^2(-\Delta q)\ \mathrm{d}x\\
&-\int q(x)G(u)\ \mathrm{d}x-\int\frac{a}{2}h\cdot\nabla\left(\frac{1}{|x|^2}\right)u^2\ \mathrm{d}x\\
&+\int\left(-\frac{1}{2}\mathrm{div}h(x)+q(x)\right)\left(-|u_t|^2+|\nabla u|^2+|u|^2-2F(u)+\frac{a}{|x|^2}u^2\right)\ \mathrm{d}x.
\end{aligned}
$$

Now, let $w:\mathbf{R}^d\to\mathbf{R}$ be a weight, and define $h(x)=\nabla w(x)$ and $q(x)=\frac{1}{2}\mathrm{div}h(x)=\frac{1}{2}\Delta w(x)$. In this case, we obtain

$$
\begin{aligned}
\partial_t M(t) &= \sum_{j,k=1}^{d}\int\partial_k u\partial_j u\partial_{jk}^2 w\ \mathrm{d}x-\frac{1}{4}\int|u|^2\Delta^2 w\ \mathrm{d}x\\
&\quad -\frac{1}{2}\int\Delta wG(u)\ \mathrm{d}x+a\int\nabla w\cdot\frac{x}{|x|^4}u^2\ \mathrm{d}x.
\end{aligned}
$$

We shall use the same weight as [**67**], where

$$
w(x)=\begin{cases}|x|^2, & |x|\le\frac{R}{2}\\ R|x|, & |x|\ge R\end{cases},
$$

and for $\frac{R}{2}<|x|<R$ we impose that

$$
\partial_r w_R\ge 0,\partial_r^2 w_R\ge 0,|\partial^\alpha w_R(x)|\lesssim_\alpha R|x|^{-|\alpha|+1}\text{ for }|\alpha|\ge 1. \tag{5.3.7}
$$

In particular, we find that $\partial_t M(t)$ may be written with some $C_1, C_2 > 0$ and $d = 4, p = 1$ as

$$2\int_{|x|<R/2} |\nabla u|^2 + a\frac{|u|^2}{|x|^2} - \frac{d}{2\cdot 3}u^3 \,\mathrm{d}x \tag{5.3.8}$$

$$+\int_{R/2<|x|<R} \sum_{j,k=1}^{d} \partial_j u \partial_k u \partial_{jk}^2 w - C_1|u|^2\Delta^2 w - C_2 u^{p+2}\Delta w \,\mathrm{d}x \tag{5.3.9}$$

$$+\int_{|x|>R} a\frac{R}{|x|^3}|u|^2 + \frac{R}{|x|}\left(|\nabla u|^2 - |\partial_r u|^2\right) - \frac{(d-1)\cdot 1}{2\cdot 3}\cdot\frac{R}{|x|}u^3 \,\mathrm{d}x. \tag{5.3.10}$$

For (6.3.43), since u is radial, we have $|\nabla u|^2 - |\partial_r u|^2 = 0$. Thus, we have with some $C_3 > 0$ that

$$(6.3.43) \geq \frac{aM(u)}{R^2} - C_3\int_{|x|>R} |u|^3 \,\mathrm{d}x.$$

For (6.3.42), the conditions for $w(x)$ in (6.3.40) ensure the summation is non-negative and also that

$$(6.3.42) \geq -C_1\frac{M(u)}{R^2} - C_2\int_{\frac{R}{2}<|x|<R} |u|^3 \,\mathrm{d}x.$$

For (6.3.41) we define a smooth cutoff function χ with support $\{x \in \mathbf{R}^4 : |x| \leq 1/2\}$ and set $\chi_R(x) := \chi(x/R)$. With the observation that

$$\int {\chi_R}^2|\nabla u|^2 \,\mathrm{d}x = \int |\nabla(\chi_R)u|^2 + \chi_R\Delta(\chi_R)|u|^2 \,\mathrm{d}x, \tag{5.3.11}$$

we find that

$$\begin{aligned}(6.3.41) \quad \geq \quad & 2\left(\|\chi_R u\|_{\dot H^1_a}^2 - \frac{d\cdot 1}{2\cdot 3}\|\chi_R u\|_3^3\right) \\ & + \int \mathcal{O}\left(\frac{1}{R^2}|u|^2\right) \,\mathrm{d}x + \int \mathcal{O}\left(\chi_R^3 - \chi_R^2\right)|u|^3 \,\mathrm{d}x.\end{aligned}$$

Next, we use (5.3.3) combined with Proposition 5.3.2, integrating over $[T, 2T]$ and discarding positive terms, we obtain

$$\int_T^{2T}\int c|\chi_R u|^3 \,\mathrm{d}x\mathrm{d}t \lesssim \sup_{t\in[T,2T]}|M(t)| + \int_T^{2T}\int_{|x|>R}|u|^3 \,\mathrm{d}x\mathrm{d}t + \frac{T}{R^2}M(u).$$

By radial Sobolev embedding,

$$\int_{|x|>R}|u|^3 \,\mathrm{d}x \lesssim \frac{1}{R^{3/2}}\|u\|_{L_t^\infty H_x^1}M(u).$$

From this, and the fact that $\sup_{t\in[T,2T]}|M(t)| \leq R$, we obtain the required result. ∎

COROLLARY 5.3.4. *Let $d = 4$ and suppose that u is a radial solution of (1.0.1) with initial data $(u_0, u_1) \in \mathcal{K}^-(a)$. Then, for any $\epsilon_0 > 0, T > 1, \tau > 0$ there exists $T_0 = T_0(\epsilon_0, T, E_a) \geq T$ such that*

$$\int_{T_0}^{T_0+\tau}\int |u(t,x)|^3 \,\mathrm{d}x\mathrm{d}t \leq \epsilon_0.$$

PROOF. See [**22**, Cor. 3.7] ∎

5.3.3. L^3 decay after large time. We shall now apply the above Corollary 5.3.4 to obtain the following smallness result needed in the proof of scattering which follows. The idea is to split the Duhamel integral over $[0,t]$ into two intervals, the first being over, say, $s \in [0, t-\tau_1]$ in which there is a gap $|t-s| > L > 0$. Thus the smallness result we obtain here shall cover the remaining interval over $[t-\tau_1, t]$.

We shall focus on the case where $-\left(\frac{d-2}{2}\right)^2 < a < 0$. In the $a > 0$ case, we have access to the $L^\infty \to L^1$ dispersive estimates which holds for the potential-free case, and therefore the argument in this case is identical to that in [**22**] by Guo-Shen. Let us consider solution to the first-order equation:

$$U(t,x) = K_a(t)U_0(x) - i\int_0^t K_a(t-s)\langle D_a\rangle^{-1}u(s,x)^2 \, \mathrm{d}s.$$

PROPOSITION 5.3.5. *Suppose that u is a radial solution of the 4D QKG(a) (1.0.1) with $a > \sigma^{-1}\left(\frac{1}{2}\right)$ and with initial data (u_0, u_1) such that $\|u_0\|_2 < \|Q\|_2$ and $E(u_0,u_1) < E(Q,0)$. For any $\epsilon_1 > 0$ and $T > 0$, there exists $\tau_1 = \tau_1(E,\epsilon_1) \geq C_E\epsilon_1^{-8}$ and $T_1 = T_1(E,\epsilon_1,T)$ such that $T < T_1 - \tau_1$ and*

$$\sup_{t\in[T_1-\tau_1,T_1]} \|U(t,x)\|_{L^3_x} \leq \epsilon_1.$$

REMARK 5.3.6. *We restrict to $a > \sigma^{-1}\left(\frac{d}{3}\right)$ in order for the L^3_x norm to make sense. The further restriction that $a > \sigma^{-1}\left(\frac{1}{2}\right)$ appears in the proof below.*

PROOF. As mentioned above, let us only discuss the $-\left(\frac{d-2}{2}\right)^2 < a < 0$ case, with the $a > 0$ being identical to the potential-free case. We first break $\|U(t,x)\|_{L^3_x}$ into parts. In particular, let $R > 0$ be chosen later. Then,

$$\|U(t,x)\|_{L^3_x} \leq \|K_a(t)U_0(x)\|_{L^3_x} \tag{5.3.12}$$

$$+ \left\|\int_0^{t-\tau_1} K_a(t-s)\langle D_a\rangle^{-1}u(s,x)^2 \, \mathrm{d}s\right\|_{L^3_x} \tag{5.3.13}$$

$$+ \left\|\int_{t-\tau_1}^{t} K_a(t-s)\langle D_a\rangle^{-1}u(s,x)^2 \, \mathrm{d}s\right\|_{L^3_x}. \tag{5.3.14}$$

First, to deal with (5.3.12), we note that by the refined radial Strichartz estimates (Theorem 3.1.1), for any $2 < q < 3$, we have $\|K_a(t)U_0(x)\|_{L^q_tL^3_x} \leq C_E$. Since

$$\|\partial_t K_a(t)U_0(x)\|_{L^2} \lesssim \|K_a(t)U_0(x)\|_{H^1_a},$$

we conclude that $K_a(t)U_0(x)$ is Lipschitz continuous, and in particular,

$$\|K_a(t)U_0(x)\|_{L^3_x} \to 0 \text{ as } t \to \infty.$$

Next, we study (5.3.13). We shall apply a frequency decomposition in order to use our time-decay estimates. Denote

$$\mathcal{I}_k := \int_0^{t-\tau_1} K_a(t-s)P^a_k\langle D_a\rangle^{-1}u(s,x)^2 \, \mathrm{d}s.$$

We have

$$(5.3.13) \leq \|\|\mathcal{I}_k\|_{L^3}\|_{\ell^2_{k\geq 0}} + \|\|\mathcal{I}_k\|_{L^3}\|_{\ell^2_{k<0}}.$$

We first consider the high-frequency case. First, we note by interpolation that

$$\|\mathcal{I}_k\|_{L^3_x} \leq \left(2^{-\frac{k}{2}}\|\mathcal{I}_k\|_{L^4_x}\right)^{\frac{2}{3}}\left(2^k\|\mathcal{I}_k\|_{L^2_x}\right)^{\frac{1}{3}}.$$

For the L^2 norm, we may use the observation that

$$\int_0^{t-\tau_1} K_a(t-s)\langle D_a\rangle^{-1}u(s,x)^2\,\mathrm{d}s = K_a(t-t+\tau_1)U(t-\tau_1) - K_a(t)U_0, \tag{5.3.15}$$

combined with H^1_a boundedness to control this piece. Thus, we have

$$\|\|\mathcal{I}_k\|_{L^3}\|_{\ell^2_{k\geq 0}} \quad \lesssim_E \quad \left\|\left(2^{-\frac{k}{2}}\,\|\mathcal{I}_k\|_{L^4_x}\right)^{\frac{2}{3}}\right\|_{\ell^2_{k\geq 0}}.$$

It remains to study the L^4 norm. We split this norm into regions $|x|<R$ and $|x|>R$, with R to be chosen later. First, we study the $|x|<R$ region using time-decay estimates (Proposition 5.1.3) with $\theta = 1-2\lambda$, where $\lambda \ll 1$. Indeed, we bound $\|K_a(t-s)P^a_k u(s,x)^2\|_{L^4_{|x|<R}}$ by

$$\begin{aligned}
&\|\left(1+|x|^{-\sigma}\right)^{\frac{3}{4}}\|_{L^8_{|x|<R}} \cdot \left\|\left(1+|x|^{-\sigma}\right)^{-\frac{3}{4}} K_a(t-s)P^a_k\langle D_a\rangle^{-1}u(s,x)^2\right\|_{L^8(\mathbf{R}^4)}\\
&\quad\lesssim R^{\frac{1}{2}}\left(|t-s|^{-2+\lambda}2^{(3-\lambda)k}\right)^{\frac{3}{4}}\|(1+|x|^{-\sigma})^{\frac{3}{4}}P^a_k\langle D_a\rangle^{-1}u^2\|_{L^{8/7}(\mathbf{R}^4)}.
\end{aligned}$$

Here, for $\|\left(1+|x|^{-\sigma}\right)^{3/4}\|_{L^8_{|x|<R}}$ to be finite, we impose that $\sigma<\frac{1}{2}$. This corresponds to further restricting the $a<0$ coefficient range from $a\in(-1,0)$ to $a\in(-\frac{3}{4},0)$. Furthermore, we need to deal with the $L^{8/7}$ norm.

Using the Hardy inequality for $\mathscr{L}_a$, Bernstein estimates, boundedness of the Paley-Littlewood operator in $L^{8/7}$, Sobolev norm equivalence (indeed, we have for $\sigma<\frac{1}{2}$ that $\max\{\frac{1}{4},\frac{\sigma}{4}\}<\frac{7}{8}<\min\{1,1-\frac{\sigma}{4}\}$) and fractional chain rule, we obtain for that $k\geq 0$

$$\begin{aligned}
\left\|(1+|x|^{-\sigma})^{\frac{3}{4}}P^a_k\langle D_a\rangle^{-1}u^2\right\|_{L^{8/7}(\mathbf{R}^4)} \quad &\lesssim \quad \|P^a_k\langle D_a\rangle^{-1}u^2\|_{L^{8/7}(\mathbf{R}^4)}\\
&\qquad + \left\||x|^{-\frac{3\sigma}{4}}P^a_k\langle D_a\rangle^{-1}u^2\right\|_{L^{8/7}}\\
&\lesssim \quad \|P^a_k\langle D_a\rangle^{-1}u^2\|_{L^{8/7}}\\
&\qquad + \left\|D_a^{\frac{3\sigma}{4}}P^a_k\langle D_a\rangle^{-1}u^2\right\|_{L^{8/7}}\\
&\lesssim \quad 2^{-(1+\frac{3\sigma}{4})k}\|P^a_k u^2\|_{L^{8/7}}\\
&\lesssim \quad 2^{-(1+\frac{3\sigma}{4})k}2^{-k}\|D_aP^a_k u^2\|_{L^{8/7}}\\
&\lesssim \quad 2^{-(1+\frac{3\sigma}{4})k}2^{-k}\|D_a u^2\|_{L^{8/7}}\\
&\lesssim \quad 2^{-(1+\frac{3\sigma}{4})k}2^{-k}\|\nabla u^2\|_{L^{8/7}}\\
&\lesssim \quad 2^{-(1+\frac{3\sigma}{4})k}2^{-k}\|u\|_{L^{8/3}}\|\nabla u\|_{L^2}\\
&\lesssim \quad 2^{-(1+\frac{3\sigma}{4})k}2^{-k}\|\nabla u\|^2_{L^2}\\
&\lesssim_E \quad 2^{-(1+\frac{3\sigma}{4})k}2^{-k}.
\end{aligned}$$

Thus, we so far have that

$$\begin{aligned}
\|\|\mathcal{I}_k\|_{L^3}\|_{\ell^2_{k\geq 0}} \quad \lesssim_E \quad &\left\|\left(2^{-\frac{k}{2}}R^{\frac{1}{2}}\int_0^{t-\tau_1}\left(|t-s|^{-2+\lambda}\,\mathrm{d}s\cdot 2^{(3-\lambda)k}\right)^{\frac{3}{4}}2^{-(1+\frac{3\sigma}{4})k}2^{-k}\right.\right.\\
&\qquad\left.\left.+2^{-\frac{k}{2}}\,\|\mathcal{I}_k\|_{L^4_{|x|>R}}\right)^{\frac{2}{3}}\right\|_{\ell^2_{k\geq 0}}.
\end{aligned}$$

The tail can be handled via the radial Sobolev embedding. That is,

$$\begin{aligned}
\|\mathcal{I}_k\|_{L^4_{|x|>R}} &\leq \left\| \int_0^{t-\tau_1} K_a(t-s)\langle D_a\rangle^{-1} u(s,x)^2 \,\mathrm{d}s \right\|_{L^\infty_{|x|>R}}^{\frac{1}{2}} \\
&\quad \cdot \left\| \int_0^{t-\tau_1} K_a(t-s)\langle D_a\rangle^{-1} u(s,x)^2 \,\mathrm{d}s \right\|_{L^2_{|x|>R}}^{\frac{1}{2}} \\
&\lesssim \frac{1}{R^{3/4}} \|K(t-t+\tau_1)U(t-\tau_1) - K(t)U_0\|_{H^1}^{\frac{1}{2}} \\
&\quad \cdot \|K(t-t+\tau_1)U(t-\tau_1) - K(t)U_0\|_{L^2}^{\frac{1}{2}} \\
&\lesssim_E \frac{1}{R^{3/4}}.
\end{aligned}$$

Thus, all together, we have

$$\begin{aligned}
\|\|\mathcal{I}_k\|_{L^3}\|_{\ell^2_{k\geq 0}} &\lesssim_E \left\| 2^{-(1+\frac{3\sigma}{4})k} 2^{-k} \left(2^{-\frac{k}{2}} R^{\frac{1}{2}} \int_0^{t-\tau_1} \left(|t-s|^{-2+\lambda} \cdot 2^{(3-\lambda)k} \right)^{\frac{3}{4}} \,\mathrm{d}s \right.\right. \\
&\qquad \left.\left. + 2^{-\frac{k}{2}} R^{-\frac{3}{4}} \right)^{\frac{2}{3}} \right\|_{\ell^2_{k\geq 0}} \\
&\lesssim_E \left(R^{\frac{1}{2}} \tau_1^{-\frac{1}{2}+\frac{3}{4}\lambda} + R^{-\frac{3}{4}} \right)^{\frac{2}{3}}.
\end{aligned}$$

We also need to deal with the low-frequency case. In this case, we also use Bernstein estimates to attain summability in $\ell^2_{k<0}$. First, define $\frac{1}{q} := \frac{1}{3} + \frac{\lambda}{4}$. Then,

$$\|\|\mathcal{I}_k\|_{L^3}\|_{\ell^2_{k<0}} \lesssim \left\| 2^{\lambda k} \|\mathcal{I}_k\|_{L^q} \right\|_{\ell^2_{k<0}}.$$

Similar to the high-frequency case, we split the L^q norm into the two regions $|x| < R$ and $|x| > R$. For the bounded $|x| < R$ region as before, we use the time-decay estimates with $\theta = 1$. Notice that we shall need to use the L^6 norm, which is allowed because we have already restricted the coefficients of the inverse-square potential to allow the use of the L^8 norm. We remark that there is no gain from the Hardy inequality in the low frequency and we have

$$\|P_k^a K_a(t-s)u(s,x)^2\|_{L^q_{|x|<R}} \lesssim R^{\frac{2}{3}} \left(|t-s|^{-2} 2^{-\sigma k} \right)^{\frac{2}{3}-\frac{\lambda}{2}} \|P_k^a u^2\|_{L^{(2q)'}}.$$

Now, using Bernstein inequality and boundedness of the Paley-Littlewood operator in $L^{(2q)'}$, we have with $\frac{1}{p} = \frac{1}{(2q)'} + \frac{1}{12} = \frac{5}{6} + \frac{\lambda}{8} + \frac{1}{12}$ and $s = \frac{1}{6} - \frac{\lambda}{4} < 1$ that

$$\begin{aligned}
\|P_k^a u^2\|_{L^{(2q)'}} &\lesssim 2^{\frac{k}{3}} \left\| D_a^{-\frac{1}{3}} P_k^a u^2 \right\|_{L^{(2q)'}} \\
&\lesssim 2^{\frac{k}{3}} \left\| u^2 \right\|_{L^p} \\
&\lesssim 2^{\frac{k}{3}} \|u\|_{L^{2p}}^2 \lesssim 2^{\frac{k}{3}} \|u\|_{H^1_a} \lesssim 2^{\frac{k}{3}} \|u\|_{H^1_a}.
\end{aligned}$$

Recall we have already restricted to $\sigma < 1/2$. This is enough for summability in $\ell^2_{k<0}$. So far, we have

$$\begin{aligned}\big\|\|\mathcal{I}_k\|_{L^3}\big\|_{\ell^2_{k<0}} &\lesssim_E \left\| 2^{\lambda k} 2^{\frac{k}{3}} R^{\frac{2}{3}} \int_0^{t-\tau_1} \left(|t-s|^{-2} 2^{-\sigma k}\right)^{\frac{2}{3}-\frac{\lambda}{2}}\, \mathrm{d}s \right\|_{\ell^2_{k<0}} \\ &\quad + \left\| 2^{\lambda k} \left\| \int_0^{t-\tau_1} K_a(t-s) P_k^a u(s,x)^2\, \mathrm{d}s \right\|_{L^q_{|x|>R}} \right\|_{\ell^2_{k<0}}.\end{aligned}$$

The tail can be dealt with using the radial Sobolev embedding. Indeed,

$$\begin{aligned}\|\mathcal{I}_k\|_{L^q_{|x|>R}} &\leq \left\| \int_0^{t-\tau_1} K_a(t-s)\langle D_a\rangle^{-1} u(s,x)^2\, \mathrm{d}s \right\|_{L^\infty_{|x|>R}}^{\frac{1}{3}-\frac{\lambda}{2}} \\ &\qquad \cdot \left\| \int_0^{t-\tau_1} K_a(t-s)\langle D_a\rangle^{-1} u(s,x)^2\, \mathrm{d}s \right\|_{L^2_{|x|>R}}^{\frac{2}{3}+\frac{\lambda}{2}} \\ &\lesssim \frac{1}{R^{\frac{1}{2}-\frac{3\lambda}{4}}} \|K(t-t+\tau_1)U(t-\tau_1) - K(t)U_0\|_{H^1}^{\frac{1}{3}-\frac{\epsilon}{2}} \\ &\qquad \cdot \|K(t-t+\tau_1)U(t-\tau_1) - K(t)U_0\|_{L^2}^{\frac{2}{3}+\frac{\epsilon}{2}} \\ &\lesssim_E \frac{1}{R^{\frac{1}{2}-\frac{3\lambda}{4}}}.\end{aligned}$$

For the tail, we have the $2^{\lambda k}$ for summability in $\ell^2_{k<0}$.

