The Fuchsian approach to global existence for hyperbolic equations

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Abstract

In this thesis we apply the global existence theory for Fuchsian equations that was developed in [1] to semi-linear wave equations. By compactifying a neighbourhood of spatial infinity in Minkowski and Schwarzschild space-times, we show through the introduction of suitable variables, that it is possible to transform a semi-linear wave equation on these space-times into a symmetric hyperbolic system of Fuchsian equations on a bounded domain. Once the wave equation is transformed into Fuchsian form, we then apply the global existence theory from [1], to obtain the existence of solutions to the wave equations on neighbourhoods of spatial infinity. We consider, in particular, three applications of this method. In the first two applications, we analyse the Cauchy problem for semi-linear wave equations in Minkowski and Schwarzschild space-times in a neighbourhood of spatial infinity with quadratic terms satisfying the *null condition*. In the third application, we investigate semi-linear wave equations in $3+1$ dimensions with semi-linear quadratic terms such that its associated asymptotic equation admits bounded solutions for suitably small choices of initial data. We call this the *bounded weak null condition* and we show that it is a special case of the *weak null condition*. In each of the three systems that we analyse here, we use the Fuchsian formalism to establish global existence of solutions along with decay estimates. The work in this thesis demonstrates the utility of the theory developed in [1] as a new method for the study of the Cauchy problem for non-linear wave equations.

Declaration

This thesis is an original work of my research and contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

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Contents

Listing of figures

To my family: Mum, Dad, Susana, Carlos, Miguel, Rocío and the little ones Emi, Santi and Migue.

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Unclose your mind. You are not a prisoner. You are a bird in flight, searching the skies for dreams.

Hard-Boiled Wonderland and the End of the World, Haruki Murakami

1

Introduction

1.1 Introduction to Fuchsian systems

Ordinary differential equations (ODEs) in the neighbourhood of singularities have been studied since Euler and Gauss. The list of mathematicians involved in the study of singular ODEs is extensive, and includes names like Riemann, Fuchs, and Frobenius, just to mention a few of them. In the theory of ordinary differential equations, points are classified as ordinary points and singular points. At an ordinary point, the coefficients of the equation are analytic, at a singular point the coefficients present singularities that are classified into regular and irregular. Consider an ordinary linear differential equation of order k^{th} defined on the complex plane $\mathbb{C}$

$$\sum_{i=0}^{k} q_i(z)u^{(i)} = 0, \tag{1.1.1}$$

where q_i are analytic coefficients defined on an open subset D of the complex plane except for a set of isolated points. These points are the poles of the function. Suppose $z = a$, such that $a \in \mathbb{C}$ is a singular point. We say that the point $z = a$ is a *regular singular point* if the coefficient q_{k-i} has a pole at most of order i. Otherwise the point is an *irregular singular point*. An ordinary differential equation is said to be Fuchsian if every singular point of the equation, including the point at infinity, is a regular singularity. These equations are named after Lazarus I. Fuchs, who studied these type of singularities. Some early interesting results include that any second order Fuchsian differential equation can be transformed by a linear fractional transformation into the so called Riemann differential equation. These equations have three pairs of characteristic exponents associated with the singular points $0, 1, \infty$, and they are a generalisation of hypergeometric equations. See [2] for an extended discussion on this topic.

Partial differential equations (PDEs) with a Fuchsian singularity, generalise the theory of linear ordinary Fuchsian equations. A Fuchsian PDE can be written as a system of the form

$$t\frac{\partial u}{\partial t} + Au = f(t, x_i, u, u_x), \tag{1.1.2}$$

where the coefficient A is a square matrix and the source term f vanishes as some power of t when $t \to 0$.

Fuchsian systems of PDEs (1.1.2) have been studied principally as Singular Initial Value Problems (SIVPs), where asymptotic data is given at the singular time $t = 0$, and the equation (1.1.2) is used to evolve the asymptotic data away from the singular time to construct solutions on time intervals $t \in (0, T]$ for some $T > 0$. Fuchsian systems with analytic coefficients viewed as SIVPs, have been studied in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]; see also [15, 16] for applications in the ODE setting.

Recent developments have extended the existence theory for Fuchsian SIVPs to Sobolev regularity by adapting local-in-time PDE techniques for hyperbolic equations. For these existence results, the asymptotic data can be large, but the existence time is expected to be small [17, 18, 19]. These techniques were applied to singular solutions of the Einstein's equations in the articles [20, 21, 22, 23, 24, 25, 26, 27].

1.2 The global initial value problem (GIVP)

An alternate approach to analysing Fuchsian systems of equations is to study the initial value problem, which was pioneered in [28]. In this approach, initial data is specified away from the singular time and evolved towards the singularity. Consider a system of symmetric hyperbolic equations in Fuchsian form

$$B^0(t,u)\partial_t u + B^i(t,u)\nabla_i u = \frac{1}{t}\mathcal{B}(t,u)\mathbb{P}u + F(t,u), \qquad \text{in}(T_1, T_0] \times \Sigma, \tag{1.2.1}$$

where Σ is a closed n-dimensional manifold, V is a N rank vector bundle, and B^0, B^i are symmetric operators on V. The unknown u is a time-dependent section of V, ∇ is a time-independent connection, $\mathcal{B}$ is a linear operator on V and $\mathbb{P}$ is a time-independent, covariantly constant, symmetric projection operator. The global initial value problem (GIVP) consists of studying (1.2.1) with suitable initial data specified at $u_0 \in T_0 \times \Sigma$, and to establish the existence of solutions to (1.2.1) in an interval that reaches the singular time at $t = 0$, that is $t \in (0, T_0]$. In this thesis, we will use a GIVP approach to study Fuchsian systems of the form (1.2.1). In our applications in Chapter 3 and 4, Fuchsian systems will arise from a conformal transformation of second order systems of wave equations.

In [28], it is established the existence of solutions to symmetric hyperbolic systems of the form (1.2.1), where the coefficients B^0, B^i, $\mathcal{B}$, and F are all regular in t as $t \searrow 0$. In the same article, it is shown that the Friedmann-Lemaître-Robertson-Walker (FLRW) solutions to the Einstein-Euler equations with a positive cosmological constant, can be cast into the Fuchsian form (1.2.1). This allows to establish the future non-linear stability of perturbations of FLRW solutions to the Einstein–Euler equations with a positive cosmological constant. The solutions exist for t on the whole interval $(0, 1]$, with suitably small initial data specified at $t = 1$. By construction, solutions to the Fuchsian GIVP yield solutions to the original system of Einstein-Euler equations. Other methods have been used to establish similar results, see for example [29, 30, 31, 32]. The theory developed in [28], has been used to establish the existence of solutions to the future for different hyperbolic systems on expanding cosmological space-times [33, 34, 35, 36, 37].

The results in [28] were generalized in [1] to apply to Fuchsian systems of the form (1.2.1), such that the coefficients B^i and source term F of the Fuchsian system (1.2.1) now are allowed to have singular behaviour on time and they can be expanded as

$$B^i(t,u) = B^i_0(t,u) + \frac{1}{t^{\frac{1}{2}}} B^i_1(t,u) + \frac{1}{t} B^i_2(t,u), \tag{1.2.2}$$

$$F(t,u) = F_0(t,u) + \frac{1}{t^{\frac{1}{2}}} F_1(t,u) + \frac{1}{t} F_2(t,u), \tag{1.2.3}$$

where the coefficients B^i_a and F^i_a, $a = 0, 1, 2$, are all regular in t as $t \searrow 0$. The main result from [1] which is Theorem 3.8, guarantees the existence of solutions to systems of the form (1.2.1) under a suitable small initial data assumption. This theorem also determines the rate at which the solutions decay as $t \searrow 0$. In this thesis we will use Theorem 3.8 to study Fuchsian systems with coefficients of the form (1.2.2)- (1.2.3).

The coefficients B^i_a, F^i_a must satisfy a set of conditions that are described in full detail in Section 2.2. These conditions are fundamental for the proof of Theorem 3.8 given in [1], and are required not only to establish the existence of solutions on time intervals of the form $(0, T_0]$, but also to obtain uniform decay estimates as $t \searrow 0$. In [38], it was shown that it is still possible to obtain existence of solutions on the time interval $(0, T_0]$ that do not decay as $t \searrow 0$ by modifying some of the assumptions on the coefficients.

1.3 A model Fuchsian equation

The aim of this section is to outline the main ideas that are behind the existence and decay estimate results from [1]. The Fuchsian method developed in [1] is largely based on the idea that, under suitable assumptions on the coefficients and initial data, the asymptotics of systems like (1.2.1) should be determined by an associated linear ODE of the form

$$\partial_t u = \frac{1}{t}\tilde{\mathcal{B}}u + F, \tag{1.3.1}$$

where

$$F = |t|^{-(1-p)}\tilde{F}(t), \quad 0 < p \le 1,$$

and $\tilde{F} \in C^0([-1, 0])$. The behaviour of the general system can be largely illustrated by studying this simple problem. We can examine explicitly the solutions of (1.3.1) and compare with a direct application of Theorem 3.8 from [1]. It can be shown that up to an arbitrarily small loss, Theorem 3.8 from [1] reproduce the behaviour of the solutions to (1.3.1) that we obtain below.

We assume for this particular example that all the fields are constant on the spatial manifold Σ, in other words, in local coordinates, they are independent of the spatial coordinates x.

We analyse (1.3.1) by restricting to an unknown with only two components $(u^1(t), u^2(t))$, one component has a possibly non-zero limit at $t = 0$, and the other decays to 0 at a fixed rate. We assume a $\tilde{\mathcal{B}}$ operator of the form

$$\tilde{\mathcal{B}} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \tag{1.3.2}$$

for some $a > 0$. Therefore, we can write the system (1.3.1) simply as

$$\begin{aligned} \partial_t u^1(t) &= |t|^{-(1-p)}\tilde{F}^1(t), \\ \partial_t u^2(t) &= \frac{a}{t}u^2(t) + |t|^{-(1-p)}\tilde{F}^2(t). \end{aligned} \tag{1.3.3}$$

Given initial data $u(-1) = (u_*, u_{**})^{\mathrm{tr}}$, we see that multiplying the second equation by $|t|^{-a}$ and integrating the system yields the unique solution given by

$$u(t) = \begin{pmatrix} u^1(t) \\ u^2(t) \end{pmatrix} = \begin{pmatrix} u_* + \int_{-1}^t |s|^{-1+p}\tilde{F}^1(s)ds \\ (-t)^a\big(u_{**} + \int_{-1}^t |s|^{-1+p-a}\tilde{F}^2(s)ds\big) \end{pmatrix}, \quad t \in [-1, 0). \tag{1.3.4}$$

From equation (1.3.4), we can see that $\lim_{t\nearrow 0} u^1(t)$, denoted $u^1(0)$, exists (since $p > 0$) and is given by

$$u^1(0) = u_* + \int_{-1}^0 |s|^{-1+p}\tilde{F}^1(s)ds, \tag{1.3.5}$$

and the decay estimates

$$|u^1(t) - u^1(0)| \lesssim |t|^p \quad \text{and} \quad |u^2(t)| \lesssim |t|^p + |t|^a \tag{1.3.6}$$

hold. This shows that if the source term F is not too singular, that is, in the case $p \geq a$, then the solutions of (1.3.1) behave like powers of $|t|$ where these powers are the eigenvalues of the matrix (1.3.2). Conversely, in the case that the source term is very singular, that is $p < a$, then there are $|t|^p$ "corrections" to these decay rates. To finalise this section, we assert that the optimal decay rates (1.3.6) of the solutions $(u^1(t), u^t(t))$ can be deduced from Theorem 3.8 in [1] up to an arbitrarily small loss, which does not significantly affect the main result.

1.4 Wave equations as Fuchsian systems

In this section we provide an informal introduction to what we call the *Fuchsian method*. The essence of the Fuchsian method as applied to wave equations, involves transforming a system of second order wave equations into a first order symmetric hyperbolic Fuchsian system. Once this is accomplished, we can apply the existence theory for Fuchsian systems from [1] provided that the Fuchsian system obtained from the wave equation satisfies the structural conditions given in [1], which we explain in detail in Section 2.2.

The class of semi-linear wave equations that we will study in Chapters 3 and 4 are of the form

$$\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu\bar{u}^K = \bar{a}^{K\mu\nu}_{IJ}\bar{\nabla}_\mu\bar{u}^I\bar{\nabla}_\nu\bar{u}^J, \tag{1.4.1}$$

were $\bar{u}^I$ denote a collection of scalar fields, with $1 \leq I \leq N$. The term $\bar{a}^K_{IJ} =$

$\bar{a}_{IJ}^{K\alpha\beta}\bar{\partial}_\alpha \otimes \bar{\partial}_\beta$, $1 \leq I, J, K \leq N$, is a prescribed smooth (2,0)-tensor field on $\mathbb{R}^4$, and $\bar{\nabla}$ is the Levi-Civita connection of the Minkowski metric $\bar{g} = \bar{g}_{\mu\nu} d\bar{x}^\mu \otimes d\bar{x}^\nu$ on $\mathbb{R}^4$. We use the coordinate chart $(\bar{x}^\mu)$ to denote spherical coordinates

$$(\bar{x}^\mu) = (\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3) = (\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi}),$$

in this coordinate system the Minkowski metric is given by

$$\bar{g} = -d\bar{t} \otimes d\bar{t} + d\bar{r} \otimes d\bar{r} + \bar{r}^2 \not{g}, \tag{1.4.2}$$

where

$$\not{g} = d\bar{\theta} \otimes d\bar{\theta} + \sin^2(\bar{\theta}) d\bar{\phi} \otimes d\bar{\phi},$$

is the canonical metric on the 2-sphere $\mathbb{S}^2$. We assume that the tensor field $\bar{a}_{IJ}^K$ is covariantly constant $\bar{\nabla}\bar{a}_{IJ}^K = 0$, this means that the components of $\bar{a}_{IJ}^K$ in Cartesian coordinates $(\hat{x}^\mu)$ are constants.

The goal in Chapters 3 and 4, is to transform the wave equation (1.4.1) into a Fuchsian system. In Chapter 3, the non-linear terms $\bar{a}_{IJ}^{K\mu\nu}\bar{\nabla}_\mu \bar{u}^I \bar{\nabla}_\nu \bar{u}^J$ satisfy the *null condition*, whereas in Chapter 4 we will consider non-linear terms satisfying the *bounded weak null condition* which is a generalization of the *weak null condition* and that we outline in the sections below. Although the transformation process is particular to each system, we can highlight 4 main steps required to transform a wave equation into a Fuchsian system:

(i) Transforming the physical manifold into a closed N-dimensional manifold whose boundary represents infinity of the physical manifold. In Chapters 3 and 4 we carry out this step by applying Friedrich's *cylinder at infinity* conformal transformation [39].

(ii) Transforming the second order wave equation into a first order symmetric hyperbolic equation via a change of variables.

(iii) A rescaling on time might be required in order to meet the coefficient assumptions (2.2) below. More specifically, the non-linear terms $\bar{a}_{IJ}^{K\alpha\beta}\bar{\nabla}_\alpha \bar{u}^I \bar{\nabla}_\beta \bar{u}^J$ after being transformed, must be able to be expanded as in (2.2.10).

(iv) Verification of the structural conditions. That means the system is symmetric hyperbolic of the form (1.2.1), the coefficients B^0, B^i, can be expanded as

(1.2.2), the non-linear terms can be expanded as (1.2.2) and the coefficients $B^0, B^i, \mathcal{B}, F$ also satisfy the coefficient assumptions from Section 2.2.

$$\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu\bar{u}^K = \bar{a}^{K\mu\nu}_{IJ}\bar{\nabla}_\mu\bar{u}^I\bar{\nabla}_\nu\bar{u}^J$$

$$B^0(t,u)\partial_t u^K + B^i(t,u)\nabla_i u^K = \frac{1}{t}\mathcal{B}(t,u)\mathbb{P}u^K + F^K(t,u),$$

Figure 1.1: Suppose we have a wave equation for which we want to prove global existence on a certain domain. We transform the wave equation in red throughout a series of steps into the Fuchsian symmetric hyperbolic system in green for which we can obtain global existence from the results reported in [1]. By construction, the solutions of the Fuchsian system in green yield solutions to the original system of wave equations.

After we have carried out successfully steps (i)-(iv) and verified that the system meets the necessary structural conditions, we can obtain global existence and decay estimates by applying the existence theory from [1] to the Fuchsian system obtained.

1.5 The null condition

The class of semi-linear wave equations that we will study in Chapter 3 are of the form (1.4.1), where the non-linear terms satisfy the so called *null condition*. Here, we briefly discuss the meaning of the *null condition* which was developed independently by Klainerman [40] and Christodoulou [41]. For an extended discussion, see [42, 43, 44, 45].

Definition 1.5.1. *The components $\bar{a}^{K\mu\nu}_{IJ}$ of the tensor field $\bar{a}^K_{IJ}$ satisfy the null condition if*

$$\bar{a}^{K\mu\nu}_{IJ}\xi_\mu\xi_\nu = 0 \tag{1.5.1}$$

for all null covectors [1] $\xi \in \mathbb{R}^4$

Given a system of wave equations of the form (1.4.1) on Minkowski space-time, with Minkowski metric given by (1.4.2) and a covariantly constant tensor field $\bar{a}^K_{IJ}$ we say that the equation (1.4.1) satisfies the *null condition* if the non linear terms satisfy the *null condition*; namely the components of the tensor field $\bar{a}^K_{IJ}$ satisfy equation (1.5.1). A typical example of an equation satisfying the null condition is given by

$$\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu u = \bar{g}^{\mu\nu}\bar{\nabla}_\mu u\bar{\nabla}_\nu u, \tag{1.5.2}$$

[1] A covector $\xi = (\xi_\mu) \in \mathbb{R}^4$ is called null if it satisfies $-\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 0$

where $\bar{g}$ is the Minkowski metric. The important feature of the null condition is that it is possible to determine that wave equations with a large class of non-linearities admit global solutions just by inspecting the non-linear terms.

To illustrate the effect of the null condition, we review an example attributed to L. Nirenberg in [40]. This example shows that the wave equation (1.5.2) in Minkowski space-time admits global solutions. In contrast, a very similar wave equation given by (1.5.5), that differs from (1.5.2) only by some terms in the non-linearities blows up in finite time. The emphasis is on the fact that the first example satisfies the null condition while the very similar second example does not. Following Nirenberg, we consider (1.5.2) with initial conditions at $t = 0$,

$$\begin{aligned} u(0) =& 0, \\ u_t(0) =& u_1. \end{aligned} \tag{1.5.3}$$

Using the change of variable $\phi = e^u$ the wave equation (1.5.2) and initial condition (1.5.3) can be transformed into

$$\Box\phi = 0 \tag{1.5.4}$$

and

$$\phi = 1, \quad \phi_t = u_1 \quad \text{at} \quad t = 0,$$

then it is not difficult to verify that the solution to (1.5.4) is given by

$$\phi = 1 + \frac{t}{2}\int_{|y|=1} u_1(x + ty)d\sigma(y).$$

where $d\sigma(y)$ is the surface element on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Inverting the transformation yields the solution

$$u = \ln\left(1 + \frac{t}{2}\int_{|y|=1} u_1(x + ty)d\sigma(y)\right),$$

to the original wave equation (1.5.2) and the initial conditions (1.5.3). Now, if the function u_1 vanishes at infinity and its size is sufficiently small, then the above solution exists globally on $\mathbb{R}^4$. If u_1 does not vanish at infinity, then the solution blows up at finite time. On the other hand, it is worth noting that every non trivial solution generated from compactly supported initial data, of the very similar wave equation

$$\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu u = (\partial_t u)^2, \tag{1.5.5}$$

which does not satisfy the *null condition*, blows up in finite time [46]. The difference between the two previous examples lies in the fact that solutions to equations of the form $\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu u = 0$ with data in C_0^∞ have gradients that decay as powers $\frac{1}{t^2}$ away from the cone $t = |x|$. The only *bad* directional derivative is the one that is transverse to the light cone. It can be shown that this *bad* derivative has a decay of the form $1/t$, see [42]. The *null condition* in the semi-linear setting excludes the possibility of having quadratic terms in the non-linearities that only involve *bad* derivatives, and consequently in each quadratic therm there is at least one good derivative, which is enough to guarantee the global existence of solutions for a sufficiently small choice of initial data..

While the null condition is a powerful tool for analysing non-linear equations, it only identifies a specific class of equations that admit global solutions. There are other non-linear wave equations that do not satisfy the null condition, and yet they have global solutions. This has sparked the question in the community if there is any other structure or condition that encompasses a large group of non-linear equations for which it is possible to obtain global solutions.

1.6 The weak null condition

The *weak null condition* was introduced by H. Lindblad and I. Rodnianski [47]. The motivation for this condition was that Einstein equations do not satisfy the null condition in wave coordinates. Moreover, it was shown by Choquet-Bruhat in [48], that there is no natural generalisation of the null condition for the Einstein equations. In [47], the authors introduced the *weak null condition* and showed that Einstein's equations satisfy such condition. In a second article [49], they used the weak null condition to prove a global existence result for Einstein's equations in wave coordinates with small initial data.

To understand the weak null condition it is necessary to introduce the asymptotic system corresponding to a given set of non-linear wave equations, see [47]. The asymptotic system can be thought of as a system were we have neglected all the terms that involve "good derivatives". In addition to this, the quadratic terms that involve at least one good derivative can be neglected along with all cubic and higher order terms. The remaining equation only involves *bad terms* which decay slower than the *good terms* or good derivatives. The asymptotic expansion found in [47] and that we reproduce here for the sake of context is based on the asymptotic expansion proposed by L. Hörmander [50, 51].

Consider the Cauchy problem for a system of non-linear wave equations in three space dimensions for the unknown $u = (u_i, \dots u_n), i = 1 \dots N$,

$$\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu u_i = \Phi_i(u, u', u''), \tag{1.6.1}$$

with initial data

$$u(0, x) = \epsilon u_0(x), \quad u_t(0, x) = \epsilon u_1(x), \tag{1.6.2}$$

where $u(0, x), u_t(0, x) \in C_0^\infty(\mathbb{R}^N)$, and u', u'' are spatial derivatives of u. The map Φ_i can be expanded as

$$\Phi_i(u, u'u'') = A^{jk}_{i\alpha\beta}\partial^\alpha u_j \partial^\beta u_k + \Psi_i(u, u', u''), \tag{1.6.3}$$

where Ψ_i vanishes at third order in the limit $(u, u', u'') \to 0$ and the derivatives in the non-linear terms are up to second order, that is $|\alpha|, |\beta| \leq 2$. For the asymptotic expansion as $|x| \to \infty$, we use the change of variable

$$q = |x| - t, \quad s = \epsilon \ln |x|, \quad w = \frac{x}{|x|}, \tag{1.6.4}$$

and

$$u(t, x) \sim \frac{\epsilon U(q, s, w)}{|x|}. \tag{1.6.5}$$

Here the symbol $\sim$ represents an equivalence relation and the functions $u(t, x)$ and $\frac{\epsilon U(q,s,w)}{|x|}$ are said to be asymptotically equivalent. Substituting (1.6.5) into equation (1.6.1) and after equating powers of order $\frac{\epsilon}{|x|^2}$ we obtain the following expression

$$2\partial_s\partial_q U_i = A^{jk}_{imn}(w)\big(\partial^m_q U_j\big)\big(\partial^n_q U_k\big), \quad U|_{s=0} = \Phi_0, \tag{1.6.6}$$

where

$$A^{jk}_{imn}(w) = \sum_{|\alpha|=m, |\beta|=n} a^{jk}_{i\alpha\beta}\widehat{w}^\alpha\widehat{w}^\beta, \quad \widehat{w} = (1, w). \tag{1.6.7}$$

Definition 1.6.1. If all the solutions of the asymptotic system (1.6.6) associated to (1.6.1) exist globally and have suitable initial data and global bounds, then the system (1.6.1) is said to satisfy the *weak null condition*. In other words, the system satisfies the *weak null condition* if the solutions as well as its derivatives have initial data decaying sufficiently fast in q and grow at most exponentially in s.

The null condition corresponds to the vanishing of the $A^{jk}_{imn}(w)$, and because of this, it is clear that the null condition leads to bounded solutions of the asymptotic equation. In this sense, the null condition is a special case of the weak null condition.

1.7 The bounded weak null condition

In Chapter 4, we study a restricted version of the weak null condition from a Fuchsian viewpoint. We use an equivalent definition to 1.6, for this purpose we define the out-going null one-form $\bar{L} = -d\bar{t} + d\bar{r}$ and we use it to define the following scalar functions

$$\bar{b}^K_{IJ} := \bar{a}^{K\mu\nu}_{IJ}\bar{L}_\mu\bar{L}_\nu = \bar{a}^{K00}_{IJ} - \bar{a}^{K01}_{IJ} - \bar{a}^{K10}_{IJ} + \bar{a}^{K11}_{IJ}, \tag{1.7.1}$$

where $\bar{a}^K_{IJ} = \bar{a}^{K\alpha\beta}_{IJ}\bar{\partial}_\alpha \otimes \bar{\partial}_\beta$, $1 \leq I, J, K \leq N$, is a prescribed smooth (2,0)-tensor field on $\mathbb{R}^4$, see [52]. The terms (1.7.1) are smooth functions on $\mathbb{S}^2$ and they will play an important role in the identification of the terms with the worse decay in the applications in Chapter 4. We use the functions (1.7.1) to define the *asymptotic equation* associated to the semi-linear wave equation on Minkowski space-time (1.4.1), see Chapter 4, which is given by

$$(2-t)\partial_t\xi = \frac{1}{t}Q(\xi), \tag{1.7.2}$$

where $\xi = (\xi^K)$ and

$$Q(\xi) = (Q^K(\xi)) := (-2\chi(\rho)\rho^m\bar{b}^K_{IJ}\xi^I\xi^J). \tag{1.7.3}$$

The coordinates, t and ρ are part of the coordinate system on an non-physical manifold that arises from the compactification of a neighbourhood of spatial infinity in Minkowski space-time; see Chapter 4, equation (4.3.2) for details. Here $\chi(\rho)$ is a smooth cut-off function and we have chosen the time coordinate t such that $0 < t \leq 1$ and $t = 0$ corresponds to future null-infinity. In this formulation the null condition is satisfied when the functions $\bar{b}^K_{IJ}$ vanish and the weak null condition becomes:

Definition 1.7.1. The *weak null condition* is a growth condition on solutions of the asymptotic equation (1.7.2), such that solutions ξ of the asymptotic system (1.7.3), satisfy a bound of the form $|\xi(t)| \lesssim t^{-C\epsilon}$ for some fixed constant $C > 0$ and initial data at $t = 1$ satisfying $|\xi(1)| \leq \epsilon \leq \epsilon_0$ for $\epsilon_0 > 0$ sufficiently small.

In the applications in Chapter 4, we will show that it is possible to obtain existence for a system of semi-linear wave equations in 3 + 1 dimensions whose associated asymptotic equation has bounded solutions for suitably small initial data on neighbourhoods of spatial infinity. We refer to this special case of the weak null condition as the *bounded weak null condition*.

Definition 1.7.2. The asymptotic equation (1.7.2) is said to satisfy the *bounded weak null condition* if there exist constants $\mathcal{R}_0 > 0$ and $C > 0$ such that solutions of the asymptotic initial value problem (IVP)

$$(2-t)\partial_t \xi = \frac{1}{t} Q(\xi), \tag{1.7.4}$$

$$\xi|_{t=1} = \mathring{\xi}, \tag{1.7.5}$$

exist for $t \in (0, 1]$ and are bounded by $\sup_{0<t\leq 1} |\xi(t)| \leq C$ for all initial data $\mathring{\xi}$ satisfying $|\mathring{\xi}| < \mathcal{R}_0$.

It is still an open conjecture, even in the semi-linear setting, to determine whether the weak null condition is enough to ensure global existence of solutions under a suitable small initial data assumption. Although the construction of our asymptotic system differs from [47, 50, 51], it is consistent with them in the sense that the asymptotic system (1.7.4) involves the non-linear terms with the worst decay in time given by $Q(\xi)$. In addition to this, the vanishing of the coefficients $\bar{b}^K_{IJ}$ is analogous to the vanishing of (1.6.7) which gives a system satisfying the null condition. It would be desirable to have a systematic way to write the asymptotic equation derived from the Fuchsian system of concern. Unfortunately, this is not an obvious task since Fuchsian systems can be quite general. We know that the asymptotic equation involves the terms with the worst decay, identifying those terms is one of the main challenges and it will depend on the particular system that we want to analyse. Moreover, we have to consider that the Fuchsian method requires a compactification of the space-time through a conformal transformation. Therefore, writing down the asymptotic system would depend intrinsically of the transformation used and the properties of the original system. We do not discard a future systematic study to obtain families of conformal transformations that are particularly useful for the Fuchsian method in the identification of the *bad terms* and the derivation of the asymptotic system.

At that same period there was also hope that the fundamental mysteries of mankind-the origin of the Library and of time-might be revealed. In all likelihood those profound mysteries can indeed be explained in words; if the language of the philosophers is not sufficient, then the multiform Library must surely have produced the extraordinary language that is required, together with the words and grammar of that language.

The Library of Babel, Jorge Luis Borges

2 The Fuchsian method and global existence

The work in this thesis consists of applications in different settings of the existence for Fuchsian equations theory developed in [1]. In this section, we will present the main ideas behind this existence theory. In [1], Fuchsian initial value problems (IVPs) of the form

$$B^0(t,u)\partial_t u + B^i(t,u)\nabla_i u = \frac{1}{t}\mathcal{B}(t,u)\mathbb{P}u + F(t,u) \qquad \text{in } [T_0,T_1)\times\Sigma, \tag{2.0.1}$$

$$u = u_0 \qquad \text{in } \{T_0\}\times\Sigma, \tag{2.0.2}$$

are analysed, where $T_0 < T_1 \leq 0$. By standard local-in-time existence and uniqueness results for symmetric hyperbolic equations, there exist a $T_1 \in \big(T_0, 0\big]$ and a unique solution $u \in C^0\Big(\big[T_0,T_1\big)H^K\Big) \cap C^1\Big(\big[T_0,T_1\big),H^{K-1}\Big)$, where T_1 is expected to be close to T_0 for generic initial data. The main existence result of [1] contained in Theorem 3.8, is to establish the existence of the solution up to the singular time $T_1 = 0$ under a small initial data assumption. In this way, (2.0.1)-(2.0.2) becomes a *global initial value problem* (GIVP). The proof of Theorem 3.8 from [1] is based on energy estimates, which we will explain in detail in Section 2.2.

We illustrate the main ideas of the proof in the Figure 2.1. The main column in purple, on the left, represents the core steps of the proof. From this column to the right, we have several bifurcations showing the intermediate steps. In the center, in green, we have the coefficient assumptions which lead to the bounds on the main operators and their projections (using the projection operator $\mathbb{P}$) as well as projections of the elements forming the source term F. Then in blue, in the central area of the diagram, we represent the preliminary estimates that are used later to obtain the L^2, and H^K energy estimates. Then, putting all these results together we can obtain the existence of solutions as well as decay estimates for systems of the form (2.0.1)- (2.0.2).

2.1 Notation

2.1.1 Spatial manifolds, coordinates, indexing and partial derivatives

Let Σ denote a n-dimensional manifold. Throughout this document, lower case Latin indices (for example i, j, k) range from 1 to n and they will be used to index coordinates associated to a local chart $x = (x^i)$ on Σ (our full indexing convention is given in Appendix A.1). Partial derivatives with respect to the coordinate system (x^i) are represented as

$$\partial_t = \frac{\partial}{\partial t} \quad \text{and} \quad \partial_i = \frac{\partial}{\partial x^i}.$$

2.1.2 Vector Bundles

We use $\pi : V \longrightarrow \Sigma$ to denote a N rank vector bundle with fibres $V_x = \pi^{-1}(\{x\})$, $x \in \Sigma$. The smooth sections of V are denoted by $\Gamma(V)$, and V is equipped with a time-independent connection ∇, that is $[\partial_t, \nabla] = 0$. We assume that V is equipped with a time-independent, compatible, positive definite metric $h \in \Gamma(T^0_2(V))$, that is,

$$\partial_t h = 0 \quad \text{and} \quad \nabla_X(h(u, v)) = h(\nabla_X u, v) + h(u, \nabla_X v) \tag{2.1.1}$$

for all $X \in \mathfrak{X}(\Sigma)$ and $u, v \in \Gamma(V)$. The vector bundle of linear operators acting on the fibres of V is given by $L(V) = \cup_{x\in\Sigma} L(V_x) \cong V \otimes V^*$. We define A^{tr}_x as the transpose of $A_x \in L(V_x)$, and A^{tr}_x is the unique element of $L(V_x)$ that satisfies

$$h(x)(A^{\mathrm{tr}}_x u_x, v_x) = h(x)(u_x, A_x v_x), \quad \forall\, u_x, v_x \in V_x.$$

The vector bundle of linear maps from the fibres of V to the fibres of W, is given by $L(V, W) = \cup_{x\in\Sigma} L(V_x, W_x) \cong W \otimes V^*$, where V, W are vector bundles over Σ. Additionally, we use π to denote the canonical projection onto Σ for any vector bundle over Σ, e.g. V, $L(V)$, $V \otimes V$.

Upper case Latin indices (for example I, J, K) range from 1 to N and they will be used to index a local basis $\{e_I\}$ associated to the vector bundle V. We can

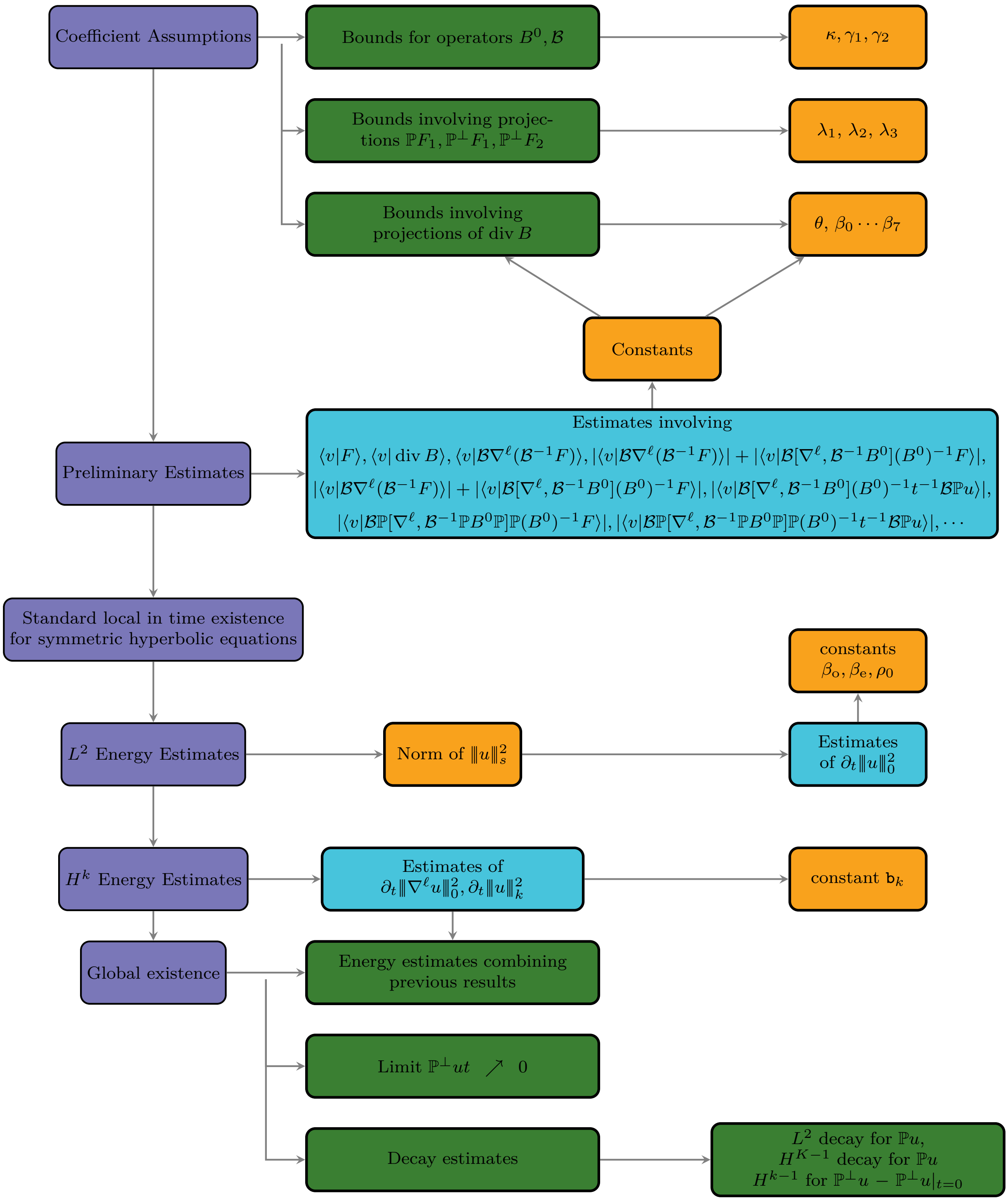

Figure 2.1: This diagram displays the core arguments used in the proof of Theorem 3.8 from [1]. The key steps are shown on the left and they split in sub-steps showing the flow of ideas in the proof. Many of these sub-steps require some preliminary estimates which we do not prove here and we refer the reader to [1] for its proof. Putting all these results together, leads to global existence of solutions and decay estimates to the GIVP (2.0.1)-(2.0.2).

represent locally $u \in \Gamma(V)$ and the inner-product h as

$$
\begin{aligned}
u &= u^I e_I, \\
h &= h_{IJ}\theta^I \otimes \theta^J,
\end{aligned}
$$

where $\{\theta^I\}$ is a local basis of V^* determined from $\{e_I\}$ by duality. Letting the local coordinates (x^i) and the local basis $\{e_I\}$ be defined on the same open region of Σ, the covariant derivative $\nabla u \in \Gamma(V \otimes T^*M)$ is given locally by

$$
\nabla u = \nabla_i u^I e_I \otimes dx^i,
$$

where

$$
\nabla_i u^I = \partial_i u^I + \omega^I_{iJ} u^J
$$

and the ω^I_{iJ} are the connection coefficients determined by

$$
\nabla_{\partial_i} e_J = \omega^I_{iJ} e_I. \tag{2.1.2}
$$

We further assume that the spatial manifold Σ is equipped with a time-independent $(\partial_t g = 0)$ Riemannian metric $g \in \Gamma(T^0_2(\Sigma))$. This metric is given in the local coordinates (x^i) by

$$
g = g_{ij} dx^i \otimes dx^j.
$$

The metric g determines the Levi-Civita connection on the tensor bundle $T^r_s(M)$ uniquely. Therefore, we can use ∇ to also denote this connection. The connection on V and the Levi-Civita connection on $T^r_s(M)$ determine a unique connection on the tensor product $V \otimes T^r_s(\Sigma)$, which we denote again by ∇. This connection is compatible with the positive definite inner-product induced on $V \otimes T^r_s(\Sigma)$ by the inner-product h on V and the Riemannian metric g on Σ. The covariant derivative of order s of a section $u \in \Gamma(V)$ defines an element of $\Gamma(V \otimes T^0_s(\Sigma))$, denoted $\nabla^s u$, that in local coordinates, is given by

$$
\nabla^s u = \nabla_{i_s} \cdots \nabla_{i_2} \nabla_{i_1} u^I e_I \otimes dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_s}.
$$

The components $\nabla_{i_s} \cdots \nabla_{i_2} \nabla_{i_1} u^I$ of $\nabla^s u$ can be further expanded in the usual way. For example, when $s = 2$, the components of $\nabla^2 u$ are given by

$$
\nabla_j \nabla_i u^I = \partial_j \nabla_i u^I - \Gamma^k_{ji} \nabla_k u^I + \omega^I_{jJ} \nabla_i u^J,
$$

where Γ^k_{ij} are the Christoffel symbols of g and w^I_{IJ} are the connection coefficients defined above by (2.1.2). We can obtain similar expressions for covariant derivatives of higher order.

2.1.3 Inner-products and operator inequalities

The norm of a vector $v \in V_x$, $x \in \Sigma$, is defined by

$$|v|^2 = h(x)(v, v).$$

For use below, we define the bundle of open balls of radius $R > 0$ in V by

$$B_R(V) = \{v \in V \mid |v| < R\}.$$

Elements of the form $v, w \in V_x \otimes T^0_s(\Sigma_x)$, can be expanded in local coordinates as

$$v = v^I_{i_1 i_2 \cdots i_s} e_I \otimes dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_s}, \quad \text{and} \quad w = w^I_{i_1 i_2 \cdots i_s} e_I \otimes dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_s},$$

respectively. With the help of this expansion, we define the inner-product of v and w by

$$(v|w) = g^{i_1 j_1} g^{i_2 j_2} \cdots g^{i_s j_s} h_{IJ} v^I_{i_1 i_2 \cdots i_s} w^J_{j_1 j_2 \cdots j_s},$$

and use

$$|v|^2 = (v|v)$$

to denote the associated norm.

For an operator on a smooth section $A \in L(V_x)$, we define the operator norm $|A|_{\text{op}}$ of A by

$$|A|_{\text{op}} = \sup\big\{\, |(w|Av)| \,\big|\, w, v \in B_1(V_x) \,\big\}.$$

We can extend this definition for linear operators in different tensor products. For example if $A \in L(V_x) \otimes T^*_x\Sigma$, we define a related operator norm $|A|_{\text{op}}$ by

$$|A|_{\text{op}} = \sup\big\{\, |(v|Aw)| \,\big|\, (v, w) \in B_1(V_x \otimes T^*_x\Sigma) \times B_1(V_x) \,\big\}.$$

In order to compare operators $A, B \in L(V_x)$, we define

$$A \leq B$$

if and only if

$$(v|Av) \leq (v|Bv), \quad \forall\, v \in V_x.$$

2.1.4 Constants, inequalities and order notation

In the following calculations, we will require constants whose explicit dependence on other quantities is not needed. These constants might change value from line to line. We will use the letter C to denote these constants and we will use the standard notation

$$a \lesssim b$$

for inequalities of the form

$$a \leq Cb.$$

In the case that the dependence of the constant on other inequalities needs to be specified, for example if the constant depends on the norm $\|u\|_{L^\infty}$, we use the notation

$$C = C(\|u\|_{L^\infty}).$$

Constants of this type will always be non-negative, non-decreasing, continuous functions of their arguments.

Given four vector bundles V, W, Y and Z defined over Σ, and maps

$$f \in C^0\big([T_0, 0), C^\infty(B_R(W) \times B_R(V), Z)\big) \quad \text{and} \quad g \in C^0\big([T_0, 0), C^\infty(B_R(V), Y)\big),$$

we say that

$$f(t, w, v) = \mathcal{O}(g(t, v))$$

if there exist a $\tilde{R} \in (0, R)$ and a map

$$\tilde{f} \in C^0\big([T_0, 0), C^\infty(B_{\tilde{R}}(W) \times B_{\tilde{R}}(V), L(Y, Z))\big)$$

such that

$$\begin{gathered} f(t, w, v) = \tilde{f}(t, w, v) g(t, v), \\ |\tilde{f}(t, w, v)| \leq 1 \quad \text{and} \quad |\nabla^s_{w,v} \tilde{f}(t, w, v)| \lesssim 1 \end{gathered}$$

for all $(t, w, v) \in [T_0, 0) \times B_{\tilde{R}}(W) \times B_{\tilde{R}}(V)$ and $s \geq 1$, where we use $\nabla_{w,v}$ to denote a covariant derivative operator on the product manifold $W \times V$. Since Σ is compact, we know that such a covariant derivative always exists. In the case that we want to bound $f(t,w,v)$ by $g(t,v)$ up to an undetermined constant of proportionality, we define

$$f(t,w,v) = \mathrm{O}(g(t,v))$$

if there exist a $\tilde{R} \in (0,R)$ and a map

$$\tilde{f} \in C^0\big([T_0,0), C^\infty(B_R(W) \times B_R(V), L(Y,Z))\big)$$

such that

$$f(t,w,v) = \tilde{f}(t,w,v)g(t,v)$$

and

$$|\nabla^s_{w,v}\tilde{f}(t,w,v)| \lesssim 1$$

for all $(t,w,v) \in [T_0,0) \times B_{\tilde{R}}(W) \times B_{\tilde{R}}(V)$ and $s \geq 0$.

2.1.5 Sobolev spaces

Let $k \in \mathbb{Z}_{\geq 0}$, the Sobolev norm $\|u\|_{W^{k,p}}$ of a section $u \in \Gamma(V)$ is defined by

$$\|u\|_{W^{k,p}} = \begin{cases} \left(\displaystyle\sum_{\ell=0}^{k} \int_\Sigma |\nabla^\ell u|^p \nu_g\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \displaystyle\max_{0\leq \ell \leq k} \sup_{x\in\Sigma} |\nabla^\ell u(x)| & \text{if } p = \infty \end{cases},$$

where $\nu_g \in \Omega^n(\Sigma)$ denotes the volume form of g. The Sobolev space $W^{k,p}(V)$ can then be defined as the completion of the space of smooth sections $\Gamma(V)$ in the norm $\|\cdot\|_{W^{k,p}}$. When $V = \Sigma \times \mathbb{R}$ or the vector bundle is clear from context, we will write $W^{k,p}(\Sigma)$ instead. We use the standard notation $H^k(V) = W^{k,2}(V)$, for the case $p = 2$ where $H^k(V)$ is a Hilbert space with the inner-product given by

$$\langle u|v\rangle_{H^k} = \sum_{\ell=0}^{k} \langle \nabla^\ell u | \nabla^\ell v\rangle,$$

where the L^2 inner-product $\langle\cdot|\cdot\rangle$ is defined by

$$\langle w|z\rangle = \int_\Sigma (w|z)\,\nu_g.$$

2.2 Coefficient assumptions

(i) We define a time-independent, covariantly constant, symmetric projection operator $\mathbb{P}$ with $\mathbb{P}\in\Gamma(L(V))$, such that it satisfies the following properties

$$\mathbb{P}^2=\mathbb{P},\quad \mathbb{P}^{\mathrm{tr}}=\mathbb{P},\quad \partial_t\mathbb{P}=0\quad\text{and}\quad \nabla\mathbb{P}=0. \tag{2.2.1}$$

the complementary projection operator is given by

$$\mathbb{P}^\perp = 1\!\!1 - \mathbb{P},$$

which is also a time-independent, covariantly constant, symmetric projection operator.

(ii) We define the maps B^0, $\mathcal{B}$, such that

$$\begin{aligned} B^0 &\in C^1\big([T_0,0),C^\infty(B_R(V),L(V))\big),\\ \mathcal{B} &\in C^0\big([T_0,0],C^\infty(B_R(V),L(V))\big),\end{aligned}$$

satisfy

$$\pi(B^0(t,v))=\pi(\mathcal{B}(t,v))=\pi(v),$$

and

$$\frac{1}{\gamma_1}\mathrm{id}_{V_{\pi(v)}}\le B^0(t,v)\le\frac{1}{\kappa}\mathcal{B}(t,v)\le\gamma_2\mathrm{id}_{V_{\pi(v)}}, \tag{2.2.2}$$

for all $(t,v)\in[T_0,0)\times B_R(V)$ and $\kappa,\gamma_1,\gamma_2>0$ are positive constants. In local coordinates $(x,v)=(x^i,v^I)$ on the vector bundle $\pi : V\longrightarrow\Sigma$, B^0 is given by a $\mathbb{S}_N$-valued map $B^0(t,x,v)=\big((B^0)^I_J(t,x,v)\big)$, while $\mathcal{B}$ is given locally by a $\mathbb{M}_{N\times N}$-valued map $\mathcal{B}(t,x,v)=\big(\mathcal{B}^I_J(t,x,v)\big)$. Here $\mathbb{S}_N$ denotes the subset of $\mathbb{M}_{N\times N}$, that is, the $N\times N$-matrices, that are symmetric with respect to the local representation of the vector bundle metric h, in other words, if h_{IJ} is the local representation of h and $(h^{IJ}):=(h_{Ij})^{-1}$ is its inverse, then A^I_J will define an element of $\mathbb{S}_N$ if and only if $h^{IJ}A^K_J=h^{KJ}A^I_J$.

The projection operator $\mathbb{P}$ and the maps $\mathcal{B}$, B^0 also satisfy

$$[\mathbb{P}(\pi(v)), \mathcal{B}(t,v)] = 0, \tag{2.2.3}$$

$$(B^0(t,v))^{\mathrm{tr}} = B^0(t,v), \tag{2.2.4}$$

$$\mathbb{P}(\pi(v))B^0(t,v)\mathbb{P}^{\perp}(\pi(v)) = \mathrm{O}\big(|t|^{\frac{1}{2}} + \mathbb{P}(\pi(v))v\big), \tag{2.2.5}$$

and

$$\mathbb{P}^{\perp}(\pi(v))B^0(t,v)\mathbb{P}(\pi(v)) = \mathrm{O}\big(|t|^{\frac{1}{2}} + \mathbb{P}(\pi(v))v\big), \tag{2.2.6}$$

for all $(t,v) \in [T_0,0)\times B_R(V)$, and there exist maps $\tilde{B}^0, \tilde{\mathcal{B}} \in C^0\big([T_0,0], \Gamma(L(V))\big)$ such that

$$[\mathbb{P}, \tilde{\mathcal{B}}] = 0, \tag{2.2.7}$$

$$B^0(t,v) - \tilde{B}^0(t,\pi(v)) = \mathrm{O}(v) \tag{2.2.8}$$

and

$$\mathcal{B}(t,v) - \tilde{\mathcal{B}}(t,\pi(v)) = \mathrm{O}(v) \tag{2.2.9}$$

for all $(t,v) \in [T_0,0) \times B_R(V)$.