Now, putting the everything back together, we have for (5.3.13) that

$$(5.3.13) \quad \lesssim_E \quad \left(R^{\frac{1}{2}} \tau_1^{-\frac{1}{2}+\frac{3}{4}\lambda} + R^{-\frac{3}{4}}\right)^{\frac{2}{3}} + R^{\frac{2}{3}} \tau_1^{-\frac{4}{3}+\lambda} + R^{-(\frac{1}{2}-\frac{3\lambda}{4})}.$$

To deal with (5.3.14), we will not exploit time-decay because for $s \in [t-\tau_1, t]$, there is not a strictly positive lower bound on $|t-s|$. Instead, smallness can be obtained via the Virial-Morawetz estimates (Corollary 5.3.4). First, we apply a frequency decomposition:

$$\begin{aligned}(5.3.14) &\lesssim \int_{t-\tau_1}^t \|K_a(t-s)\langle D_a\rangle^{-1} u(s,x)^2\|_{L^3(\mathbf{R}^4)}\, \mathrm{d}s \\ &\lesssim \int_{t-\tau_1}^t \big\|\|P_k^a K_a(t-s)\langle D_a\rangle^{-1} u(s,x)^2\|_{L^3(\mathbf{R}^4)}\big\|_{\ell^2_{k\geq 0}}\, \mathrm{d}s \\ &\qquad + \int_{t-\tau_1}^t \big\|\|P_k^a K_a(t-s) u(s,x)^2\|_{L^3(\mathbf{R}^4)}\big\|_{\ell^2_{k<0}}\, \mathrm{d}s.\end{aligned}$$

In the high frequency case, we have

$$\begin{aligned}\|P_k^a K_a(t-s)\langle D_a\rangle^{-1} u^2\|_{L^3(\mathbf{R}^4)} &\lesssim 2^{k\left(\frac{4}{2}-\frac{4}{3}\right)} \|P_k^a K_a(t-s)\langle D_a\rangle^{-1} u^2\|_{L^2(\mathbf{R}^4)} \\ &= 2^{k\left(\frac{4}{2}-\frac{4}{3}\right)} \|P_k^a \langle D_a\rangle^{-1} u^2\|_{L^2(\mathbf{R}^4)} \\ &\lesssim \|P_k^a \langle D_a\rangle^{-1/3} u^2\|_{L^2(\mathbf{R}^4)}.\end{aligned}$$

By Sobolev norm equivalence (Proposition 2.4.1) and fractional chain rule

$$\begin{aligned}\big\|\|P_k^a K_a(t-s)\langle D_a\rangle^{-1}u(s,x)^2\|_{L^3(\mathbf{R}^4)}\big\|_{\ell^2_{k\geq 0}} &\lesssim \Big\|\|P_k^a\langle D_a\rangle^{-1/3}u^2\|_{L^2(\mathbf{R}^4)}\Big\|_{\ell^2_{k\geq 0}}\\ &\lesssim \|u^2\|_{H_a^{-\frac{1}{3}}}\\ &\lesssim \|u^2\|_{W_a^{\frac{3}{2},\frac{1}{3}}}\\ &\lesssim \|u\|_{L^3}\|u\|_{W^{\frac{3}{2},\frac{1}{3}}}\\ &\lesssim \|u\|_{L^3}\|u\|_{H_a^1}.\end{aligned}$$

The low-frequency case can be handled similarly. Hence, by Hölder inequality,

$$\begin{aligned}(5.3.14) &\lesssim_E \int_{t-\tau_1}^t \|u\|_{L^3}\,\mathrm{d}s\\ &\lesssim \left(\int_{t-\tau_1}^t \|u\|_{L^3}^3\,\mathrm{d}s\right)^{\frac{1}{3}}\left(\int_{t-\tau_1}^t \mathrm{d}s\right)^{\frac{2}{3}}\\ &\lesssim \tau_1^{\frac{2}{3}}\left(\int_{t-\tau_1}^t\int_{\mathbf{R}^4}|u(s,x)|^3\,\mathrm{d}x\mathrm{d}s\right)^{\frac{1}{3}}.\end{aligned}$$

Thus, when we put these back together we obtain

$$\begin{aligned}\|U(t,x)\|_{L^3} \lesssim_E\ & (5.3.12) + \left(R^{\frac{1}{2}}\tau_1^{-\frac{1}{2}+\frac{3}{4}\lambda}+R^{-\frac{3}{4}}\right)^{\frac{2}{3}} + R^{\frac{2}{3}}\tau_1^{-\frac{4}{3}+\lambda} + R^{-(\frac{1}{2}-\frac{3\lambda}{4})}\\ &+\tau_1^{\frac{2}{3}}\left(\int_{t-\tau_1}^t\int_{\mathbf{R}^4}|u(s,x)|^3\,\mathrm{d}x\mathrm{d}s\right)^{\frac{1}{3}}.\end{aligned}$$

Now, we choose $R=\tau_1^{\frac{1}{4}}$. For any $\epsilon_1<0$ and $T>0$, there exists $\widetilde{T}=\widetilde{T}(\epsilon_1,T)>T$ and $\tau_1=\tau_1(E,\epsilon_1)\geq C_E\epsilon_1^{-9}$ such that if $t>\widetilde{T}$, then

$$(5.3.12) + \left(R^{\frac{1}{2}}\tau_1^{-\frac{1}{2}+\frac{3}{4}\lambda}+R^{-\frac{3}{4}}\right)^{\frac{2}{3}} + R^{\frac{2}{3}}\tau_1^{-\frac{4}{3}+\lambda} + R^{-(\frac{1}{2}-\frac{3\lambda}{4})} \leq \frac{1}{2}\epsilon_1.$$

Then, by Corollary 5.3.4, we can conclude that for the above $\widetilde{T}$, and choosing $\tau=2\tau_1$ and $\epsilon_0\leq C_E\tau_1^{-2}\epsilon_1^3$ there exists $\widetilde{T}_0=\widetilde{T}_0(\epsilon,E,T)\geq\widetilde{T}$ and $t\in[\widetilde{T}_0+\tau_1,\widetilde{T}_0+2\tau_1]$, so that

$$(5.3.14)\lesssim\tau_1^{\frac{2}{3}}\left(\int_{t-\tau_1}^t\int_{\mathbf{R}^4}|u(s,x)|^3\,\mathrm{d}x\mathrm{d}s\right)^{\frac{1}{3}}\leq\frac{1}{2}\epsilon_1.$$

Finally, with $T_1=\widetilde{T}_0+2\tau_1$, the result follows. □

5.3.4. Normal-form transform. Next, we shall obtain some estimates we need in the proof of scattering. First, define the space

$$S_a(I)=\left(0,\frac{1}{2},0|1\right)\cap\left(\frac{1}{2},\frac{5}{14}-\kappa,\frac{3}{7}-4\kappa|\frac{11}{14}+\kappa\right),$$

and also the space

$$Z_a(I)=\left(0,\frac{1}{4},0|s_H\right).$$

where $\min\{-\frac{1}{3}+\delta, -\frac{1}{4}-\frac{3}{4}\sigma\}$. Finally, define the weak space $\tilde{S}_a(I)$ with norm

$$\begin{aligned}\|U\|_{\tilde{S}_a(I)} &= \left\|P^a_{\geq 0}U\right\|_{(\frac{1}{2}-\epsilon,\frac{1}{4}+\epsilon,7\epsilon)\cap(\frac{1}{2}-\epsilon,\frac{1}{4}+3\epsilon,\frac{2}{7})} \\ &\quad + \left\|P^a_{\leq 0}U\right\|_{(\frac{1}{2}-\epsilon,\frac{1}{4}-\epsilon,\epsilon)\cap L^3_tL^6_x\cap(\epsilon,2\epsilon(\frac{5}{14}-\kappa)+(1-4\epsilon)\frac{1}{2},1)}.\end{aligned}$$

In contrast to the 3D radial small energy problem (Chapter 3), here we need the space $Z_a(I)$ to provide some 'extra room'. The choice of exponents for $S_a(I)$ and $\tilde{S}_a(I)$ are the same as in [**22**] (though the spaces are different because we are using Littlewood-Paley projectors adapted to $\mathscr{L}_a$), while for $Z_a(I)$, we have chosen the L^4_x based space rather than the L^∞_x space, which is the best exponent possible to define the Littlewood-Paley projectors without reducing the range of a (the coefficient of the inverse-square potential).

The control of $\tilde{S}_a(I)$ by interpolation of $S_a(I)$ and $Z_a(I)$ is the same as in [**22**]. For clarity, let us check the case when $\sigma = 1/2$. Here, $s_H = \min\{-\frac{1}{3}+\delta, -\frac{5}{8}\} = -\frac{5}{8}$.

Lemma 5.3.7. *Let $u \in S_a(I)\cap Z_a(I)$. Then,*

$$\|u\|_{\tilde{S}_a(I)} \lesssim \|U\|^{1-2\epsilon}_{S_a(I)}\|U\|^{2\epsilon}_{\tilde{S}_a(I)}. \tag{5.3.16}$$

Proof. We need to check the five norms in the definition of $\tilde{S}_a(I)$. Note that ϵ and κ are sufficiently small. We start with the high frequency case. First, we have

$$\begin{aligned}\left\|P^a_{\geq 0}U\right\|_{(\frac{1}{2}-\epsilon,\frac{1}{4}+\epsilon,7\epsilon)} &\lesssim \left\|P^a_{\geq 0}U\right\|_{(\frac{1}{2}-\epsilon,(1-2\epsilon)(\frac{5}{14}-\kappa)+2\epsilon\cdot\frac{1}{4},s)} \\ &\lesssim \left\|P^a_{\geq 0}U\right\|_{(\frac{1}{2}-\epsilon,(1-2\epsilon)(\frac{5}{14}-\kappa)+2\epsilon\cdot\frac{1}{4},(1-2\epsilon)(\frac{11}{14}+\kappa)+2\epsilon(-\frac{5}{8}))} \\ &\lesssim \|U\|^{1-2\epsilon}_{S_a(I)}\|U\|^{2\epsilon}_{\tilde{S}_a(I)},\end{aligned}$$

where $s = 4((1-2\epsilon)(\frac{5}{14}-\kappa)+2\epsilon\cdot\frac{1}{4}-\frac{1}{4}-\epsilon)+7\epsilon$. Next, we have

$$\begin{aligned}\left\|P^a_{\geq 0}U\right\|_{(\frac{1}{2}-\epsilon,\frac{1}{4}+3\epsilon,\frac{2}{7})} &\lesssim \left\|P^a_{\geq 0}U\right\|_{(\frac{1}{2}-\epsilon,(1-2\epsilon)(\frac{5}{14}-\kappa)+2\epsilon\cdot\frac{1}{4},s)} \\ &\lesssim \left\|P^a_{\geq 0}U\right\|_{(\frac{1}{2}-\epsilon,(1-2\epsilon)(\frac{5}{14}-\kappa)+2\epsilon\cdot\frac{1}{4},(1-2\epsilon)(\frac{11}{14}+\kappa)+2\epsilon(-\frac{5}{8}))} \\ &\lesssim \|U\|^{1-2\epsilon}_{S_a(I)}\|U\|^{2\epsilon}_{\tilde{S}_a(I)},\end{aligned}$$

where $s = 4((1-2\epsilon)(\frac{5}{14}-\kappa)+2\epsilon\cdot\frac{1}{4}-\frac{1}{4}-3\epsilon)-\frac{2}{7}$.

Now, we check the low frequency case. First,

$$\begin{aligned}\left\|P^a_{\leq 0}U\right\|_{(\frac{1}{2}-\epsilon,\frac{1}{4}-\epsilon,\epsilon)} &\lesssim \left\|P^a_{\leq 0}U\right\|_{(\frac{1}{2}-\epsilon,(1-2\epsilon)(\frac{5}{14}-\kappa)+2\epsilon\cdot\frac{1}{4},s)} \\ &\lesssim \left\|P^a_{\leq 0}U\right\|_{(\frac{1}{2}-\epsilon,(1-2\epsilon)(\frac{5}{14}-\kappa)+2\epsilon\cdot\frac{1}{4},(1-2\epsilon)(\frac{4}{7}-4\kappa))} \\ &\lesssim \|U\|^{1-2\epsilon}_{S_a(I)}\|U\|^{2\epsilon}_{\tilde{S}_a(I)},\end{aligned}$$

where $s = 4((1-2\epsilon)(\frac{5}{14}-\kappa)+2\epsilon\cdot\frac{1}{4}-\frac{1}{4}+\epsilon)+\epsilon$. Next,

$$\begin{aligned}\left\|P^a_{\leq 0}U\right\|_{L^3_tL^6_x} &\lesssim \left\|P^a_{\leq 0}U\right\|_{(\frac{1}{3},\frac{2}{3}(\frac{5}{14}-\kappa)+(\frac{1}{3}-\epsilon)\frac{1}{2}+\epsilon\cdot\frac{1}{4},s)} \\ &\lesssim \left\|P^a_{\leq 0}U\right\|_{(\frac{1}{3},\frac{2}{3}(\frac{5}{14}-\kappa)+(\frac{1}{3}-\epsilon)\frac{1}{2}+\epsilon\cdot\frac{1}{4},\frac{2}{3}(\frac{3}{7}-4\kappa))} \\ &\lesssim \|U\|^{1-2\epsilon}_{S_a(I)}\|U\|^{2\epsilon}_{\tilde{S}_a(I)},\end{aligned}$$

where $s = 4(\frac{2}{3}(\frac{5}{14}-\kappa)+(\frac{1}{3}-\epsilon)\frac{1}{2}+\epsilon\cdot\frac{1}{4}-\frac{1}{6})$. Finally

$$\begin{aligned}\left\|P^a_{\le 0}U\right\|_{(\epsilon,2\epsilon(\frac{5}{14}-\kappa)+(1-4\epsilon)\frac{1}{2},1)} &\lesssim \left\|P^a_{\le 0}U\right\|_{(\epsilon,2\epsilon(\frac{5}{14}-\kappa)+(1-4\epsilon)\frac{1}{2}+2\epsilon\cdot\frac{1}{4},s)}\\ &\lesssim \left\|P^a_{\le 0}U\right\|_{(\epsilon,2\epsilon(\frac{5}{14}-\kappa)+(1-4\epsilon)\frac{1}{2}+2\epsilon\cdot\frac{1}{4},2\epsilon(\frac{3}{7}-4\kappa))}\\ &\lesssim \|U\|^{1-2\epsilon}_{S_a(I)}\|U\|^{2\epsilon}_{\tilde{S}_a(I)},\end{aligned}$$

where $s = 2\epsilon+1$. □

We have the following control for $Z_a(I)$:

LEMMA 5.3.8. *Let $u \in L^3 \cap H^1_a$. Then,*

$$\|u\|_{\dot{B}_4^{(0|s_H)}} \lesssim \|u\|^{1-3\delta}_{L^3}\|u\|^{3\delta}_{H^1_a}. \tag{5.3.17}$$

PROOF. In the high-frequency case, using Bernstein's estimate yields

$$\|P^a_k u\|_{L^4} \lesssim 2^{\left(\frac{4}{3}-1\right)k}\|P^a_k u\|_{L^3}$$

and

$$\|P^a_k u\|_{L^4} \lesssim \|P^a_k u\|_{H^1_a}.$$

Therefore, we also obtain the bound

$$\|P^a_k u\|_{L^4} \lesssim \left(2^{\left(\frac{4}{3}-1\right)k}\|P^a_k u\|_{L^3}\right)^{1-\alpha}\|P^a_k u\|^{\alpha}_{H^1_a}.$$

If we choose $s_H+\frac{1}{3}(1-\alpha)\le 0$, we have

$$\begin{aligned}\sum_{k\ge 0}2^{2s_Hk}\|P^a_k u\|^2_{L^4} &\lesssim \sum_{k\ge 0}2^{2s_Hk}\left(2^{2\left(\frac{4}{3}-1\right)k}\|P^a_k u\|^2_{L^3}\right)^{1-\alpha}\|P^a_k u\|^{2\alpha}_{H^1_a}\\ &\lesssim \sum_{k\ge 0}2^{2\left(s_H+\frac{1}{3}(1-\alpha)\right)k}\left(\|P^a_k u\|^{1-\alpha}_{L^3}\|P^a_k u\|^{\alpha}_{H^1_a}\right)^2\\ &\lesssim \sum_{k\ge 0}\left(\|P^a_k u\|^{1-\alpha}_{L^3}\|P^a_k u\|^{\alpha}_{H^1_a}\right)^2\\ &\lesssim \left(\|u\|^{1-\alpha}_{L^3}\|u\|^{\alpha}_{H^1_a}\right)^2.\end{aligned}$$

Now, let us write $\alpha = 3\delta$ for convenience, and choose $s_H = \min\{-\frac{1}{3}+\delta, -\frac{1}{4}-\frac{3}{4}\sigma\}$. In the low-frequency case, we use the same argument. Indeed, by Bernstein estimate, we have

$$\|P^a_k u\|_{L^4} \lesssim 2^{\left(\frac{4}{3}-1\right)k}\|P^a_k u\|_{L^3} \lesssim \|P^a_k u\|_{L^3}.$$

Now, using the Sobolev embedding $H^1_a \hookrightarrow L^3$, we may conclude for any $\alpha\in[0,1]$ that

$$\|P^a_{\le 0}u\|_{\dot{B}^0_4} \lesssim \|u\|^{1-\alpha}_{L^3}\|u\|^{\alpha}_{H^1_a}.$$

□

Furthermore, we have the following estimates for the various terms of the normal form transform for $d=4$:

LEMMA 5.3.9. *Let U, U' and U'' be radial. Then we have:*

(i) *(Resonance term)*

$$\left\|\int_0^t K_a(t-s)\langle D_a\rangle^{-1}(UU')_{HH+HL+LH}\,\mathrm{d}s\right\|_{S_a(I)} \lesssim_\beta \|U\|_{\tilde{S}_a(I)}\|U'\|_{\tilde{S}_a(I)}$$

(ii) *(Boundary term) There exists $\theta > 0$ such that*

$$\|\langle D_a\rangle^{-1}\Omega(U,U')\|_{S_a(I)} \lesssim 2^{-\theta\beta}\|U\|_{S_a(I)}^{1-2\epsilon}\|U\|_{Z(I)}^{2\epsilon}\|U'\|_{S_a(I)}^{1-2\epsilon}\|U'\|_{Z(I)}^{2\epsilon}$$

(ii*) *(Refined estimate for boundary term) For $0 < \kappa \ll \epsilon \ll 1$,*

$$\|\langle D_a\rangle^{-1}\Omega(U,U')\|_{\tilde{S}_a(I)} \lesssim 2^{-\beta}\|U\|_{(0,\frac{1}{2},0|1)}\|U'\|_{\tilde{S}_a(I)} + \|U'\|_{(0,\frac{1}{2},0|1)}\|U\|_{\tilde{S}_a(I)}$$

(iii) *(Trilinear term)*

$$\left\|\int_0^t K(t-s)\langle D_a\rangle^{-1}\Omega(\langle D_a\rangle^{-1}(UU'',U')\;\mathrm{d}s\right\|_{S_a(I)} \lesssim_\beta \|U\|_{\tilde{S}_a(I)}\|U'\|_{\tilde{S}_a(I)}\|U'''\|_{\tilde{S}_a(I)}$$

PROOF. Same as in [**22**]. □

These estimates can also be used to obtain 4D small data scattering for a suitable range of a. The argument is similar to the 3D case discussed previously – see [**22**].