(iii) The map $F \in C^0\big([T_0,0), C^{\infty}(B_R(V),V)\big)$ can be expanded as

$$F(t,v) = \tilde{F}(t,\pi(v)) + F_0(t,v) + |t|^{-\frac{1}{2}}F_1(t,v) + |t|^{-1}F_2(t,v) \tag{2.2.10}$$

where $\tilde{F} \in C^0\big([T_0,0],\Gamma(V)\big)$, and $F_0, F_1, F_2 \in C^0\big([T_0,0], C^{\infty}(B_R(V),V)\big)$. In local coordinates $(x,v) = (x^i, v^I)$ on the vector bundle $\pi : V \longrightarrow \Sigma$, $\tilde{F}$, F and F_a are given by $\mathbb{R}^N$-valued maps $\tilde{F}(t,x) = (\tilde{F}^I(t,x))$, $F(t,x,v) = (F^I(t,x,v))$ and $F_a(t,x,v) = (F^I_a(t,x,v))$, respectively.

Moreover, the maps F_0, F_1, F_2 satisfy

$$\pi(F_a(t,v)) = \pi(v), \quad a = 0,1,2,$$

and

$$\mathbb{P}(\pi(v))F_2(t,v) = 0 \tag{2.2.11}$$

for all $(t,v) \in [T_0,0] \times B_R(V)$, and there exist constants $\lambda_a \geq 0$, $a = 1,2,3$, such that

$$F_0(t,v) = \mathrm{O}(v), \tag{2.2.12}$$

$$\mathbb{P}(\pi(v))F_1(t,v) = \mathcal{O}(\lambda_1 v), \tag{2.2.13}$$

$$\mathbb{P}^{\perp}(\pi(v))F_1(t,v) = \mathcal{O}(\lambda_2\mathbb{P}(\pi(v))v) \tag{2.2.14}$$

and

$$\mathbb{P}^{\perp}(\pi(v))F_2(t,v) = \mathcal{O}\biggl(\frac{\lambda_3}{R}\mathbb{P}(\pi(v))v \otimes \mathbb{P}(\pi(v))v\biggr) \tag{2.2.15}$$

for all $(t,v) \in [T_0,0) \times B_R(V)$.

(iv) The map $B \in C^0\bigl([T_0,0), C^\infty(B_R(V), L(V)\otimes T\Sigma)\bigr)$ satisfies

$$\pi(B(t,v)) = \pi(v)$$

and

$$\bigl[\sigma(\pi(v))(B(t,v))\bigr]^{\mathrm{tr}} = \sigma(\pi(v))(B(t,v))$$

for all $(t,v) \in [T_0,0) \times B_R(V)$ and $\sigma \in \mathfrak{X}^*(\Sigma)$, where we are using the notation $\sigma(A)$ to denote the natural action of a differential 1-form $\sigma \in \mathfrak{X}^*(\Sigma)$ on an element of $A \in \Gamma(L(V)\otimes T\Sigma)$, which if we express A and σ locally as

$$A = A^{iI}_J \theta^J \otimes e_I \otimes \partial_i \quad \text{and} \quad \sigma = \sigma_i dx^i,$$

is defined by

$$\sigma(A) = \sigma_i A^{iI}_J \theta^J \otimes e_I.$$

The map, B can be expanded as

$$B(t,v) = B_0(t,v) + |t|^{-\frac{1}{2}}B_1(t,v) + |t|^{-1}B_2(t,v) \tag{2.2.16}$$

where $B_0, B_1, B_2 \in C^0\bigl([T_0,0], C^\infty(B_R(V), L(V)\otimes T\Sigma)\bigr)$.

Additionally, the maps B_0, B_1, B_2 satisfy

$$\pi(B_a(t,v)) = \pi(v), \quad a = 0,1,2,$$

for all $(t,v) \in [T_0,0] \times B_R(V)$, and there exist a constant $\alpha \geq 0$ and a map

$\tilde{B}_2 \in C^0\big([T_0,0],\Gamma(L(V)\otimes T\Sigma)\big)$ such that

$$\mathbb{P}(\pi(v))B_1(t,v)\mathbb{P}(\pi(v)) = \mathrm{O}(1), \tag{2.2.17}$$
$$\mathbb{P}(\pi(v))B_1(t,v)\mathbb{P}^{\perp}(\pi(v)) = \mathcal{O}(\alpha), \tag{2.2.18}$$
$$\mathbb{P}^{\perp}(\pi(v))B_1(t,v)\mathbb{P}(\pi(v)) = \mathcal{O}(\alpha), \tag{2.2.19}$$
$$\mathbb{P}^{\perp}(\pi(v))B_1(t,v)\mathbb{P}^{\perp}(\pi(v)) = \mathrm{O}(\mathbb{P}(\pi(v))v), \tag{2.2.20}$$
$$\mathbb{P}(\pi(v))B_2(t,v)\mathbb{P}^{\perp}(\pi(v)) = \mathrm{O}(\mathbb{P}(\pi(v))v), \tag{2.2.21}$$
$$\mathbb{P}^{\perp}(\pi(v))B_2(t,v)\mathbb{P}(\pi(v)) = \mathrm{O}(\mathbb{P}(\pi(v))v), \tag{2.2.22}$$
$$\mathbb{P}^{\perp}(\pi(v))B_2(t,v)\mathbb{P}^{\perp}(\pi(v)) = \mathrm{O}\big(\mathbb{P}(\pi(v))v\otimes\mathbb{P}(\pi(v))v\big) \tag{2.2.23}$$

and

$$\mathbb{P}(\pi(v))(B_2(t,v)-\tilde{B}_2(t,\pi(v)))\mathbb{P}(\pi(v)) = \mathrm{O}(v) \tag{2.2.24}$$

for all $(t,v)\in[T_0,0)\times B_R(V)$.

In local coordinates $(x,v)=(x^i,v^I)$ on the vector bundle $\pi : V \longrightarrow \Sigma$, the maps B, $\tilde{B}_2$ and B_a can be expressed as

$$B = B^i(t,x,v)\partial_i, \quad \tilde{B}_2 = \tilde{B}_2^i(t,x)\partial_i \quad \text{and} \quad B_a = B_a^i(t,x,v)\partial_i,$$

respectively, where $B^i(t,x,v) = ((B^i)^I_J(t,x,v))$, $\tilde{B}^i(t,x) = ((\tilde{B}^i_2)^I_J(t,x))$ and $B^i_a(t,x,v) = ((B^i_a)^I_J(t,x,v))$ are $\mathbb{S}_N$-valued maps. Then expressing $\sigma\in\mathfrak{X}^*(\Sigma)$ locally as

$$\sigma = \sigma_i(x)dx^i,$$

we see that

$$\sigma(B) = B^i(t,x,v)\sigma_i(x), \quad \sigma(\tilde{B}_2) = \tilde{B}^i_2(x,t)\sigma_i(x) \quad \text{and} \quad \sigma(B_a) = B^i_a(t,x,v)\sigma_i(x).$$

Since $u(t,x)$ is a time-dependent section of the vector bundle V, it can be represented in local coordinates as

$$u(t,x) = (x,\hat{u}(t,x)),$$

where $\hat{u}(t,x) = (\hat{u}^J(t,x))$ is $\mathbb{R}^N$-valued. Using this and the expressions for B^0,

$\mathcal{B}$, B, in local coordinates we can write the local version of (2.0.1) as

$$\begin{aligned} B^0(t,x,\hat{u}(t,x))\partial_t\hat{u}(t,x)+&B^i(t,x,\hat{u}(t,x))\big(\partial_i\hat{u}(t,x)+\omega_i(x)\hat{u}(t,x)\big)\\ &=\frac{1}{t}\mathcal{B}(t,x,\hat{u}(t,x))\mathbb{P}(x)\hat{u}(t,x)+F(t,x,\hat{u}(t,x)), \end{aligned} \tag{2.2.25}$$

where the $\omega_i = (\omega_{iI}^J)$ are the connection coefficients (2.1.2). The assumptions on the coefficients described before guarantee that (2.2.25) defines a symmetric hyperbolic system.

(v) We define the map

$$\operatorname{div}B \;:\; [T_0,0)\times B_R(V\otimes V\otimes T^*\Sigma)\longrightarrow L(V)$$

which in local coordinates is given by

$$\begin{aligned} \operatorname{div}B(t,x,v,w) &= \partial_t B^0(t,x,v) + D_v B^0(t,x,v)\cdot(B^0(t,x,v))^{-1}\Big[-B^i(t,x,v)\cdot w_i\\ &+\frac{1}{t}\mathcal{B}(t,x,v)\mathbb{P}(x)v+F(t,x,v)\Big]+\partial_i B^i(t,x,v)+D_v B^i(t,x,v)\cdot(w_i-\omega_i(x)v)\\ &+\Gamma^i_{ij}(x)B^j(t,x,v)+\omega_i(x)B^i(t,x,v)-B^i(t,x,v)\omega_i(x), \end{aligned} \tag{2.2.26}$$

where $v=(v^J)$, $w=(w_i)$, $w_i=(w_i^J)$, $\omega_i=(\omega_{iI}^J)$, and $B^i=(B_I^{iJ})$. There exist constants θ and $\beta_a\geq 0$, $a=0,1,\ldots,7$, such that the map $\operatorname{div}B$ satisfies

$$\mathbb{P}(\pi(v))\operatorname{div}B(t,v,w)\mathbb{P}(\pi(v)) = \mathcal{O}\big(\theta+|t|^{-\frac{1}{2}}\beta_0+|t|^{-1}\beta_1\big), \tag{2.2.27}$$

$$\mathbb{P}(\pi(v))\operatorname{div}B(t,v,w)\mathbb{P}^\perp(\pi(v)) = \mathcal{O}\left(\theta+|t|^{-\frac{1}{2}}\beta_2+\frac{|t|^{-1}\beta_3}{R}\mathbb{P}(\pi(v))v\right), \tag{2.2.28}$$

$$\mathbb{P}^\perp(\pi(v))\operatorname{div}B(t,v,w)\mathbb{P}(\pi(v)) = \mathcal{O}\left(\theta+|t|^{-\frac{1}{2}}\beta_4+\frac{|t|^{-1}\beta_5}{R}\mathbb{P}(\pi(v))v\right) \tag{2.2.29}$$

$$\begin{aligned} &\mathbb{P}^\perp(\pi(v))\operatorname{div}B(t,v,w)\mathbb{P}^\perp(\pi(v))\\ &\qquad=\mathcal{O}\left(\theta+\frac{|t|^{-\frac{1}{2}}\beta_6}{R}\mathbb{P}(\pi(v))v+\frac{|t|^{-1}\beta_7}{R^2}\mathbb{P}(\pi(v))v\otimes\mathbb{P}(\pi(v))v\right). \end{aligned}$$

One can also verify that

$$\operatorname{div} B(t,u(t,x),\nabla u(t,x)) = \partial_t(B^0(t,u(t,x)) + \nabla_i(B^i(t,u(t,x))) \tag{2.2.30}$$

for solutions $u(t,x)$ of (2.0.1).

It is not difficult to verify that $(B^0)^{-1}$ satisfies the following relations

$$\mathbb{P}(\pi(v))(B^0)^{-1}(t,v)\mathbb{P}^\perp(\pi(v)) = \mathrm{O}\big(|t|^{\frac{1}{2}} + \mathbb{P}(\pi(v))v\big) \tag{2.2.31}$$

and

$$\mathbb{P}^\perp(\pi(v))(B^0)^{-1}(t,v)\mathbb{P}(\pi(v)) = \mathrm{O}\big(|t|^{\frac{1}{2}} + \mathbb{P}(\pi(v))v\big) \tag{2.2.32}$$

for all $(t,v) \in [T_0,0)\times B_R(V)$, these relations can be deduced from (2.2.5) and (2.2.6). From (2.2.2) we deduce that there exist constants $0 < \tilde{\gamma}_1 \leq \gamma_1$ and $\tilde{\kappa} \geq \kappa > 0$ such that the maps $B^0, \mathcal{B}$ satisfy

$$\frac{1}{\tilde{\gamma}_1}\mathbb{P}(\pi(v)) \leq \mathbb{P}(\pi(v))B^0(t,v)\mathbb{P}(\pi(v)) \leq \frac{1}{\tilde{\kappa}}\mathcal{B}(t,v)\mathbb{P}(\pi(v)) \leq \gamma_2\mathbb{P}(\pi(v)) \tag{2.2.33}$$

for all $(t,v) \in [T_0,0) \times B_R(V)$.

2.3 Global existence and asymptotics

The following theorem is the main result of [1] and cornerstone of this thesis. The theorem guarantees the existence of solutions to the GIVP (2.0.1)-(2.0.2) on time intervals of the form $[T_0,0)$ under a suitable small initial data hypothesis, as well as establishing decay estimates for those solutions. In this section, we review the essential parts and core ideas of the proof given in [1]. We do not include the preliminary estimates of section 3.2 of [1] in our review; we refer the interested reader to [1] for their proof. Chapters 3 and 4 are dedicated to applications of this theorem in different settings.

Theorem 2.3.1. *Suppose $k \in \mathbb{Z}_{>n/2+3}$, $\sigma > 0$, $u_0 \in H^k(\Sigma)$, assumptions (i)-(v) from Section 2.2 are fulfilled, and the constants κ, γ_1, λ_3, β_0, β_1, β_3, β_5, β_7 from Section 2.2 satisfy*

$$\kappa > \frac{1}{2}\gamma_1 \max\Big\{\sum_{a=0}^{3}\beta_{2a+1} + 2\lambda_3, \beta_1 + 2k(k+1)\mathtt{b}\Big\} \tag{2.3.1}$$

where

$$\mathtt{b} = \sup_{T_0\le t<0}\Big(\big\||\mathbb{P}\tilde{\mathcal{B}}(t)\nabla(\tilde{\mathcal{B}}(t)^{-1}\tilde{B}^0(t))\tilde{B}^0(t)^{-1}\mathbb{P}\tilde{B}_2(t)\mathbb{P}|_{\mathrm{op}}\big\|_{L^\infty} + \big\||\mathbb{P}\tilde{\mathcal{B}}(t)\nabla(\tilde{\mathcal{B}}(t)^{-1}\tilde{B}_2(t))\mathbb{P}|_{\mathrm{op}}\big\|_{L^\infty}\Big).$$

Then there exists $\delta > 0$ such that if

$$\max\Big\{\|u_0\|_{H^k}, \sup_{T_0\le\tau<0}\|\tilde{F}(\tau)\|_{H^k}\Big\} \le \delta,$$

then there exists a unique solution

$$u \in C^0\big([T_0,0),H^k(\Sigma)\big)\cap L^\infty\big([T_0,0),H^k(\Sigma)\big)\cap C^1\big([T_0,0),H^{k-1}(\Sigma)\big)$$

of the IVP (2.0.1)-(2.0.2) *with $T_1 = 0$ such that the limit $\lim_{t\nearrow 0}\mathbb{P}^\perp u(t)$, denoted $\mathbb{P}^\perp u(0)$, exists in $H^{k-1}(\Sigma)$.*

Moreover, for $T_0 \le t < 0$, the solution u satisfies the energy estimate

$$\|u(t)\|^2_{H^k} + \sup_{T_0\le\tau<0}\|\tilde{F}(\tau)\|^2_{H^k} - \int_{T_0}^t \frac{1}{\tau}\|\mathbb{P}u(\tau)\|^2_{H^k}\,d\tau \le C(\delta,\delta^{-1})\Big(\|u_0\|^2_{H^k} + \sup_{T_0\le\tau<0}\|\tilde{F}(\tau)\|^2_{H^k}\Big)$$

and the decay estimates

$$\|\mathbb{P}u(t)\|_{H^{k-1}} \lesssim \begin{cases} |t| + (\lambda_1+\alpha)|t|^{\frac{1}{2}} & \text{if } \zeta > 1 \\ |t|^{\zeta-\sigma} + (\lambda_1+\alpha)|t|^{\frac{1}{2}} & \text{if } \frac{1}{2} < \zeta \le 1 \\ |t|^{\zeta-\sigma} & \text{if } 0 < \zeta \le \frac{1}{2} \end{cases}$$

and

$$\|\mathbb{P}^\perp u(t) - \mathbb{P}^\perp u(0)\|_{H^{k-1}} \lesssim \begin{cases} |t|^{\frac{1}{2}} + |t|^{\zeta-\sigma} & \text{if } \zeta > \frac{1}{2} \\ |t|^{\zeta-\sigma} & \text{if } \zeta \le \frac{1}{2} \end{cases},$$

where

$$\zeta = \tilde{\kappa} - \frac{1}{2}\tilde{\gamma}_1\big(\beta_1 + (k-1)k\tilde{\mathtt{b}}\big) \tag{2.3.2}$$

and

$$\tilde{\mathtt{b}} = \sup_{T_0\le t<0}\Big(\big\||\mathbb{P}\tilde{\mathcal{B}}(t)\nabla(\tilde{\mathcal{B}}(t)^{-1}\mathbb{P}\tilde{B}^0(t)\mathbb{P})\mathbb{P}\tilde{B}^0(t)^{-1}\tilde{B}_2(t)\mathbb{P}|_{\mathrm{op}}\big\|_{L^\infty} + \big\||\mathbb{P}\tilde{\mathcal{B}}(t)\nabla(\tilde{\mathcal{B}}(t)^{-1}\tilde{B}_2(t))\mathbb{P}|_{\mathrm{op}}\big\|_{L^\infty}\Big).$$

Remark 2.3.2. By (2.2.33), we note that the constants $\tilde{\kappa},\kappa,\gamma_1,\tilde{\gamma}_1$ satisfy $\tilde{\kappa}\ge\kappa>0$ and $\gamma_1 \ge \tilde{\gamma}_1 > 0$, and we further observe from the definitions of $\mathtt{b}$ and $\tilde{\mathtt{b}}$ that $\mathtt{b}\ge\tilde{\mathtt{b}}\ge 0$. As a consequence, we have that, $\zeta = \tilde{\kappa} - \frac{1}{2}\tilde{\gamma}_1\big(\beta_1+(k-1)k\tilde{\mathtt{b}}\big) > 0$ since we have assumed that

$$\kappa - \frac{1}{2}\gamma_1 \max\left\{\sum_{a=0}^{3}\beta_{2a+1} + 2\lambda_3, \beta_1 + 2k(k+1)\mathtt{b}\right\} > 0. \tag{2.3.3}$$

Proof. By assumption, we have that $k \in \mathbb{Z}_{>n/2+3}$ and we know by standard local-in-time existence and uniqueness results for symmetric hyperbolic equations [53, Ch.16 §1], that there exists a solution $u \in C^0([T_0, T^*), H^k) \cap C^1([T_0, T^*), H^{k-1})$ to (2.0.1)-(2.0.2) for some time $T^* \in (T_0, 0]$ that we can take to be maximal. We choose the constants $R > 0$ and δ such that

$$\delta \in (0, \tfrac{1}{4}\mathcal{R}), \quad \mathcal{R} = \min\left\{\tfrac{3R}{4C_{\text{Sob}}}, \tfrac{3R}{4}\right\},$$

and the initial data satisfies

$$\|u(T_0)\|_{H^k} < \delta. \tag{2.3.4}$$

From here, we see that two situations can happen: either $\|u(t)\|_{H^k} < \mathcal{R}$ for all $t \in [T_0, T^*)$ or there exists a time $T_* \in (T_0, T^*)$ such that

$$\|u(T_*)\|_{H^k} = \mathcal{R} \leq \tfrac{3}{4}R.$$

If the first case holds, we set $T_* = T^*$ and so, in either case, we have by Sobolev's inequality that

$$\max\left\{\|\nabla u(t)\|_{L^\infty},\ \|u(t)\|_{L^\infty},\ \|u(t)\|_{H^k}\right\} \leq \tfrac{3}{4}R, \quad T_0 \leq t < T_*. \tag{2.3.5}$$

Now, we apply on the left the operator $\mathcal{B}\nabla^\ell\mathcal{B}^{-1}$, $0 \leq \ell \leq k$, to equation (2.0.1) and we get

$$\mathcal{B}\nabla^\ell\mathcal{B}^{-1}\big(B^0(t,u)\partial_t u\big) + \mathcal{B}\nabla^\ell\mathcal{B}^{-1}\big(B^i(t,u)\nabla_i u\big) = \frac{\mathcal{B}\nabla^\ell\mathcal{B}^{-1}}{t}\mathcal{B}(t,u)\mathbb{P}u + \mathcal{B}\nabla^\ell\mathcal{B}^{-1}F(t,u). \tag{2.3.6}$$

We can write this as

$$\begin{aligned} B^0\partial_t\nabla^\ell u + \mathcal{B}[\nabla^\ell, \mathcal{B}^{-1}B^0]\partial_t u + \mathcal{B}[\nabla^\ell, \mathcal{B}^{-1}B^i]\nabla_i u + B^i[\nabla^\ell, \nabla_i]u + \\ B_i\nabla_i\nabla^\ell u = \frac{1}{t}\mathcal{B}\nabla^\ell\mathbb{P}u + \mathcal{B}\nabla^\ell(\mathcal{B}^{-1}F), \end{aligned} \tag{2.3.7}$$

which leads to

$$B^0\partial_t\nabla^\ell u + B^i\nabla_i\nabla^\ell u = \frac{1}{t}\mathcal{B}\nabla^\ell\mathbb{P}u - \mathcal{B}[\nabla^\ell, \mathcal{B}^{-1}B^0]\partial_t u - \mathcal{B}[\nabla^\ell, \mathcal{B}^{-1}B^i]\nabla_i u - \\ B^i[\nabla^\ell, \nabla_i]u + \mathcal{B}\nabla^\ell(\mathcal{B}^{-1}F). \tag{2.3.8}$$

Next, we note from (2.0.1) that

$$\partial_t u = -B^{0^{-1}}B^i\nabla_i u + \frac{1}{t}B^{0^{-1}}\mathcal{B}\mathbb{P}u + B^{0^{-1}}F, \tag{2.3.9}$$

substituting this into (2.3.8) gives

$$\begin{aligned} B^0\partial_t\nabla^\ell u + B^i\nabla_i\nabla^\ell u =& \frac{1}{t}\Big[\mathcal{B}\mathbb{P}\nabla^\ell u - \mathcal{B}[\nabla^\ell, \mathcal{B}^{-1}B^0](B^0)^{-1}\mathcal{B}\mathbb{P}u\Big] + \\ & \mathcal{B}[\nabla^\ell, \mathcal{B}^{-1}B^0](B^0)^{-1}B^i\nabla_i u - \mathcal{B}[\nabla^\ell, \mathcal{B}^{-1}B^i]\nabla_i u - B^i[\nabla^\ell, \nabla_i]u \\ & - \mathcal{B}[\nabla^\ell, \mathcal{B}^{-1}B^0](B^0)^{-1}F + \mathcal{B}\nabla^\ell(\mathcal{B}^{-1}F). \end{aligned} \tag{2.3.10}$$

In the next sections of the proof, we will use energy estimates derived from the expansion (2.3.10) that are well behaved in the limit $t \nearrow 0$. Then we will use these energy estimates to deduce the global existence of solutions as well as decay estimates assuming suitable small initial data.

2.4 L^2 ENERGY ESTIMATE

We derive a L^2-energy identity by taking the inner product of (2.0.1) with u, which after simplifying gives

$$\frac{1}{2}\partial_t\langle u|B^0u\rangle = \frac{1}{t}\langle u|\mathcal{B}\mathbb{P}u\rangle + \frac{1}{2}\langle u|\,\mathrm{div}Bu\rangle + \langle u|F\rangle, \tag{2.4.1}$$

where

$$\mathrm{div}B = \mathrm{div}B(t, x, u(t, x), \nabla u(t, x))$$

and $\mathrm{div}B(t, x, u, w)$ is as defined above by (2.2.26), see also (2.2.30). We then define the energy norm by

$$\|\!|u|\!\|_s^2 = \sum_{\ell=0}^{s}\langle\nabla^\ell u|B^0\nabla^\ell u\rangle, \tag{2.4.2}$$

and notice, with the help of (2.2.2) and the fact that $t<0$, that the inequalities

$$\frac{2}{t}\langle v|\mathcal{B}v\rangle \leq \frac{2\kappa}{t}\|v\|_0^2, \quad \text{and} \quad \|v\|_{L^2} \leq \sqrt{\gamma_1}\|v\|_0 \tag{2.4.3}$$

hold for any $v\in L^2(V)$. We also recall the estimates

$$\begin{aligned}
&|\langle u|F\rangle| \leq \|u\|_{L^2}\|\tilde{F}\|_{L^2} + C\|u\|_{L^2}\|u\|_{H^k} + |t|^{-\frac{1}{2}}(\lambda_1+\lambda_2)\|u\|_{L^2}\|\mathbb{P}u\|_{L^2} + |t|^{-1}\lambda_3\|\mathbb{P}u\|_{L^2}^2,\\
&|\langle \mathbb{P}u|\mathbb{P}F\rangle| \leq \|\tilde{F}\|_{L^2}\|\mathbb{P}u\|_{L^2} + C\|u\|_{L^2}\|\mathbb{P}u\|_{L^2} + |t|^{-\frac{1}{2}}\lambda_1\|u\|_{L^2}\|\mathbb{P}u\|_{L^2},
\end{aligned} \tag{2.4.4}$$

and

$$\begin{aligned}
|\langle v|\operatorname{div}Bv\rangle| \leq& 4\theta\|v\|_{L^2}^2 + |t|^{-\frac{1}{2}}\Big((\beta_0+\beta_2+\beta_4)\|v\|_{L^2}\|\mathbb{P}v\|_{L^2} + \frac{\beta_6}{R}\||v|^2|\mathbb{P}u|\|_{L^1}\Big)\\
&+|t|^{-1}\Big(\beta_1\|\mathbb{P}v\|_{L^2}^2 + \frac{\beta_3+\beta_5}{R}\||v||\mathbb{P}v||\mathbb{P}u|\|_{L^1} + \frac{\beta_7}{R^2}\||v|^2|\mathbb{P}u|^2\|_{L^1}\Big),
\end{aligned} \tag{2.4.5}$$

from Proposition 3.4 of [1]. Estimates (2.4.4)-(2.4.5) can be calculated using the expansion (2.2.10), Cauchy-Schwartz inequalities, and the properties of the projection operators $\mathbb{P}$, $\mathbb{P}^\perp$. Using (2.4.3), (2.4.4), (2.4.5), (2.3.5), (2.4.1) and Sobolev's inequalities, we obtain, after some calculation, the estimate

$$\begin{aligned}
\partial_t\|u\|_0^2 \leq& \frac{\big(2\kappa-\gamma_1\big[\beta_{\mathrm{o}}+2\lambda_3\big]\big)}{t}\|\mathbb{P}u\|_0^2 + \frac{\sqrt{\gamma_1}\big(\beta_{\mathrm{e}}+2(\lambda_1+\lambda_2)\big)}{|t|^{\frac{1}{2}}}\|\mathbb{P}u\|_0\|u\|_{L^2}\\
&+C(\|u\|_{H^k})\|u\|_{L^2}\|u\|_{H^k} + 2\|u\|_{L^2}\|\tilde{F}\|_{L^2}, \quad T_0\leq t<T_*,
\end{aligned} \tag{2.4.6}$$

where we have set

$$\beta_{\mathrm{o}} = \sum_{a=0}^{3}\beta_{2a+1} \quad \text{and} \quad \beta_{\mathrm{e}} = \sum_{a=0}^{3}\beta_{2a}.$$

The constant $C(\|u\|_k)$ depends implicitly on θ, which will be fixed throughout the proof. We do not indicate the dependence of any of the constants on θ, for example λ_a, β_a, γ_2, κ, and so on. The estimate (2.4.6) together with Young's inequality

$$ab \leq \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2, \tag{2.4.7}$$

for $a, b \geq 0$ and $\epsilon > 0$, gives

$$\partial_t \|\!|u|\!\|_0^2 \leq \frac{\rho_0}{t}\|\!|\mathbb{P}u|\!\|_0^2 + (1+\epsilon^{-1})C(\|\!|u|\!\|_k)\|\!|u|\!\|_0\|\!|u|\!\|_k + 2\sqrt{\gamma_1}\|\!|u|\!\|_0\|\tilde{F}\|_{L^2}, \quad T_0 \leq t < T_*, \tag{2.4.8}$$

where

$$\rho_0 = 2\kappa - \gamma_1\big[\beta_{\mathrm{o}} + 2\lambda_3 + \epsilon(\beta_{\mathrm{e}} + 2\lambda_1 + 2\lambda_2)\big],$$

which holds for any $\epsilon > 0$. Since we have assumed that $2\kappa - \gamma_1\big[\beta_{\mathrm{o}} + 2\lambda_3\big] > 0$, we can choose ϵ small enough to ensure that

$$\rho_0 > 0. \tag{2.4.9}$$

2.5 H^k energy estimate

Before continuing the proof, we first note the equivalence between the norm $\|\!|u|\!\|_k$ and the standard Sobolev norm $\|u\|_{H^k}$. This equivalence is a consequence of the assumption

$$\frac{1}{\gamma_1}\mathrm{id}_{V_{\pi(v)}} \leq B^0(t,v) \leq \frac{1}{\kappa}\mathcal{B}(t,v) \leq \gamma_2 \mathrm{id}_{V_{\pi(v)}}. \tag{2.5.1}$$

Indeed, that multiplying (2.5.1) by $\nabla^l u$, and taking the inner product of the result with $\nabla^l u$, it follows, after summing from $\ell = 0$ up to s that

$$\frac{1}{\sqrt{\gamma_1}}\|\cdot\|_{H^s} \leq \|\!|\cdot|\!\|_s \leq \sqrt{\gamma_2}\|\cdot\|_{H^s}, \tag{2.5.2}$$

which establishes the equivalence. In the following calculations we use either $\|\!|\cdot|\!\|_s$ and $\|\!|\cdot|\!\|_{H^s}$ interchangeably and without comment. Applying the L^2 energy identity (2.4.1) to (2.3.10) gives

$$\frac{1}{2}\partial_t\langle\nabla^\ell u|B^0\nabla^\ell u\rangle = \frac{1}{t}\langle\nabla^\ell u|\mathcal{B}\mathbb{P}\nabla^\ell u\rangle + \frac{1}{2}\langle\nabla^\ell u|\,\mathrm{div}B\nabla^\ell u\rangle + \langle\nabla^\ell u|G_\ell\rangle, \qquad 0 \leq \ell \leq k, \tag{2.5.3}$$

where

$$\begin{aligned} G_\ell =& |t|^{-1}\mathcal{B}[\nabla^\ell, \mathcal{B}^{-1}B^0](B^0)^{-1}\mathcal{B}\mathbb{P}u + \mathcal{B}[\nabla^\ell, \mathcal{B}^{-1}B^0](B^0)^{-1}B^i\nabla_i u \\ &- \mathcal{B}[\nabla^\ell, \mathcal{B}^{-1}B^i]\nabla_i u - B^i[\nabla^\ell, \nabla_i]u - \mathcal{B}[\nabla^\ell, \mathcal{B}^{-1}B^0](B^0)^{-1}F + \mathcal{B}\nabla^\ell(\mathcal{B}^{-1}F). \end{aligned}$$

Using the properties of the projection operator (2.2.1), the estimates (2.4.4), (2.4.5), the bound (2.3.5), the energy identity (2.5.3) and Sobolev's inequality, we get

$$\begin{aligned}\partial_t \|\!|\nabla^\ell u|\!\|_0^2 \leq & \frac{2\kappa - \gamma_1\beta_1}{t}\|\!|\nabla^\ell \mathbb{P}u|\!\|_0^2 - \frac{\gamma_1(\beta_3+\beta_5+\beta_7)}{t}\|\!|\mathbb{P}u|\!\|_k\|\!|\mathbb{P}u|\!\|_{k-1} \\ & + \frac{\gamma_1\beta_{\mathrm{e}}}{|t|^{\frac{1}{2}}}\|\!|\mathbb{P}u|\!\|_k\|\!|u|\!\|_k + 4\theta\gamma_1\|\!|\nabla^\ell u|\!\|_{L^2}^2 + 2\langle\nabla^\ell u|G_\ell\rangle, \qquad T_0 \leq t < T_*.\end{aligned} \tag{2.5.4}$$

Equation (2.5.4) gives us a bound for $\partial_t\|\!|\nabla^\ell u|\!\|_0^2$, and so, it only remains to obtain an estimate for $\langle\nabla^\ell u|G_\ell\rangle$. We obtain this estimate with the help of Proposition 3.6 from [1] which we recall here: Assuming $k \in \mathbb{Z}_{>n/2+2}$, $1 \leq \ell \leq k$, $v \in L^2(V\otimes T^0_\ell(\Sigma))$, $u \in B_{C^{-1}_{\mathrm{Sob}}R}\big(H^k(V)\big)$, $\mathcal{B} = \mathcal{B}(t, u(x))$, $B^0 = B^0(t,u(x))$ and $B = B(t,u(x))$, we have that

$$\begin{aligned}|\langle v|\mathcal{B}\nabla^\ell(\mathcal{B}^{-1}F)\rangle| + |\langle v|\mathcal{B}[\nabla^\ell, \mathcal{B}^{-1}B^0](B^0)^{-1}F\rangle| &\leq C\big(\|v\|_{L^2}\|\tilde{F}\|_{H^k} + \Xi\big), \\ |\langle v|\mathcal{B}[\nabla^\ell, \mathcal{B}^{-1}B^0](B^0)^{-1}t^{-1}\mathcal{B}\mathbb{P}u\rangle| &\leq C\big(|t|^{-1}\|\mathbb{P}v\|_{L^2}\|\mathbb{P}u\|_{H^{k-1}} + \Xi\big), \\ |\langle v|B^i[\nabla^\ell, \nabla_i]u\rangle| &\leq C\big(|t|^{-1}\|\mathbb{P}v\|_{L^2}\|\mathbb{P}u\|_{H^{k-1}} + \Xi\big),\end{aligned} \tag{2.5.5}$$

and

$$\begin{aligned}|\langle v|\mathcal{B}[\nabla^\ell, \mathcal{B}^{-1}B^i]\nabla_i u\rangle| + |\langle v|\mathcal{B}[\nabla^\ell, \mathcal{B}^{-1}B^0](B^0)^{-1}B^i\nabla_i u\rangle| \leq |t|^{-1}\Big(&\ell\mathtt{b}\|\mathbb{P}v\|_{L^2}\|\mathbb{P}u\|_{H^k} + \\ &C\|\mathbb{P}v\|_{L^2}\|\mathbb{P}u\|_{H^{k-1}}\Big) + C\Xi,\end{aligned} \tag{2.5.6}$$

where $C = C(\|u\|_{H^k})$ and the constants $\mathtt{b}, \Xi$, are given by

$$\begin{aligned}\Xi &= \|v\|_{L^2}\|u\|_{H^k} + |t|^{-\frac{1}{2}}\Big(\|v\|_{L^2}\|\mathbb{P}u\|_{H^k} + \|\mathbb{P}v\|_{L^2}\|u\|_{H^k}\Big) + \\ &\qquad |t|^{-1}\Big(\|v\|_{L^2}\|\mathbb{P}u\|_{H^k}^2 + \|\mathbb{P}v\|_{L^2}\|u\|_{H^k}\|\mathbb{P}u\|_{H^k}\Big), \\ \mathtt{b} &= \sup_{T_0\leq t<0}\Big(\big\||\mathbb{P}\tilde{\mathcal{B}}(t)\nabla(\tilde{\mathcal{B}}(t)^{-1}\tilde{B}^0(t))\tilde{B}^0(t)^{-1}\mathbb{P}\tilde{B}_2(t)\mathbb{P}|_{\mathrm{op}}\big\|_{L^\infty} + \\ &\qquad \big\||\mathbb{P}\tilde{\mathcal{B}}(t)\nabla(\tilde{\mathcal{B}}(t)^{-1}\tilde{B}_2(t))\mathbb{P}|_{\mathrm{op}}\big\|_{L^\infty}\Big).\end{aligned} \tag{2.5.7}$$

Then, using (2.5.5)-(2.5.6), it can be shown that $\langle\nabla^\ell u|G_\ell\rangle$ is bounded by

$$\langle\nabla^\ell u|G_\ell\rangle \leq -\frac{1}{t}\Big[\ell\mathtt{b}\|\mathbb{P}u\|_{H^k}^2 + C(\|u\|_{H^k})\big(\|\mathbb{P}u\|_{H^k}\|\mathbb{P}u\|_{H^{k-1}} + \|u\|_{H^k}\|\mathbb{P}u\|_{H^k}^2\big)\Big]$$

$$+\frac{1}{|t|^{\frac{1}{2}}}C(\|u\|_{H^k})\|u\|_{H^k}\|\mathbb{P}u\|_{H^k}+C(\|u\|_{H^k})\Big[\|u\|^2_{H^k}+\|u\|_{H^k}\|\tilde{F}\|_{H^k}\Big], \tag{2.5.8}$$

and therefore, from the estimates (2.5.4) and (2.5.8), we obtain

$$\begin{aligned}\partial_t\|\nabla^\ell u\|_0^2 \leq & \frac{2\kappa-\gamma_1\beta_1}{t}\|\nabla^\ell \mathbb{P}u\|_0^2-\frac{1}{t}\Big[2\gamma_1\ell\mathtt{b}\|\mathbb{P}u\|_k^2+C(\|u\|_k)\big(\|\mathbb{P}u\|_k\|\mathbb{P}u\|_{k-1}+\|u\|_k\|\mathbb{P}u\|_k^2\big)\Big]\\ &+\frac{1}{|t|^{\frac{1}{2}}}C(\|u\|_k)\|u\|_k\|\mathbb{P}u\|_k+C(\|u\|_k)\big(\|u\|_k^2+\|\tilde{F}\|^2_{H^k}\big), \qquad T_0\leq t<T_*.\end{aligned}$$

Summing this inequality from $\ell=0$ to k, applying Young's inequality and Ehrling's lemma (Lemma A.2.5) we obtain the estimate

$$\begin{aligned}\partial_t\|u\|_k^2 \leq & \frac{2\kappa-\gamma_1(\beta_1+2\mathtt{b}_k)-C(\|u\|_k)(\epsilon+\|u\|_k)}{t}\|\mathbb{P}u\|_k^2\\ &-\frac{1}{t}c(\|u\|_k,\epsilon^{-1})\|\mathbb{P}u\|_0^2+C(\|u\|_k,\epsilon^{-1})\big(\|u\|_k^2+\|\tilde{F}\|^2_{H^k}\big),\end{aligned} \tag{2.5.9}$$

for any $\epsilon>0$, where we have set

$$\mathtt{b}_k=\mathtt{b}\sum_{\ell=1}^{k}\ell=\frac{1}{2}k(k+1)\mathtt{b}.$$

2.6 Global existence on $[T_0,0)\times\Sigma$

In this section, we focus our attention to prove the global existence of solutions to (2.0.1) on $[T_0,0)\times\Sigma$. From our initial assumptions, we have initial data satisfying $\|u(T_0)\|_{H^k}<\delta$, combining this with (2.5.2) leads to

$$\|u(T_0)\|_k\leq\sqrt{\gamma_2}\|u(T_0)\|_{H^k}<\delta\sqrt{\gamma_2}.$$

Then, we define the time $T_\delta\in(T_0,T_*]$ as the first time when $\|u(T_\delta)\|_k=2\delta\sqrt{\gamma_2}$, where δ satisfies

$$0<\delta\leq\min\left\{\frac{\mathcal{R}}{2\sqrt{\gamma_1\gamma_2}},\frac{\mathcal{R}}{4}\right\}. \tag{2.6.1}$$

In the case that such time does not exist, we can define the maximal time of existence as $T_\delta = T^*$. In both cases, we have that

$$\|u(t)\|_k \leq 2\delta\sqrt{\gamma_2}, \qquad T_0 \leq t < T_\delta. \tag{2.6.2}$$

Combining (2.6.2) with (2.5.2) leads to

$$\|u(t)\|_{H^k} \leq \sqrt{\gamma_1}\|u(t)\|_k \leq 2\delta\sqrt{\gamma_1\gamma_2} \leq \mathcal{R}, \qquad T_0 \leq t < T_\delta \leq T_* \leq T^*,$$

and notice that choosing ϵ such that $\epsilon = \delta\sqrt{\gamma_2}$, then we can write (2.5.9) as

$$\partial_t\|u\|_k^2 \leq \frac{\rho_k}{t}\|\mathbb{P}u\|_k^2 - \frac{1}{t}c(\delta,\delta^{-1})\|\mathbb{P}u\|_0^2 + C(\delta,\delta^{-1})\big(\|u\|_k^2 + \|\tilde{F}\|_{H^k}^2\big), \qquad T_0 \leq t < T_\delta, \tag{2.6.3}$$

where

$$\rho_k = 2\kappa - \gamma_1(\beta_1 + 2\mathtt{b}_k) - C(\delta)\delta. \tag{2.6.4}$$

By assumption, we have that $2\kappa - \gamma_1(\beta_1 + 2\mathtt{b}_k) > 0$. Noting that $\lim_{\delta\searrow 0} C(\delta)\delta = 0$, we therefore can ensure

$$\rho_k > 0 \tag{2.6.5}$$

by choosing $\delta > 0$ sufficiently small. Adding $\rho_0^{-1}c(\delta,\delta^{-1})$ times (2.4.8) to (2.6.3) yields the differential energy inequality

$$\partial_t\big(\|u\|_k^2 + \rho_0^{-1}c(\delta,\delta^{-1})\|u\|_0^2\big) \leq \frac{\rho_k}{t}\|\mathbb{P}u\|_k^2 + C(\delta,\delta^{-1})\big(\|u\|_k^2 + \|\tilde{F}\|_{H^k}^2\big), \qquad T_0 \leq t < T_\delta.$$

Setting

$$E_k(t) = \|u(t)\|_k^2 + \rho_0^{-1}c(\delta,\delta^{-1})\|u(t)\|_0^2 - \int_{T_0}^{t}\frac{\rho_k}{\tau}\|\mathbb{P}u(\tau)\|_k^2\,d\tau + \sup_{T_0\leq\tau<0}\|\tilde{F}(\tau)\|_{H^k}^2 \tag{2.6.6}$$

we then have

$$\partial_t E_k \leq C(\delta,\delta^{-1})E_k, \qquad T_0 \leq t < T_\delta,$$

and so, applying Gronwall's inequality, we conclude that

$$E_k(t) \leq e^{C(\delta,\delta^{-1})(t-T_0)}E_k(T_0), \qquad T_0 \leq t < T_\delta. \tag{2.6.7}$$

We now consider $\delta_0 \in (0, \delta)$ with δ fixed such that the equations (2.4.9), (2.6.1), and (2.6.5) are satisfied, and we assume that

$$\|u(T_0)\|_{H^k} \leq \delta_0 \quad \text{and} \quad \sup_{T_0 \leq \tau < 0} \|\tilde{F}(\tau)\|^2_{H^k} \leq \delta_0. \tag{2.6.8}$$

Then, using (2.6.6) and (2.6.7), we determine the bound

$$|\!|\!| u(t) |\!|\!|_k \leq e^{C(\delta,\delta^{-1})(-T_0)}\big(2 + \rho_0^{-1} c(\delta, \delta^{-1})\big)\delta_0, \qquad T_0 \leq t < T_\delta, \tag{2.6.9}$$

and notice that, choosing δ_0 small enough, we can write inequality (2.6.9) as

$$|\!|\!| u(t) |\!|\!|_k < \delta\sqrt{\gamma_2}, \quad T_0 \leq t < T_\delta,$$

where

$$0 < \delta_0 < \delta \leq \min\Big\{\frac{\mathcal{R}}{2\sqrt{\gamma_1\gamma_2}}, \frac{\mathcal{R}}{4}\Big\}. \tag{2.6.10}$$

We conclude that $T_\delta = T_* = T^* = 0$ from the definition of T_δ and the maximality of T^*, and consequently, this establishes the global existence of solutions on $[T_0, 0) \times \Sigma$ that are uniformly bounded by

$$\|u(t)\|_{H^k} \leq \sqrt{\gamma_1} |\!|\!| u(t) |\!|\!|_k \leq \hat{\delta} := 2\delta\sqrt{\gamma_1\gamma_2}, \quad T_0 \leq t < 0. \tag{2.6.11}$$

Moreover, by (2.6.6), (2.6.7) and the equivalence of norms $\|\cdot\|_{H^k}$ and $|\!|\!| \cdot |\!|\!|_k$, it follows that the energy estimate

$$\|u(t)\|^2_{H^k} + \sup_{T_0 \leq \tau < 0} \|\tilde{F}(\tau)\|^2_{H^k} - \int_{T_0}^t \frac{1}{\tau}\|\mathbb{P}u(\tau)\|^2_{H^k}\, d\tau \leq C(\delta) e^{C(\delta,\delta^{-1})(t-T_0)}\Big(\|u(T_0)\|^2_{H^k} + \sup_{T_0 \leq \tau < 0} \|\tilde{F}(\tau)\|^2_{H^k}\Big), \tag{2.6.12}$$

holds for all $t \in [T_0, 0)$.

2.7 Limit of $\mathbb{P}^\perp u$ as $t \nearrow 0$

Noting that we can write (2.0.1) as

$$\partial_t u = (B^0)^{-1}\Big(-B^i \nabla_i u + \frac{1}{t}\mathcal{B}\mathbb{P}u + F\Big),$$

we see, after multiplying on the left by $\mathbb{P}^\perp$, that

$$\partial_t \mathbb{P}^\perp u = \mathbb{P}^\perp (B^0)^{-1}\Big(-B^i\nabla_i u + \frac{1}{t}\mathcal{B}\mathbb{P}u + F\Big). \tag{2.7.1}$$

Integrating this in time and taking the H^{k-1} norm yields

$$\|\mathbb{P}^\perp u(t_2) - \mathbb{P}^\perp u(t_1)\|_{H^{k-1}} \leq \int_{t_1}^{t_2} \Big\|\mathbb{P}^\perp (B^0(\tau))^{-1}\Big(-B^i(\tau)\nabla_i u(\tau) + \frac{1}{\tau}\mathcal{B}(\tau)\mathbb{P}u(\tau) + F(\tau)\Big)\Big\|_{H^{k-1}} d\tau, \tag{2.7.2}$$

for any t_1, t_2 satisfying $T_0 \leq t_1 < t_2$. We then estimate the integrand of the right hand side of (2.7.2) using Proposition 3.2 from [1] to get

$$\begin{aligned}\|\mathbb{P}^\perp u(t_2) - \mathbb{P}^\perp u(t_1)\|_{H^{k-1}} \leq \int_{t_1}^{t_2} C(\|u(\tau)\|_{H^k})\Big(&\|\tilde{F}(\tau)\|_{H^k} + 1 + \frac{1}{|\tau|^{\frac{1}{2}}}\|\mathbb{P}u(\tau)\|_{H^k} \\ &- \frac{1}{\tau}\|\mathbb{P}u(\tau)\|_{H^k}\|\mathbb{P}u(\tau)\|_{H^{k-1}}\Big) d\tau.\end{aligned} \tag{2.7.3}$$

From this estimate, (2.6.8) and the bounds (2.6.8), (2.6.10) and (2.6.12), we have

$$\|\mathbb{P}^\perp u(t_2) - \mathbb{P}^\perp u(t_1)\|_{H^{k-1}} \leq C(\delta)\Big(|t_1 - t_2| - \int_{t_1}^{t_2} \frac{1}{\tau}\|\mathbb{P}u(\tau)\|^2_{H^k}\, d\tau\Big) = \mathrm{o}(|t_1 - t_2|). \tag{2.7.4}$$

By choosing δ_0 small enough to make the right hand side of (2.6.9) comparable to δ, we can make the right hand side of (2.6.12) comparable to δ. This then allows us to conclude that $\lim_{t\nearrow 0}\mathbb{P}^\perp u(t)$ exists in $H^{k-1}(V)$ and that

$$\mathbb{P}^\perp u \in C^0([T_0, 0], H^{k-1}(V)).$$

2.8 L^2 decay estimate for $\mathbb{P}u$

We now proceed to obtain decay estimates for $\mathbb{P}u$ with the help of an L^2 energy identity. We can construct this identity by multiplying equation (2.0.1) on the left by $\mathbb{P}$, using the properties of the projection operator (2.2.1), and setting

$$\mathcal{F} = -\mathbb{P}B^0\mathbb{P}^\perp\partial_t\mathbb{P}^\perp u - \mathbb{P}B^i\mathbb{P}^\perp\nabla_i u + \mathbb{P}F \tag{2.8.1}$$

we get that

$$\mathbb{P}B^0\mathbb{P}\partial_t\mathbb{P}u + \mathbb{P}B^i\mathbb{P}\nabla_i\mathbb{P}u = \frac{1}{t}\mathcal{B}\mathbb{P}u + \mathcal{F}. \tag{2.8.2}$$

Then we take the L^2 inner-product of (2.8.2) with $\mathbb{P}u$, and we obtain the energy identity

$$\frac{1}{2}\partial_t\langle \mathbb{P}u|B^0\mathbb{P}u\rangle = \frac{1}{t}\langle \mathbb{P}u|\mathcal{B}\mathbb{P}u\rangle + \frac{1}{2}\langle u|\mathbb{P}\,\mathrm{div}B\,\mathbb{P}u\rangle + \langle \mathbb{P}u|\mathbb{P}\mathcal{F}\rangle. \tag{2.8.3}$$

Taking the L^2 inner product of (2.2.33) with $\mathbb{P}u$ in conjunction with the definition (2.4.2), we see that

$$\frac{1}{t}\langle \mathbb{P}u|\mathcal{B}\mathbb{P}u\rangle \le \frac{\tilde{\kappa}}{t}|\!|\!|\mathbb{P}u|\!|\!|_0^2. \tag{2.8.4}$$