5.3.5. Proof of large-energy scattering (Theorem 1.1.4). We shall now prove the 4D large-energy scattering result. Similar to before, we shall only study the $-\left(\frac{d-2}{2}\right)^2 < a < 0$ case in detail because the $a > 0$ case is identical to the potential-free case. Consider the simplified equation with non-linear term U^2 so that $U(t,x)$ is equal to

$$\begin{aligned}
&K_a(t)(U_0 + i\langle D_a\rangle\Omega(U,U)(0)) - i\langle D_a\rangle\Omega(U,U)\\
&-i\int_0^{T_2-\tau_2} K_a(t-s)\langle D_a\rangle^{-1}\left((UU)_{LH+HL+HH} + 2\Omega(-i\langle D_a\rangle^{-1}U^2,U)\right)\;\mathrm{d}s\\
&-i\int_{T_2-\tau_2}^t K_a(t-s)\langle D_a\rangle^{-1}\left((UU)_{LH+HL+HH} + 2\Omega(-i\langle D_a\rangle^{-1}U^2,U)\right)\;\mathrm{d}s.
\end{aligned}$$

We shall write the two integrals above as $I_1 + I_2$. The goal here is to show that for any $\epsilon_1 > 0$, there exists T_2 such that

$$\|U\|_{\tilde{S}_a(T_2,\infty)} \lesssim C_E\epsilon_1^{\frac{9}{32}\epsilon^2}.$$

Firstly, using the Strichartz estimates, we obtain that

$$\|K_a(t)(U_0 + i\langle D_a\rangle^{-1}\Omega(U,U)(0))\|_{\tilde{S}_a(\mathbf{R})} \lesssim \|U_0\|_{H_a^1} + \|U_0\|_{H_a^1}^2.$$

Therefore, for any $\epsilon_1 > 0$, there exists $\widetilde{T} = \widetilde{T}(\epsilon_1) > 0$ such that for all $T > \widetilde{T}$, we have $\|K_a(t)(U_0 + i\langle D_a\rangle^{-1}\Omega(U,U)(0))\|_{\tilde{S}_a(T_2,\infty)} \le \epsilon_1$. Also, using the refined bound (Lemma 5.3.9 *(ii*)*), we have

$$\|\Omega(U,U)\|_{\tilde{S}_a(T_2,\infty)} \le 2^{-\beta}C_E\|U\|_{\tilde{S}_a(T_2,\infty)}.$$

Next, we consider the $(0, T_2 - \tau_2)$ integral I_1. First, we have that

$$\|I_1\|_{\tilde{S}_a(T_2,\infty)} \le \|I_1\|_{S(T_2,\infty)}^{1-2\epsilon}\|I_1\|_{(0,\frac{1}{4},0|-\frac{1}{3}+\delta)}^{2\epsilon}.$$

From the observation (5.3.15) and boundedness of the S_a norm, it remains to study the Z_a norm. In particular, we first estimate

$$\left\|\int_0^{T_2-\tau_2} K_a(t-s)\langle D_a\rangle^{-1}((UU)_{LH+HL+HH} + 2\Omega(-i\langle D_a\rangle^{-1}U^2,U))\;\mathrm{d}s\right\|_{\dot{B}_4^{\left(0|-\frac{1}{3}+\delta\right)}}$$

by frequency decomposition

$$\left\|\|\mathcal{I}_L\|_{\dot{B}_0^4}\right\|_{\ell^2_{k<0}} + \left\|\|\mathcal{I}_H\|_{\dot{B}_4^{s_H}}\right\|_{\ell^2_{k\ge 0}},$$

where

$$\mathcal{I}_L := \int_0^{T_2-\tau_2} K_a(t-s)P_k^a((UU)_{LH+HL+HH} + 2\Omega(-i\langle D_a\rangle^{-1}U^2, U))\,\mathrm{d}s,$$
$$\mathcal{I}_H := \int_0^{T_2-\tau_2} K_a(t-s)P_k^a\langle D_a\rangle^{-1}((UU)_{LH+HL+HH} + 2\Omega(-i\langle D_a\rangle^{-1}U^2, U))\,\mathrm{d}s.$$

Let us first study the low-frequency part $\left\|\|\mathcal{I}_L\|_{\dot{B}_0^4}\right\|_{\ell^2_{k<0}}$. To deal with the summability in $\ell^2_{k<0}$, we use Bernstein estimates. Thus, define $\frac{1}{q} = \frac{1}{4} + \frac{\epsilon}{4}$. We have

$$\begin{aligned}
&\left\|\int_0^{T_2-\tau_2} K_a(t-s)P_k^a((UU)_{LH+HL+HH} + 2\Omega(-i\langle D_a\rangle^{-1}U^2, U))\,\mathrm{d}s\right\|_{\dot{B}_4^0}\\
\lesssim\ & 2^{\lambda k}\left\|\int_0^{T_2-\tau_2} K_a(t-s)P_k^a((UU)_{LH+HL+HH} + 2\Omega(-i\langle D_a\rangle^{-1}U^2, U))\,\mathrm{d}s\right\|_{\dot{B}_q^0}\\
\lesssim\ & 2^{\lambda k}\left\|\int_0^{T_2-\tau_2} K_a(t-s)P_k^a((UU)_{LH+HL+HH} + 2\Omega(-i\langle D_a\rangle^{-1}U^2, U))\,\mathrm{d}s\right\|_{L^q}
\end{aligned}$$

The factor of $2^{\lambda k}$ is used to deal with the $\ell^2_{k<0}$ summation later, and so we focus on the L^q norm. Indeed, we split the norm into regions $|x| < R$ and $|x| > R$:

$$\begin{aligned}
&\left\|\int_0^{T_2-\tau_2} K_a(t-s)P_k^a((UU)_{LH+HL+HH} + 2\Omega(-i\langle D_a\rangle^{-1}U^2, U))\mathrm{d}s\right\|_{L^q_{|x|<R}}\\
&\quad+\left\|\int_0^{T_2-\tau_2} K_a(t-s)P_k^a((UU)_{LH+HL+HH} + 2\Omega(-i\langle D_a\rangle^{-1}U^2, U))\,\mathrm{d}s\right\|_{L^q_{|x|>R}}\\
\lesssim\ & \int_0^{T_2-\tau_2} \left\|K_a(t-s)P_k^a((UU)_{LH+HL+HH} + 2\Omega(-i\langle D_a\rangle^{-1}U^2, U))\right\|_{L^q_{|x|<R}}\mathrm{d}s\\
&\quad+\left\|\int_0^{T_2-\tau_2} K_a(t-s)P_k^a((UU)_{LH+HL+HH} + 2\Omega(-i\langle D_a\rangle^{-1}U^2, U))\mathrm{d}s\right\|_{L^q_{|x|>R}}.
\end{aligned}$$

For the $|x| < R$ region, we may apply the dispersive estimate with $\theta = 1/2$. Thus, we bound

$$\left\|K_a(t-s)P_k^a((UU)_{LH+HL+HH} + 2\Omega(-i\langle D_a\rangle^{-1}U^2, U))\right\|_{L^q_{|x|<R}} \lesssim I\cdot II,$$

where I is given by

$$\left\|(1+|x|^{-\sigma})^{1-\frac{2}{2q}}\right\|_{L^{2q}_{|x|<R}},$$

and II is given by

$$\left\|\left(\frac{1}{1+|x|^{-\sigma}}\right)^{1-\frac{2}{2q}} K_a(t-s)P_k^a((UU)_{LH+HL+HH} + 2\Omega(-i\langle D_a\rangle^{-1}U^2, U))\right\|_{L^{2q}}.$$

Now, $I \cdot II$ can be bounded by

$$\begin{aligned}&\left(|t-s|^{-\frac{4-1+1/2}{2}} 2^{(1-1/2-\sigma)k}\right)^{1-\frac{2}{2q}} R^{\frac{4}{2q}}\\&\quad\cdot\| \left(1+|x|^{-\sigma}\right)^{1-\frac{2}{2q}} P_k^a((UU)_{LH+HL+HH}+2\Omega(-i\langle D_a\rangle^{-1}U^2,U))\|_{L^{(2q)'}}.\end{aligned}$$

Furthermore, similar to before, for low-frequency case, we have

$$\begin{aligned}&\| \left(1+|x|^{-\sigma}\right)^{1-\frac{2}{2q}} P_k^a((UU)_{LH+HL+HH}+2\Omega(-i\langle D_a\rangle^{-1}U^2,U))\|_{L^{(2q)'}}\\&\lesssim \|P_k^a((UU)_{LH+HL+HH}+2\Omega(-i\langle D_a\rangle^{-1}U^2,U))\|_{L^{(2q)'}}.\end{aligned}$$

Thus, it suffices to study the term

$$\|P_k^a\left((UU)_{LH+HL+HH}+2\Omega(-i\langle D_a\rangle^{-1}U^2,U)\right)\|_{L^{(2q)'}}. \tag{5.3.18}$$

In the low-frequency case, since the exponent $2^{(1-1/2-\sigma)k}$ is a negative power (recall that $\sigma < 1/2$ from previous restrictions), we already have summability in $\ell^2_{k<0}$, whence it is enough to use the estimate

$$(5.3.18) \lesssim \|U\|^2_{L^{2(2q)'}} + \|U\|^3_{L^{3(2q)'}}.$$

Next, we need to deal with the tail. In this case, by using the observation (5.3.15), we have

$$\begin{aligned}&\left\|\int_0^{T_2-\tau_2} K_a(t-s)P_k^a((UU)_{LH+HL+HH}+2\Omega(-i\langle D_a\rangle^{-1}U^2,U))\,\mathrm{d}s\right\|_{L^q_{|x|>R}}\\ \lesssim\ &\left\|\int_0^{T_2-\tau_2} K_a(t-s)P_k^a((UU)_{LH+HL+HH}+2\Omega(-i\langle D_a\rangle^{-1}U^2,U))\,\mathrm{d}s\right\|^{\frac12-\frac{\epsilon}{2}}_{L^2_{|x|>R}}\\ &\cdot\left\|\int_0^{T_2-\tau_2} K_a(t-s)P_k^a((UU)_{LH+HL+HH}+2\Omega(-i\langle D_a\rangle^{-1}U^2,U))\,\mathrm{d}s\right\|^{\frac12+\frac{\epsilon}{2}}_{L^\infty_{|x|>R}}\\ \lesssim_E\ & R^{-\frac32\cdot\left(\frac12-\frac{\epsilon}{2}\right)}.\end{aligned}$$

The high-frequency part $\|\mathcal{I}_H\|_{\dot{B}_4^{s_H}}$ may be handled in a similar manner. We first split the problem into the $|x| < R$ and $|x| > R$ regions again. We first use Bernstein inequality to take out a factor of $2^{-\frac13+\delta}$. This reduces the study of the norm

$$\left\|\int_0^{T_2-\tau_2} K_a(t-s)\langle D_a\rangle^{-1}P_k^a((UU)_{LH+HL+HH}+2\Omega(-i\langle D_a\rangle^{-1}U^2,U))\mathrm{d}s\right\|_{\dot{B}_4^{s_H}}$$

to the study of the L^4 norm

$$\left\|\int_0^{T_2-\tau_2} K_a(t-s)\langle D_a\rangle^{-1}P_k^a((UU)_{LH+HL+HH}+2\Omega(-i\langle D_a\rangle^{-1}U^2,U))\mathrm{d}s\right\|_{L^4(\mathbf{R}^4)}.$$

Next, we split this L^4 norm into the bounded and tail regions $I + II$, where I is given by

$$\left\|\int_0^{T_2-\tau_2} K_a(t-s)\langle D_a\rangle^{-1}P_k^a((UU)_{LH+HL+HH}+2\Omega(-i\langle D_a\rangle^{-1}U^2,U))\,\mathrm{d}s\right\|_{L^4_{|x|<R}}$$

and II is given by

$$\left\|\int_0^{T_2-\tau_2} K_a(t-s)\langle D_a\rangle^{-1}P_k^a((UU)_{LH+HL+HH}+2\Omega(-i\langle D_a\rangle^{-1}U^2,U))\,\mathrm{d}s\right\|_{L^4_{|x|>R}}.$$

Finally, for piece I, we move the integral outside for the bounded region. Hence, we shall estimate $A+II$ where now A is given by

$$\int_0^{T_2-\tau_2}\left\|K_a(t-s)\langle D_a\rangle^{-1}P_k^a((UU)_{LH+HL+HH}+2\Omega(-i\langle D_a\rangle^{-1}U^2,U))\right\|_{L^4_{|x|<R}}\,\mathrm{d}s.$$

Once again, we apply the dispersive estimate for the $|x|<R$ region (i.e. piece A) with $\theta=1/2$

$$\begin{aligned}
&\left\|K_a(t-s)\langle D_a\rangle^{-1}P_k^a((UU)_{LH+HL+HH}+2\Omega(-i\langle D_a\rangle^{-1}U^2,U))\right\|_{L^4_{|x|<R}}\\
\lesssim\;& R^{\frac{4}{8}}\left(|t-s|^{-\frac{4-1+1/2}{2}}2^{\frac{4+1+1/2}{2}k}\right)^{1-\frac{2}{8}}\\
&\cdot\left\|\left(1+|x|^{-\sigma}\right)^{\frac{3}{4}}\langle D_a\rangle^{-1}P_k^a((UU)_{LH+HL+HH}+2\Omega(-i\langle D_a\rangle^{-1}U^2,U))\right\|_{L^{8/7}}\\
\lesssim\;& 2^{\frac{3}{4}\sigma}R^{\frac{4}{8}}\left(|t-s|^{-\frac{4-1+1/2}{2}}2^{\frac{4+1+1/2}{2}k}\right)^{1-\frac{2}{8}}\\
&\cdot\left\|\langle D_a\rangle^{-1}P_k^a((UU)_{LH+HL+HH}+2\Omega(-i\langle D_a\rangle^{-1}U^2,U))\right\|_{L^{8/7}}
\end{aligned}$$

For the tail (piece II), again we use the radial Sobolev embedding and the observation (5.3.15). Indeed, we bound

$$\left\|\int_0^{T_2-\tau_2}\langle D_a\rangle^{-1}K_a(t-s)P_k^a((UU)_{LH+HL+HH}+2\Omega(-i\langle D_a\rangle^{-1}U^2,U))\mathrm{d}s\right\|_{L^4_{|x|>R}}$$

by $X\cdot Y$ where X is given by

$$\left\|\int_0^{T_2-\tau_2}\langle D_a\rangle^{-1}K_a(t-s)P_k^a((UU)_{LH+HL+HH}+2\Omega(-i\langle D_a\rangle^{-1}U^2,U))\mathrm{d}s\right\|_{L^2_{|x|>R}}^{\frac{1}{2}}$$

and Y is given by

$$\left\|\int_0^{T_2-\tau_2}K_a(t-s)P_k^a((UU)_{LH+HL+HH}+2\Omega(-i\langle D_a\rangle^{-1}U^2,U))\mathrm{d}s\right\|_{L^\infty_{|x|>R}}^{\frac{1}{2}}.$$

Now, $X\cdot Y$ is bounded (up to a constant that depends on the energy E) by $2^{s_Hk}R^{-\frac{3}{4}}$ (recall that $s_H=-\min\{-\frac{1}{3}+\delta,-\frac{1}{4}-\frac{3}{4}\sigma\}$).

In the high-frequency case, since the exponent $2^{\frac{33k}{16}}2^{\frac{3}{4}\sigma}2^{s_Hk}2^{-k}$ is not a negative power, we need to use a more refined estimate for this $L^{8/7}$ norm. First, we note that

$$P_k^a\Omega(-i\langle D_a\rangle^{-1}U^2,U))=0$$

for $\beta>0$ sufficiently large. By Bernstein inequality, we also obtain the bilinear estimate

$$\max_{k\geq 0}2^{\frac{7}{8}k}\|P_k^a((UU)_{HH+HL+LH})\|_{L^{8/7}}\lesssim\|P_{\geq 0}U\|_{H^1_a}\|U\|_{L^2}+\|P_{\geq 0}U\|^2_{H^{1/2}}\leq C_E.$$

Indeed, s_H was chosen so that $\frac{33}{16}+\frac{3}{4}\sigma+s_H-1-\frac{7}{8}<0$. Thus, putting everything together, and choosing $R=\tau_1^{1/8}$, we find that

$$\begin{aligned}\|I_1\|_{(0,\frac14,0|s_H} &\lesssim_E R^{\frac12+\frac{\epsilon}{2}}\int_0^{T_2-\tau_2}|t-s|^{-\frac74(\frac34-\frac{\epsilon}{4})}\,\mathrm{d}s+R^{-\frac34+\frac34\epsilon}\\ &\quad+R^{\frac12}\int_0^{T_2-\tau_2}|t-s|^{-\frac74\cdot\frac34}\,\mathrm{d}s+R^{-\frac34}\\ &\lesssim_E \tau_2^{-\frac{3}{32}+\frac{3}{32}\epsilon}.\end{aligned}$$

Therefore,

$$\|I_1\|_{\tilde{S}_a(T_2,\infty)} \lesssim_E \tau_2^{-\frac{3}{16}\epsilon+\frac{3}{16}\epsilon^2}.$$

Now, we study I_2. Using radial Strichartz estimate, the Sobolev embedding $H_a^1\hookrightarrow L^4$ and the variational result $\|U\|^2_{L_t^\infty H_a^1}\sim E$, we have that for any interval $I\subset R$,

$$\|U\|_{S_a(I)}\le C_E+C\|u^2\|_{L_t^1L_x^2(I\times\mathbf{R}^4)}\le C_E\langle|I|\rangle.$$

Note that by Proposition 5.3.5, for $\epsilon>0$ and $\widetilde{T}$ above, there exists $\tilde\tau_1=C_E\epsilon_1^{-9}$ and T_2 such that

$$\|U\|_{L_t^\infty(T_2-\tilde\tau_1,T_2;L_x^3)}\le\epsilon_1.$$

Thus, let $\tau_2=\epsilon_1^{-3\epsilon/2}$. In this case, we have $[T_2-\tau_2]\subset[T_2-\tilde\tau_1,T_2]$ and

$$\begin{aligned}\|I_2\|_{\tilde{S}_a(T_2,\infty)} &\le C_E\left(\|U\|^{2-4\epsilon}_{S(T_2-\tau_2,t)}\|U\|^{4\epsilon}_{Z(T_2-\tau_2,t)}+\|U\|^{3-6\epsilon}_{S(T_2-\tau_2,t)}\|U\|^{6\epsilon}_{Z(T_2-\tau_2,t)}\right)\\ &\le C_E\left(\langle\tau_2\rangle^{2-4\epsilon}\|U\|^{4\epsilon(1-3\delta)}_{L_t^4(T_2-\tau_2,T_2;L_x^3)}+\langle\tau_2\rangle^{3-6\epsilon}\|U\|^{6\epsilon(1-3\delta)}_{L_t^4(T_2-\tau_2,t;L_x^3)}\right)\\ &\le C_E\left(\langle\tau_2\rangle^{2-4\epsilon}\epsilon_1^{4\epsilon(1-3\delta)}+\langle\tau_2\rangle^{3-6\epsilon}\epsilon_1^{6\epsilon(1-3\delta)}\right)\\ &\le C_E\epsilon^{2\epsilon}.\end{aligned}$$

Thus, we have

$$\|U\|_{\tilde{S}_a(T_2,\infty)}\le C_E\left(\epsilon_1+2^{-\beta}\|U\|_{\tilde{S}_a(T_2,\infty)}+\epsilon_1^{\frac{9}{32}\epsilon^2}\right).$$

Now, we apply a bootstrap argument to show that for $T_2=T_2(\epsilon_1)$,

$$\|U\|_{\tilde{S}_a(T_2,\infty)}\le C_E\epsilon_1^{\frac{9}{32}\epsilon^2}$$

Thus, we have $\|U\|_{\tilde{S}_a(0,\infty)}\le C_E$ for some constant C_E.

Finally, we can prove large-energy scattering. The argument is the same as in [**22**]. Indeed, the goal is now to show that $K_a(-t)U(t)$ has a limit in H_a^1 as $t\to\infty$, which shall be done by verifying that the sequence is Cauchy. First, we write $K_a(-t)U(t)$ explicitly as

$$\begin{aligned}K_a(-t)U(t) &= U_0+i\langle D_a\rangle^{-1}\Omega(U,U)(0)-iK_a(-t)\langle D_a\rangle^{-1}\Omega(U,U)\\ &\quad -i\int_0^t K(-s)\langle D_a\rangle^{-1}T_{\mathrm{Res}}(U,U)\,\mathrm{d}s\\ &\quad -2i\int_0^t K(-s)\langle D_a\rangle^{-1}\left(\Omega\left(-i\langle D_a\rangle^{-1}U^2,U\right)\right)\,\mathrm{d}s.\end{aligned}$$

First, using Strichartz estimates, we obtain

$$\left\|\int_{t_1}^{t_2}K(-s)\langle D_a\rangle^{-1}T_{\mathrm{Res}}(U,U)\,\mathrm{d}s\right\|_{H_a^1}\lesssim\|U\|^2_{\tilde{S}_a(t_1,t_2)}$$

and

$$\left\| \int_{t_1}^{t_2} K(-s)\langle D_a\rangle^{-1} \left(\Omega\left(-i\langle D_a\rangle^{-1}U^2, U\right)\right)\,\mathrm{d}s \right\|_{H_a^1} \lesssim \|U\|_{\tilde{S}_a(t_1,t_2)}^3.$$

Thus, we conclude that

$$-i\int_0^t K(-s)\langle D_a\rangle^{-1} T_{\mathrm{Res}}(U,U)\,\mathrm{d}s - 2i\int_0^t K(-s)\langle D_a\rangle^{-1}\left(\Omega\left(-i\langle D_a\rangle^{-1}U^2, U\right)\right)\,\mathrm{d}s.$$

has a limit in H_a^1. Finally, we claim that $K_a(-t)\langle D_a\rangle^{-1}\Omega(U,U) \to 0$ in H_a^1 as $t \to \pm\infty$, from which we can obtain scattering. First, by Bernstein estimates, we have that

$$\|\Omega(U,U)\|_{H_a^1} \lesssim \|P_{\le 0}^a U\|_{L^2} \|P_{\le 0}^a U\|_{L^6}.$$

Thus, it remains to show that $\lim_{t\to\pm\infty} \|P_{\le 0}^a U(t)\|_{L_x^6} = 0$. This follows from the fact that $\|P_{\le 0}^a U(t)\|_{L_x^6}$ is Lipschitz continuous in t (see [**22**] for details). Finally, this verifies the scattering result Theorem 1.1.4.

CHAPTER 6

NLS and NLKG with exponential nonlinearity and inverse-square potential

In this chapter, we study scattering for the 2D non-linear Schrödinger (NLS) and non-linear Klein-Gordon (NLKG) with inverse-square potential and with exponential-type non-linearity:

$$
\begin{cases} i\partial_t u - \Delta u + \frac{a}{|x|^2} u = f(u) \\ u(0,x) = u_0(x) \end{cases} \tag{6.0.1}
$$

and

$$
\begin{cases} \partial_t^2 u - \Delta u + \frac{a}{|x|^2} u + u = f(u) \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x) \end{cases} \tag{6.0.2}
$$

where $u : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{C}$, $f(u) := \lambda \left(e^{\kappa_0 |u|^2} - 1 - \kappa_0 |u|^2 \right) u$, $\kappa_0 > 0$, $\lambda \in \{1, -1\}$ and $a \geq -\left(\frac{d-2}{2}\right)^2$. Throughout this chapter, we take $d = 2$ unless otherwise specified.