Note that the estimate

$$|\langle \mathbb{P}u|\mathbb{P}B^i\mathbb{P}^\perp\nabla_i u\rangle| \le C(\delta)\|u\|_{H^k}\|\mathbb{P}u\|_{L^2}+|t|^{-\frac{1}{2}}\alpha\|\nabla\mathbb{P}^\perp u\|_{L^2}\|\mathbb{P}u\|_{L^2}+|t|^{-1}C(\delta)\|\nabla\mathbb{P}^\perp u\|_{L^2}\|\mathbb{P}u\|_{L^2}^2 \tag{2.8.5}$$

can be computed using the calculus inequalities A.2, the expansion (2.2.16) for $B=(B^i)$, and the properties (2.2.18) and (2.2.21) of the expansions coefficients $\mathbb{P}B_a\mathbb{P}^\perp$, where $a=1,2$. Then, taking the H^{k-1} norm of (2.7.1), integrating in time, and following similar arguments used to arrive to (2.7.4), we obtain the estimate

$$\int_{T_0}^t \|\partial_t\mathbb{P}^\perp u(\tau)\|_{H^{k-1}}\,d\tau \le C(\delta), \qquad T_0\le t<0. \tag{2.8.6}$$

Then using Proposition 3.2 from [1], (2.6.11) and (2.7.1) it can be shown that

$$\||t|^{\frac{1}{2}}\partial_t\mathbb{P}^\perp u(t)\|_{L^2} \le C(\delta)\biggl(1+\frac{1}{|t|^{\frac{1}{2}}}\|\mathbb{P}u\|_{L^2}\biggr), \qquad T_0\le t<0. \tag{2.8.7}$$

From the energy identity (2.8.3), the estimates (2.8.4), (2.8.5) and (2.8.7), the coefficient bounds (2.2.5), (2.2.12), (2.2.13), (2.2.27) and (2.2.33); the expansion (2.2.10) for F along with (2.2.11), and (2.4.4), (2.4.5), (2.6.11) and the calculus inequalities from Appendix A.2, we obtain the differential energy inequality

$$\begin{aligned}\frac{1}{2}\partial_t|\!|\!|\mathbb{P}u|\!|\!|_0^2 \le \biggl[\frac{1}{t}\biggl(\tilde{\kappa}-\tilde{\gamma}_1\biggl(\frac{\beta_1}{2}+\|\nabla\mathbb{P}^\perp u\|_{L^2}C(\delta)\biggr)\biggr)+C(\delta)\biggl(1+\frac{1}{|t|^{\frac{1}{2}}}+\|\partial_t\mathbb{P}^\perp u\|_{H^{k-1}}\biggr)\biggr]|\!|\!|\mathbb{P}u|\!|\!|_0^2 \\ +\biggl[C(\delta)+\frac{\bigl(\|u\|_{L^2}\lambda_1+\|\nabla\mathbb{P}^\perp u\|_{L^2}\alpha\bigr)\sqrt{\tilde{\gamma}_1}}{|t|^{\frac{1}{2}}}\biggr]|\!|\!|\mathbb{P}u|\!|\!|_0.\end{aligned}$$

Dividing through $|\!|\!|\mathbb{P}u|\!|\!|_0$ gives

$$\partial_t|\!|\!|\mathbb{P}u|\!|\!|_0 \le \biggl[\frac{\hat{\rho}}{t}+C(\delta)\biggl(1+\frac{1}{|t|^{\frac{1}{2}}}+\|\partial_t\mathbb{P}^\perp u\|_{H^{k-1}}\biggr)\biggr]|\!|\!|\mathbb{P}u|\!|\!|_0+C(\delta)\biggl[1+\frac{(\lambda_1+\alpha)}{|t|^{\frac{1}{2}}}\biggr], \tag{2.8.8}$$

where

$$\hat{\rho} = \tilde{\kappa} - \tilde{\gamma}_1\left(\frac{\beta_1}{2} + C(\delta)\delta\right).$$

By assumption $\tilde{\kappa} - \frac{\tilde{\gamma}_1\beta_1}{2} > 0$, and therefore, we can arrange, by shrinking δ if necessary, that

$$\hat{\rho} > 0.$$

Now, suppose that $x(t)$ satisfies $x'(t) \leq a(t)x(t) + h(t)$, $t \geq T_0$. Then by Gronwall's inequality, we have

$$x(t) \leq x(T_0)e^{A(t)} + \int_{T_0}^{t} e^{A(t)-A(\tau)}h(\tau)d\tau, \tag{2.8.9}$$

where

$$A(t) = \int_{T_0}^{t} a(\tau)d\tau. \tag{2.8.10}$$

In the particular case that $x(T_0) \geq 0$ and $a(t) = \frac{\lambda}{t} + b(t)$, where $\lambda \in \mathbb{R}$ and $|\int_{T_0}^{t} b(\tau)\, d\tau| \leq r$, this becomes

$$x(t) \leq e^r x(T_0)\left(\frac{t}{T_0}\right)^{\lambda} + e^{2r}(-t)^{\lambda}\int_{T_0}^{t} \frac{|h(\tau)|}{(-\tau)^{\lambda}}d\tau, \tag{2.8.11}$$

for $T_0 \leq t < 0$. With the help of the integral formula

$$(-t)^{\lambda}\int_{T_0}^{t} \frac{1}{(-\tau)^{\lambda+\mu}}\,d\tau = \begin{cases} \frac{1}{\lambda+\mu-1}|t|^{1-\mu} + \frac{|T_0|^{1-(\lambda+\mu)}}{1-(\lambda+\mu)}|t|^{\lambda} & \text{if } \lambda+\mu \neq 1 \\ -|t|^{\lambda}\ln\left(\frac{t}{T_0}\right) & \text{if } \lambda+\mu = 1 \end{cases}, \tag{2.8.12}$$

we then deduce from (2.8.6), (2.8.8), the L^2 decay estimate

$$\|\mathbb{P}u(t)\|_{L^2} \lesssim \begin{cases} |t| + (\lambda_1+\alpha)|t|^{\frac{1}{2}} & \text{if } \hat{\rho} > 1 \\ -|t|\ln\left(\frac{t}{T_0}\right) + (\lambda_1+\alpha)|t|^{\frac{1}{2}} & \text{if } \hat{\rho} = 1 \\ |t|^{\hat{\rho}} + (\lambda_1+\alpha)|t|^{\frac{1}{2}} & \text{if } \frac{1}{2} < \hat{\rho} < 1 \\ |t|^{\frac{1}{2}} - (\lambda_1+\alpha)|t|^{\frac{1}{2}}\ln\left(\frac{t}{T_0}\right) & \text{if } \hat{\rho} = \frac{1}{2} \\ |t|^{\hat{\rho}} & \text{if } 0 < \hat{\rho} < \frac{1}{2} \end{cases}, \qquad T_0 \leq t < 0. \tag{2.8.13}$$

The H^{k-1} decay estimate for $\mathbb{P}u$ can be obtained using similar steps as in the derivation of the L^2 decay estimate for $\mathbb{P}u$. As a first step, assume $0 \leq \ell \leq k-1$. Then by applying $\mathcal{B}\mathbb{P}\nabla^\ell\mathcal{B}^{-1}\mathbb{P}$ to (2.8.2), we see that

$$\begin{aligned}\mathbb{P}B^0\mathbb{P}\partial_t\nabla^\ell\mathbb{P}u+\mathbb{P}B^i\mathbb{P}\nabla_i\nabla^\ell\mathbb{P}u &= \frac{1}{t}\mathcal{B}\nabla^\ell\mathbb{P}u - \mathcal{B}\mathbb{P}[\nabla^\ell,\mathcal{B}^{-1}\mathbb{P}B^0\mathbb{P}]\mathbb{P}\partial_t u \\ &\quad - \mathcal{B}\mathbb{P}[\nabla^\ell,\mathcal{B}^{-1}\mathbb{P}B^i\mathbb{P}]\nabla_i\mathbb{P}u - \mathbb{P}B^i\mathbb{P}[\nabla^\ell,\nabla_i]\mathbb{P}u + \mathcal{B}\mathbb{P}\nabla^\ell(\mathcal{B}^{-1}\mathbb{P}\mathcal{F}).\end{aligned}$$

Using (2.0.1), we can write this as

$$\begin{aligned}\mathbb{P}B^0\mathbb{P}\partial_t\nabla^\ell\mathbb{P}u+\mathbb{P}B^i\mathbb{P}\nabla_i\nabla^\ell\mathbb{P}u &= \frac{1}{t}\Big[\mathcal{B}\mathbb{P}\nabla^\ell\mathbb{P}u - \mathcal{B}\mathbb{P}[\nabla^\ell,\mathcal{B}^{-1}\mathbb{P}B^0\mathbb{P}]\mathbb{P}(B^0)^{-1}\mathcal{B}\mathbb{P}u\Big] \\ &\quad + \mathcal{B}\mathbb{P}[\nabla^\ell,\mathcal{B}^{-1}\mathbb{P}B^0\mathbb{P}]\mathbb{P}(B^0)^{-1}B^i\nabla_i u - \mathcal{B}\mathbb{P}[\nabla^\ell,\mathcal{B}^{-1}\mathbb{P}B^i\mathbb{P}]\nabla_i\mathbb{P}u \\ &\quad - \mathbb{P}B^i\mathbb{P}[\nabla^\ell,\nabla_i]\mathbb{P}u - \mathcal{B}\mathbb{P}[\nabla^\ell,\mathcal{B}^{-1}\mathbb{P}B^0\mathbb{P}]\mathbb{P}(B^0)^{-1}F + \mathcal{B}\mathbb{P}\nabla^\ell(\mathcal{B}^{-1}\mathcal{F}).\end{aligned} \tag{2.9.1}$$

The aim is to obtain via Propositions 3.2, and 3.7 from [1], appropriate bounds for the energy identity obtained from taking the L^2 inner-product of (2.9.1) with $\nabla^\ell\mathbb{P}u$ which gives

$$\begin{aligned}\frac{1}{2}\partial_t\langle\nabla^\ell\mathbb{P}u|B^0\nabla^\ell\mathbb{P}u\rangle &= \frac{1}{t}\langle\nabla^\ell\mathbb{P}u|\mathcal{B}\nabla^\ell\mathbb{P}u\rangle + \frac{1}{2}\langle\nabla^\ell\mathbb{P}u|\mathbb{P}\,\mathrm{div}B\,\mathbb{P}\nabla^\ell\mathbb{P}u\rangle + \langle\nabla^\ell\mathbb{P}u|\mathcal{G}_\ell\rangle \\ &\quad + \langle\nabla^\ell\mathbb{P}u|\mathcal{B}\mathbb{P}\nabla^\ell(\mathcal{B}^{-1}\mathcal{F})\rangle,\end{aligned} \tag{2.9.2}$$

where

$$\begin{aligned}\mathcal{G}_\ell =&|t|^{-1}\mathcal{B}\mathbb{P}[\nabla^\ell,\mathcal{B}^{-1}\mathbb{P}B^0\mathbb{P}]\mathbb{P}(B^0)^{-1}\mathcal{B}\mathbb{P}u + \mathcal{B}\mathbb{P}[\nabla^\ell,\mathcal{B}^{-1}\mathbb{P}B^0\mathbb{P}]\mathbb{P}(B^0)^{-1}B^i\nabla_i u \\ &- \mathcal{B}\mathbb{P}[\nabla^\ell,\mathcal{B}^{-1}\mathbb{P}B^i\mathbb{P}]\nabla_i\mathbb{P}u - \mathbb{P}B^i\mathbb{P}[\nabla^\ell,\nabla_i]\mathbb{P}u - \mathcal{B}\mathbb{P}[\nabla^\ell,\mathcal{B}^{-1}\mathbb{P}B^0\mathbb{P}]\mathbb{P}(B^0)^{-1}F.\end{aligned}$$

Then, after some calculations, it can be verified that $\partial_t|||\mathbb{P}u|||_{k-1}$, satisfies the differential inequality

$$\begin{aligned}\partial_t|||\mathbb{P}u|||_{k-1} \leq& \left[\frac{\tilde{\rho}}{t} + C(\delta)\left(1+\frac{1}{|t|^{\frac{1}{2}}} + \|\partial_t\mathbb{P}^\perp u\|_{H^{k-1}}\right)\right]|||\mathbb{P}u|||_{k-1} \\ &+ C(\delta)\left(1+\frac{\lambda_1+\alpha}{|t|^{\frac{1}{2}}}\right) - \frac{C(\delta,\delta^{-1})}{t}\|\mathbb{P}u\|_{L^2},\end{aligned} \tag{2.9.3}$$

where
$$\tilde{\rho} = \tilde{\kappa} - \tilde{\gamma}_1\left(\frac{\beta_1}{2} + \tilde{\mathtt{b}}_k\right) - C(\delta)\delta$$
and
$$\tilde{\mathtt{b}}_k = \tilde{\mathtt{b}}\sum_{\ell=1}^{k-1}\ell = \frac{(k-1)k}{2}\tilde{\mathtt{b}}.$$
By assumption,$\tilde{\kappa} - \tilde{\gamma}_1\left(\frac{\beta_1}{2} + \tilde{\mathtt{b}}_k\right) > 0$ therefore we can choose $\delta > 0$ small enough, that
$$\tilde{\rho} > 0.$$
Putting it all together, we obtain from (2.8.6), (2.8.13), (2.8.12), (2.9.3) and Gronwall's inequality the decay estimate
$$\|\mathbb{P}u(t)\|_{H^{k-1}} \lesssim \begin{cases} |t| + (\lambda_1+\alpha)|t|^{\frac{1}{2}} & \text{if } \tilde{\rho} > 1 \\ -|t|\ln\left(\frac{t}{T_0}\right) + (\lambda_1+\alpha)|t|^{\frac{1}{2}} & \text{if } \tilde{\rho} = 1 \\ |t|^{\tilde{\rho}} + (\lambda_1+\alpha)|t|^{\frac{1}{2}} & \text{if } \frac{1}{2} < \tilde{\rho} < 1 \\ |t|^{\frac{1}{2}} - (\lambda_1+\alpha)|t|^{\frac{1}{2}}\ln\left(\frac{t}{T_0}\right) & \text{if } \tilde{\rho} = \frac{1}{2} \\ |t|^{\tilde{\rho}} & \text{if } 0 < \tilde{\rho} < \frac{1}{2} \end{cases}, \qquad T_0 \le t < 0.$$
From this inequality, it follows that we can choose $\delta > 0$ small enough such that for any $\sigma > 0$ the decay estimate
$$\|\mathbb{P}u(t)\|_{H^{k-1}} \lesssim \begin{cases} |t| + (\lambda_1+\alpha)|t|^{\frac{1}{2}} & \text{if } \zeta > 1 \\ |t|^{\zeta-\sigma} + (\lambda_1+\alpha)|t|^{\frac{1}{2}} & \text{if } \frac{1}{2} < \zeta \le 1 \\ |t|^{\zeta-\sigma} & \text{if } 0 < \zeta \le \frac{1}{2} \end{cases}, \tag{2.9.4}$$
holds for $T_0 \le t \le 0$, where
$$\zeta = \tilde{\kappa} - \tilde{\gamma}_1\left(\frac{\beta_1}{2} + \tilde{\mathtt{b}}_k\right).$$

2.10 H^{k-1} decay estimate for $\mathbb{P}^\perp u - \mathbb{P}^\perp u|_{t=0}$

Using the Cauchy Schwartz inequality, we see that the expression
$$\int_{t_1}^{t_2} -\frac{1}{\tau}\|\mathbb{P}u(\tau)\|_{H^{k-1}}\|\mathbb{P}u(\tau)\|_{H^k}\,d\tau \le \left(\int_{t_1}^{t_2} -\frac{1}{\tau}\|\mathbb{P}u(\tau)\|_{H^{k-1}}^2\,d\tau\right)^{\frac{1}{2}}\left(\int_{t_1}^{t_2} -\frac{1}{\tau}\|\mathbb{P}u(\tau)\|_{H^k}^2\,d\tau\right)^{\frac{1}{2}},$$

holds for all $T_0 \leq t_1 < t_2 < 0$. As a consequence of this inequality and (2.6.12), (2.7.3), (2.9.4), we obtain the estimate

$$\|\mathbb{P}^\perp u(t_2) - \mathbb{P}^\perp u(t_1)\|_{H^{k-1}} \lesssim \begin{cases} \Big(-((-t_2)^{2(\zeta-\sigma)} - t_2\big) + \big((-t_1)^{2(\zeta-\sigma)} - t_1)\big)\Big)^{\frac{1}{2}} & \text{if } \zeta > \frac{1}{2} \\ \Big(-(-t_2)^{2(\zeta-\sigma)} + (-t_1)^{2(\zeta-\sigma)}\Big)^{\frac{1}{2}} & \text{if } 0 < \zeta \leq \frac{1}{2} \end{cases}.$$

Taking the limit $t_2 \nearrow 0$ yields

$$\|\mathbb{P}^\perp u(t) - \mathbb{P}^\perp u(0)\|_{H^{k-1}} \lesssim \begin{cases} |t|^{\frac{1}{2}} + |t|^{\zeta-\sigma} & \text{if } \zeta > \frac{1}{2} \\ |t|^{\zeta-\sigma} & \text{if } \zeta \leq \frac{1}{2} \end{cases}, \qquad T_0 \leq t < 0,$$

which completes the proof. $\square$

Don't ask for guarantees. And don't look to be saved in any one thing, person, machine, or library. Do your own bit of saving, and if you drown, at least die knowing you were heading for shore.

Fahrenheit 451, Ray Bradbury

3

Wave equation on Minkowski and Schwarzschild space-times near spatial infinity

In this chapter we focus our attention on semi-linear wave equations on Minkowski and Schwarzschild space-times with a source term satisfying the null condition (1.5.1). In Section 1.4 of Chapter 1, we summarized in four main steps the transformation process to obtain a Fuchsian system from a second order wave equation. Here, we elaborate on those steps. As a first step, we transform the physical manifold into a closed N-dimensional manifold. In this application our physical manifolds are Minkowski and Schwarzschild space-times and we employ the *Friedrich's cylinder at infinity* construction to compactify them into non-physical bounded manifolds. Then, in Sections 3.2, 3.4, we transform a system of wave equations in Minkowski and Schwarzschild space-times respectively into a first order symmetric hyperbolic system via a change of variables. The change of variables for each system includes a rescaling by powers of t that are chosen to ensure that the resulting systems are symmetric hyperbolic and satisfies the expansion (2.2.10). Then we define the extended systems respectively, which are defined on a bounded manifold which is needed to apply the existence theory developed in [1]. As a final step we apply Theorem 2.3.1 to these systems which yield GIVP to (3.1.1) and (3.3.1) as stated in the Theorems 3.2.1 and 3.5.1.

3.1 The cylinder at infinity in Minkowski space-time

In this section our main goal is to transform a semi-linear wave equation of the form

$$\bar{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta\bar{u}^K = q^K_{IJ}(\bar{u}^L)\bar{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{u}^I\bar{\nabla}_\beta\bar{u}^J, \tag{3.1.1}$$

into a first order Fuchsian system of the form

$$B^0(t,u)\partial_t u + B^i(t,u)\nabla_i u = \frac{1}{t}\mathcal{B}(t,u)\mathbb{P}u + F(t,u) \qquad \text{in } [T_0,T_1)\times\Sigma \tag{3.1.2}$$

$$u = u_0 \qquad \text{in}\{T_0\}\times\Sigma, \tag{3.1.3}$$

where the u^I, $1 \leq I \leq N$, are a collection of scalar fields, $q^K_{IJ} \in C^\infty(\mathbb{R}^N)$, and $[T_0,T_1)\times\Sigma$ arises from compactifying a neighbourhood of spatial infinity and $\bar{\nabla}$ is the Levi-Civita connection of the Minkowski metric $\bar{g} = \bar{g}_{\alpha\beta}d\bar{x}^\alpha \otimes d\bar{x}^\beta$ on $\mathbb{R}^4$. After obtaining system (3.1.2)-(3.1.3) we can apply Theorem 2.3.1 to obtain global existence and decay estimates.

In general relativity, conformal transformations are useful since they preserve the causal structure of space-times. Roughly speaking, the purpose of the transformation is to obtain an unphysical, bounded manifold, whose boundary represents an "infinity" of the physical manifold, and the interior preserves the causal structure of the original space-time. We refer the reader to Appendix A.3 where we have collected a number of results on how geometric quantities transform under a conformal transformation. The particular conformal transformation of the Minkowski space-time that we use is based on *Friedrich's cylinder at infinity*. The cylinder at infinity was initially considered for the study of Friedrich's conformal version of the Einstein field equations near spatial infinity see [54, 55, 56]. Other applications of the cylinder at infinity construction include spin-2 equations on Minkowski space-time, see for example [57, 58, 59, 60, 61].

Friedrich's cylinder at infinity construction in our setting starts with the Minkowski metric on $\mathbb{R}^4$ given by

$$\bar{g} = -d\bar{t}\otimes d\bar{t} + d\bar{r}\otimes d\bar{r} + \bar{r}\not{g}, \tag{3.1.4}$$

where $\not{g}$ is the canonic metric on the 2-sphere $\mathbb{S}^2$ and $(\bar{t},\bar{r},\bar{\theta},\bar{\phi})$ are spherical coordinates. The region of Minkowski space-time we will consider is given by the manifold

$$\bar{M} = \{\, (\bar{t},\bar{r},(\bar{\theta},\bar{\phi})) \in (-\infty,\infty)\times(0,\infty)\times\mathbb{S}^2 \mid -\bar{t}^2+\bar{r}^2 > 0\,\}, \tag{3.1.5}$$

and by using the dipheomorphism

$$\psi \; : \; \bar{M} \longrightarrow M \; : \; (\bar{x}^\mu) = (\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi}) \longmapsto (x^\mu) = \left(1 - \frac{\bar{t}}{\bar{r}}, \frac{\bar{r}}{-\bar{t}^2 + \bar{r}^2}, \bar{\theta}, \bar{\phi}\right), \tag{3.1.6}$$

where (x^μ) are the coordinates on the new *non-physical* manifold, we obtain the conformal metric

$$\tilde{g} = \Omega^2 g, \tag{3.1.7}$$

such that

$$g = -dt \otimes dt + \frac{1-t}{r}(dt \otimes dr + dr \otimes dt) + \frac{(2-t)t}{r^2} dr \otimes dr + \not{g}, \tag{3.1.8}$$

where the conformal factor is given by

$$\Omega = \frac{1}{r(2-t)t}, \tag{3.1.9}$$

and our new manifold M, is the region given by

$$M = (0,2) \times (0,\infty) \times \mathbb{S}^2. \tag{3.1.10}$$

We are following a similar approach to the cylinder at infinity transformation given in [59], but it is important to note that we have inverted the direction of time and shifted the time interval by a unit. In [59], the time interval is of the form $-1 < \tau < 1$, while our time coordinate satisfies $0 < t < 2$, in our case we are interested in the singular time $t \searrow 0$.

In geometric terms, $\bar{M}$ is the interior of the space-like cone with vertex at the origin in $\mathbb{R}^4$. The diffeomorphism (3.1.6) transforms this region into the manifold M with a boundary composed by

$$\partial M = \mathscr{I}^+ \cup i^0 \cup \mathscr{I}^-$$

where

$$\mathscr{I}^+ = \{0\} \times (0,\infty) \times \mathbb{S}^2, \quad \mathscr{I}^- = \{2\} \times (0,\infty) \times \mathbb{S}^2 \quad \text{and} \quad i^0 = [0,2] \times \{0\} \times \mathbb{S}^2,$$

the compatification defined by (3.1.6) leads to the interpretation of $\mathscr{I}^\pm$ as portions of $(+)$ future and $(-)$ past null-infinity, respectively, and i^0 as spatial infinity. We further note that the space-like hypersurface

$$\Sigma = \{1\} \times (0, \infty) \times \mathbb{S}^2 \subset M,$$

corresponds to the constant time hypersurface

$$\bar{\Sigma} = \{ (\bar{x}^\mu) \in \mathbb{R}^4 \mid \bar{g}_{\mu\nu}\bar{x}^\mu\bar{x}^\nu > 0,\ \bar{x}^0 = 0 \} \subset \bar{M} \subset \mathbb{R}^4,$$

in Minkowski space-time.

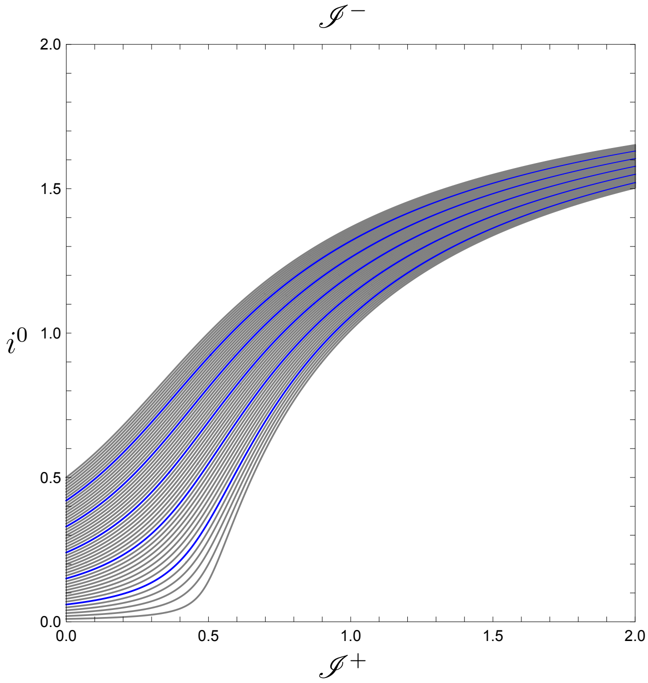

Figure 3.1: In this diagram we plot a family of space-like geodesics from Minkowski space time represented in the (t, r, θ, ϕ) coordinates. The family of geodesics from Minkowski space that we are considering are of the form $\bar{t} = a\bar{r} + b$ with $b = 1$ and we take $0 < a < 1$ to identify different elements from the same family; after applying (3.1.6) we obtain the curves plotted here . We have drawn some curves in blue only to emphasize how these geodesics evolve. Note also that all these curves have their endpoints at $r = 0$, it is not difficult to see that the spatial infinity region i^0 is given in the (t, r) coordinates by the region $i^0 = \{(t, r, \theta, \phi) \mid t \in (0, 2),\ r = 0,\ (\theta, \phi) \in \mathbb{S}^2\}$.

To visualize the compactification of space-time we have drawn two diagrams in Figures 3.1, 3.2. The first one shows the trajectories of space-like geodesics taken from Minkowski space-time and plotted in the new coordinates after applying our conformal transformation. Note from Figure 3.1 that all space-like geodesics end on the line $r = 0$, then by using the conformal transformation (3.1.6) we have blown up spatial infinity i^0. Therefore, to analyse wave equations on the space-like infinity

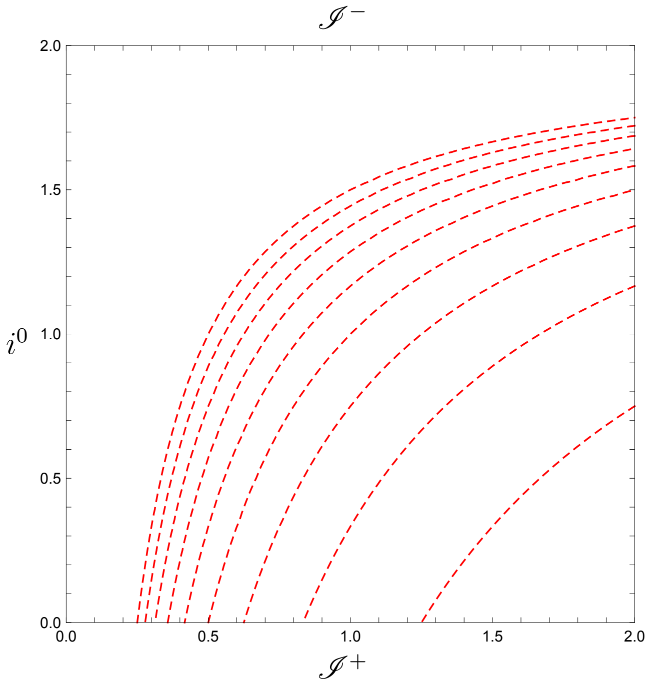

Figure 3.2: This diagram shows dashed curves in red which are future directed null geodesics from Minkowski space time represented in the (t, r, θ, ϕ) coordinates. Note that in the (x^μ) coordinate representation, the future null infinity region $\mathscr{I}^+$ is given by $\mathscr{I}^+ = \{(t, r, \theta, \phi) \mid t = 0,\ r \in (0, \infty),\ (\theta, \phi) \in \mathbb{S}^2\}$.

region of Minkowski space-time, we can focus our attention at the cylindrical region in the neighbourhood of $r = 0$ of the unphysical manifold.

In Figure 3.2 we have plotted future directed null geodesics from Minkowski space-time, all of them ending on the line $t = 0$. The transformation (3.1.6) has compactified the spatial infinity into a cylinder-type region with the right structure required to apply Theorem 2.3.1. We use this construction in our analysis of wave equations in Minkowsky and Schwarzschild space-time. It will be interesting to study a similar construction on a time-like infinity region, unfortunately it is out of the scope of this thesis and we leave it for a future work.

For use below, we observe that the Ricci scalar curvature of $\tilde{g}$ satisfies

$$\tilde{R} = 0, \tag{3.1.11}$$

by virture of $\tilde{g}$ being flat. The same is also true for the Ricci scalar curvature of the metric g, that is,

$$R = 0, \tag{3.1.12}$$

as can be verified via a straightforward calculation using (3.1.8).

3.2 Semi-linear wave equations on Minkowski space-time near spatial infinity

With the metric defined by (3.1.8), we are ready to write the wave equation explicitly. We recall that we use lower case Greek letters, e.g. μ, ν, γ, to label space-time coordinate indices that run from 0 to 3, upper case Latin indices, e.g. I, J, K, run from 1 to N, while upper case Greek letter, e.g. Λ, Σ, Γ, will be reserved to label spherical coordinate indices that run from 2 to 3, see Appendix A.1 for the full index convention.

The class of semi-linear wave equations on Minkowski space-time that we consider are systems of N-coupled scalar wave equations of the form (3.1.1), and we will be interested in solving this type of system of wave equations on domains of the form

$$\bar{M}_{\bar{r}_0} = \{\, (\bar{t}, \bar{r}) \in (0, \infty) \times (0, \infty) \mid -\bar{t} + \bar{r} > \bar{r}_0 \,\} \times \mathbb{S}^2, \qquad \bar{r}_0 > 0. \tag{3.2.1}$$

It is important to remark here that the non-linear terms $q^K_{IJ}(\bar{u}^L)\bar{g}^{\alpha\beta}\bar{\nabla}_\alpha \bar{u}^I \bar{\nabla}_\beta \bar{u}^J$ satisfy the null condition of Klainerman. Global existence results, under a small initial data condition, for systems of wave equation of the form (3.1.1) have been known since the pioneering work of Klainerman [40] and Christodoulou [41]. Therefore, the results that we present are not new. Despite of this, we believe that the method used here to establish global existence on regions of the form (3.2.1) brings a valuable new perspective to global existence problems for systems of non-linear wave equations. The analysis carried out here is also of interest because it demonstrates the utility of Friedrich's cylinder at spatial infinity construction for solving non-linear wave equations near spatial infinity.

Rather than attempt to solve (3.1.1) directly, we use the diffeomorphism (3.1.6) to push (3.1.1) forward to obtain the system (see Appendix A.3)

$$\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha \tilde{\nabla}_\beta \tilde{u}^K = q^K_{IJ}(\tilde{u}^L)\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha \tilde{u}^I \tilde{\nabla}_\beta \tilde{u}^J \tag{3.2.2}$$

on the domain

$$M_{r_0} := \bar{\psi}(\bar{M}_{\bar{r}_0}) = \left\{ (t, r) \in (0, 1) \times (0, r_0) \,\middle|\, t > 2 - \frac{r_0}{r} \right\} \times \mathbb{S}^2, \qquad r_0 = \frac{1}{\bar{r}_0}, \tag{3.2.3}$$

where $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{g}$ and $\tilde{u}^K$ is related to $\bar{u}^K$ via

$$\tilde{u}^K = \bar{\psi}_* \bar{u}^K. \tag{3.2.4}$$

The relation (3.2.4) establishes an equivalence between the wave equations (3.1.1) on $\bar{M}_{\bar{r}_0}$ and (3.2.2) on M_{r_0}, since the diffeomorphism (3.1.6) is invertible. Therefore we are free to restrict our attention to wave equations of the form (3.2.2) on M_{r_0}. We further note that the constant time hypersurface

$$\bar{\Sigma}_{\bar{r}_0} = \{0\} \times (\bar{r}_0, \infty) \times \mathbb{S}^2 \tag{3.2.5}$$

forms the bottom of $\bar{M}_{\bar{r}_0}$ and it gets mapped via the diffeomorphism (3.1.6) to the constant time hypersurface

$$\Sigma_{r_0} := \bar{\psi}(\bar{\Sigma}_{\bar{r}_0}) = \{1\} \times (0, r_0) \times \mathbb{S}^2$$

that forms the top of M_{r_0}. Using

$$\tilde{f}^K = q^K_{IJ}(\tilde{u}^L)\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha \tilde{u}^I \tilde{\nabla}_\beta \tilde{u}^J,$$

to denote the nonlinear terms that appear in (3.2.2), it follows from (3.3.4), (3.3.8) and the formulas (A.3.39)-(A.3.40) and (A.3.42)-(A.3.43), with $n = 4$, from Appendix A.3 that the wave equations (3.2.2) transform, under the conformal transformation (3.1.7), into

$$g^{\alpha\beta}\nabla_\alpha \nabla_\beta u^K = f^K \tag{3.2.6}$$

where ∇ is the Levi-Civita connection of g

$$\tilde{u}^K = rt(2-t)u^K,$$

and

$$f^K = q^K_{IJ}\big(rt(2-t)u^L\big)\Bigg(rt(2-t)g^{\mu\nu}\nabla_\mu u^I \nabla_\nu u^J + 2g^{\mu\nu}\nabla_\mu\big(rt(2-t)\big)\nabla_\nu u^{(I}u^{J)} + \frac{1}{rt(2-t)}g^{\mu\nu}\nabla_\mu\big(rt(2-t)\big)\nabla_\nu(rt(2-t))u^I u^J\Bigg). \tag{3.2.7}$$

With the help of equation (3.1.8), a routine computation shows that the conformal wave equations (3.2.6) can be expressed as

$$(-2+t)t\partial_t^2 u^K + r^2\partial_r^2 u^K + 2r(1-t)\partial_r\partial_t u^K + \not{g}^{\Lambda\Sigma}\not{\nabla}_\Lambda\not{\nabla}_\Sigma u^K + 2(t-1)\partial_t u^K = f^K \quad (3.2.8)$$

where $\not{\nabla}_\Lambda$ is the Levi-Civita connection of the metric $\not{g}$ on $\mathbb{S}^2$. Using (3.1.8), we compute the non- linear terms (3.2.7) which in the local coordinates are given by

$$\begin{aligned} f^K =& q^K_{IJ}(rt(2-t)u^L)\Big(2r(2-2t+t^2)u^{(I}r\partial_r u^{J)} - r(2-t)^2t^2\partial_t u^I\partial_t u^J + r(2-t)tr\partial_r u^I r\partial_r u^J \\ &+ 2r(2-3t+t^2)t(\partial_t u^{(I}r\partial_r u^{J)} - \partial_t u^{(I}u^{J)}) + r(2-t)t\not{g}^{\Lambda\Sigma}\not{\nabla}_\Lambda u^I\not{\nabla}_\Sigma u^J + r(2-t)tu^I u^J\Big). \end{aligned}$$

To proceed, we define new variables U_0^J, U_1^J, U_2^J, U_3^J and U_4^J by setting

$$U_0^J = t^{\lambda+\frac{1}{2}}\partial_t u^J, \quad U_1^J = t^\lambda r\partial_r u^J, \quad U_\Lambda^J = t^\lambda\not{\nabla}_\Lambda u^J \quad \text{and} \quad U_4^J = t^{\lambda-\frac{1}{2}}u^J \qquad (3.2.9)$$

where $\lambda \in \mathbb{R}$ is a constant to be fixed later. Substituting this change of variables into the wave equation (3.1.1), we can write the conformal system of wave equations (3.2.8) into first order form as follows:

$$B^0\partial_t U + B^1 r\partial_r U + B^\Gamma\not{\nabla}_\Gamma U = \frac{1}{t}\mathcal{B}U + F \qquad (3.2.10)$$

where

$$U = \begin{pmatrix} U_0^J \\ U_1^J \\ U_\Sigma^J \\ U_4^J \end{pmatrix}, \qquad (3.2.11)$$

$$B^0 = \begin{pmatrix} (2-t)\delta_J^K & 0 & 0 & 0 \\ 0 & \delta_J^K & 0 & 0 \\ 0 & 0 & \delta_\Lambda^\Sigma\delta_J^K & 0 \\ 0 & 0 & 0 & \delta_J^K \end{pmatrix}, \qquad (3.2.12)$$

$$B^1 = \begin{pmatrix} \frac{2(t-1)}{t}\delta_J^K & -\frac{1}{t^{\frac{1}{2}}}\delta_J^K & 0 & 0 \\ -\frac{1}{t^{\frac{1}{2}}}\delta_J^K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (3.2.13)$$

$$B^\Gamma = \begin{pmatrix} 0 & 0 & -\dfrac{1}{t^{\frac{1}{2}}}\not{g}^{\Gamma\Sigma}\delta^K_J & 0 \\ 0 & 0 & 0 & 0 \\ -\dfrac{1}{t^{\frac{1}{2}}}\delta^\Gamma_\Lambda\delta^K_J & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.2.14}$$

$$\mathcal{B} = \begin{pmatrix} 2\left(\lambda - \dfrac{1}{2}\right)\delta^K_J & 0 & 0 & 0 \\ 0 & \lambda\delta^K_J & 0 & 0 \\ 0 & 0 & \lambda\delta^\Sigma_\Lambda\delta^K_J & 0 \\ \delta^K_J & 0 & 0 & \left(\lambda - \dfrac{1}{2}\right)\delta^K_J \end{pmatrix}, \tag{3.2.15}$$

and

$$F = \begin{pmatrix} F^K_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.2.16}$$

with

$$\begin{aligned} F^K_0 = \left(\frac{3}{2} - \lambda\right)U^K_0 + \frac{1}{t}\Big[& -t^{\frac{1}{2}}U^K_1 + q^K_{IJ}\big(r(2-t)t^{\frac{3}{2}-\lambda}U^L_4\big)\Big(-2r(2-2t+t^2)t^{1-\lambda}U^{(I}_4U^{J)}_1 + \\ & r(t-2)^2t^{\frac{3}{2}-\lambda}U^I_0U^J_0 + r(t-2)t^{\frac{3}{2}-\lambda}U^I_1U^J_1 + 2r(2-3t+t^2)\big(t^{\frac{3}{2}-\lambda}U^{(I}_4U^{J)}_0 \\ & - t^{1-\lambda}U^{(I}_0U^{J)}_1\big) + r(t-2)t^{\frac{5}{2}-\lambda}U^I_4U^J_4 + r(t-2)t^{\frac{3}{2}-\lambda}\not{g}^{\Lambda\Sigma}U^I_\Lambda U^J_\Sigma\Big)\Big]. \end{aligned}$$

It will be convenient to expand (3.2.16) as follows

$$F = \begin{pmatrix} \left(\frac{3}{2} - \lambda\right)U^K_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{t}\big(t^{\frac{1}{2}}\mathscr{C} + \mathscr{B}\big)U \tag{3.2.17}$$

where

$$\mathscr{C} = \begin{pmatrix} 0 & -\delta^K_J & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathscr{B} = \begin{pmatrix} \mathscr{B}^{0K}_J & \mathscr{B}^{1K}_J & \mathscr{B}^{\Sigma K}_J & \mathscr{B}^{4K}_J \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with

$$\begin{aligned} \mathscr{B}^{0K}_J &= q^K_{PQ}\big(r(2-t)t^{\frac{3}{2}-\lambda}U^L_4\big)\delta^{(P}_I\delta^{Q)}_J\big[r(t-2)^2t^{\frac{3}{2}-\lambda}U^I_0 - 2r(2-3t+t^2)t^{1-\lambda}U^I_1\big], \\ \mathscr{B}^{1K}_J &= q^K_{PQ}\big(r(2-t)t^{\frac{3}{2}-\lambda}U^L_4\big)\delta^{(P}_I\delta^{Q)}_J\big[-2r(2-2t+t^2)t^{1-\lambda}U^I_4 + r(t-2)t^{\frac{3}{2}-\lambda}U^I_1\big], \\ \mathscr{B}^{\Sigma K}_J &= q^K_{PQ}\big(r(2-t)t^{\frac{3}{2}-\lambda}U^L_4\big)\delta^{(P}_I\delta^{Q)}_J r(t-2)t^{\frac{3}{2}-\lambda}\not{g}^{\Lambda\Sigma}U^I_\Lambda \end{aligned}$$

and

$$\mathscr{B}^{4K}_J = q^K_{PQ}\big(r(2-t)t^{\frac{3}{2}-\lambda}U^L_4\big)\delta^{(P}_I\delta^{Q)}_J\big[r(t-2)t^{\frac{5}{2}-\lambda}U^I_4 + 2r(2-3t+t^2)t^{\frac{3}{2}-\lambda}U^I_0\big].$$

From these formulas, it is clear that

$$\mathscr{B} = \mathrm{O}(U) \quad \text{for } \lambda \le 1. \tag{3.2.18}$$

Now we define a new variable

$$V = \begin{pmatrix} V^J_0 \\ V^J_1 \\ V^J_\Sigma \\ V^J_4 \end{pmatrix}, \tag{3.2.19}$$

and we introduce a positive definite inner-product h by

$$h(U,V) = \delta_{IJ}\big(U^I_0V^J_0 + U^I_1V^J_1 + \not{g}^{\Lambda\Sigma}U^I_\Lambda V^J_\Sigma + U^I_4V^J_4\big). \tag{3.2.20}$$

It is then not difficult to see that B^0, B^1 and B^Λ are all symmetric with respect to this inner-product and that

$$h(V, B^0V) \ge h(V,V)$$

as long as $t \in [0,1]$. This implies that the system of equations (3.2.10) is symmetric hyperbolic.

To proceed, we introduce a new radial coordinate via

$$r = \rho^m, \quad m \in \mathbb{Z}_{\geq 1}. \tag{3.2.21}$$

Using the transformation law

$$r\partial_r = r\frac{d\rho}{dr}\partial_\rho = \frac{\rho}{m}\partial_\rho,$$

we can express (3.2.10) as

$$B^0\partial_t U + \frac{\rho}{m}B^1\partial_\rho U + B^\Gamma \nabla\!\!\!\!/_\Gamma U = \frac{1}{t}\mathcal{B}U + F, \tag{3.2.22}$$

where now any r appearing in F is replaced using (3.2.21). We further observe that the space-time region (3.2.3) can be expressed in terms of the radial coordinate ρ as

$$M_{r_0} = \left\{ (t,\rho) \in (0,1)\times(0,\rho_0) \,\middle|\, t > 2 - \frac{\rho_0^m}{\rho^m} \right\} \times \mathbb{S}^2, \quad \rho_0 = (r_0)^{\frac{1}{m}}. \tag{3.2.23}$$

3.2.1 The extended system

Next, we let $\hat{\chi}(\rho)$ denote a smooth cut-off function that satisfies

$$\hat{\chi} \geq 0, \quad \hat{\chi}|_{[-1,1]} = 1 \quad \text{and} \quad \text{supp}(\hat{\chi}) \subset (-2,2),$$

and we define

$$\chi(\rho) = \hat{\chi}\left(\frac{\rho}{\rho_0}\right), \tag{3.2.24}$$

which is easily seen to satisfy

$$\chi \geq 0, \quad \chi|_{[-\rho_0,\rho_0]} = 1 \quad \text{and} \quad \text{supp}(\chi) \subset (-2\rho_0, 2\rho_0).$$

We then consider an extended version of (3.2.22) given by

$$B^0\partial_t U + \frac{\chi\rho}{m}B^1\partial_\rho U + B^\Gamma \nabla\!\!\!\!/_\Gamma U = \frac{1}{t}\mathcal{B}U + \chi F \tag{3.2.25}$$

that is well-defined on the extended space-time region

$$(0,1)\times T^1_{3\rho_0}\times \mathbb{S}^2 \tag{3.2.26}$$

where $T^1_{3\rho_0} \cong \mathbb{S}^1$ is the 1-dimensional torus obtained from identifying the end points of the interval $[-3\rho_0, 3\rho_0]$. By construction, (3.2.25) agrees with (3.2.22) when restricted to (3.2.23). Noting that the boundary of the region (3.2.23) can be decomposed as

$$\partial M_{r_0} = \Sigma_{r_0} \cup \Sigma^+_{r_0} \cup \Gamma^- \cup \Gamma^+$$

where

$$\Sigma_{r_0} = \{1\} \times (0, \rho_0) \times \mathbb{S}^2, \quad \Sigma^+_{r_0} = \{0\} \times \Big(0, \frac{\rho_0}{2^{\frac{1}{m}}}\Big) \times \mathbb{S}^2,$$
$$\Gamma^- = [0,1] \times \{0\} \times \mathbb{S}^2 \quad \text{and} \quad \Gamma^+ = \left\{ (t,r) \in [0,1] \times (0,\rho_0) \,\middle|\, t = 2 - \frac{\rho_0^m}{\rho^m} \right\} \times \mathbb{S}^2,$$

we see immediately that

$$n^- = -d\rho \quad \text{and} \quad n^+ = -dt + m\frac{\rho_0^m}{\rho^{m+1}}d\rho$$

define outward pointing co-normals to Γ^- and Γ^+, respectively. Furthermore, from (3.2.12)-(3.2.14), we get that

$$\Big(n^-_0 B^0 + n^-_1 \frac{\chi\rho}{m} B^1 + n^-_\Gamma B^\Gamma\Big)\Big|_{\Gamma^-} = 0$$

$$(n^+_0 B^0 + n^+_1 \frac{\chi\rho}{m} B^1 + n^+_\Gamma B^\Gamma)|_{\Gamma^+} = \begin{pmatrix} \Big(-(2-t) + \frac{\rho_0^m}{\rho^m}\frac{2(t-1)}{t}\Big)\delta^K_J & -\frac{\rho_0^m}{\rho^m t^{\frac{1}{2}}}\delta^K_J & 0 & 0 \\ -\frac{\rho_0^m}{\rho^m t^{\frac{1}{2}}}\delta^K_J & -\delta^K_J & 0 & 0 \\ 0 & 0 & -\delta^\Sigma_\Lambda \delta^K_J & 0 \\ 0 & 0 & 0 & -\delta^K_J \end{pmatrix}\Bigg|_{\Gamma^+},$$

where we note that

$$\Big(-(2-t) + \frac{\rho_0^m}{\rho^m}\frac{2(t-1)}{t}\Big)\Big|_{\Gamma^+} = \Big(-(2-t) + \frac{\rho_0^m}{\rho^m} + \frac{\rho_0^m}{\rho^m}\frac{(t-2)}{t}\Big)\Big|_{\Gamma^+} = -\frac{\rho_0^{2m}}{\rho^{2m} t}\Big|_{\Gamma^+}$$

since $2 - t = \frac{\rho_0^m}{\rho^m}$ on Γ^+. From these expressions and the definition of the inner-product (3.2.20), we deduce that

$$h\Big(V, \Big(n^\pm_0 B^0 + n^-_1 \frac{\rho}{m} B^1 + n^-_\Gamma B^\Gamma\Big)V\Big)\Big|_{\Gamma^-} = 0$$

and

$$h\Big(V,\Big(n_0^+B^0+n_1^+\frac{\rho}{m}B^1+n_\Gamma^+B^\Gamma\Big)V\Big)\Big|_{\Gamma^+}=$$
$$-\delta_{IJ}\bigg(\frac{\rho_0^mV_0^I}{\rho^mt^{\frac{1}{2}}}+V_1^I\bigg)\bigg(\frac{\rho_0^mV_0^J}{\rho^mt^{\frac{1}{2}}}+V_1^J\bigg)-\delta_{IJ}\not{g}^{\Lambda\Sigma}V_\Lambda^IV_\Sigma^J-\delta_{IJ}V_4^IV_4^J\leq 0.$$

This implies that the surfaces $\Gamma^\pm$ are weakly space-like for the symmetric hyperbolic system (3.2.25). See the definition in [62, §4.3],. The importance of this is that it will guarantee that any solution of the extended system (3.2.25) on the extended spacetime (3.2.26) will also yield by restriction a solution of the original system (3.2.22) on the region (3.2.23) that is uniquely determined by the restriction of the initial data to $\{1\}\times(0,\rho_0)\times\mathbb{S}^2$. From this property and the above arguments, we conclude that the existence of solutions to the system of semi-linear wave equations (3.1.1) on the regions of the form (3.2.1) in Minkowski space-time can be obtained from solving the initial value problem

$$B^0\partial_tU+\frac{\chi\rho}{m}B^1\partial_\rho U+B^\Gamma\not{\nabla}_\Gamma U=\frac{1}{t}\mathcal{B}U+\chi F\qquad\text{in }(0,1)\times T^1_{3\rho_0}\times\mathbb{S}^2,\tag{3.2.27}$$
$$U=\mathring{U}\qquad\text{in }\{1\}\times T^1_{3\rho_0}\times\mathbb{S}^2,\tag{3.2.28}$$

where the solutions generated this way are independent of the particular form of the initial data $\mathring{U}$ on $(\{1\}\times T^1_{3\rho_0}\times\mathbb{S}^2)\setminus(\{1\}\times(0,\rho_0)\times\mathbb{S}^2)$.