We shall study various settings in which global solutions to (6.0.1) and (6.0.2) respectively approach solutions to the free NLS equation

$$
i\partial_t u - \Delta u + \frac{a}{|x|^2} u = 0, \tag{6.0.3}
$$

and free NLKG equation

$$
\partial_t^2 u - \Delta u + \frac{a}{|x|^2} u + u = 0. \tag{6.0.4}
$$

as $t \to \infty$. Define $F(u) : \mathbf{C} \to \mathbf{R}$ so that $F(0) = 0$ and $\partial_{\overline{u}} F(u) = f(u)$. More explicitly, we have

$$
F(u) = \frac{\lambda}{\kappa_0} \left(e^{\kappa_0 |u|^2} - 1 - \kappa_0 |u|^2 - \frac{\kappa_0^2}{2} |u|^4 \right). \tag{6.0.5}
$$

Then, the NLS (6.0.1) has conserved energy

$$
E_S(u(t)) = \frac{1}{2} \int_{\mathbf{R}^2} |\nabla u(t,x)|^2 + \frac{a}{|x|^2} |u|^2 - F(u(t,x)) \, \mathrm{d}x
$$

and mass

$$
M(u(t)) = \int_{\mathbf{R}^2} |u(t,x)|^2 \, \mathrm{d}x,
$$

while the NLKG (6.0.2) has conserved energy

$$
E_{KG,a}(u(t)) = \frac{1}{2} \int_{\mathbf{R}^2} |\nabla u(t,x)|^2 + \frac{a}{|x|^2} |u|^2 + |u(t,x)|^2 + |u_t(t,x)|^2 - F(u(t,x)) \, \mathrm{d}x.
$$

We shall omit the dependence on the coefficient a for ease of notation.

In the higher dimensional setting (i.e. $d \geq 3$), the analysis adapted to the operator $\mathscr{L}_a$ is well understood and has been used to obtain scattering results for dispersive PDEs with inverse-square potential, as we have seen previously.

For $d \geq 3$ the equivalence of fractional Sobolev norms $\|(-\Delta)^s u\|_p \sim \|(\mathscr{L}_a)^s u\|_p$ for a suitable range of values s, p and a is a central tool. This equivalence allows us to use the fractional chain rule result which holds for $(-\Delta)^s$ by switching norms when required. Heat kernel bounds for $\mathscr{L}_a$ available for $d \geq 3$ were used to obtain this result. We remark that there were earlier results in some special cases. Indeed, the sharp Hardy inequality

$$\int_{\mathbf{R}^d} \frac{1}{|x|^2}|u|^2 \, \mathrm{d}x \leq \left(\frac{2}{d-2}\right)^2 \int |\nabla u|^2 \, \mathrm{d}x \tag{6.0.6}$$

can be used to obtain this equivalence for $p = 2$ and $s = 1$ (see Burq–Planchon–Stalker–Tahvildar-Zadeh [**6**]). In the case $s = 1$, this equivalence corresponds to the L^p boundness of the Riesz transform $(-\Delta)\mathscr{L}_a^{-1}$. Notice that in the free case (i.e. when $a = 0$), the boundedness of the Riesz transform holds for $p \in (1, \infty)$, and yet if we were to take the limit $|a| \to 0$, one would find that the limiting range is $(1, d)$. Indeed, this was remarked by Hassel (see Zhang-Zheng [**66**]).

The two-dimensional setting is somewhat different. In this setting, it is natural to restrict to $a \geq 0$. We notice that in this setting, the Hardy inequality (6.0.6) breaks down. From this, we immediately can see that $H_a^1(\mathbf{R}^2)$ is strictly smaller than $H^1(\mathbf{R}^2)$ for $a \neq 0$. A Hardy-type inequality can be recovered in $d = 2$ if we restrict to functions orthogonal to radial functions, as noted in [**6**]. The equivalence can be recovered for a restricted range of $p \in (1, \infty)$ via the $W^{s,p}$ boundedness of conjugation operators, for which results are known in dimension two (see below). Time-decay and Strichartz estimates for the Schrödinger propagator are well-established. For the Klein-Gordon propagator, similar estimates are available, albeit in frequency-localised pieces (localised with respect to $\mathscr{L}_a$) which need to be carefully put back together.

We also review work on the exponential-type non-linearity. Indeed, the (defocusing) NLS with power nonlinearity (i.e. $i\partial_t - \Delta u = |u|^p u$) is energy subcritical for all $p > 1$. Colliander-Ibrahim-Majdoub-Masmoudi [**11**] identified the NLS with exponential-type non-linearity as being *the* energy critical problem in 2D. Here, the notion of energy criticality is given in terms of a well-posedness/ill-posedness trichotomy with respect to the Hamiltonian. Prior to this result, Nakamura-Ozawa [**45**] had obtained small energy global-well posedness and scattering for the NLS problem. We also mention the work of Cazenave [**8**] for decreasing exponential-type non-linearities. For increasing exponential-type non-linearities, the failure of the embedding $H^1(\mathbf{R}^2) \hookrightarrow L^\infty(\mathbf{R}^2)$ means that we still need a growth condition. Hence, the non-embedding can be replaced by the Moser-Trudinger inequality to give such a condition.

Scattering with respect to an energy trichotomy was subsequently studied for the NLS and NLKG by Ibrahim-Majdoub-Masmoudi-Nakanishi [**29**]. Guo-Shen [**23**] revisited scattering for 2D NLS and NLKG by extending the methods of Dodson-Murphy to the two-dimensional setting. Furthermore, by using the radially refined Strichartz estimates for the NLKG combined with $L^6_{t,x}$ smallness of the solution, Guo-Shen were able to give a simpler proof. Their proof is similar to the NLS case of Ibrahim et al., who used the $L_t^4 L_x^8$ smallness of the solution established independently by Planchon-Vega [**51**] and Colliander-Grillakis-Tzikaris [**10**].

In this chapter, we obtain the following result:

THEOREM 6.0.1. *Suppose that* $u_0 \in H^1_a(\mathbf{R}^2)$, (α, β) *satisfies conditions (6.3.1),* $m_{\alpha,\beta}$ *is defined by (6.3.7) and* κ^*_a *is a constant defined in Proposition 6.1.8 below. Recall also* $\kappa_0 > 0$ *and* $f(u) := \lambda\left(e^{\kappa_0|u|^2} - 1 - \kappa_0|u|^2\right)u$. *Then,*

(a) *If* $\lambda = -1$, *the solution to (6.0.1) exists globally and scatters provided* $E_S(u_0) < \frac{\kappa^*_a}{2\kappa_0}$.
(b) *If* $\lambda = 1$, *the solution to (6.0.1) exists globally and scatters provided that* $E_S(u_0) + M(u)/2 < m_{\alpha,\beta}$ *and* $K_{\alpha,\beta}(u_0) > 0$, *and* $a > 1$ *or sufficiently close to zero.*
(c) *If* $\lambda = -1$, *the solution to (6.0.2) exists globally and scatters provided* $E_K(u_0) < \frac{\kappa^*_a}{2\kappa_0}$.
(d) *If* $\lambda = 1$, *the solution to (6.0.2) exists globally and scatters provided that* $E_K(u_0) < m_{\alpha,\beta}$ *and* $K_{\alpha,\beta}(u_0) > 0$ *for* $a > 1$ *or sufficiently close to zero.*

The restriction in the coefficients a of the inverse-square potential comes from the proof of scattering and the L^p theory as seen above. Indeed, for $a > 1$, we are able to use the method of Guo-Shen [**23**] in combination with the L^p continuity of conjugation operators. In particular, we are able to avoid using the double logarithmic inequality to obtain scattering, as is needed in papers such as [**29**] and [**30**]. For smaller values of a, we need to use the double logarithmic inequality. We did not pursue the optimising the double logarithmic inequality for the inverse-square potential. The issues are similar to characterising the threshold for the Moser-Trudinger equation. For instance, rearrangement techniques cannot be applied in this setting (recall that $a \geq 0$ in the 2D context). As a consequence, we need to stay close to the potential-free case due to the requirements in the double logarithmic inequality. If the double logarithmic inequality was optimised, then we could deal with the remaining values for the coefficient using the same argument.

6.1. Preliminaries

6.1.1. Inverse square potential in 2D. We first review some important estimates related to the operator $\mathscr{L}_a$ in two dimensions. In particular we have the following estimates for the heat kernel, Riesz kernel and Littlewood-Paley theory. These results are analogous to the higher dimensional setting (i.e. $d \geq 3$) as discussed above, as well as in the literature.

LEMMA 6.1.1 (Heat kernel bounds in 2D, [**31**]). *Let* $a \geq 0$. *Then, there exists constants* $c, C > 0$ *such that for* $x, y \in \mathbf{R}^2\backslash\{0\}$,

$$0 \leq e^{-t\mathscr{L}_a}(x,y) \leq C\left(1 \vee \frac{\sqrt{t}}{|x|}\right)^{\sigma}\left(1 \vee \frac{\sqrt{t}}{|y|}\right)^{\sigma} t^{-1} e^{-\frac{|x-y|^2}{ct}}. \tag{6.1.1}$$

The following Riesz kernel estimates and Littlewood-Paley theory can then be obtained using the above heat kernel bound as in [**33**]:

LEMMA 6.1.2 (Riesz kernel). *Let* $x, y \in \mathbf{R}^2\backslash\{0\}$, $s \in (0,2)$ *and* $2 - s - 2\sigma > 0$. *Then, the Riesz kernel satisfies*

$$\mathscr{L}_a^{-s/2}(x,y) := \frac{1}{\Gamma(s/2)}\int_0^\infty e^{-t\mathscr{L}_a}(x,y)t^{s/2}\frac{\mathrm{d}t}{t} \leq |x-y|^{s-2}\left(\frac{|x|}{|x-y|} \wedge \frac{|y|}{|x-y|} \wedge 1\right)^{-\sigma}.$$

LEMMA 6.1.3 (Littlewood-Paley theory in 2D). *Let $a \geq 0$, $1 < p \leq q \leq \infty$, $s \in \mathbf{R}$ and $k \in \mathbf{Z}$. Define the Littlewood-Paley operators $P^a_{\leq k}$, P^a_k and $P^a_{>k}$ as in Chapter 2. Then,*

(a) *$P^a_{\leq k}$ and P^a_k are bounded on L^p,*

(b) *$P^a_{\leq k}$ and P^a_k are bounded from L^p to L^q with norm $O\left(2^{k\left(\frac{2}{p}-\frac{2}{q}\right)}\right)$, and*

(c) $2^{ks}\left\|P^a_k f\right\|_{L^p} \sim \left\|\mathscr{L}_a^{s/2} P^a_k f\right\|_{L^p}$.

6.1.2. Boundedness of conjugation operators. We define the conjugation operator $\mathcal{K}_{\nu,\mu} := \mathcal{H}_\nu \mathcal{H}_\mu$ and also its inverse $\mathcal{K}_{\mu,\nu}$ as in [**49**]. We have that $A_\nu \mathcal{K}_{\nu,\mu} = \mathcal{K}_{\nu,\mu} A_\mu$. The continuity of these operators on $\dot{W}^{s,p}$ (that is, $\|\mathcal{K}_{\nu,\mu} u\|_{\dot{W}^{s,p}} \lesssim \|u\|_{\dot{W}^{s,p}}$) will be important for our purposes. We shall also need similar results for the exchange operator $B^s_{\mu,\nu} := A_\mu^{s/2} A_\nu^{-s/2}$ on L^p_{rad}. Indeed, from [**49**], we have the following result:

PROPOSITION 6.1.4. *Let the operators $K_{\mu,\nu}$ and $B^s_{\mu,\nu}$ be as defined above. We have that*

(a) *The conjugation operator $\mathcal{K}_{\mu,\nu}$ is continuous on L^p_{rad} if*

$$\max\left\{\frac{\lambda-\mu}{d}, 0\right\} < \frac{1}{p} < \min\left\{\frac{\lambda+2+\mu}{d}, 1\right\}. \tag{6.1.2}$$

(b) *The conjugation operator $\mathcal{K}_{\lambda,\nu}$ is continuous on $\dot{W}^{s,p}_{\mathrm{rad}}$ if*

$$\max\left\{0, \frac{\lambda-\nu}{d}, \frac{s}{d}\right\} < \frac{1}{p} < \min\left\{\frac{\lambda+\nu+2}{d}, \frac{\lambda+\nu+2+s}{d}, 1\right\}, \tag{6.1.3}$$

while its inverse $\mathcal{K}_{\nu,\lambda}$ is continuous on $\dot{W}^{s,p}_{\mathrm{rad}}$ if

$$\max\left\{0, \frac{\lambda-\nu}{d}, \frac{\lambda-\nu+s}{d}\right\} < \frac{1}{p} < \min\left\{\frac{\lambda+\nu+2}{d}, 1, 1+\frac{s}{d}\right\}. \tag{6.1.4}$$

(c) *The exchange operator $B^s_{\mu,\nu}$ is continuous on L^p if*

$$\max\{\lambda-\nu+s, \lambda-\mu\} < \frac{d}{p} < \min\{\lambda+\mu+2+s, \lambda+\nu+2\}. \tag{6.1.5}$$

6.1.3. Strichartz estimates. We have the following Strichartz estimates in the potential-free case. In previous chapters, we used Besov-type spaces in order to study the quadratic non-linearity. In contrast, we shall use Sobolev spaces instead. This means that we may use the conjugation operators above to obtain the corresponding results in the inverse-square potential case. This shall be the approach that we use for this chapter.

PROPOSITION 6.1.5 (Strichartz estimates, [**37**]). *We have*

(a) *If (q,r) satisfies $2 \leq q, r \leq \infty, (q,r) \neq (2,\infty)$ and $\frac{1}{q}+\frac{1}{r}=\frac{1}{2}$. Then, we have*

$$\left\|e^{it\Delta}u_0\right\|_{L^q_t L^r_x} \lesssim \|u_0\|_{L^2}.$$

(b) *We also have for (q,r) satisfying $2 \leq q, r \leq \infty, (q,r) \neq (2,\infty)$ that*

$$\left\|e^{it\langle\nabla\rangle}u_0\right\|_{L^q_t B^{\beta(q,r)}_{r,2}} \lesssim \|u_0\|_{L^2}.$$

where

$$\beta(q,r) = \begin{cases} -\frac{1}{q} + \frac{1}{r} - \frac{1}{2}, & \frac{d-1}{2}(\frac{1}{2} - \frac{1}{r}) \le \frac{1}{q} \le \frac{d}{2}(\frac{1}{2} - \frac{1}{r}) \\ \frac{1}{q} + \frac{d}{r} - \frac{d}{2}, & \frac{1}{q} \le \frac{d-1}{2}(\frac{1}{2} - \frac{1}{r}) \end{cases} .$$

and $B^s_{r,2}$ is the standard Besov space.

6.1.4. Logarithmic estimates. We shall also need the following logarithmic inequality from [**28**] in our study of (6.0.1) and (6.0.2). First, we define the following spaces for $0 < \alpha < 1$ and $0 < \mu \le 1$:

$$\|u\|_{\mathcal{C}^\alpha} := \sup_{x \ne y} \frac{|u(x) - u(y)|}{|x-y|^\alpha}, \tag{6.1.6}$$

and

$$\|u\|_{H^1_{\mu,a}} := \|u\|^2_{H^1_a} + \mu \|u\|^2_2. \tag{6.1.7}$$

Proposition 6.1.6 (Logarithmic estimates for $H^1_a(\mathbf{R}^2)$, [**28**]). *Let $0 < \alpha < 1$, $\lambda > \frac{1}{2\pi\alpha}$ and $0 < \mu \le 1$. There exists a constant $C_\lambda > 0$ depending on λ such that for any function $u \in H^1_a(\mathbf{R}^2) \cap \mathcal{C}^\alpha(\mathbf{R}^2)$ we have*

$$\|u\|^2_{L^\infty} \le \lambda \|u\|^2_{H^1_{\mu,a}} \log\left(C_\lambda + \frac{8^\alpha \mu^{-\alpha} \|u\|_{\mathcal{C}^\alpha}}{\|u\|_{H^1_{\mu,a}}} \right).$$

Proof. This follows from the estimate in the $a = 0$ case from [**28**], the fact that $H^1_a(\mathbf{R}^2) \hookrightarrow H^1(\mathbf{R}^2)$, and the fact that $x \mapsto x^2 \ln(C_\lambda + c/x)$ is increasing. □

6.1.5. Moser-Trudinger inequality. We now discuss the Moser-Trudinger inequality. In the radial setting for the inverse-square potential, we shall see that there is an improvement to the threshold (see below). This improvement is a corollary of the equivalence between the Moser-Trudinger inequality and the Galgliardo-Nirenberg inequality as observed by Ozawa [**47**]. Thus, we shall first state some results related to the Gagliaro-Nirenberg inequality. Here, the Gagliardo-Nirenberg inequality is written with explict reference to the growth rate.

Proposition 6.1.7 (Gagliardo-Nirenberg estimate for $H^1_a(\mathbf{R}^2)$. *Let $q \in (2, \infty)$ and $a \ge 0$.*

(a) *There exists a constant $C > 0$ independent of q such that*

$$\|u\|_{L^q_x} \le C q^{\frac{1}{2}} \|u\|^{1-\frac{2}{q}}_{\dot{H}^1_a} \|u\|^{\frac{2}{q}}_{L^2_x}. \tag{6.1.8}$$

(b) *Furthermore, the best constant in (6.1.8) given by*

$$C_{\mathrm{GN},a} := \sup \left\{ \frac{\|u\|_q}{q^{\frac{1}{2}} \|u\|^{1-\frac{2}{q}}_{\dot{H}^1_a} \|u\|^{\frac{2}{q}}_2} : u \in H^1_a(\mathbf{R}^2) \backslash \{0\} \right\} \tag{6.1.9}$$

is equal to $C_{\mathrm{GN},0}$ and is only attained for $a = 0$. In this case, the best constant is attained by a radial solution of

$$-\Delta Q_0 + Q_0 = -Q_0^{q-1}. \tag{6.1.10}$$

(c) *In the radial setting, the best constant* $C_{\mathrm{GN},a,\mathrm{rad}}$ *defined as in (6.1.9) restricted to* $H^1_{a,\mathrm{rad}}$ *is attained by radial solution of*

$$\left(-\Delta + \frac{a}{|x|^2}\right) Q_{a,\mathrm{rad}} + Q_{a,\mathrm{rad}} = -Q^{q-1}_{a,\mathrm{rad}}. \tag{6.1.11}$$

Furthermore, $C_{\mathrm{GN},a,\mathrm{rad}} < C_{\mathrm{GN},a}$.

Proof. The proof is exactly as in [**13**, Thereom 4.1]. □

Proposition 6.1.8 (Moser-Trudinger estimate for $H^1_a(\mathbf{R}^2)$). *Let* $a \geq 0$.

(a) *For all* $\kappa \leq 4\pi$, *we have*

$$\sup_{\|u\|_{H^1_a(\mathbf{R}^2)} \leq 1} \int_{\mathbf{R}^2} \exp\left(\kappa|u|^2\right) - 1 \,\mathrm{d}x < c. \tag{6.1.12}$$

Furthermore, this threshold is sharp in the sense that for all $\kappa > 4\pi$, *there exists a sequence of functions* $(u_n) \subset H^1_a(\mathbf{R}^2)$ *such that* $\|u_n\|_{H^1_a} \leq 1$ *and*

$$\int_{\mathbf{R}^2} \exp(\kappa|u_n|^2) - 1 \,\mathrm{d}x \to \infty$$

as $n \to \infty$.

(b) *For all* $\kappa \leq \kappa^*_a$, *we have*

$$\sup_{\|u\|_{H^1_{a,\mathrm{rad}}(\mathbf{R}^2)} \leq 1} \int_{\mathbf{R}^2} \exp\left(\kappa|u|^2\right) - 1 \,\mathrm{d}x < c(a). \tag{6.1.13}$$

Furthermore, this threshold is sharp in the above sense, except with a radial sequence.

Proof. The estimate (6.1.12) follows from the embedding $H^1_a \hookrightarrow H^1$ and the corresponding Moser-Trudinger inequality in the $H^1(\mathbf{R}^2)$ case (see Ruf [**52**]). All that remains to show is the sharpness of the thresholds in the two respective settings (non-radial and radial). This follows from the equivalence of the Gagliardo-Nirenberg and Moser-Trudinger inequalities (see below). □

Let us consider the radial setting, as this will be the setting for which we shall study the problems (6.0.1) and (6.0.2). There are many equivalent forms of the Moser-Trudinger inequality. First, we consider the form discussed in [**47**]. The assumption that $\|u\|_{\dot{H}^1_a} \leq 1$ can be removed by replacing u with $u/\|u\|_{\dot{H}^1_a}$. In this case, the Moser-Trudinger inequality in the form

$$\int_{\mathbf{R}^2} \exp\left(\kappa^*_a|u|^2\right) - 1 \,\mathrm{d}x \leq c\|u\|_2^2 \tag{6.1.14}$$

is equivalent to

$$\int_{\mathbf{R}^2} \exp\left(\frac{\kappa^*_a|u|^2}{\|u\|^2_{\dot{H}^1_a}}\right) - 1 \,\mathrm{d}x \leq \frac{c\|u\|_2^2}{\|u\|^2_{\dot{H}^1_a}}. \tag{6.1.15}$$

Furthermore, the reverse implication (i.e. showing that (6.1.15) implies (6.1.14)) is non-trivial as noted by Ozawa [**47**], and can also be proved via the equivalence with the Gagliardo-Nirenberg inequality, which we discuss below.