The next step is to verify two structural conditions. First we show that the inequality

$$h(V,\mathcal{B}V)\geq\kappa h(V,B^0V)\tag{3.2.29}$$

holds where λ is chosen so that

$$\kappa=\lambda-\frac{1}{4}(2+\sqrt{2})>0,\tag{3.2.30}$$

which we note is compatible with the condition $\lambda\leq 1$ that is needed to ensure that (3.2.18) holds. To see the validity of (3.2.29), we note, with the help of the inequality $|\delta_{IJ}V_0^IV_4^J|\leq\frac{\epsilon^2}{2}\delta_{IJ}V_0^IV_0^J+\frac{1}{2\epsilon^2}\delta_{IJ}V_4^IV_4^J$, $\epsilon>0$, that by using (3.2.15) (3.2.20)

$$\begin{aligned}h(V,\mathcal{B}V)&=\delta_{IJ}\Big(2\Big(\lambda-\frac{1}{2}\Big)V_0^IV_0^J+V_0^IV_4^J+\Big(\lambda-\frac{1}{2}\Big)V_4^IV_4^J+\lambda V_1^IV_1^J+\lambda\not{g}^{\Lambda\Sigma}V_\Lambda^IV_\Sigma^J\Big)\\&\geq\delta_{IJ}\Big(\Big(2\Big(\lambda-\frac{1}{2}\Big)-\frac{\epsilon^2}{2}\Big)V_0^IV_0^J+\Big(\lambda-\frac{1}{2}-\frac{1}{2\epsilon^2}\Big)V_4^IV_4^J+\lambda V_1^IV_1^J+\lambda\not{g}^{\Lambda\Sigma}V_\Lambda^IV_\Sigma^J\Big).\end{aligned}\tag{3.2.31}$$

We then fix ϵ by demanding that $\frac{1}{2}\Big(2\Big(\lambda-\frac{1}{2}\Big)-\frac{\epsilon^2}{2}\Big)=\Big(\lambda-\frac{1}{2}-\frac{1}{2\epsilon^2}\Big)$. Solving this yields

$$\epsilon^2=\sqrt{2}, \tag{3.2.32}$$

which in turns gives

$$\frac{1}{2}\Big(2\Big(\lambda-\frac{1}{2}\Big)-\frac{\epsilon^2}{2}\Big)=\Big(\lambda-\frac{1}{2}-\frac{1}{2\epsilon^2}\Big)=\lambda-\frac{1}{4}(2+\sqrt{2})=\kappa. \tag{3.2.33}$$

Assuming that (3.2.30) holds, we see, after substituting (3.2.32) and (3.2.33) into (3.2.31) and recalling (3.2.12) and (3.2.20), that the inequality

$$h(V,\mathcal{B}V)\geq\kappa\delta_{IJ}\big(2V_0^IV_0^J+V_1^IV_1^J+\not{g}^{\Lambda\Sigma}V_\Lambda^IV_\Sigma^J+V_4^IV_4^J\big)\geq\kappa h(V,B^0V)$$

holds for $t\in[0,1]$.

The second structural condition, which is related to the constants $\mathtt{b}$ and $\tilde{\mathtt{b}}$, involves bounding the size of the matrix[1] $\partial_\rho\big(t\frac{\rho\chi}{m}B^1\big)$. From the bound

$$\Big|\partial_\rho\Big(t\frac{\rho\chi}{m}B^1\Big)\Big|_{\mathrm{op}}\leq\max_{0\leq t\leq 1}|tB^1(t)|_{\mathrm{op}}\|\partial_\rho(\rho\chi)\|_{L^\infty\left(T^1_{3\rho_0}\right)}\frac{1}{m},$$

it is clear, given any $\sigma>0$, that there exists a positive integer $m=m(\sigma)$ such that

$$\Big|\partial_\rho\Big(t\frac{\rho\chi}{m}B^1\Big)\Big|_{\mathrm{op}}<\sigma\quad\text{ in }(0,1)\times T^1_{3\rho_0}\times\mathbb{S}^2. \tag{3.2.34}$$

3.2.2 Global existence

Having established that (3.2.27) is symmetric hyperbolic, we can appeal to the Cauchy stability property satisfied by symmetric hyperbolic systems to conclude, for any given $t_0\in(0,1)$, the existence of a unique solution

$$U\in C^0\big((t_0,1],H^k(T^1_{3\rho_0}\times\mathbb{S}^2)\big)\cap L^\infty\big((t_0,1],H^k(T^1_{3\rho_0}\times\mathbb{S}^2)\big)\cap C^1\big((t_0,1],H^{k-1}(T^1_{3\rho_0}\times\mathbb{S}^2)\big)$$

to (3.2.27) provided that $k\in\mathbb{Z}_{>3/2}$ and the initial data $U|_{t=1}=\mathring{U}\in H^k(T^1_{3\rho_0}\times\mathbb{S}^2)$ is chosen small enough. Furthermore, by standard results, this solution will satisfy an energy estimate of the form

$$\|U(t)\|^2_{H^k}+\int_t^1\frac{1}{\tau}\|U(\tau)\|^2_{H^k}\,d\tau\leq C\big(\|U\|_{L^\infty((t_0,0],H^k)}\big)\|\mathring{U}\|^2_{H^k},\qquad t_0<t\leq 1,$$

[1]Note that the spatial derivatives (e.g. $\not{\nabla}_\Lambda\mathcal{B}$, $\partial_\rho\mathcal{B}$, $\not{\nabla}_\Lambda\mathscr{C}$, $\partial_\rho\mathscr{C}$, $\not{\nabla}_\Lambda\big(\frac{\rho\chi}{m}B^1\big)$, $\not{\nabla}_\Lambda B^\ell$ and $\partial_\rho B^\ell$ for $\ell=0,2,3$) of all the other U-independent matrices appearing in (3.2.27) (see also (3.2.17)) vanish.

and the norm $\|U(t_0)\|_{H^k}$ can be made as small as we like by choosing the initial data $\mathring{U}$ at $t=1$ suitably small.

To continue this solution from $t=t_0$ to $t=0$, we now assume that $k\in\mathbb{Z}_{>9/2}$ and choose the initial data $\mathring{U}$ at $t=1$ small enough so that $\|U(t_0)\|_{H^k}$ and t_0 can be taken to be sufficiently small. If we further assume that

$$\lambda \in \Big(\frac{1}{4}(2+\sqrt{2}),1\Big], \tag{3.2.35}$$

then from (3.2.12)-(3.2.15), (3.2.17), (3.2.18), (3.2.29), (3.2.34) and the simple time transformation $t\mapsto -t$, it is not difficult to verify that (3.2.27) will satisfy all the assumptions from Section 2.2 with $\mathbb{P}=\mathbb{1}$ on the region $(-t_0,0)\times T^1_{3\rho_0}\times\mathbb{S}^2$. Moreover, by (3.2.34), we can always choose the integer m large enough so that the constants β_1 and $\mathtt{b}=\tilde{\mathtt{b}}$ from Theorem 2.3.1 can be made as small as we like while the constants λ_1 and α vanish. Thus, for any $\upsilon>0$, we can apply Theorem 2.3.1 to (3.2.27) with initial data given by $U(t_0)$ at $t=t_0$ to obtain the existence of a unique solution

$$U\in C^0\big((0,t_0],H^k(T^1_{3\rho_0}\times\mathbb{S}^2)\big)\cap L^\infty\big((0,t_0],H^k(T^1_{3\rho_0}\times\mathbb{S}^2)\big)\cap C^1\big((0,t_0],H^{k-1}(T^1_{3\rho_0}\times\mathbb{S}^2)\big)$$

that satisfies the energy

$$\|U(t)\|^2_{H^k}+\int_t^1\frac{1}{\tau}\|U(\tau)\|^2_{H^k}\,d\tau\le C\big(\|U\|_{L^\infty((1,t_0],H^k)}\big)\|U(t_0)\|^2_{H^k}$$

and decay

$$\|U(t)\|_{H^{k-1}}\lesssim t^{\kappa-\upsilon}$$

estimates for $0<t\le t_0$. This establishes the existence of a unique solution of the IVP (3.2.27)-(3.2.28), and completes the proof of the following theorem.

Theorem 3.2.1. *Suppose $k\in\mathbb{Z}_{>9/2}$, $\rho_0>0$, $\upsilon>0$, $\lambda\in\big(\frac{1}{4}(2+\sqrt{2}),1\big]$ and $\kappa=\lambda-\frac{1}{4}(2+\sqrt{2})$. Then there exist $m\in\mathbb{Z}_{\ge 1}$ and $\delta>0$ such that if $\mathring{U}\in H^k(T^1_{3\rho_0}\times\mathbb{S}^2)$ is chosen so that $\|\mathring{U}\|_{H^k}<\delta$, then there exists a unique solution*

$$U\in C^0\big((0,1],H^k(T^1_{3\rho_0}\times\mathbb{S}^2)\big)\cap L^\infty\big((0,1],H^k(T^1_{3\rho_0}\times\mathbb{S}^2)\big)\cap C^1\big((0,1],H^{k-1}(T^1_{3\rho_0}\times\mathbb{S}^2)\big)$$

of the IVP (3.2.27)-(3.2.28) that satisfies the energy

$$\|U(t)\|^2_{H^k}+\int_t^1\frac{1}{\tau}\|U(\tau)\|^2_{H^k}\,d\tau\le C\big(\|U\|_{L^\infty((1,0],H^k)}\big)\|\mathring{U}\|^2_{H^k}$$

and decay

$$\|U(t)\|_{H^{k-1}} \lesssim t^{\kappa-\upsilon}$$

estimates for $0 < t \leq 1$.

3.3 Wave equations near spatial infinity on Schwarzschild space-times

In this section, we generalize the global existence results for semi-linear wave equations on Miknowksi space-time to semilinear wave equations on Schwarzschild space-time. In the previous section we employed the cylinder at spatial infinity to compactify Minkowski space-time, here we will use the same argument applied to a Schwarzschild space-time of mass $\mu > 0$. In the following, we consider the same class of semi-linear wave equations as we did in the previous section, namely systems of wave equations of the type

$$\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\tilde{u}^K = q^K_{IJ}(\tilde{u}^L)\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{u}^I\tilde{\nabla}_\beta\tilde{u}^J \tag{3.3.1}$$

where $q^K_{IJ} \in C^\infty(\mathbb{R}^N)$, but $\tilde{g}$ is now the Schwarzschild metric defined by (3.3.3) and $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{g}$. The existence results we establish in this section are not totally new since solutions to scalar semi-linear wave equations of the form (3.3.1) have been previously established in [63]. In particular, see Theorem 1.5 from [63], solutions under a small initial data assumption to scalar semi-linear wave equations on Kerr space-time, which includes the rotation of the body generating the Schwarzschild space-time. Despite of this, we believe that the new method used here is a valuable new perspective to global existence problems. In particular we show the usefulness of the method for systems of non-linear wave equations on Schwarzschild space-times. The analysis carried out in this section is in essence the same analysis carried out above for wave equations on Minkowski space-time. Consider the coordinate transformation given by

$$(\tilde{t}, \tilde{r}, \theta, \phi) \longmapsto (x^\mu) = \left(\frac{\mu\left(1 - t - r^2 t(1-t)\right)}{2rt} - 4\mu \ln\left(\frac{t^{\frac{1}{2}}(1-r)}{1-rt}\right), \frac{\mu}{2rt}, \theta, \phi\right) \tag{3.3.2}$$

where $(t, r) \in (0,1) \times (0,1)$. Applying (3.3.2) to the Schwarzschild space-time given by

$$\tilde{g} = -\left(\frac{1 - \frac{\mu}{2\tilde{r}}}{1 + \frac{\mu}{2\tilde{r}}}\right)^2 d\tilde{t} \otimes d\tilde{t} + \left(1 + \frac{\mu}{2\tilde{r}}\right)^4 (d\tilde{r} \otimes d\tilde{r} + \not{g})$$

it is not difficult to see that

$$\tilde{g} = \Omega^2 g, \tag{3.3.3}$$

where the conformal factor is given by

$$\Omega = \frac{\mu(1+rt)^2}{2rt}, \tag{3.3.4}$$

and

$$g = \frac{1}{r}A(dr \otimes dt + dt \otimes dr) + \frac{t}{r^2}A\,(2 - tA)\,dr \otimes dr + \not{g} \tag{3.3.5}$$

is the conformal Lorentzian metric, and we have defined

$$A = \frac{(1+r)^3(1-rt)}{(1-r)(1+rt)^3}. \tag{3.3.6}$$

Our new bounded manifold M is given by

$$M = (0,1) \times (0,1) \times \mathbb{S}^2$$

and similar to the previous section we can decompose the boundary of M corresponding to spatial infinity and future null-infinity, respectively,

$$i^0 = [0,1] \times \{0\} \times \mathbb{S}^2 \quad \text{and} \quad \mathscr{I}^+ = \{0\} \times (0,1) \times \mathbb{S}^2$$

and the boundary component

$$\Sigma = \{1\} \times (0,1) \times \mathbb{S}^2$$

defines a space-like hypersurface in M. For more applications of the cylinder at spatial infinity construction in Schwarzschild space-times to linear wave equations, see the articles [64, 65, 66, 67].

Note that, the Ricci scalar of $\tilde{g}$ vanishes since $\tilde{g}$ coincides with a Schwarzschild metric which has vanishing Ricci curvature and therefore

$$\tilde{R} = 0. \tag{3.3.7}$$

It is also not difficult to verify via a straightforward calculation that the Ricci scalar curvature of the metric g, defined by (3.3.5), is given by

$$R = \frac{24rt}{(1+rt)^2}. \tag{3.3.8}$$

3.4 First order Fuchsian system

In the following, we consider semi-linear wave equations of the form (3.3.1) with $\tilde{g}$ the Schwarzschild metric defined by (3.3.3). We will solve these type of wave equations on domains of the form

$$\mathcal{M}_{r_0} = (0,1) \times (0, r_0) \times \mathbb{S}^2, \qquad 0 < r_0 < 1, \tag{3.4.1}$$

that define a "neighbourhood" of spatial infinity in the Schwarzschild space-time of mass $\mu > 0$. Using (3.3.7), we can write the system of wave equations (3.3.1) as

$$\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\tilde{u}^K = \tilde{f}^K \tag{3.4.2}$$

where

$$\tilde{f}^K = q^K_{IJ}(\tilde{u}^L)\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{u}^I\tilde{\nabla}_\beta\tilde{u}^J.$$

From (3.3.4), (3.3.8) and the formulas (A.3.33)-(A.3.34) and (A.3.42)-(A.3.43) from Appendix A.3, we then see that under the conformal transformation (3.3.3) the wave equations (3.4.2) transforms into

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta u^K - \frac{R}{6}u^K = f^K \tag{3.4.3}$$

where ∇ is the Levi-Civita connection of the metric g, defined by (3.3.5), the unknown $\tilde{u}^K$ is given by

$$\tilde{u}^K = \frac{2rt}{\mu(1+rt)^2}u^K$$

and

$$\begin{aligned} f^K = q^K_{IJ}\left(\frac{2rt}{\mu(1+rt)^2}u^L\right)\Bigg(&\frac{2rt}{\mu(1+rt)^2}g^{\alpha\beta}\nabla_\alpha u^I\nabla_\beta u^J + 2g^{\alpha\beta}\nabla_\alpha\left(\frac{2rt}{\mu(1+rt)^2}\right)\nabla_\beta u^{(I}u^{J)} \\ &+ \frac{\mu(1+rt)^2}{2rt}g^{\alpha\beta}\nabla_\alpha\left(\frac{2rt}{\mu(1+rt)^2}\right)\nabla_\beta\left(\frac{2rt}{\mu(1+rt)^2}\right)u^I u^J\Bigg). \end{aligned} \tag{3.4.4}$$

A straightforward calculation using the metric (3.3.5), shows that the system of wave equations (3.4.3), after multiplication by A, can be expressed as

$$-t(2-tA)\partial_t^2 u^K+2r\partial_r\partial_t u^K-2\mathscr{A}\partial_t u^K+A\not{g}^{\Lambda\Sigma}\not{\nabla}_\Lambda\not{\nabla}_\Sigma u^K-\frac{4rtA}{(1+rt)^2}u^K = Af^K \quad (3.4.5)$$

where

$$\mathscr{A} = 1-\frac{1-2rt}{1-(rt)^2}tA, \quad (3.4.6)$$

$$\begin{aligned} Af^K = q^K_{IJ}\left(\frac{2rt}{\mu(1+rt)^2}u^L\right)\Bigg(&-\frac{2t^2r(2-tA)}{\mu(1+rt)^2}\partial_t u^I\partial_t u^J+\frac{4rt}{\mu(1+rt)^2}r\partial_r u^{(I}\partial_t u^{J)} \\ &+\frac{2rtA\not{g}^{\Lambda\Sigma}}{\mu(1+rt)^2}\not{\nabla}_\Lambda u^I\not{\nabla}_\Sigma u^J+\frac{4rAt(t-1)\left[1-r-t(r+6r^2+r^3)+t^2(r^4-r^3)\right]}{\mu(1+r)^3(1+rt)^3}u^{(I}\partial_t u^{J)} \\ &+\frac{4r(1-rt)}{\mu(1+rt)^3}u^{(I}r\partial_r u^{J)}+\frac{2tr(1-rt)^2A}{\mu(1+rt)^4}u^I u^J\Bigg). \end{aligned}$$

The next step is to write the system (3.4.5) into first order form, taking care that the source term can be expanded as in (2.2.10) and the system is symmetric hyperbolic. In Section 3.1, it was not difficult to show that the change of variable (3.2.9) transforms the system (3.1.1) into first order symmetric hyperbolic form such that the non linear terms (3.2.7) can be expanded as in (2.2.10). Here, we first do a change of variable for the system (3.4.3) followed by a rescaling on time.

Let us start by seeking a change of variables of the form

$$\alpha X_0^I+\beta X_1^I = \partial_t u^I, \quad (3.4.7)$$

and

$$\begin{aligned} X_1^I &= r\partial_r u^I, \\ X_\Sigma^I &= \not{\nabla}_\Sigma u^I, \\ X_4^I &= u^I, \end{aligned} \quad (3.4.8)$$

with α,β constants. Computing the time derivative of the system (3.4.7)-(3.4.8) yields

$$\begin{aligned} \partial_t X_1^I &= \alpha r\partial_r X_0^I+\beta r\partial_r X_1^I, \\ \partial_t X_\Lambda^I &= \alpha\not{\nabla}_\Lambda X_0^I+\beta\not{\nabla}_\Lambda X_1^I, \\ \partial_t X_4^I &= \alpha X_0^I+\beta X_1^I, \end{aligned} \quad (3.4.9)$$

and note that we can write the second derivative of u^I with respect to time as

$$\partial_t^2 u^I = \alpha\partial_t X_0^I + \beta\alpha r\partial_r X_0^I + \beta^2 r\partial_r X_1^I. \tag{3.4.10}$$

Substituting (3.4.7)-(3.4.10) into equation (3.4.5) we get the system

$$\begin{aligned}
&t\left(\alpha\partial_t X_0^K + \beta\alpha r\partial_r X_0^K + \beta^2 r\partial_r X_1^K\right) - \frac{2\alpha}{(2-tA)}r\partial_r X_0^K - \frac{2\beta}{(2-tA)}r\partial_r X_1^K + \\
&\frac{2\mathscr{A}\left(\alpha X_0^K + \beta X_1^K\right)}{(2-tA)} - \frac{A}{(2-tA)}\not{g}^{\Lambda\Sigma}\not{\nabla}_\Lambda X_\Sigma^K + \frac{4rtA}{(2-tA)(1+rt)^2}X_4^K = -\frac{A}{(2-tA)}f^K,
\end{aligned}$$

$$\begin{aligned}
&\partial_t X_1^K - \alpha r\partial_r X_0^K - \beta r\partial_r X_1^K = 0, \\
&\partial_t X_\Lambda^K - \alpha\not{\nabla}_\Lambda U_0 - \beta r\partial_r X_\Lambda^K = 0, \\
&\partial_t X_4^K = \alpha X_0^K + \beta X_1^K,
\end{aligned} \tag{3.4.11}$$

which can be written as follows

$$\begin{aligned}
&\alpha t\partial_t X_0^K + \frac{2\alpha\beta t - 2\alpha - \alpha\beta t^2 A}{2-tA}r\partial_r X_0^K + \frac{2\beta^2 t - 2\beta - \beta^2 t^2 A}{2-tA}r\partial_r X_1^K + \frac{2\mathscr{A}\left(\alpha X_0^K + \beta X_1^K\right)}{(2-tA)} - \\
&\frac{A}{(2-tA)}\not{g}^{\Lambda\Sigma}\not{\nabla}_\Lambda X_\Sigma + \frac{4rtA}{(2-tA)(1+rt)^2}X_4^K = -\frac{A}{(2-tA)}f^K,
\end{aligned} \tag{3.4.12}$$

and

$$\begin{aligned}
&-\frac{2\beta^2 t - 2\beta - \beta^2 t^2 A}{\alpha(2-tA)}\partial_t X_1^K + \frac{2\beta^2 t - 2\beta - \beta^2 t^2 A}{2-tA}r\partial_r X_0^K + \frac{2\beta^3 t - 2\beta^2 - \beta^3 t^2 A}{\alpha(2-tA)}r\partial_r X_1^K = 0, \\
&\frac{A}{\alpha(2-tA)}\partial_t X_\Sigma^K - \frac{A}{(2-tA)}\not{\nabla}_\Lambda X_0^K - \frac{\beta A}{\alpha(2-tA)}r\partial_r X_\Lambda^K = 0.
\end{aligned} \tag{3.4.13}$$

Now, let us perform a time rescaling

$$\begin{aligned}
X_0^K &= t^{-\lambda-\frac{1}{2}}U_0^K, \\
X_1^K &= t^{-\lambda}U_1^K, \\
X_\Lambda^K &= t^{-\lambda}U_\Lambda^K, \\
X_4^K &= t^{-\lambda+\frac{1}{2}}U_4^K.
\end{aligned} \tag{3.4.14}$$

Taking the time derivative of (3.4.14) we obtain

$$\begin{aligned}
\partial_t X_0^K &= t^{-\lambda-\frac{1}{2}}\left(\partial_t U_0^K - \frac{\lambda+\frac{1}{2}}{t}U_0^K\right),\\
\partial_t X_1^K &= t^{-\lambda}\left(\partial_t U_1^K - \frac{\lambda}{t}U_1^K\right),\\
\partial_t X_\Lambda^K &= t^{-\lambda}\left(\partial_t U_\Lambda^K - \frac{\lambda}{t}U_\Lambda^K\right),\\
\partial_t X_4^K &= t^{-\lambda+\frac{1}{2}}\left(\frac{-\lambda+\frac{1}{2}}{t}U_4^K + \partial_t U_4^K\right).
\end{aligned} \tag{3.4.15}$$

Substituting (3.4.14)- (3.4.15) into (3.4.12)-(3.4.13) and multiplying by $t^{\lambda-\frac{1}{2}}$ yields

$$\begin{aligned}
&\alpha\partial_t U_0^K + \frac{2\alpha\beta t - 2\alpha - \alpha\beta t^2 A}{t(2-tA)} r\partial_r U_0^K + \frac{2\beta^2 t - 2\beta - \beta^2 t^2 A}{t^{\frac{1}{2}}(2-tA)} r\partial_r U_1^K - \frac{A}{t^{\frac{1}{2}}(2-tA)}\not{g}^{\Lambda\Sigma}\not{\nabla}_\Lambda U_\Sigma^K = \\
&-\frac{t^{\lambda-\frac{1}{2}}A}{(2-tA)}f^K + \frac{\alpha\left(\lambda+\frac{1}{2}\right)(2-tA) - 2\mathscr{A}\alpha}{t(2-tA)}U_0^K - \frac{2\mathscr{A}\beta}{t^{\frac{1}{2}}(2-tA)}U_1^K + \frac{4rtA}{(2-tA)(1+rT)^2}U_4^K,
\end{aligned} \tag{3.4.16}$$

$$\begin{aligned}
&-\frac{2\beta^2 t - 2\beta - \beta^2 t^2 A}{\alpha(2-tA)}\partial_t U_1^K + \frac{2\beta^2 t - 2\beta - \beta^2 t^2 A}{t^{\frac{1}{2}}(2-tA)} r\partial_r U_0^K + \\
&\qquad \frac{2\beta^3 t - 2\beta^2 - \beta^3 t^2 A}{\alpha(2-tA)} r\partial_r U_1^K = -\lambda\frac{2\beta^2 t - 2\beta - \beta^2 tA}{\alpha t(2-tA)}U_1^K,
\end{aligned} \tag{3.4.17}$$

$$\frac{A}{\alpha(2-tA)}\partial_t U_\Lambda^K - \frac{A}{t^{\frac{1}{2}}(2-tA)}\not{\nabla}_\Lambda U_0^K - \frac{\beta A}{\alpha(2-tA)} r\partial_r U_\Lambda^K = \lambda\frac{A}{\alpha t(2-tA)}U_\Lambda^K$$

and

$$\partial_t U_4^K = \frac{\lambda-\frac{1}{2}}{t}U_4^K + \frac{\alpha}{t}U_0^K + \frac{\beta}{t^{\frac{1}{2}}}U_1^K. \tag{3.4.18}$$