We also mention the following form of the Moser-Trudinger inequality for $u \in H_a^1$ such that $\|u\|_{\dot{H}_a^1} < 1$ given by

$$\int_{\mathbf{R}^2} \exp\left(|u|^2\right) - 1 \, \mathrm{d}x \lesssim \frac{\|u\|_2^2}{\kappa_a^* - \|u\|_{\dot{H}_a^1}^2}. \tag{6.1.16}$$

The fact that (6.1.16) implies (6.1.12) follows immediately when we apply the condition that $\|u\|_{H_a^1} \leq 1$. Now, to prove the opposite direction, we shall prove something more general. We follow the proof from [**23**]. Indeed, let $c \geq 1$. Then, we have that

$$\sup_{\|\phi\|_{H_a^1}^2 \leq 1} \int_{\mathbf{R}^2} \left(e^{\kappa_a^* c^{-1}|\phi|^2} - 1\right)^c \, \mathrm{d}x \leq C. \tag{6.1.17}$$

Let $u \in H_a^1$ such that $\|u\|_{\dot{H}_a^1} < \sqrt{\kappa_a^*/c}$. We note that for $u_\lambda(x) := u(\lambda x)$, we have

$$\|u_\lambda\|_{H_a^1}^2 = \|u_\lambda\|_{\dot{H}_a^1}^2 + \lambda^{-2}\|u_\lambda\|_2^2.$$

Hence, we may choose $\lambda > 0$ such that $\|u_\lambda\|_{H_a^1}^2 = \kappa_a^*/c$. Then, applying (6.1.17) to $\phi = u_\lambda/\sqrt{\kappa_a^* c^{-1}}$, we obtain

$$\int_{\mathbf{R}^2} \left(e^{|u_\lambda(x)|^2} - 1\right)^c \, \mathrm{d}x \leq C.$$

Finally, after a change of variables, we use the choice of λ to obtain

$$\int_{\mathbf{R}^2} \left(\exp\left(|u(x)|^2\right) - 1\right)^c \, \mathrm{d}x \lesssim \lambda^2 = \frac{\|\phi_\lambda\|^2}{\kappa_a^* c^{-1} - \|\phi\|_{\dot{H}_a^1}^2}.$$

Proposition 6.1.9 (Equivalence of Gagliardo-Nirenberg and Moser-Trudinger inequalities, [**47**]). *Define the following optimal constants*

$$\begin{aligned}
\kappa_a &:= \sup\left\{\kappa : \sup_{\|u\|_{\dot{H}_a^1(\mathbf{R}^2)} \leq 1} \int_{\mathbf{R}^2} \exp\left(\kappa|u|^2\right) - 1 \, \mathrm{d}x < C\|u\|_2^2 < \infty\right\}, \\
M_0 &:= \inf\left\{M : \exists\, r = r(M) \text{ s.t. (6.1.8) holds for all } u \in H_a^1(\mathbf{R}^2) \text{ and } r \leq q < \infty\right\} \\
\beta_0 &:= \limsup_{q \to \infty} \frac{\|u\|_q}{q^{\frac{1}{2}} \|u\|_{\dot{H}_a^1}^{1-\frac{2}{q}} \|u\|_2^{\frac{2}{q}}},
\end{aligned}$$

Then, the estimates (6.1.8) and (6.1.12) are equivalent and $1/\kappa_a = 2eM_0^2 = 2e\beta_0^2$.

Proof. Since $\beta_0 \leq M_0$, it suffices to prove that (6.1.8) implies (6.1.12) with $1/\kappa_a \leq 2e\beta_0^2$ and also to prove that (6.1.12) implies (6.1.8) with $1/\kappa_a \geq 2eM_0^2$. We first show that (6.1.8) implies (6.1.12). Using the Taylor series expansion, the fact that $\|u\|_{\dot{H}_a^1} \leq 1$ and the monotone convergence theorem, we obtain for any $\epsilon > 0$ that

$$\begin{aligned}
\int_{\mathbf{R}^2} \exp\left(\kappa|u|^2\right) - 1 \, \mathrm{d}x &= \sum_{j \geq 1} \frac{\kappa^j}{j!} \|u\|_{2j}^{2j} \\
&\leq \sum_{j \geq 1} \frac{\kappa^j}{j!} (\beta_0 + \epsilon)^{2j} (2j)^j \|u\|_2^2 \\
&\leq \sum_{j \geq 1} \frac{j^j}{j!} (2(\beta_0 + \epsilon)^2 \kappa)^j \|u\|_2^2
\end{aligned}$$

as long as the final series is finite, which is satisfied if $0 \leq \kappa < 1/2e(\beta_0+\epsilon)^2$. Thus, we have that (6.1.8) implies (6.1.12) for any κ such that $0 \leq \kappa \leq 1/2e(\beta_0+\epsilon)^2$. Therefore, we also have that $\kappa_a \geq 1/2e\beta_0^2$.

Next, we show that (6.1.12) implies (6.1.8). Indeed, for $0 < \epsilon < \kappa_a$, we have, as remarked before, that

$$\int_{\mathbf{R}^2} \exp\left(\frac{(\kappa_a-\epsilon)|u|^2}{\|u\|_{\dot{H}_a^1}^{2j}}\right) - 1 \,\mathrm{d}x \leq C(\epsilon)\frac{\|u\|_2^2}{\|u\|_{\dot{H}_a^1}^{2j}},$$

for some constant $C(\epsilon) > 0$. Thus, expanding the left-hand side using Taylor series expansion, we see that for each $j \geq 1$, we have

$$\frac{1}{j!}\frac{(\kappa_a-\epsilon)^{2j}\|u\|_{2j}^{2j}}{\|u\|_{\dot{H}_a^1}^{2j}} \leq C(\epsilon)\frac{\|u\|_2^2}{\|u\|_{\dot{H}_a^1}^{2j}}$$

Therefore,

$$\|u\|_{2j} \leq \frac{(C(\epsilon)\cdot j!)^{1/2j}}{(\kappa_a-\epsilon)^{1/2j}}\|u\|_{\dot{H}_a^1}^{1-1/j}\|u\|_2^{1/j} \tag{6.1.18}$$

Let $q > 2$ and such that $2j \leq q < 2(j+1)$. Interpolating the above (6.1.18) with $p = 2j$ and $p = 2(j+1)$ we have that

$$\|u\|_q \leq \left(C(\epsilon)\Gamma\left(\frac{q}{2}+2\right)\right)^{1/2j}(\kappa_a-\epsilon)^{-1/2}\|u\|_{\dot{H}_a^1}^{1-2/q}\|u\|_2^{2/q} \tag{6.1.19}$$

where Γ is the Gamma function, and we note that $(j+1)! \leq \Gamma(q/2+2)$. Now, using Stirling's formula, the fact that $2j \geq q-2$, we obtain (6.1.8) for any $\delta > 0$ and for some $r = r(\delta) > 0$ sufficiently large, we have (6.1.8) with $C = (2e(\kappa_a-\epsilon))^{-1/2}+\delta$ for all $q \geq r$. Thus, we also have $M_0 \leq (2e\kappa_a)^{-1/2}$. $\square$

6.2. Local and global well-posedness

6.2.1. Local existence. We begin our study of the (6.0.1) and (6.0.2) with the local existence theory. The proofs here are similar to those in [**11**] for the NLS case and [**27**] for the NLKG case, so we shall place emphasis on adjustments we make compared to the potential-free case studied in these papers.

PROPOSITION 6.2.1. *Let $\sigma \in \{-1,1\}$, $u_0 \in H_a^1(\mathbf{R}^2)$ and $\|u_0\|_{H_a^1} < \frac{\kappa_a^*}{\kappa_0}$.*

(a) *There exists a time $T > 0$ and a unique solution to (6.0.1) in the space $C_T(H_a^1(\mathbf{R}^2)) \cap L_{t\in[0,T]}^{2(-\eta)}W_a^{1,\infty(\eta)}$. Furthermore, the solution satisfies the conservation laws $M(u(t,\cdot)) = M(u_0)$ and $E_S(u(t,\cdot)) = E_S(u_0)$.*

(b) *There exists a time $T > 0$ and a unique solution to (6.0.2) in the space $C_T(H_a^1(\mathbf{R}^2) \cap C_T^1(L^2(\mathbf{R}^2)) \cap L_{t\in[0,T]}^{2(\eta)}W_a^{1/2,\infty(\eta/2)}(\mathbf{R}^2)$. Furthermore, the solution satisfies the conservation law $E_{KG}(u(t,\cdot)) = E_{KG}(u_0)$.*

PROOF. We shall study the NLS case, as the idea for the NLKG case follows from [**27**] and the extra steps we discuss below for the inverse-square potential. The spaces chosen match the scattering proof later. We shall study the local existence theory in the following space:

$$\mathrm{Str}_a([0,T]) := L_{t\in[0,T]}^\infty H_a^1 \cap L_{t\in[0,T]}^{2(-\eta)}W_a^{1,\infty(\eta)}(I\times\mathbf{R}^2), \tag{6.2.1}$$

Note that for $a > 1$, we have $\mathrm{Str}_a([0,T]) \hookrightarrow \mathrm{Str}_0([0,T])$, since $H^1_a(\mathbf{R}^2) \hookrightarrow H^1(\mathbf{R}^2)$ and since $A_\mu^{\frac{1}{2}} A_\nu^{-\frac{1}{2}}$ is continuous on $L^{\infty(\eta)}(\mathbf{R}^2)$. Here, $q(\epsilon)$ is defined for small $\epsilon > 0$ via

$$\frac{1}{q(\epsilon)} = \frac{1}{q} + \epsilon.$$

First, we define the map with $S_a(t) := e^{it\mathscr{L}_a}$ by

$$\Phi(v_1) = \int_0^t S_a(t-s) f((v_0+v_1)(s))\, \mathrm{d}s. \tag{6.2.2}$$

where v_0 solves the following free Schrödinger equation

$$\begin{cases} i\partial_t v_0 - \Delta v_0 + \frac{a}{|x|^2} v_0 = 0 \\ \\ v_0(0,x) = u_0(x) \end{cases}. \tag{6.2.3}$$

The idea is to show that Φ is a contraction on $X_T := \mathrm{Str}_a([0,T])$ with metric $d(u,v) := \|u-v\|_{\mathrm{Str}_a([0,T])}$. Then, by construction $u = w + v$ solves (6.0.1). First, we check that Φ maps X_T to itself. By Strichartz estimates and continuity of the conjugation operators $\mathcal{K}^+ := \mathcal{H}_\nu \mathcal{H}_\lambda$ and $\mathcal{K}^- := \mathcal{H}_\lambda \mathcal{H}_\nu$, we have

$$\begin{aligned} \|\Phi(v_1)\|_{\mathrm{Str}_a([0,T])} &= \left\| \int_0^t S_a(t-s) f(v_0+v_1)\, \mathrm{d}s \right\|_{L^\infty_{t\in[0,T]} H^1_a \cap L^{2(\eta)}_{L_{t\in[0,T]}} W_x^{1,\infty(-\eta)}} \\ &= \left\| \int_0^t S_a(t-s) \langle D_a \rangle f(v_0+v_1)\, \mathrm{d}s \right\|_{L^\infty_{t\in[0,T]} L^2_x \cap L^{2(\eta)}_{t\in[0,T]} L_x^{\infty(-\eta)}} \\ &= \left\| \mathcal{K}^+ \int_0^t S(t-s) \langle \nabla \rangle \mathcal{K}^- f(v_0+v_1)\, \mathrm{d}s \right\|_{L^\infty_{t\in[0,T]} L^2_x \cap L^{2(\eta)}_{t\in[0,T]} L_x^{\infty(-\eta)}} \\ &\lesssim \left\| \int_0^t S(t-s) \langle \nabla \rangle \mathcal{K}^- f(v_0+v_1)\, \mathrm{d}s \right\|_{L^\infty_{t\in[0,T]} L^2_x \cap L^{2(\eta)}_{t\in[0,T]} L_x^{\infty(-\eta)}} \\ &\lesssim \|\langle \nabla \rangle \mathcal{K}^- f(v_0+v_1)\|_{L^{2(-\eta)}_{t\in[0,T]} L_x^{1(\eta)}} \lesssim \|\langle \nabla \rangle f(v_0+v_1)\|_{L^{2(-\eta)}_{t\in[0,T]} L_x^{1(\eta)}}. \end{aligned}$$

Now, we use the assumption that $\|u_0\|_{\dot{H}^1_a} < \frac{\kappa_a^*}{\kappa_0}$. Indeed, since v solves (6.2.3), we also have $\|v\|_{\dot{H}^1_a} < \frac{\kappa_a^*}{\kappa_0}$. Hence, $\|v_0 + v_1\|_{\dot{H}^1_a} < \frac{\kappa_a^*}{\kappa_0}$ for a sufficiently small choice δ. Thus, we may apply the Moser-Trudinger inequality, continuity of the conjugation operators and Hölder's inequality to obtain

$$\begin{aligned} \|\langle \nabla \rangle f(v_0+v_1)\|_{L_t^{2(-\eta)} L_x^{1(\eta)}} &\lesssim \left\| \left(e^{\kappa_0 |v_0+v_1|^2} - 1 \right) \kappa_0 |v_0+v_1|^2 \langle \nabla \rangle (v_0+v_1) \right\|_{L_t^{2(\eta)} L_x^{1(\eta)}} \\ &\lesssim \left\| e^{\kappa_0|v_0+v_1|^2} - 1 \right\|_{L_t^\infty L_x^{1(49\eta)}} \left\| |v_0+v_1|^2 \langle \nabla \rangle (v_0+v_1) \right\|_{L_t^{2(\eta)} L_x^{\infty(48\eta)}} \\ &\lesssim \|\langle \nabla \rangle (v_0+v_1)\|_{L_t^{2(-\eta)} L_x^{\infty(\eta)}} \cdot \|v_0+v_1\|^2_{L_t^{\infty(\eta/2)} L_x^{\infty(47\eta/2)}} \\ &\lesssim \sum_{i,j\in\{0,1\}} \|\langle \nabla \rangle v_i\|_{L_t^{2(-\eta)} L_x^{\infty(\eta)}} \cdot \|v_j\|^2_{L_t^{\infty(\eta/2)} L_x^{\infty(47\eta/2)}} \\ &\lesssim \sum_{i,j\in\{0,1\}} \|\langle D_a \rangle v_i\|_{L_t^{2(-\eta)} L_x^{\infty(\eta)}} \cdot \|v_j\|^2_{L_t^{\infty(\eta/2)} L_x^{\infty(47\eta/2)}} \\ &\lesssim T^{\eta/2} (\delta + \|u_0\|_{H^1})^2. \end{aligned}$$

Thus, $\Phi : X_T \to X_T$ for a sufficiently small choice of $T > 0$. Now, using a similar argument above, we also obtain that Φ is indeed a contraction if we choose M and T sufficiently small. By the contraction mapping theorem, we obtain the required result. $\square$

6.2.2. Global well-posedness for defocusing case. As a corollary of the above study, we have the following results in the subcritical regime:

PROPOSITION 6.2.2. *The above local existence results can also be extended to global existence results. That is,*

(a) *Assume that* $E_S(u_0) < \frac{\kappa_a^*}{2\kappa_0}$. *Then, the defocusing problem to (6.0.1) has a unique global solution in space* $C(\mathbf{R}, H_a^1(\mathbf{R}^2)) \cap L_{t\in\mathbf{R}}^{2(-\eta)} W_a^{1,\infty(\eta)}$.

(b) *Assume that* $E_{KG}(u_0) < \frac{\kappa_a^*}{2\kappa_0}$. *Then, the defocusing problem to (6.0.2) has a unique global solution in the space* $C(\mathbf{R}, H_a^1(\mathbf{R}^2)) \cap C^1(\mathbf{R}, L^2(\mathbf{R}^2)) \cap L_{t\in\mathbf{R}}^{2(\eta)} W_a^{1/2,\infty(\eta/2)}(\mathbf{R}^2)$.

PROOF. We shall consider the NLS case, as the proof for the NLKG case is similar. Indeed, let $u(t)$ be the solution to (6.0.1) with maximal time of existence T. Assume for contradiction that $T < \infty$. By the conservation law in the local theory, combined with our assumptions, we have that

$$\sup_{t\in[0,T]} \|u\|_{H_a^1} \leq E_S(u_0) < 1.$$

Now, let $s \in [0, T]$ and consider the Cauchy problem

$$\begin{cases} i\partial_t v - \Delta v + \frac{a}{|x|^2} v = f(v) \\ v(s, x) = u(s, x) \in H_a^1(\mathbf{R}^2) \end{cases}$$

Then, applying the argument from the local existence result, we obtain a time $\tau > 0$ and a unique solution v to the above Cauchy problem on the time interval $[s, s+\tau]$. Now, choosing s sufficiently close to T (in particular, choosing s such that $T - s < \tau$), we are able to extend the solution $u(t)$ beyond T, whence we obtain a contradiction. $\square$

6.3. Variational analysis

6.3.1. Variational results for the focusing case. In this section we discuss the variational setting for the study of the focusing cases of (6.0.1) and (6.0.2). Indeed, we review the following notation from [**30**]. Let $(\alpha, \beta) \in \mathbf{R}^2$ such that

$$\alpha \geq 0,\ 2\alpha + d\beta \geq 0,\ 2\alpha + (d-2)\beta \geq 0 \text{ and } (\alpha, \beta) \neq (0, 0). \tag{6.3.1}$$

For $c \geq 0$ and $\phi \in H_a^1(\mathbf{R}^d)$, define the static energy

$$J^{(c)}(\phi) := \frac{1}{2} \int_{\mathbf{R}^d} |\nabla\phi|^2 + \frac{a}{|x|^2}|\phi|^2 \, \mathrm{d}x + \frac{c}{2} \int_{\mathbf{R}^d} |\phi|^2 \, \mathrm{d}x - \frac{1}{2} \int_{\mathbf{R}^d} F(\phi) \, \mathrm{d}x. \tag{6.3.2}$$

Let

$$\phi_{\alpha,\beta}^{\lambda}(x) := e^{\alpha\lambda} \phi \left(e^{-\beta\lambda} x\right), \tag{6.3.3}$$

and

$$K^{(c)}_{\alpha,\beta}(\phi) := \mathcal{L}_{\alpha,\beta}J(\phi^{(c)}_{\alpha,\beta}) \tag{6.3.4}$$

$$:= \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} j^{(c)}_{\alpha,\beta}(\lambda)$$

$$\begin{aligned} &= \frac{2\alpha+(d-2)\beta}{2}\int_{\mathbf{R}^d}|\nabla\phi|^2+\frac{a}{|x|^2}|\phi|^2\,\mathrm{d}x \\ &\quad+\frac{2\alpha+d\beta}{2}c\int_{\mathbf{R}^d}|\phi|^2\,\mathrm{d}x \\ &\quad-\frac{1}{2}\int_{\mathbf{R}^d}2\alpha\Re\left(\partial_\phi F(\phi)\phi\right)+d\beta F(\phi)\,\mathrm{d}x. \end{aligned} \tag{6.3.5}$$

If $c=1$, we omit the superscript c. Furthermore, define the quadratic part of $K_{\alpha,\beta}(\phi)$ (i.e. the linear energy of the sign functional with $c=1$) by

$$K^Q_{\alpha,\beta}(\phi) := \frac{2\alpha+(d-2)\beta}{2}\int_{\mathbf{R}^d}|\nabla\phi|^2+\frac{a}{|x|^2}|\phi|^2\,\mathrm{d}x+\frac{2\alpha+d\beta}{2}c\int_{\mathbf{R}^d}|\phi|^2\,\mathrm{d}x. \tag{6.3.6}$$

We shall consider the minimisation problem

$$m_{\alpha,\beta} = \inf\left\{J(\phi) : \phi\in H^1_{a,\mathrm{rad}}(\mathbf{R}^d), \phi\neq 0, K_{\alpha,\beta}(\phi)=0\right\}. \tag{6.3.7}$$

We need to include the radial assumption in the above minimisation problem because symmetrisation methods do not decrease the H^1_a norm for $a>0$. From the variational problem, we define the following subsets of the energy space:

$$\begin{aligned} \mathcal{K}^+_{\alpha,\beta} &= \{(u_0,u_1)\in H^1_a(\mathbf{R}^2)\times L^2(\mathbf{R}^2) : E(u_0,u_1)<m_{\alpha,\beta}, K_{\alpha,\beta}(u_0)\geq 0\}, \\ \mathcal{K}^-_{\alpha,\beta} &= \{(u_0,u_1)\in H^1_a(\mathbf{R}^2)\times L^2(\mathbf{R}^2) : E(u_0,u_1)<m_{\alpha,\beta}, K_{\alpha,\beta}(u_0)< 0\}. \end{aligned}$$

Furthermore, we restate the Moser-Trudinger inequality. First, we define

$$C^A_{MT}(G) := \sup\left\{\frac{2G(\phi)}{\|\phi\|_2^2} : \phi\in H^1_a(\mathbf{R}^2), \phi\neq 0, \|u\|_{\dot H^1_a}\leq A\right\}. \tag{6.3.8}$$

Next, define

$$\mathfrak{M}(G) := \sup\{A>0 : C^A_{MT}(G)<\infty\}. \tag{6.3.9}$$

Finally, denote

$$C^*_{MT}(G) := C^{\mathfrak{M}(G)}_{MT}(G). \tag{6.3.10}$$

Thus, the Moser-Trudinger inequality gives

$$\mathfrak{M}(\mathcal{L}_{\alpha,\beta}\mathcal{F}) = \mathfrak{M}(\mathcal{F}) = \sqrt{\frac{\kappa^*_a}{\kappa_0}}. \tag{6.3.11}$$

Here, we have used the fact that for a functional $H(\phi)$ of the form $H(\phi)=\int_{\mathbf{R}^d}h(\phi)\,\mathrm{d}x$, we have

$$\mathcal{L}_{\alpha,\beta}H(\phi) = \int_{\mathbf{R}^d}\alpha\phi h'(\phi)+\beta d h(\phi)\,\mathrm{d}x. \tag{6.3.12}$$

We collect some variational results analogous to those by Ibrahim-Masmoudi-Nakanishi [**30**].

LEMMA 6.3.1 (Minimisation problem). *Recall that*

$$F(u) = \frac{\lambda}{\kappa_0}\left(e^{\kappa_0|u|^2} - 1 - \kappa_0|u|^2 - \frac{\kappa_0^2}{2}|u|^4\right) \tag{6.3.13}$$

and that the pair (α,β) *satisfies (6.3.1). Furthermore, suppose that* $\phi \in H^1_a(\mathbf{R}^2)$. *Then, the minimisation problem (6.3.7) is equivalent to*

$$m_{\alpha,\beta} = \inf\left\{H_{\alpha,\beta}(\phi) : \phi \neq 0,\ \phi \in H^1_a,\ \phi \text{ is radial, and } K_{\alpha,\beta}(\phi) \leq 0\right\}. \tag{6.3.14}$$

where

$$H_{\alpha,\beta} = \left(1 - \frac{\mathcal{L}_{\alpha,\beta}}{\overline{\mu}}\right) J. \tag{6.3.15}$$

REMARK 6.3.2. *If* $(\alpha,\beta) = (1,0)$, *then we see that*

$$H_{1,0}(\phi) = \frac{1}{2}\|\phi\|^2_{\dot{H}^1_a}.$$

PROOF. The proof is similar to [**30**, Lemma 2.3] with minor alterations. □

LEMMA 6.3.3 (Compactness via dominated convergence). *Let* $g, h : \mathbf{R} \to \mathbf{R}$ *be continuous functions satisfying*

$$\lim_{u\to\pm\infty} \frac{|g(u)|}{h(u)} = 0, \quad \lim_{u\to 0} \frac{|g(u)|}{|u|^2} = 0. \tag{6.3.16}$$

Let $(\phi_n)_n$ *be a sequence of radial functions such that* $\phi_n \rightharpoonup \phi$ *weakly in* $H^1_a(\mathbf{R}^2)$ *and* $(h(\phi_n))_n$ *is bounded in* $L^1(\mathbf{R}^2)$. *Then,* $g(\phi_n) \to g(\phi)$ *strongly in* $L^1(\mathbf{R}^2)$.

PROOF. We shall follow the proof of [**30**, Lemma 2.7] (see also [**42**]). We want to show that

$$\int_{\mathbf{R}^2} |g(\phi_n) - g(\phi)|\,\mathrm{d}x \to 0$$

as $n \to \infty$. First, by assumption (6.3.16), we have that for any $\epsilon > 0$, there exists an $L = L(\epsilon) > 0$ such that if $|u| > L$ then $|g(u)| < \epsilon h(u)$. Therefore,

$$\int_{|\phi_n|>L} |g(\phi_n)|\,\mathrm{d}x \lesssim \epsilon \int h(\phi_n)\,\mathrm{d}x \lesssim \epsilon.$$

Furthermore, from the radial Sobolev inequality, we also have that $|\phi_n| \to 0$ uniformly as $|x| \to \infty$. Let B_R denote the ball centred at the origin of radius R. Then, together with (6.3.16), there exists an $R = R(\epsilon) > 0$ such that

$$\int_{\mathbf{R}^2\setminus B_R} g(\phi_n)\,\mathrm{d}x \lesssim \int_{\mathbf{R}^2\setminus B_R} |u|^2\,\mathrm{d}x \lesssim \epsilon.$$

From the assumptions, the weak convergence also implies that $\phi_n(x) \to \phi(x)$ for $x \neq 0$. Therefore, by Fatou's lemma, we have

$$\int_{|\phi|>L} |g(\phi)|\,\mathrm{d}x \lesssim \epsilon.$$

Now, we split

$$\int_{\mathbf{R}^2} |g(\phi_n) - g(\phi)|\,\mathrm{d}x \leq \int_{\mathbf{R}^2\setminus B_R} |g(\phi_n) - g(\phi)|\,\mathrm{d}x$$

into

$$\int_{\mathbf{R}^2\setminus B_R} |g(\phi_n) - g(\phi)|\,\mathrm{d}x \tag{6.3.17}$$

$$+ \int_{\mathbf{R}^2\cap\{|\phi_n|>L\}} |g(\phi_n) - g(\phi)|\,\mathrm{d}x \tag{6.3.18}$$

$$+ \int_{\mathbf{R}^2\cap\{|\phi_n|\le L\}} |g(\phi_n) - g(\phi)|\,\mathrm{d}x. \tag{6.3.19}$$

We already have that (6.3.17) + (6.3.18) $\lesssim \epsilon$. Finally, we define

$$g^L(t) := \begin{cases} g(t), & |t| \le L \\ g(L), & |t| \ge L \end{cases}.$$

Then, by the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{n\to\infty} \int_{\mathbf{R}^2} |g(\phi_n) - g(\phi)| \lesssim \epsilon + \lim_{n\to\infty} \int_{B_R} |g^L(\phi_n) - g^L(\phi)|\,\mathrm{d}x \lesssim \epsilon.$$

Finally, noting that ϵ is arbitrary, we obtain the required result. □

LEMMA 6.3.4 (Ground state). *Let f and (α, β) be as above in Lemma 6.3.1. Furthermore, define $c := \min\{1, C^*_{MT}(\mathcal{F})\}$. Then,*

(a) *The minimal mass $m_{\alpha,\beta}$ is independent of (α, β).*

(b) *If $C^*_{MT}(\mathcal{F}) < 1$, we have $0 < m_{\alpha,\beta} < \frac{1}{2}\cdot\frac{\kappa^*_a}{\kappa_0}$. If $C^*_{MT}(\mathcal{F}) > 1$, then $m_{\alpha,\beta} = \frac{1}{2}\cdot\frac{\kappa^*_a}{\kappa_0}$*

(c) *The minimal mass is attained by some $Q \in H^1_a$ which solves*

$$\mathscr{L}_a Q + cQ = f(Q). \tag{6.3.20}$$

That is, $m_{\alpha,\beta} = J^{(c)}(Q)$.

PROOF. We shall follow the proof of [**30**, Lemma 2.6] which uses the compactness via dominated convergence and Moser-Trudinger inequality.

We proceed in three steps – first, we show (b) and (c) for the exceptional case $\alpha = 0$ and the two cases for $c = \min\{1, C^*_{MT}(\mathcal{F})\}$. Then, showing independence of parameters (a) completes the proof.

Thus, we begin by considering the exceptional case with $\alpha = 0$ and $C^*_{MT}(\mathcal{F}) > 1$. By assumption, there exists a non-zero function $\phi \in H^1_a(\mathbf{R}^2)$ such that $\|\phi\|_{\dot H^1_a} \le \mathfrak{M}(\mathcal{F})$ and $\mathcal{F}(\phi) > \|\phi\|_2^2/2$. Thus, we have $K_{0,1}(\phi) = \|\phi\|_2^2 - 2\mathcal{F}(\phi) < 0$. Fix a small $\epsilon > 0$. Then, $K_{0,1}((1-\epsilon)\phi) < 0$ and by Lemma 6.3.1, we have that

$$m_{0,1} \le H_{0,1}((1-\epsilon)\phi) = \frac{1}{2}\|(1-\epsilon)\phi\|^2_{\dot H^1_a} < \frac{1}{2}\mathfrak{M}(\mathcal{F})^2 = \frac{1}{2}\cdot\frac{\kappa^*_a}{\kappa_0}. \tag{6.3.21}$$

This proves (b) in this case.

Now, we study the variational problem (c). We take a minimising sequence

$$(\phi)_n \subseteq H^1_{a,\mathrm{rad}} \text{ such that } H_{0,1}(\phi_n) = \|\phi\|^2_{\dot H^1_a}/2 \searrow m_{0,1} \text{ and } K_{0,1}(\phi_n) \le 0.$$

First, we note that

$$H_{0,1}(\phi(e^{-\lambda}x)) = H_{0,1}(\phi) \text{ and } K_{0,1}(\phi(e^{-\lambda}x)) = e^{2\lambda}K_{0,1}(\phi). \tag{6.3.22}$$

Thus, by rescaling, we may take $\|\phi_n\|_2 = 1$. Furthermore, notice that for $\nu \in (0,1)$, $K^Q(\nu\phi) = \nu^2 K^Q(\phi)$ while $|K^N(\nu\phi)| = o(\nu^4)$. Therefore $K(\nu\phi) = K^Q(\nu\phi) - K^N(\nu\phi) > 0$ for small ν. Thus, if $K(\phi) < 0$, then there exists an $\nu \in (0,1)$ such that $K(\nu\phi) = 0$. Furthermore, we have $H(\nu\phi) \leq H(\phi)$. Thus, we may consider a minimising sequence

$$(\phi)_n \subseteq H^1_{a,\mathrm{rad}} \text{ such that } H_{0,1}(\phi_n) = \|\phi\|^2_{\dot{H}^1_a}/2 \searrow m_{0,1} \text{ and } 1-2\mathcal{F}(\phi_n) = K_{0,1}(\phi_n) = 0.$$

All together, we have $\phi_n \to \phi$ in H^1_a. We may now apply the compactness result (Lemma 6.3.3) with ϕ_n, $g := F$ and $h(u) := \exp\left(\kappa|u|^2\right) - 1$ where $\kappa \in (\kappa_0, 2\pi/m_{0,1})$ to obtain $\mathcal{F}(\phi_n) \to \mathcal{F}(\phi)$. Thus, ϕ attains $m_{0,1}$. Furthermore, for some Lagrange multiplier $\eta \in \mathbf{R}$, we have $J'(\phi) = \eta K'(\phi)$ where J' and K' are the Fréchet derivatives of J and K respectively. Thus, we have

$$0 = K_{0,1}(\phi) = \mathcal{L}_{0,1}J(\phi) = \langle J'(\phi)|\mathcal{L}_{0,1}\phi\rangle = \eta\langle K'(\phi)|\mathcal{L}_{0,1}\phi\rangle = \eta\mathcal{L}^2 J(\phi). \tag{6.3.23}$$

Furthermore, we show that $\mathcal{L}^2 J(\phi) < 0$, from which we can obtain that $\eta = 0$ and thus the minimiser ϕ satisfies $\mathscr{L}_a\phi + \phi = f(\phi)$. Indeed, the same computations as in [**30**, Lemma 2.2] show that for (α,β) satisfying (6.3.1), $\overline{\mu} := \max\{2\alpha + d\beta, 2\alpha + (d-2)\beta\}$ and $\underline{\mu} := \min\{2\alpha + d\beta, 2\alpha + (d-2)\beta\}$, we have

$$-(\mathcal{L}_{\alpha,\beta} - \overline{\mu})(\mathcal{L}_{\alpha,\beta} - \underline{\mu})J(\phi) \geq \frac{2\alpha\epsilon\overline{\mu}}{d+1}G(\phi).$$

Rearranging this inequality and using $(\alpha,\beta) = (1,0)$, we obtain the required result.

Now, we move on to the next case. Suppose that $(\alpha,\beta) = (1,0)$ and $C^*_{MT}(\mathcal{F}) < 1$. First, we verify (b). We notice that if $\phi \in H^1_a$ and $\|\phi\|^2_{\dot{H}^1_a} < \mathfrak{M}(\mathcal{F})$ then $K_{0,1}(\phi) = \|\phi\|^2_2 - 2\mathcal{F}(\phi) > 0$. Thus, $m_{0,1} = \inf\{\|u\|^2_{\dot{H}^1_a}/2 : K_{0,1}(\phi) < 0\} \geq \mathfrak{M}(\mathcal{F})^2/2$ and consequently $m_{0,1} = \mathfrak{M}(\mathcal{F})^2/2$ as required. However, this means there is no room to use the Moser-Trudinger inequality to close the compactness argument as in the previous case. The idea is to instead consider the variational problem for $c_n := C^{\mathfrak{M}(\mathcal{F})-1/n}_{MT}(\mathcal{F})$. Indeed, take a maximising sequence for c_n:

$$\|\phi^k_n\|_{\dot{H}^1_a} \leq \mathfrak{M}(\mathcal{F}) - \tfrac{1}{n}, \quad \mathcal{F}(\phi^k_n) \nearrow \tfrac{c_n}{2} \quad \text{and} \quad \|\phi^k_n\|_2 = 1.$$

Again, we may take $\|\phi^k_n\|_2 = 1$ as in (6.3.22). Thus, we may extract a subsequence such that $\phi^k_n \rightharpoonup \phi_n$ weakly in H^1_a. Now, we apply Lemma 6.3.3 with ϕ_n, $g := F$ and $h(u) := \exp\left(\kappa|u|^2\right) - 1$ where $\kappa \in (\kappa_0, \kappa^*_a/(\mathfrak{M}(\mathcal{F}) - 1/n)^2)$. Hence $\mathcal{F}(\phi^k_n) \to \mathcal{F}(\phi_n) = c_n/2$. Thus, ϕ_n is a maximiser for c_n with $\|\phi_n\|_2 = 1$. We now obtain the Euler-Lagrange

$$\eta\mathscr{L}_a\phi_n = -c_n\phi_n + f(\phi_n). \tag{6.3.24}$$

Furthermore, multiplying the above equation with ϕ_n and using the fact that $c_n = 2\mathcal{F}(\phi_n)$ and $\|\phi_n\|_{L^2} = 1$, we have

$$\begin{aligned}\eta\|\phi_n\|_{\dot{H}^1_a} &= \int DF(\phi_n)\,\mathrm{d}x - c_n\|\phi_n\|^2_{L^2} \\ &= \int (D-2)F(\phi_n)\,\mathrm{d}x > 0.\end{aligned}$$

Thus, $\eta > 0$ and we may define $Q_n(x) := \phi_n(\eta^{1/2}x)$ such that $\|Q_n\|_{\dot{H}^1_a} \leq \mathfrak{M}(\mathcal{F}) - 1/n$ and

$$\mathscr{L}_a Q_n + c_n Q_n = f(Q_n). \tag{6.3.25}$$

Now, we want to consider the limit $n \to \infty$. Multiplying the equation (6.3.25) with Q_n and $x \cdot \nabla Q_n$ we obtain

$$c_n \|Q_n\|_2^2 = 2\mathcal{F}(Q_n), \quad \|Q_n\|_{\dot H_a^1}^2 = 2\int_{\mathbf{R}^2} (D-2)F(Q_n)\,\mathrm{d}x \geq 4\mathcal{F}(Q_n).$$

Furthermore, using the facts that $\|Q_n\|_{\dot H_a^1}$ is bounded, and c_n is a positive non-decreasing sequence, we conclude that $\|Q_n\|_{L^2}$ and $D\mathcal{F}(Q_n)$ are bounded. Thus, we extract a subsequence such that $Q_n \rightharpoonup Q_a$ weakly in H_a^1. Now, we apply Lemma 6.3.3 with $\phi_n := Q_n, g := f$ and $h := D\mathcal{F}$ to get $f(Q_n) \to f(Q)$ in L^1. Now, taking the Euler-Lagrange as before we get $\mathscr{L}_a Q_a + cQ = f(Q)$ where $c = C_{MT}^*(\mathcal{F})$. Furthermore, we have that $K_{0,1}^{(c)}(Q_a) = \langle J^{(c)\prime}(Q_a) | \mathcal{L}_{0,1}(Q_a)\rangle = 0$. Thus, $c\|Q_a\|_{L^2}^2 = 2\mathcal{F}(Q_a)$. Furthermore, Q_a is a maximiser of $C_{MT}^{\mathfrak{M}(\mathcal{F})}(\mathcal{F})$ with non-zero Lagrange multiplier and $\|Q_a\|_{H_a^1} = \mathfrak{M}(\mathcal{F})$. Finally, we also have that $J^{(c)}(Q_a) = \mathfrak{M}(\mathcal{F})^2/2$.

It remains to verify that $m_{\alpha,\beta}$ is independent of the parameters (α, β). If $m_{0,1} < \mathfrak{M}(\mathcal{F})^2/2$, then the ground state Q_a satisfies $K_{\alpha,\beta}(Q_a) = 0$ for all (α, β) since

$$K_{\alpha,\beta}(Q_a) = \langle J'(Q_a) | \mathcal{L}_{\alpha,\beta} Q_a \rangle = 0. \tag{6.3.26}$$

Thus, $m_{\alpha,\beta} \leq J(Q_a) = m_{0,1}$. Next, if $m_{0,1} = \mathfrak{M}(\mathcal{F})^2/2 = \mathfrak{M}(\mathcal{L}_{\alpha,\beta}\mathcal{F})^2/2$ then for all $A > \mathfrak{M}(\mathcal{F})^2/2$ there exists a sequence $\phi_n \in H^1_{a,\mathrm{rad}}$ such that $\|\phi_n\|_{\dot H_a^1} \leq A, \|\phi_n\|_2 \to 0$ and $\mathcal{L}_{\alpha,\beta}\mathcal{F}(\phi_n) \to \infty$. Replacing $\phi_n(x)$ by $\phi_n(x/\nu_n)$ where $\nu_n \to 0$ as $n \to \infty$, we may study the sequence $\|\phi_n\|_{\dot H_a^1} \leq A, \|\phi_n\|_2 \to 0$ and $K_{0,1}(\phi_n) = 0$. Thus, $m_{\alpha,\beta} \leq \liminf_{n\to\infty} J(\phi_n) \leq A^2/2$. Therefore, $m_{\alpha,\beta} \leq m_{0,1}$ in both cases.

Finally, suppose for contradiction that $m_{\alpha,\beta} < m_{0,1} \leq \mathfrak{M}(\mathcal{F})^2/2$. Here, we take a minimising sequence (ϕ_n) in $H^1_{a,\mathrm{rad}}$ such that $K_{\alpha,\beta}(\phi_n) = 0$ and $H_{\alpha,\beta}(\phi_n) \searrow m_{\alpha,\beta}$. Thus, we extract a subsequence $\phi_n \rightharpoonup \phi$ weakly in H_a^1. Next, we recall (6.3.12) and apply Lemma 6.3.3 with $\phi_n, g(u) := \alpha u f(u) + 2\beta F(u)$ and $h(u) := \exp(\kappa|u|^2) - 1$ with $\kappa \in (\kappa_0, 2\pi/m_{\alpha,\beta})$ to obtain that $\mathcal{LF}(\phi_n) \to \mathcal{LF}(\phi)$ as $n \to \infty$. Thus, ϕ is a minimiser of $m_{\alpha,\beta}$. Thus, we obtain a ground state Q such that $J(Q) = m_{\alpha,\beta} < m_{0,1}$, which is a contradiction. Hence, we finally obtain $m_{0,1} = m_{\alpha,\beta}$. □

LEMMA 6.3.5 (Splitting independent of (α, β)). *Let F and (α, β) be as above in Lemma 6.3.1. Then, $\mathcal{K}^{\pm}_{\alpha,\beta}$ (as defined in (6.3.8) and (6.3.8)) are independent of (α, β).*

PROOF. The proof is the same as for [**30**, Lemma 2.9]. □

LEMMA 6.3.6. *Let F and (α, β) be as above in Lemma 6.3.1. Furetheremore, suppose that $(u_0, u_1) \in H_a^1(\mathbf{R}^2) \times L^2(\mathbf{R}^2)$ satisfies $(u_0, u_1) \in \mathcal{K}^+$. Then, we have the following estimates:*

(a) *(Free energy equivalence)*

$$J(u_0) \leq \frac{1}{2}\|u_0\|_{H_a^1}^2 \leq \left(1 + \frac{d}{2}\right) J(u_0). \tag{6.3.27}$$

(b) *(Subcritical bound in $\mathcal{K}^+$)*

$$\|u_0\|_{\dot H_a^1}^2 + \|u_1\|_2^2 < 2m_{\alpha,\beta} \leq \mathfrak{M}(\mathcal{F})^2 = \frac{\kappa_a^*}{\kappa_0}. \tag{6.3.28}$$

(c) *(Variational estimate) If we also assume that* $(\alpha,\beta)\neq(0,1)$ *and* $J(\phi)<m_{\alpha,\beta}$*, then*

$$K_{\alpha,\beta}(\phi)\geq\min\left\{C(m_{\alpha,\beta}-J(\phi)),CK^{Q}_{\alpha,\beta}(\phi)\right\}. \tag{6.3.29}$$

PROOF. The proof is the same as for [**30**, Lemma 2.10, Lemma 2.11, Lemma 2.12]. □

6.3.2. Global well-posedness for focusing case. We shall first apply the above results to obtain global well-posedness results for (6.0.1) and (6.0.2) in the focusing case. This complements the global well-posedness results obtained the defocusing case previously.

PROPOSITION 6.3.7. *Let* (α,β) *satisfy (6.3.1). Then,*

(a) *Let* $u(t)$ *be the solution of (6.0.1) with* $\lambda=1$*,* $E_S(u_0)+M(u_0)/2<m_{\alpha,\beta}$ *and* $K_{\alpha,\beta}(u_0)>0$*. Then* $u(t)\in C(\mathbf{R},H^1_a(\mathbf{R}^2))$*.*
(b) *Let* $u(t)$ *be the solution of (6.0.2) with* $\lambda=1$*,* $E_{KG}(u_0)<m_{\alpha,\beta}$ *and* $K_{\alpha,\beta}(u_0)>0$*. Then* $u(t)\in C(\mathbf{R},H^1_a(\mathbf{R}^2))\cap C^1(\mathbf{R},L^2(\mathbf{R}^2))$*.*

PROOF. We shall consider the NLS case. The argument for the NLKG case is similar. From the local-in-time theory, we take $u(t)$ to be the solution of (6.0.1) with maximal existence interval I. From the conservation law, we have that $J_a(u(t))=E_S(u(t))+M(u(t))/2<m_{\alpha,\beta}$. We first show that $u(t)\in\mathcal{K}^+=\{(u_0,u_1)\in H^1_a(\mathbf{R}^2)\times L^2(\mathbf{R}^2):E(u_0,u_1)<m_{\alpha,\beta},K_{\alpha,\beta}(u_0)\geq 0\}$ for all $t\in I$. Indeed, supposed for contradiction that there exists a $t^*\in I$ such that $K(u(t^*))=0$. Then, by definition of K, we have that $u(t^*)=0$, whence $u(t^*)\in\mathcal{K}^+$. Furthermore, since $\mathcal{K}^+$ is an open set (see [**30**]) and also $u(t)\in C(I,H^1_a(\mathbf{R}^2))$, we have $u(t)\in\mathcal{K}^+$ for all $t\in I$.

Finally, since $u(t)\in\mathcal{K}^+$ for all $t\in I$, applying the identities (6.3.27) and (6.3.28) in Lemma 6.3.6 to $u(t)$, we obtain that $I=\mathbf{R}$. □

6.3.3. Morawetz estimates: part 1. Consider the Morawetz quantity

$$M(t):=\begin{cases}\frac{1}{2}\Im\int_{\mathbf{R}^2}uh\cdot\nabla\overline{u}\,\mathrm{d}x, & \text{NLS case}\\[2ex] \Im\int_{\mathbf{R}^2}u_t(h\cdot\nabla\overline{u}+q\overline{u})\,\mathrm{d}x, & \text{NLKG case}\end{cases}. \tag{6.3.30}$$

First, we obtain

LEMMA 6.3.8 (Morawetz estimate in the NLS case). *Let* $\partial_{\overline{u}}F(u)=f(u)$ *and* $G(u):=\Re\left(\bar{u}f(u)-2F(u)\right)$*.*

(a) *If* $u\in C(\mathbf{R},H^1_a)$ *is a global solution to the NLS (6.0.1). Then,*

$$\begin{aligned}\partial_tM(t)\quad&=\quad\sum_{j,k=1}^{d}\Re\int\partial_ku\partial_kh_j\partial_j\overline{u}\,\mathrm{d}x-\int qG(u)\,\mathrm{d}x\\&\quad+\frac{1}{2}\int\left(-\Delta q+\frac{2a}{|x|^2}q\right)|u|^2\,\mathrm{d}x+\frac{1}{2}\Re\int h\cdot\nabla\left(\frac{a}{|x|^2}\right)|u|^2\,\mathrm{d}x\\&\quad+\Re\int\left(q-\frac{1}{2}\mathrm{div}h\right)\left(iu_t\overline{u}+|\nabla u|^2+\frac{a}{|x|^2}|u|^2-F(u)\right).\end{aligned}$$

(b) *If $u \in C(\mathbf{R}, H^1_a)$ is a global solution to the NLKG (6.0.2). Then,*

$$\begin{aligned}\partial_t M(t) &= \sum_{j,k=1}^{d} \Re \int \partial_k u \partial_k h_j \partial_j \overline{u}\, \mathrm{d}x - \int qG(u)\, \mathrm{d}x \\ &\quad -\frac{1}{2}\int \Delta q|u|^2 + \frac{1}{2}\Re \int h\cdot\nabla\left(\frac{a}{|x|^2}\right)|u|^2\, \mathrm{d}x \\ &\quad + \Re\int\left(q - \frac{1}{2}\mathrm{div}h\right)\left(-|u_t|^2 + |u|^2 + |\nabla u|^2 + \frac{a}{|x|^2}|u|^2 - F(u)\right).\end{aligned}$$

Proof. We shall prove the NLS case. The NLKG case is similar. First, we have that

$$\partial_t M(t) = \frac{1}{2}\Im \int u_t h\cdot\nabla\overline{u} + uh\cdot\nabla\overline{u_t}\, \mathrm{d}x. \tag{6.3.31}$$

We claim that

$$\frac{1}{2}\Im\int uh\cdot\nabla\overline{u_t}\, \mathrm{d}x = \frac{1}{2}\Im\int u_t h\cdot\nabla\overline{u}\, \mathrm{d}x - \frac{1}{2}\Re\int \mathrm{div}h\cdot iu_t\overline{u}\, \mathrm{d}x. \tag{6.3.32}$$

Indeed,

$$\begin{aligned}-\frac{1}{2}\Re\int \mathrm{div}h\cdot iu_t\overline{u}\, \mathrm{d}x &= \frac{1}{2}\Im\int \mathrm{div}h\cdot u_t\overline{u}\, \mathrm{d}x \\ &= -\frac{1}{2}\Im\int h\cdot\nabla(u_t\overline{u})\, \mathrm{d}x \\ &= -\frac{1}{2}\Im\int u_t h\cdot\nabla\overline{u} + \overline{u}h\cdot\nabla u_t\, \mathrm{d}x \\ &= \frac{1}{2}\Im\int -u_t h\cdot\nabla\overline{u} + uh\cdot\nabla\overline{u_t}\, \mathrm{d}x.\end{aligned}$$

Thus, combining (6.3.31) and (6.3.32) we have

$$\partial_t M(t) = \Im\int u_t h\cdot\nabla\overline{u}\, \mathrm{d}x - \frac{1}{2}\Re\int \mathrm{div}h\cdot iu_t\overline{u}\, \mathrm{d}x. \tag{6.3.33}$$

Consider the first integral. Using the fact that $u_t = -i\left(\Delta u - \frac{a}{|x|^2} + f(u)\right)$, we have that

$$\begin{aligned}\Im\int u_t h\cdot\nabla\overline{u}\, \mathrm{d}x &= \Im\int -i\left(\Delta u - \frac{a}{|x|^2} + f(u)\right)h\cdot\nabla\overline{u}\, \mathrm{d}x \\ &= -\Re\int \Delta uh\cdot\nabla\overline{u}\, \mathrm{d}x && (6.3.34)\\ &\quad -\Re\int f(u)h\cdot\nabla\overline{u}\, \mathrm{d}x && (6.3.35)\\ &\quad +\Re\int \frac{a}{|x|^2}uh\cdot\nabla\overline{u}\, \mathrm{d}x && (6.3.36)\end{aligned}$$

First, for (6.3.34) we have

$$\begin{aligned}
-\int \Delta u h\cdot\nabla\overline{u}\,\mathrm{d}x &= \int \nabla u\cdot\nabla(h\cdot\nabla\overline{u})\,\mathrm{d}x\\
&= \sum_{j,k=1}^{d}\int \partial_k u\partial_k(h_j\partial_j\overline{u})\,\mathrm{d}x\\
&= \sum_{j,k=1}^{d}\int \partial_k u\partial_k h_j\partial_j\overline{u}\,\mathrm{d}x+\sum_{j,k=1}^{d}\int \partial_k u h_j\partial_{kj}\overline{u}\,\mathrm{d}x
\end{aligned}$$

Next we have that

$$\begin{aligned}
\sum_{j,k=1}^{d}\int \partial_k u h_j\partial_{kj}\overline{u}\,\mathrm{d}x &= -\sum_{j,k=1}^{d}\int \partial_j(\partial_k u h_j)\partial_k\overline{u}\,\mathrm{d}x.\\
&= -\sum_{j,k=1}^{d}\int \partial_{jk} u h_j\partial_k\overline{u}\,\mathrm{d}x-\sum_{j,k=1}^{d}\int \partial_k u\partial_j h_j\partial_k\overline{u}\,\mathrm{d}x.
\end{aligned}$$

Therefore,

$$\sum_{j,k=1}^{d}\Re\int \partial_k u h_j\partial_{kj}\overline{u}\,\mathrm{d}x=-\frac{1}{2}\Re\int \mathrm{div}h|\nabla u|^2\,\mathrm{d}x.$$

Next, for (6.3.35),we have

$$\Re\int f(u)h\cdot\nabla\overline{u}\,\mathrm{d}x=\Re\int h\cdot\nabla F(u)\,\mathrm{d}x=-\Re\int \mathrm{div}hF(u)\,\mathrm{d}x.$$

Finally, for the last term (6.3.36) use the fact that

$$\Re h\cdot\nabla\left(\frac{a}{|x|^2}u\overline{u}\right)=\Re h\cdot\nabla\left(\frac{a}{|x|^2}\right)|u|^2+2\Re\frac{a}{|x|^2}uh\cdot\nabla\overline{u}$$

to obtain

$$\Re\int\frac{a}{|x|^2}uh\cdot\nabla\overline{u}\,\mathrm{d}x=-\frac{1}{2}\Re\int h\cdot\frac{a}{|x|^2}|u|^2\,\mathrm{d}x-\frac{1}{2}\Re\int \mathrm{div}h\cdot\nabla\left(\frac{a}{|x|^2}\right)|u|^2\,\mathrm{d}x.$$

□

6.3.4. Virial-Morawetz estimate. In this subsection, we use the above calculations to obtain the following virial-Morawetz estimate

PROPOSITION 6.3.9. *Assume $d=2$ and $u\in C(\mathbf{R},H^1_a)$ is a global solution to (6.0.1) or (6.0.2). Then, for any $R>0$ and $T_2>T_1>0$, we have*

$$\int_{T_1}^{T_2}\int |G(u)|\,\mathrm{d}x\mathrm{d}t\lesssim R+(T_2-T_1)R^{-2}. \tag{6.3.37}$$

Furthermore for any $\delta>0$ and $T>0$ we have

$$\int_{T}^{\infty} t^{-\frac{1}{3}-\delta}\int |G(u)|\,\mathrm{d}x\,\mathrm{d}t\lesssim T^{-\delta}. \tag{6.3.38}$$

PROOF. Again, we shall prove the result for the NLS case as the NLKG case is similar. Let $w : \mathbf{R}^d \to \mathbf{R}$ be a weight, and define $h(x) = \nabla w(x)$ and $q(x) = \frac{1}{2}\mathrm{div} h(x) = \frac{1}{2}\Delta w(x)$. With these choices, we obtain

$$\begin{aligned}\partial_t M(t) &= \sum_{j,k=1}^{d} \Re \int \partial_k u \partial_{jk}^2 w \partial_j \overline{u} \, \mathrm{d}x - \frac{1}{2}\int \Delta w G(u) \, \mathrm{d}x \\ &\quad + \frac{1}{2}\int -\frac{1}{2}|u|^2 \Delta^2 w + \frac{a}{|x|^2}\Delta w |u|^2 \, \mathrm{d}x + a\int \nabla w \cdot \frac{x}{|x|^4}|u|^2 \, \mathrm{d}x\end{aligned}$$

Now, fix some $R > 0$ and define

$$w(x) = \begin{cases} |x|^2, & |x| \le R/2 \\ R|x|, & |x| \ge R \end{cases} \tag{6.3.39}$$

and for $\frac{R}{2} < |x| < R$ we impose that

$$\partial_r w_R \ge 0, \partial_r^2 w_R \ge 0, |\partial^\alpha w_R(x)| \lesssim_\alpha R|x|^{-|\alpha|+1} \text{ for } |\alpha| \ge 1. \tag{6.3.40}$$

Since $\Delta w \ge 0$ for all $x \in \mathbf{R}^2$ for this choice of $w(x)$, we have $\int \frac{a}{|x|^2}\Delta w|u|^2 \, \mathrm{d}x \ge 0$. Thus, we may remove this term to obtain for some constants $C_1, C_2 > 0$ that $\partial_t M(t)$ is bounded below by

$$2\int_{|x|<R/2} |\nabla u|^2 + \frac{a}{|x|^2}|u|^2 - \frac{d}{2}G(u) \, \mathrm{d}x \tag{6.3.41}$$

$$+ \int_{R/2<|x|<R} \partial_k u \partial_{jk}^2 w \partial_j \overline{u} - C_1|u|^2\Delta^2 w \tag{6.3.42}$$

$$-C_2 G(u)\Delta w + a\nabla w \cdot \frac{x}{|x|^4}|u|^2 \, \mathrm{d}x \tag{6.3.43}$$

$$+ \int_{|x|>R} a\frac{R}{|x|^3}|u|^2 + \frac{R}{|x|}\left(|\nabla u|^2 - |\partial_r u|^2\right) - (d-1)\frac{R}{|x|}G(u) \, \mathrm{d}x. \tag{6.3.44}$$

For (6.3.44), since u is radial, we have $|\nabla u|^2 - |\partial_r u|^2 = 0$. Thus, we have with some $C_3 > 0$ that

$$(6.3.44) \ge \frac{a\|u\|_2^2}{R^2} - C_3\int_{|x|>R} G(u) \, \mathrm{d}x.$$

For (6.3.42), the conditions for $w(x)$ in (6.3.40) ensure the summation is non-negative and also that $\nabla w \cdot \frac{x}{|x|^4}|u|^2 = \frac{1}{r^2}\partial_r w|u|^2 \ge 0$. Thus,

$$(6.3.42) + (6.3.43) \ge -C_1\frac{\|u\|_2^2}{R^2} - C_2\int_{\frac{R}{2}<|x|<R} G(u) \, \mathrm{d}x.$$

For (6.3.41) we define a smooth cutoff function χ with support $\{x \in \mathbf{R}^2 : |x| \le 1/2\}$ and set $\chi_R(x) := \chi(x/R)$. With the observation that

$$\int {\chi_R}^2|\nabla u|^2 \, \mathrm{d}x = \int |\nabla(\chi_R)u|^2 + \chi_R\Delta(\chi_R)|u|^2 \, \mathrm{d}x, \tag{6.3.45}$$

we find that

$$\begin{aligned}(6.3.41) \quad &\ge \quad 2\left(\|\chi_R u\|_{\dot{H}_a^1} - \frac{d}{2}\int G(\chi_R u) \ \mathrm{d}x\right) - \frac{d}{2}\int G(u) - G(\chi_R u) \, \mathrm{d}x \\ &\quad + \int \mathcal{O}\left(\frac{1}{R^2}|u|^2\right) \, \mathrm{d}x + \int (1-\chi_R^2)G(u) \, \mathrm{d}x.\end{aligned}$$

To obtain the estimate (6.3.37), we need two more ingredients.

First, we use the radial Sobolev embedding to obtain the following estimate (see [**23**, Proposition 3.7])

$$\left|\int G(u) - G(\chi_R u)\,\mathrm{d}x\right| \lesssim \frac{1}{R^2}, \tag{6.3.46}$$

$$\left|\int (1-\chi_R^2) G(u)\,\mathrm{d}x\right| \lesssim \frac{1}{R^2}. \tag{6.3.47}$$

Furthermore, to deal with (6.3.41), in the defocusing case ($\lambda = -1$), we have the estimate

$$\|\chi_R u\|_{\dot{H}^1_a}^2 - \frac{d}{2}\int G(\chi_R u)\,\mathrm{d}x \gtrsim \int |G(\chi_R u)|\,\mathrm{d}x.$$

For the focusing case ($\lambda = 1$), we follow the argument in [**23**, Proposition 2.6] to obtain that a more general result that

$$K_{\alpha,\beta}(\chi_R u(t)) \gtrsim \int G(\chi_R(u(t))\,\mathrm{d}x. \tag{6.3.48}$$

In this case, we first claim that there exists an $R_0 > 0$ depending on mass and energy of the initial data such that for any $R > R_0$, we have $\sup_t J(\chi_R u(t)) < m_{\alpha,\beta}$. By assumption, we have $\sup_t J(u(t)) < m_{\alpha,\beta}$. Using the fact that $\chi_R \leq 1$, together with identity (6.3.45), we find that

$$\begin{aligned} J(\chi_R u(t)) &= \frac{1}{2}\|\chi_R u(t)\|_{H^1_a}^2 - \frac{1}{2}\int F(\chi_R u)\,\mathrm{d}x \\ &\leq J(u(t)) + CR^{-2}. \end{aligned}$$

We now choose R_0 such that

$$CR_0^{-2} < \frac{1}{2}(m_{\alpha,\beta} - J(u_0)). \tag{6.3.49}$$

Next, consider the continuous orbit $\{\chi_R u(t) : R > R_0\}$ in $\{J(\phi) < m_{\alpha,\beta}\}$. We note that $u(t) \in \mathcal{K}^+$ is a limit point for this set. Furthermore, since $\mathcal{K}^+$ is open and connected, we conclude that for all $R > R_0$ and $t \in \mathbf{R}$, we have $K(\chi_R u(t)) > 0$. Furthermore, from the choice of R_0 in (6.3.49), we have

$$m_{\alpha,\beta} - \sup_{t\in\mathbf{R}} J(\chi_R u(t)) \geq \frac{1}{2}(m_{\alpha,\beta} - J(u_0)) = C.$$

Next, using the variational estimate (6.3.29), we have

$$\begin{aligned} K(\chi_R u(t)) &\geq C\min\left\{m_{\alpha,\beta} - \sup_{t\in\mathbf{R}} J(\chi_R u(t)), K^Q(\chi_R u(t))\right\} \\ &\geq C\min\left\{C, K^Q(\chi_R u(t))\right\}. \end{aligned}$$

Now, to obtain (6.3.48), we consider the two cases. First, suppose that $K^Q(\chi_R u(t)) \geq C$. Then, by combining the global-in-time theory and the Moser-Trudinger inequality, we obtain that

$$\int G(\chi_R u(t))\,\mathrm{d}x \leq \int G(u(t))\,\mathrm{d}x \lesssim 1,$$

and therefore

$$K(\chi_R u(t)) \geq C \geq C\int G(\chi_R(u(t))\,\mathrm{d}x.$$

Thus, we obtain (6.3.48) for this case. Otherwise, suppose that $K^Q(\chi_R u(t)) < C$. By definition, $K(\chi_R u(t)) > 0$ gives that

$$K^Q(\chi_R u(t)) > \int G(\chi_R u(t))\,\mathrm{d}x.$$

Therefore,

$$K(\chi_r u(t)) \geq C K^Q(\chi_R u(t)) \geq C \int G(\chi_R u(t))\,\mathrm{d}x.$$

Thus, we have obtained (6.3.48) in both cases.

Finally, putting everything together, integrating over $[T_1, T_2]$ and discarding positive terms, we obtain

$$c\int_{T_1}^{T_2}\int G(u)\,\mathrm{d}x\mathrm{d}t \lesssim \sup_{t\in[T_1,T_2]}|M(t)| + \int_{T_1}^{T_2}\int_{|x|>R} G(u)\,\mathrm{d}x\mathrm{d}t + \frac{T_2-T_1}{R^2}\|u\|_2^2.$$

Now, combining this with (6.3.46) and the fact that $\sup_{t\in[T_1,T_2]}|M(t)| \leq R$, we obtain the required result for $R > R_0$. In the case $R \leq R_0$, we can simply use that

$$\int |\chi_R G(u(t))|\,\mathrm{d}x \lesssim C\int e^{\kappa_0|u|^2} - 1\,\mathrm{d}x \lesssim C\left(\|u\|_{L_t^\infty H_a^1}\right).$$

Thus, we obtain for all $0 < T_1 < T_2$ that

$$\int_{T_1}^{T_2}\int |G(u)|\,\mathrm{d}x \lesssim R + \frac{T_2-T_1}{R^2}.$$

This proves (6.3.37). Finally, to obtain (6.3.38), we follow the argument in [**23**, Lemma 2.6] (i.e. let $T_1 = 2^k T$, $T_2 = 2^{k+1}T$ and $R = (2^k T)^{1/3}$ and then sum up these integrals). □

Corollary 6.3.10. *Let $d = 2$ and let $\delta > 0$ be sufficiently small. Define $\alpha := 1/3 + \delta$ and $\beta := 1/2 + \delta$. We have*

(a) *for $t > 0$,*

$$\int_T^\infty t^{-\beta}\int |f(u)|\,\mathrm{d}x\mathrm{d}t \lesssim T^{-\delta}, \tag{6.3.50}$$

and also

(b) *For any $\epsilon > 0, T > 0$, there exists a $T_0 = T_0(\epsilon, T) > T$ such that*

$$\int_{T_0 - T_0^{1-\alpha}/10}^{T_0}\int |G(u)|\,\mathrm{d}x\mathrm{d}t \lesssim \epsilon. \tag{6.3.51}$$

Proof. See [**23**, Lemma 2.9]. □

6.4. Proof of scattering

We shall now prove scattering. We split the proof into two cases: where the coefficient a is sufficiently large (here, we take $a > 1$) and where the coefficient is small ($0 < a \leq 1$). The issue is that required continuity results for the conjugation operator (Proposition 6.1.4) in dimension two only hold in the range $p \in (1, 2/(s+\sigma))$. Since we study scattering at regularity $s = 1$, we see that this range becomes $p \in (1, 2/(1+\sigma))$. Recall that $\sigma = -\nu < 0$, thus using larger coefficients improves the range. Indeed, for $a > 1$, this range covers exponents arbitrarily close to $p = \infty$, while for small a, there is only a small amount of room above $p = 2$. Guo-Shen

[**23**] uses the Strichartz admissible space $L^{2(-\eta)}W^{1,\infty(\eta)}$, which is only controlled by $L^{2(-\eta)}W_a^{1,\infty(\eta)}$ if $a > 1$. Recall the notation $q(\epsilon)$ is defined for small $\epsilon > 0$ via

$$\frac{1}{q(\epsilon)} = \frac{1}{q} + \epsilon.$$

6.4.1. Proof of scattering – NLS case. We recall the strong Strichartz space for the NLS (6.0.1)

$$\mathrm{Str}_a(I) := L_t^\infty H_a^1 \cap L_t^{2(\eta)} W_a^{1,\infty(-\eta)}(I \times \mathbf{R}^2). \tag{6.4.1}$$

We also define the weak Strichartz space

$$W(I) = L^6_{t,x}(I \times \mathbf{R}^2). \tag{6.4.2}$$

Recall that for $a > 1$, we have

$$\mathrm{Str}_a(I) \hookrightarrow \mathrm{Str}_0(I). \tag{6.4.3}$$

In order to prove the scattering result, we shall show that for all $\epsilon > 0$, there exists a $T > 0$ such that

$$\|S_a(t-T)u(T)\|_{W_T} < \epsilon. \tag{6.4.4}$$

Recall the notation $S_a(t) := e^{it\mathscr{L}_a}$. Firstly, we have that

$$\begin{aligned} & S_a(t-T)u(T) \\ = \;& S_a(t)u_0 + \int_0^T S_a(t-s)f(u)\,\mathrm{d}s \\ = \;& S_a(t)u_0 + \int_0^{T-\tau} S_a(t-s)f(u)\,\mathrm{d}s + \int_{T-\tau}^T S_a(t-s)f(u)\,\mathrm{d}s \\ = \;& I + II + III, \end{aligned}$$

For the term I, there exists a $T > 0$ such that $\|I\|_{W_T} < \epsilon$. Next, for the term II, we use the fact that

$$\int_{T_1}^{T_2} S_a(t-s)f(u)\,\mathrm{d}s = S_a(t-T_2)u(T_2) - S_a(t-T_1)u(T_1)$$

as well as Strichartz estimates and the triangle inequality to obtain$\|II\|_{S_T} \lesssim 1$. Thus to show (6.4.4) for II, it suffices to show that $\|II\|_{L^\infty_{t,x}} < \epsilon$ and then interpolate. Indeed, using dispersive estimates, we obtain

$$\|II\|_{L^\infty_{T,x}} \lesssim \left\| \int_0^1 |t-s|^{-1} \|f(u)\|_{L^1_x}\,\mathrm{d}s \right\|_{L^\infty_T} + \left\| \int_1^{T-\tau} |t-s|^{-1} \|f(u)\|_{L^1_x}\,\mathrm{d}s \right\|_{L^\infty_T}.$$

For the first term, we use the L^1 control $\|f(u)\|_{L^1} \lesssim 1$ to find that for sufficiently large $T > 0$,

$$\left\| \int_0^1 |t-s|^{-1} \|f(u)\|_{L^1_x}\,\mathrm{d}s \right\|_{L^\infty_T} \lesssim \left\| \int_0^1 |t-s|^{-1}\,\mathrm{d}s \right\|_{L^\infty_T} \lesssim \left\| \log_e \left(1 - \frac{1}{t}\right) \right\|_{L^\infty_T} \lesssim \frac{1}{\tau}.$$

For the second term, we use Corollary 6.3.10 to obtain

$$\begin{aligned}\left\|\int_1^{T-\tau}|t-s|^{-1}\|f(u)\|_{L^1_x}\,\mathrm{d}s\right\|_{L^\infty_T} &= \left\|\int_1^{T-\tau}|t-s|^{-1}s^\beta s^{-\beta}\|f(u)\|_{L^1_x}\,\mathrm{d}s\right\|_{L^\infty_T}\\ &\lesssim T^\beta\tau^{-1}\int_1^{T-\tau}s^\beta\|f(u)\|_{L^1_x}\,\mathrm{d}s\\ &\lesssim T^\beta\tau^{-1}.\end{aligned}$$

Thus, we obtain $\|II\|_{L^\infty_{T,x}}<\epsilon$. Finally, we need to estimate the term III. Recall the integral equation

$$u(t)=S_a(t-T+\tau)u(T-\tau)-i\int_{T-\tau}^t S_a(t-s)f(u)\,\mathrm{d}s. \tag{6.4.5}$$

In order to obtain $\|III\|_{W_T}<\epsilon$, we shall first show that $\|u\|_{S_a([T-\tau,T])}\lesssim 1$. For this task, it remains to estimate the integral in (6.4.5). By conjugation and Strichartz estimates, we obtain

$$\begin{aligned}&\left\|\int_{T-\tau}^t S_a(t-s)f(u)\,\mathrm{d}s\right\|_{L^\infty_t H^1_a\cap L^{2(-\eta)}_t W^{1,\infty(\eta)}_x}\\ =\ &\left\|\int_{T-\tau}^t S_a(t-s)\langle D_a\rangle f(u)\,\mathrm{d}s\right\|_{L^\infty_t L^2_x\cap L^{2(-\eta)}_t L^{\infty(\eta)}_x}\\ =\ &\left\|\mathcal{K}^+\int_{T-\tau}^t S(t-s)\langle\nabla\rangle\mathcal{K}^-f(u)\,\mathrm{d}s\right\|_{L^\infty_t L^2_x\cap L^{2(-\eta)}_t L^{\infty(\eta)}_x}\\ \lesssim\ &\left\|\int_{T-\tau}^t S(t-s)\langle\nabla\rangle\mathcal{K}^-f(u)\,\mathrm{d}s\right\|_{L^\infty_t L^2_x\cap L^{2(-\eta)}_t L^{\infty(\eta)}_x}\\ \lesssim\ &\|\langle\nabla\rangle\mathcal{K}^-f(u)\|_{L^{2(-\eta)}_t L^{1(\eta)}_x}\lesssim\|\langle\nabla\rangle f(u)\|_{L^{2(-\eta)}_t L^{1(\eta)}_x},\end{aligned}$$

Furthermore, we have for $a>1$ that

$$\begin{aligned}\|\langle\nabla\rangle f(u)\|_{L^{2(-\eta)}_t L^{1(\eta)}_x} &\lesssim \left\|\left(e^{\kappa_0|u|^2}-1\right)\kappa_0|u|^2\langle\nabla\rangle u\right\|_{L^{2(-\eta)}_t L^{1(\eta)}_x}\\ &\lesssim \left\|e^{\kappa_0|u|^2}-1\right\|_{L^\infty_t L^{1(49\eta)}_x}\left\||u|^2\langle\nabla\rangle u\right\|_{L^{2(-\eta)}_t L^{\infty(48\eta)}_x}\\ &\lesssim \|\langle\nabla\rangle u\|_{L^{2(-\eta)}_t L^{\infty(\eta)}_x}\|u\|^2_{L^{\infty(\eta/2)}_t L^{\infty(47\eta/2)}_x}\\ &\lesssim \|u\|^\theta_{L^6_{t,x}}\|u\|^{1-\theta}_{S(I)}\lesssim\|u\|^\theta_{L^6_{t,x}}\|u\|^{1-\theta}_{S_a(I)},\end{aligned}$$

These estimates combined with (6.4.5) give

$$\|u\|_{S_a([T-\tau,T])}\lesssim 1+\|u\|^\theta_{L^6_{t,x}([T-\tau,T]\times\mathbf{R}^2)}\|u\|^{1-\theta}_{S_a([T-\tau,T])}, \tag{6.4.6}$$

Thus, using a continuity argument and (6.3.51), which implies that $\|u\|_{L^6_{t,x}([T-\tau,T]\times\mathbf{R}^2)}<\epsilon$, we obtain $\|u\|_{S_a([T-\tau,T])}\lesssim 1$. Next, we note that $L^6_{T,x}$ is controlled by $L^4_{T,x}$ and $S_{a,T}$ by interpolation. Thus, we now show that $\|III\|_{L^4_{T,x}}\leq\epsilon$. Indeed, since $(4,4)$ is admissible, by Strichartz estimates and the calculations above, we have that

$$\|III\|_{L^4_{T,x}}\lesssim\|u\|^\theta_{L^6_{t\in[T-\tau,T],x}}\|u\|^{1-\theta}_{S([T-\tau,T])}<\epsilon.$$

Thus, we have (6.4.4) and using the integral equation (6.4.5), we obtain the estimates

$$\begin{aligned}\|u\|_{W_T} &\lesssim \epsilon + \|u\|_{W_T}^a \|u\|_{S_T}^b\\ \|u\|_{S_T} &\lesssim 1 + \|u\|_{W_T}^{a'} \|u\|_{S_T}^{b'}.\end{aligned}$$

Hence, we obtain $\|u\|_{S_a(\mathbf{R})} < \infty$. Finally, standard arguments show that scattering follows for $a > 1$.

Now, we deal with the case when a is close to zero. Choose η small enough so that $2(-\eta)$ is in the range $(1, 2/(1+\sigma))$. Then, we have

$$\begin{aligned}\|\langle\nabla\rangle f(u)\|_{L_t^{2(-\eta)} L_x^{1(\eta)}} &\lesssim \left\|\left(e^{\kappa_0|u|^2} - 1\right)\kappa_0|u|^2\langle\nabla\rangle u\right\|_{L_t^{2(\eta)} L_x^{1(-\eta)}}\\ &\lesssim \left\|e^{\kappa_0|u|^2} - 1\right\|_{L_t^2 L_x^{2(-3\eta/2)}} \left\||u|^2\right\|_{L_t^{\infty(\eta/2)} L_x^{\infty(\eta)}} \|\langle\nabla\rangle u\|_{L_t^{\infty(\eta/2)} L_x^{2(-\eta/2)}}.\end{aligned}$$

Thus, we can control the $|u|^2$ term by $L_{t,x}^6$ and $S_a(I)$ as above, and control the $\langle\nabla\rangle u$ by $S_a(I)$. It remains to control the exponential term. For this term, we have

$$\left\|e^{\kappa_0|u|^2} - 1\right\|_{L_x^{2(-3\eta/2)}} \lesssim \left\|e^{\kappa_0|u|^2} - 1\right\|_{L_x^1}^{1/2-3\eta/2} \left\|e^{\kappa_0|u|^2} - 1\right\|_{L_x^\infty}^{1/2+3\eta/2}$$

Since $\|u\|_{\dot H_a^1} < \frac{\kappa_a^*}{\kappa_0}$ (see Proposition 6.3.7), the first term is bounded by the Moser-Trudinger inequality. Furthermore, there exists a $\Theta \in (0,1)$ such that $\|u\|_{\dot H_a^1} < \Theta\frac{\kappa_a^*}{\kappa_0}$. Thus, there exists a $\mu > 0$ such that we have

$$\|u\|_{H_{\mu,a}^1} < \Theta'\frac{\kappa_a^*}{\kappa_0},$$

where $\Theta' = (1+\Theta)/2$ and $\|u\|_{H_{\mu,a}^1}^2 := \|u\|_{H_a^1}^2 + \mu\|u\|_2^2$. Thus, it remains to control the L^∞ term. We shall use the logarithmic inequality (Proposition 6.1.6). Indeed, choose $\alpha = \theta(1-2\eta)$. Thus, we need to choose

$$\lambda > 1/2\pi\theta(1-2\eta)$$

in the statement of Proposition 6.1.6. Furthermore, we choose λ such that we also have

$$\kappa_a^*\left(\frac{1}{2} + \frac{3\eta}{2}\right)\lambda\Theta' = \frac{2}{\theta(1/2-\eta)}.$$

Next, for the choice $\alpha = \theta(1-2\eta)$, define $s_\theta = 1 - \theta\eta$ and $r_\theta = 2(-\theta(1-\eta)/2)$ so that we have the following embeddings:

$$W_a^{s_\theta, r_\theta} \hookrightarrow W^{s_\theta, r_\theta} \hookrightarrow B_{r_\theta,2}^{s_\theta} \hookrightarrow B_{\infty,\infty}^\alpha = \mathcal{C}^\alpha.$$

We now consider two cases. First, suppose that $\|u\|_{\mathcal{C}^\alpha} \gtrsim \|u\|_{L^\infty} \gtrsim 1$. Then, also using the fact that $x \mapsto x^2 \ln(C_\lambda + c/x)$ is increasing, we apply the logarithmic inequality to obtain

$$\begin{aligned}e^{\kappa_0(1/2+3\eta/2)\|u\|_{L^\infty}^2} &\lesssim \left(1 + \frac{\|u\|_{\mathcal{C}^\alpha}}{\|u\|_{H_{\mu;a}}}\right)^{\kappa_0(1/2+3\eta/2)\lambda\|u\|_{H_a^1}^2}\\ &\lesssim \left(1 + \frac{\|u\|_{\mathcal{C}^\alpha}}{\Theta'\cdot\kappa_a^*/\kappa_0}\right)^{\kappa_a^*\left(\frac{1}{2}+\frac{3\eta}{2}\right)\lambda\Theta'}\\ &\lesssim \|u\|_{C^\alpha}^{\frac{2}{\theta(1/2-\eta)}}.\end{aligned}$$

Finally, by interpolation and Sobolev embedding, this gives us with $q_\theta = \infty(\theta(1/2-\eta))$

$$\begin{aligned}\left\|e^{\kappa_0|u|^2}-1\right\|_{L_t^2 L_x^{2(-3\eta/2)}} &\lesssim \left\|\|u\|_{C^\alpha}^{\frac{2}{\theta(1/2-\eta)}}\right\|_{L^2} \\ &\lesssim \|\langle\nabla\rangle^{s_\theta} u\|_{L_t^{q_\theta} L_x^{r_\theta}}^{\frac{2}{\theta(1/2-\eta)}} \\ &\lesssim \|u\|_{S_a(I)}^{\frac{2}{\theta(1/2-\eta)}}.\end{aligned}$$

In the other case where $\|u\|_{L^\infty} \lesssim 1$, we have that $|\langle\nabla\rangle f(u)| \lesssim |u|^4 |\langle\nabla\rangle u|$, thus we can handle this case with only Sobolev embeddings.

6.4.2. Proof of scattering – NLKG case. We now consider scattering for the NLKG case. The proof is similar to the NLS case. Indeed, first define the strong Strichartz space for the NLKG (6.0.2)

$$\mathrm{Str}_a(I) := L_t^\infty H_a^1 \cap L_t^{2(\eta)} W_a^{1/2,\infty(\eta/2)}(I \times \mathbf{R}^2), \tag{6.4.7}$$

and the weak Strichartz space

$$W(I) = L_{t,x}^6(I \times \mathbf{R}^2). \tag{6.4.8}$$

Define

$$K_a(t) := \frac{\sin\left(t\sqrt{-\Delta + \frac{a}{|x|^2}}\right)}{\sqrt{-\Delta + \frac{a}{|x|^2}}}.$$

Similar to the NLS case, the goal is now to show that for all $\epsilon > 0$, there exists a $T > 0$ such that

$$\left\|\dot{K}_a(t-T)u(T) + K_a(t-T)u_t(T)\right\|_{W_T} < \epsilon.$$

Firstly, we have that

$$\begin{aligned}& \dot{K}_a(t-T)u(T) + K(t-T)u_t(T) \\ = \;& \dot{K}_a(t)u_0 + K_a(t)u_1 + \int_0^T K_a(t-s)f(u)\,\mathrm{d}s \\ = \;& \left(\dot{K}_a(t)u_0 + K_a(t)u_1\right) + \int_0^{T-\tau} K_a(t-s)f(u)\,\mathrm{d}s + \int_{T-\tau}^T K_a(t-s)f(u)\,\mathrm{d}s \\ = \;& I + II + III.\end{aligned}$$

We can deal with terms I and II in a similar way to the NLS case, where we instead use the radial Strichartz estimate. For term III, we shall obtain $\|III\|_{W_T} < \epsilon$ by a similar argument as we did for the NLS case and also using the fractional chain rule. Thus, we have

$$\begin{aligned}
&\left\|\int_{T-\tau}^{t} K_a(t-s)f(u)\,\mathrm{d}s\right\|_{L_t^\infty H_a^1\cap L_t^{2(\eta)}W_x^{1/2,\infty(-\eta)}}\\
= &\left\|\int_{T-\tau}^{t} K_a(t-s)\langle D_a\rangle^{1/2}f(u)\,\mathrm{d}s\right\|_{L_t^\infty H_a^1\cap L_t^{2(-\eta)}L_x^{\infty(\eta)}}\\
= &\left\|\mathcal{K}^+\int_{T-\tau}^{t} K_a(t-s)\langle\nabla\rangle^{1/2}\mathcal{K}^-f(u)\,\mathrm{d}s\right\|_{L_t^\infty H_a^1\cap L_t^{2(-\eta)}L_x^{\infty(\eta)}}\\
\lesssim &\left\|\int_{T-\tau}^{t} K_a(t-s)\langle\nabla\rangle\mathcal{K}^-f(u)\,\mathrm{d}s\right\|_{L_t^\infty H_a^1\cap L_t^{2(-\eta)}W_x^{-1/2,\infty(\eta)}}\\
\lesssim &\left\|\langle\nabla\rangle\mathcal{K}^-f(u)\right\|_{L_t^{2(-\eta)}W_x^{-1/2,1(\eta)}}\lesssim\|f(u)\|_{L_t^{2(-\eta)}W_x^{1/2,1(\eta)}}.
\end{aligned}$$

Next, we use the fractional chain rule to obtain

$$\begin{aligned}
\|f(u)\|_{L_t^{2(-\eta)}W_x^{1/2,1(\eta)}} &\lesssim \left\|\left(e^{\kappa_0|u|^2}-1\right)\kappa_0|u|^2\langle\nabla\rangle^{1/2}u\right\|_{L_t^{2(-\eta)}L_x^{1(\eta)}}\\
&\lesssim \left\|e^{\kappa_0|u|^2}-1\right\|_{L_t^\infty L_x^{1(49\eta)}}\left\||u|^2\right\|_{L_t^{\infty(\eta)}L_x^{\infty(95\eta/2)}}\left\|\langle\nabla\rangle^{1/2}u\right\|_{L_t^{2(-\eta)}L_x^{\infty(\eta/2)}}\\
&\lesssim \|u\|_{\mathrm{Str}_a(I)}\|u\|^2_{L_t^{(\infty(\eta/2)}L_x^{(95\eta/4)}}\\
&\lesssim \|u\|^\theta_{L^6_{t,x}(I\times\mathbf{R}^2)}\|u\|^{3-\theta}_{\mathrm{Str}_a(I)}.
\end{aligned}$$

By the same argument as in the NLS case, we obtain scattering. Finally, we deal with the case when a is close to zero. In this case, we have

$$\begin{aligned}
\|f(u)\|_{L_t^{2(-\eta)}W_x^{1/2,1(\eta)}} &\lesssim \left\|\left(e^{\kappa_0|u|^2}-1\right)\kappa_0|u|^2\langle\nabla\rangle^{1/2}u\right\|_{L_t^{2(-\eta)}L_x^{1(\eta)}}\\
&\lesssim \left\|e^{\kappa_0|u|^2}-1\right\|_{L_t^\infty L_x^{\infty(\eta/2)}}\left\||u|^2\right\|_{L_t^{\infty(\eta/2)}L_x^{\infty(\eta)}}\left\|\langle\nabla\rangle^{1/2}u\right\|_{L_t^{\infty(\eta/2)}L_x^{2(-\eta/2)}}.
\end{aligned}$$

The last two terms are controlled as in the NLS case. Thus, we finally need to control the L^∞ term. We shall again use the logarithmic inequality (Proposition 6.1.6). We take Θ such that $\|u\|_{\dot H_a^1}<\Theta\frac{\kappa_a^*}{\kappa_0}$ as well as a constant $\mu>0$ such that we have

$$|u\|^2_{H^1_{\mu,a}}<\Theta'\frac{\kappa_a^*}{\kappa_0},$$

where $\Theta'=(1+\Theta)/2$ and $\|u\|_{H^1_{\mu,a}}:=\|u\|^2_{H_a^1}+\mu\|u\|_2^2$ as before. Next, we choose $\alpha=\theta(1/2-\eta)$. Thus, we need to choose

$$\lambda>\frac{1}{2\pi\theta(1/2-\eta)}$$

in the statement of Proposition 6.1.6. We also set

$$\kappa_a^*\left(\frac{1}{2}+\frac{3\eta}{2}\right)\lambda\Theta'=\frac{2}{\theta(1/2-\eta)}.$$

Next, with this choice of α and $s_\theta=1-\theta/2$, $r_\theta=2(\theta(1-\eta)/2)$,

$$W_a^{s_\theta,r_\theta}\hookrightarrow W^{s_\theta,r_\theta}\hookrightarrow B^{s_\theta}_{r_\theta,2}\hookrightarrow B^\alpha_{\infty,\infty}=\mathcal{C}^\alpha.$$

Again, we consider two cases. Suppose that $\|u\|_{L^\infty} \gtrsim 1$. Then we have

$$\begin{aligned} e^{\kappa_0(1/2+3\eta/2)\|u\|^2_{L^\infty}} &\lesssim \left(1+\frac{\|u\|_{\mathcal{C}^\alpha}}{\|u\|_{H_{\mu;a}}}\right)^{\kappa_0(1/2+3\eta/2)\|u\|^2_{H^1_a}} \\ &\lesssim \left(1+\frac{\|u\|_{\mathcal{C}^\alpha}}{\Theta'\cdot\kappa_a^*/\kappa}\right)^{\kappa_a^*(1/2+3\eta/2)\lambda\Theta'} \\ &\lesssim \|u\|_{C^\alpha}^{\frac{2}{\theta(1/2-\eta)}}. \end{aligned}$$

Finally, this gives us

$$\begin{aligned} \left\|e^{\kappa_0|u|^2}-1\right\|_{L^2_tL^{2(-3\eta/2)}_x} &\lesssim \left\|\|u\|_{C^\alpha}^{\frac{2}{\theta(1/2-\eta)}}\right\|_{L^2} \\ &\lesssim \|\langle\nabla\rangle^{s_\theta}u\|^{\frac{2}{\theta(1/2-\eta)}}_{L_t^{\infty(\theta(1/2-\eta))}L_x^{2(-\theta(1-\eta)/2)}} \\ &\lesssim \|u\|^{2/\nu}_{S_a(I)}. \end{aligned}$$

The case where $\|u\|_{L^\infty} \lesssim 1$ can be dealt with using the observation made in the NLS case. We may then obtain scattering as in the NLS case.

Conclusion

In this thesis, we obtained scattering results for some nonlinear dispersive PDEs with inverse-square potential. These results were generalisations of the analogous result in the potential-free case. By applying the ideas of [**6**], we found that the Hankel transform could be used effectively to study radial problems. Indeed, we used the Hankel transform to gain an understanding of the linear theory of the Klein-Gordon flow in Chapter 3, as well as the Schroödinger flow in Chapter 6 – for instance, to obtain the relevant Strichartz estimates.

In our application of this linear theory to the non-linear scattering problems, we saw that a major issue that one has to deal with is the fact that many L^p estimates fail outside a range too far away from $p = 2$, if the coefficient of the inverse-square potential is negative. We also saw the importance of the equivalence of Sobolev norms in order to use the fractional chain rule associated with ∇. Furthermore, we saw that other tools such as Virial-Morawetz estimates could be applied in much the same way as in the potential-free case. Overall, many standard techniques employed to study the potential-free case can also be applied with the inverse-square potential.

Finally, let us also remark on some aspects of this thesis that could be explored in future research. We need other tools to study non-radial data. For instance, in Chapter 4, while we were able to obtain bilinear Strichartz estimates for radial initial data, there was an obstacle in the non-radial setting. In particular, we had to decompose non-radial data using a spherical decomposition in order to apply the Hankel transform, but we were unable to add these pieces back together to obtain a satisfactory bilinear estimate. For this problem, perhaps other methods such as physical space methods may be more effective in obtaining such estimates. Furthermore, a better understanding of the thresholds of the double logarithmic inequality and Moser-Trudinger inequality in Chapter 6 could be gained in future research. This would improve on the scattering result that was obtained.

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