We now express the system (3.4.16)-(3.4.18) in matrix form as

$$B^0\partial_T U + B^1 r\partial_r U + B^\Gamma\not{\nabla}_\Gamma U = \frac{1}{T}\mathcal{B}U + F, \tag{3.4.19}$$

where

$$U = \begin{pmatrix} U_0^J \\ U_1^J \\ U_\Lambda^J \\ U_4^J \end{pmatrix},$$

$$B^0 = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & -\frac{2\beta^2 t - 2\beta - \beta^2 t^2 A}{\alpha(2-tA)}\delta_J^K & 0 & 0 \\ 0 & 0 & \frac{A}{\alpha(2-tA)}\delta_J^K & 0 \\ 0 & 0 & 0 & \delta_J^K \end{pmatrix},$$

$$B^1 = \begin{pmatrix} \frac{2\alpha\beta t - 2\alpha - \alpha\beta t^2 A}{t(2-tA)}\delta_J^K & \frac{2\beta^2 t - 2\beta - \beta^2 t^2 A}{t^{\frac{1}{2}}(2-tA)}\delta_J^K & 0 & 0 \\ \frac{2\beta^2 t - 2\beta - \beta^2 t^2 A}{t^{\frac{1}{2}}(2-tA)}\delta_J^K & \frac{2\beta^3 t - 2\beta^2 - \beta^3 t^2 A}{\alpha(2-tA)}\delta_J^K & 0 & 0 \\ 0 & 0 & -\frac{\beta A \delta_\Lambda^\Sigma}{\alpha(2-tA)}\delta_J^K & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B^2 = \begin{pmatrix} 0 & 0 & -\frac{A\not{g}^{\Gamma\Sigma}}{t^{\frac{1}{2}}(2-tA)}\delta_J^K & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{A\not{g}^{\Gamma\Sigma}}{t^{\frac{1}{2}}(2-tA)}\delta_J^K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.4.20}$$

$$\mathcal{B} = \begin{pmatrix} \frac{\alpha\left(\lambda + \frac{1}{2}\right)(2-tA) - 2\mathscr{A}\alpha}{2-tA} & 0 & 0 & 0 \\ 0 & -\lambda\frac{2\beta^2 t - 2\beta - \beta^2 tA}{\alpha(2-tA)} & 0 & 0 \\ 0 & 0 & \lambda\frac{A\not{g}^{ab}}{\alpha(2-tA)} & 0 \\ \alpha & 0 & 0 & \lambda - \frac{1}{2} \end{pmatrix},$$

$$F = \begin{pmatrix} -\frac{t^{\lambda-\frac{1}{2}}A}{(2-tA)}f^K + \frac{2\mathscr{A}\beta W_1}{\sqrt{t}(2-tA)} + \frac{4rtA}{(2-tA)(1+rt)^2}W_4^K \\ 0 \\ 0 \\ 0 \\ \frac{\beta}{t^{\frac{1}{2}}}W_1^K \end{pmatrix}.$$

Since we require positive eigenvalues for the matrix $\mathcal{B}$ it is necessary to impose the following conditions on the constants α, β

$$0 < \alpha \leq 1, \quad \text{and} \quad \frac{1+\sqrt{\alpha}}{2} < \lambda \leq 1, \quad \beta > 0. \tag{3.4.21}$$

For simplicity, we choose $\alpha = \frac{1}{2}, \beta = 1$, which satisfies with (3.4.21), the constant $\lambda \in \mathbb{R}$ will be fixed below. Now we can put together into a single change of variables the transformations (3.4.7)(3.4.8), and (3.4.14) using the variables U_0^J, U_1^J, U_2^J, U_3^J, U_4^J and the the relations

$$\frac{1}{2}t^{-\lambda-\frac{1}{2}}U_0^J + t^{-\lambda}U_1^J = \partial_t u^J, \quad t^{-\lambda}U_1^J = r\partial_r u^J, \quad t^{-\lambda}U_\Lambda^J = \not\nabla_\Lambda u^J \quad \text{and} \quad t^{-\lambda+\frac{1}{2}}U_4^J = u^J. \tag{3.4.22}$$

This change of variables summarizes the calculations (3.4.7)-(3.4.15) above and the system to (3.4.19) becomes

$$B^0\partial_t U + B^1 r\partial_r U + B^\Gamma \not\nabla_\Gamma U = \frac{1}{t}\mathcal{B}U + F \tag{3.4.23}$$

where

$$U = \begin{pmatrix} U_0^J \\ U_1^J \\ U_\Sigma^J \\ U_4^J \end{pmatrix}, \tag{3.4.24}$$

$$B^0 = \begin{pmatrix} \frac{1}{2}\delta_J^K & 0 & 0 & 0 \\ 0 & -\dfrac{2t-2-tA}{\frac{1}{2}(2-tA)}\delta_J^K & 0 & 0 \\ 0 & 0 & \dfrac{A\delta_\Lambda^\Sigma}{\frac{1}{2}(2-tA)}\delta_J^K & 0 \\ 0 & 0 & 0 & \delta_J^K \end{pmatrix}, \tag{3.4.25}$$

$$B^1 = \begin{pmatrix} \dfrac{2t-2-t^2A}{2t(2-tA)}\delta_J^K & \dfrac{2t-2-t^2A}{t^{\frac{1}{2}}(2-tA)}\delta_J^K & 0 & 0 \\ \dfrac{2t-2-t^2A}{t^{\frac{1}{2}}(2-tA)}\delta_J^K & \dfrac{2t-2-t^2A}{\frac{1}{2}(2-tA)}\delta_J^K & 0 & 0 \\ 0 & 0 & -\dfrac{A\delta_\Lambda^\Sigma}{\frac{1}{2}(2-tA)}\delta_J^K & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.4.26}$$

$$
B^{\Gamma} = \begin{pmatrix} 0 & 0 & -\frac{A\not{g}^{\Gamma\Sigma}}{t^{\frac{1}{2}}(2-tA)}\delta^K_J & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{A\delta^{\Gamma}_{\Lambda}}{t^{\frac{1}{2}}(2-tA)}\delta^K_J & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.4.27}
$$

$$
\mathcal{B} = \begin{pmatrix} \frac{\frac{1}{2}\left(\lambda+\frac{1}{2}\right)(2-tA)-\mathscr{A}}{2-tA}\delta^K_J & 0 & 0 & 0 \\ 0 & -\lambda\frac{2t-2-tA}{\frac{1}{2}(2-tA)}\delta^K_J & 0 & 0 \\ 0 & 0 & \lambda\frac{A\delta^{\Sigma}_{\Lambda}}{\frac{1}{2}(2-tA)}\delta^K_J & 0 \\ \frac{1}{2}\delta^K_J & 0 & 0 & \left(\lambda-\frac{1}{2}\right)\delta^K_J \end{pmatrix}, \tag{3.4.28}
$$

and

$$
F = \begin{pmatrix} -\frac{t^{\lambda-\frac{1}{2}}A}{(2-tA)}f^K - \frac{2\mathscr{A}U^K_1}{t^{\frac{1}{2}}(2-tA)} - \frac{4rtAU^K_4}{(2-tA)(1+rt)^2} \\ 0 \\ 0 \\ 0 \\ \frac{U_1}{t^{\frac{1}{2}}} \end{pmatrix}. \tag{3.4.29}
$$

We further observe that F can be expressed as

$$
F = \begin{pmatrix} -\frac{4rtA}{(2-tA)(1+rt)^2}U^K_4 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{t}(t^{\frac{1}{2}}\mathscr{C}+\mathscr{B})U \tag{3.4.30}
$$

where

$$
\mathscr{C} = \begin{pmatrix} 0 & -\frac{2\mathscr{A}}{(2-tA)}\delta^K_J & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \delta^K_J & 0 & 0 \end{pmatrix} \tag{3.4.31}
$$

and

$$
\mathscr{B} = \begin{pmatrix} \mathscr{B}^{0K}_J & \mathscr{B}^{1K}_J & \mathscr{B}^{\Sigma K}_J & \mathscr{B}^{4K}_J \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

with

$$\begin{aligned}
\mathscr{B}_J^{0K} =& q_{PQ}^K\left(\frac{2rt^{\frac{3}{2}-\lambda}U_4^L}{\mu(1+rt)^2}\right)\delta_I^{(P}\delta_J^{Q)}\left(-\frac{\frac{1}{2}rt^{-\lambda+\frac{3}{2}}}{\mu(1+rt)^2}U_0^I-\frac{2rt^{-\lambda+1}(1-2t-t^2A)}{\mu(1+rt)^2(2-tA)^2}U_1^I\right),\\
\mathscr{B}_J^{1K} =& q_{PQ}^K\left(\frac{2rt^{\frac{3}{2}-\lambda}U_4^L}{\mu(1+rt)^2}\right)\delta_I^{(P}\delta_J^{Q)}\left(-\frac{2t^{\frac{3}{2}-\lambda}r(2-2t+t^2A)}{\mu(2-tA)(1+tr)^2}U_1^I\right.\\
&\left.-\frac{4rt^{1-\lambda}[(1-rt)(1+r)^3+At(t-1)(1-r-t(r+6r^2+r^3)+t^2(r^4-r^3))]}{\mu(2-tA)(1+r)^3(1+tr)^3}U_4^I\right),\\
\mathscr{B}_J^{\Sigma K} =& q_{PQ}^K\left(\frac{2rt^{\frac{3}{2}-\lambda}U_4^L}{\mu(1+rt)^2}\right)\delta_I^{(P}\delta_J^{Q)}\left(-\frac{2rAt^{\frac{3}{2}-\lambda}}{\mu(2-tA)(1+rt)^2}\not{g}^{\Sigma\Lambda}U_\Lambda^I\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{B}_J^{4K} =& q_{PQ}^K\left(\frac{2rt^{\frac{3}{2}-\lambda}U_4^L}{\mu(1+rt)^2}\right)\delta_I^{(P}\delta_J^{Q)}\left(-\frac{2t^{\frac{5}{2}-\lambda}r(1-tr)^2A}{\mu(2-tA)(1+rt)^4}U_4^I\right.\\
&\left.-\frac{4rAt^{\frac{3}{2}-\lambda}(t-1)[1-r-t(r+6r^2+r^3)+t^2(r^4-r^3)]}{\mu(2-tA)(1+r)^3(1+rt)^3}U_0^I\right).
\end{aligned} \tag{3.4.32}$$

From (3.3.6) and the above formulas, it is clear that

$$\mathscr{B}=\mathrm{O}(U) \quad \text{for } \lambda\leq 1. \tag{3.4.33}$$

Moreover, we see, using the change of radial coordinate (3.2.21), that we can express (3.4.23) as

$$B^0\partial_t U+\frac{\rho}{m}B^1\partial_\rho U+B^\Gamma\nabla\!\!\!/_\Gamma U=\frac{1}{t}\mathcal{B}U+F \tag{3.4.34}$$

where now any r appearing in the coefficients is replaced using (3.2.21).

3.4.1 The extended system

Proceeding in a similar fashion as above for wave equations on Minkowski spacetime, we consider an extended version of (3.4.34) given by

$$\tilde{B}^0\partial_t U+\frac{\chi\rho}{m}\tilde{B}^1\partial_\rho U+\tilde{B}^\Gamma\nabla\!\!\!/_\Gamma U=\frac{1}{t}\tilde{\mathcal{B}}U+\chi F \tag{3.4.35}$$

where χ is the cut-off function defined above by (3.2.24),

$$\tilde{B}^\mu = B_*^\mu+\chi(B^\mu-B_*^\mu), \qquad \mu=0,1,2,3,4, \tag{3.4.36}$$

$$\tilde{\mathcal{B}}=\mathcal{B}_*+\chi(\mathcal{B}-\mathcal{B}_*), \tag{3.4.37}$$

and we are employing the notation

$$(\cdot)_* = (\cdot)|_{\rho=0}. \tag{3.4.38}$$

Assuming that

$$0 < \rho_0 < \frac{1}{3},$$

$|\rho| < 3\rho_0$ implies, via (3.2.21), that $|r| < 1$ and we see from the definitions (3.4.36), (3.4.37), and the formulas (3.2.24), (3.3.6), (3.4.6), and (3.4.25)-(3.4.29) that the extended system (3.4.35) is well-defined on the extended space-time $(0,1) \times T^1_{3\rho_0} \times \mathbb{S}^2$ (see (3.2.26)) and agrees with the original system (3.4.34) when restricted to the region $\mathcal{M}_{\rho_0} = (0,1) \times (0,\rho_0) \times \mathbb{S}^2$.

Assuming that

$$m \in \mathbb{N}_{\geq 2} \quad \text{and} \quad 0 < \eta < 1,$$

it then follows from (3.2.21), (3.3.6), (3.4.6), (3.4.25), (3.4.28) and Taylor's Theorem that there exists a constant

$$C = C(m,\eta) > 0, \tag{3.4.39}$$

such that

$$|B^0 - B^0_*| \leq C|\rho|^2, \tag{3.4.40}$$

$$|\partial_\rho B^0| \leq C|\rho|, \tag{3.4.41}$$

$$|\mathcal{B} - \mathcal{B}_*| \leq C|\rho|^2 \tag{3.4.42}$$

and

$$|\partial_\rho \mathcal{B}| \leq C|\rho| \tag{3.4.43}$$

for all $(t,\rho,x^\Lambda) \in (0,1) \times (-\eta,\eta) \times \mathbb{S}^2$. Fixing

$$\sigma > 0,$$

we can, by (3.4.36), (3.4.37), (3.4.40) and (3.4.42), ensure that

$$|\tilde{B}^0 - B^0_*| \leq |\chi||B^0 - B^0_*| \leq |B^0 - B^0_*| < \sigma|\rho| < \sigma \tag{3.4.44}$$

and

$$|\tilde{\mathcal{B}} - \mathcal{B}_*| \leq |\chi||\mathcal{B} - \mathcal{B}_*| \leq |\mathcal{B} - \mathcal{B}_*| < \sigma|\rho| < \sigma \tag{3.4.45}$$

for all $(t, \rho, x^\Lambda) \in (0, 1) \times (-3\rho_0, 3\rho_0) \times \mathbb{S}^2$ by choosing ρ_0 so that

$$0 < \rho_0 < \min\left\{\frac{\eta}{3}, \frac{\sigma}{3C(m,\eta)}\right\}. \tag{3.4.46}$$

Moreover, evaluating (3.4.25) and (3.4.28) at $\rho = 0$ yields, we find, with the help of (3.2.21), (3.3.6) and (3.4.6), that

$$B^0_* = \begin{pmatrix} \frac{1}{2}\delta^K_J & 0 & 0 & 0 \\ 0 & 2\delta^K_J & 0 & 0 \\ 0 & 0 & \frac{2}{2-t}\delta^\Sigma_\Lambda \delta^K_J & 0 \\ 0 & 0 & 0 & \delta^K_J \end{pmatrix} \tag{3.4.47}$$

and

$$\mathcal{B}_* = \begin{pmatrix} \frac{\frac{1}{2}\left(\lambda + \frac{1}{2}\right)(2-t) - (1-t)}{2-t}\delta^K_J & 0 & 0 & 0 \\ 0 & 2\lambda\delta^K_J & 0 & 0 \\ 0 & 0 & \lambda\frac{2}{2-t}\delta^\Sigma_\Lambda\delta^K_J & 0 \\ \frac{1}{2}\delta^K_J & 0 & 0 & \left(\lambda - \frac{1}{2}\right)\delta^K_J \end{pmatrix}. \tag{3.4.48}$$

From (3.2.20) and (3.4.47), we then have

$$h(V, B^0_* V) \geq \frac{1}{2}h(V, V)$$

for all V of the form (3.2.19). By choosing $\sigma > 0$ sufficiently small, we deduce from the above inequality and the estimate (3.4.45) that

$$h(V, \tilde{B}^0 V) \geq \frac{1}{4}h(V, V) \tag{3.4.49}$$

on $(0, 1) \times T^1_{3\rho_0} \times \mathbb{S}^2$. From this inequality and the obvious symmetry of the $\tilde{B}^\mu$, $\mu = 0, 1, 2, 3, 4$, with respect to the inner-product (3.2.20), we conclude that the extended system (3.4.35) is symmetric hyperbolic. Furthermore, decomposing the boundary of $\mathcal{M}_{\rho_0} = (0,1)\times(0,\rho_0)\times\mathbb{S}^2$ as

$$\mathcal{M}_{\rho_0} = \Sigma^- \cup \Sigma^+ \cup \Gamma^- \cup \Gamma^+,$$

where

$$\Sigma^- = \{1\}\times(0,\rho_0)\times\mathbb{S}^2, \quad \Sigma^+ = \{0\}\times(0,\rho_0)\times\mathbb{S}^2,$$
$$\Gamma^- = [0,1]\times\{0\}\times\mathbb{S}^2 \quad \text{and} \quad \Gamma^+ = [0,1]\times\{\rho_0\}\times\mathbb{S}^2,$$

it is clear that

$$n^\pm = \pm d\rho$$

define outward pointing co-normals to $\Gamma^\pm$,

$$\Big(n_0^-\tilde{B}^0 + n_1^-\frac{\chi\rho}{m}\tilde{B}^1 + n_\Gamma^-\tilde{B}^\Gamma\Big)\Big|_{\Gamma^-} = 0$$

and

$$\Big(n_0^+\tilde{B}^0 + n_1^+\frac{\chi\rho}{m}\tilde{B}^1 + n_\Gamma^+\tilde{B}^\Gamma\Big)\Big|_{\Gamma^+} \overset{(3.4.36)}{=} \frac{\rho_0}{m}B^1|_{\Gamma^+}.$$

From these expression and (3.4.26), we deduce that

$$h\Big(V, \Big(n_0^-\tilde{B}^0 + n_1^-\frac{\chi\rho}{m}\tilde{B}^i + n_\Gamma^-\tilde{B}^\Gamma\Big)V\Big)\Big|_{\Gamma^-} = 0$$

and

$$\begin{aligned} &h\Big(V, \Big(n_0^+\tilde{B}^0 + n_1^+\frac{\chi\rho}{m}\tilde{B}^1 + n_\Gamma^+\tilde{B}^\Gamma\Big)V\Big)\Big|_{\Gamma^+} = \frac{\rho_0}{m}h(V,B^1V)|_{\Gamma^+} \\ &\quad= \frac{\rho_0}{m}\left[\frac{2t-2-t^2A}{2-tA}\delta_{IJ}\Big(\frac{1}{\sqrt{2}}\frac{V_0^I}{t^{\frac{1}{2}}} + \sqrt{2}V_1^I\Big)\Big(\frac{1}{\sqrt{2}}\frac{V_0^J}{t^{\frac{1}{2}}} + \sqrt{2}V_1^J\Big) - \frac{2A}{2-tA}\rlap{/}g^{\Lambda\Sigma}\delta_{IJ}V_\Lambda^I V_\Sigma^J\right]\Bigg|_{\Gamma^+} \leq 0. \end{aligned}$$

By definition, see [62, §4.3], this shows that the surfaces $\Gamma^\pm$ are weakly spacelike for the extended system (3.4.35), and hence that any solution of (3.4.35) on the extended spacetime $(0,1)\times T^1_{3\rho_0}\times\mathbb{S}^2$ will determine a solution of the original system (3.4.34) on the region $(0,1)\times(0,\rho_0)\times\mathbb{S}^2$ via restriction that is uniquely determined by initial data on $\{1\}\times(0,\rho_0)\times\mathbb{S}^2$. We therefore conclude that solutions to the system of semilinear wave equations (3.3.1) on the regions of the form (3.4.1) in a Schwarzschild spacetime can be obtained from solving the initial value problem

$$\tilde{B}^0\partial_t U + \frac{\chi\rho}{m}\tilde{B}^1\partial_\rho U + \tilde{B}^\Gamma\rlap{/}\nabla_\Gamma U = \frac{1}{t}\tilde{\mathcal{B}}U + \chi F \qquad \text{in } (0,1)\times T^1_{3\rho_0}\times\mathbb{S}^2, \tag{3.4.50}$$

$$U = \mathring{U} \qquad \text{in } \{1\}\times T^1_{3\rho_0}\times\mathbb{S}^2, \tag{3.4.51}$$

where the solution to (3.3.1) generated this way are independent of the particular form of the initial data $\mathring{U}$ on the region $(\{1\} \times T^1_{3\rho_0} \times \mathbb{S}^2) \setminus (\{1\} \times (0, \rho_0) \times \mathbb{S}^2)$.

We now want to conclude existence of solutions to the IVP (3.4.50)-(3.4.51) via an application of Theorem 2.3.1. However, in order to do this, we must first verify a number of structural conditions. We proceed by noting from (3.4.48) that

$$h(V, \mathcal{B}_* V) = \delta_{IJ}\biggl(\biggl(\frac{1}{2}\biggl(\lambda + \frac{1}{2}\biggr) - \frac{1-t}{2-t}\biggr)V_0^I V_0^J + \frac{1}{2}V_0^I V_4^J + 2\lambda V_1^I V_1^J + \frac{2\lambda}{2-t}\not{g}^{\Lambda\Sigma}V_\Lambda^I V_\Sigma^J + \biggl(\lambda - \frac{1}{2}\biggr)V_4^I V_4^J\biggr).$$

From this and the inequality $\bigl|\frac{1}{2}\delta_{IJ}V_0^I V_4^J\bigr| \leq \frac{\epsilon^2}{2}\delta_{IJ}\frac{V_0^I}{2}\frac{V_0^J}{2} + \frac{1}{2\epsilon^2}\delta_{IJ}V_4^I V_4^J$, $\epsilon > 0$, we obtain

$$h(V, \mathcal{B}_* V) \geq \delta_{IJ}\biggl(\biggl(\frac{1}{2}\biggl(\lambda + \frac{1}{2}\biggr) - \frac{1-t}{2-t} - \frac{\epsilon^2}{8}\biggr)V_0^I V_0^J + 2\lambda V_1^I V_1^J + \frac{2\lambda}{2-t}\not{g}^{\Lambda\Sigma}V_\Lambda^I V_\Sigma^J + \biggl(\lambda - \frac{1}{2} - \frac{1}{2\epsilon^2}\biggr)V_4^I V_4^J\biggr),$$

and hence, by setting $\epsilon = 2^{\frac{1}{4}}$, that

$$h(V, \mathcal{B}_* V) \geq \delta_{IJ}\biggl(\frac{1}{2}\biggl(\kappa + 1 - \frac{2(1-t)}{2-t}\biggr)V_0^I V_0^J + 2\lambda V_1^I V_1^J + \frac{2\lambda}{2-t}\not{g}^{\Lambda\Sigma}V_\Lambda^I V_\Sigma^J + \kappa V_4^I V_4^J\biggr),$$

where κ is as defined previously by (3.2.30). But $\frac{2(1-t)}{2-t} \leq 1$ for $0 \leq t \leq 1$ and $\lambda > \kappa$, and so we conclude from the above inequality, (3.2.20) and (3.4.47) that

$$h(V, \mathcal{B}_* V) \geq \kappa h(V, B^0_* V)$$

on $(0,1) \times (-1,1) \times \mathbb{S}^3$. Fixing $\tilde{\sigma} > 0$, it follows from this inequality and the estimates (3.4.44) and (3.4.45) that we can guarantee that

$$h(V, \tilde{\mathcal{B}} V) \geq (\kappa - \tilde{\sigma}) h(V, \tilde{B}^0 V) \tag{3.4.52}$$

on $(0,1) \times T^1_{3\rho_0} \times \mathbb{S}^2$ by choosing $\sigma > 0$ small enough.

Next, setting $\mu = 0$ in (3.4.36) and differentiating with respect to ρ shows, with the help of (3.2.24), that

$$\partial_\rho \tilde{B}^0 = \hat{\chi}'\biggl(\frac{\rho}{\rho_0}\biggr)\frac{B^0 - B^0_*}{\rho_0} + \chi \partial_\rho B^0.$$

Using (3.4.41) and (3.4.44), we obtain from the above expression the estimate

$$|\partial_\rho \tilde{B}^0| = \|\hat{\chi}'\|_{L^\infty(\mathbb{R})}\frac{|B^0 - B^0_*|}{\rho_0} + |\chi||\partial_\rho B^0| < \bigl(3\|\hat{\chi}'\|_{L^\infty(\mathbb{R})} + 1\bigr)\sigma \tag{3.4.53}$$

that holds for all $(t,\rho,x^\Lambda) \in (0,1)\times(-3\rho_0,3\rho_0)\times\mathbb{S}^2$. Additionally, we find, using similar arguments this time starting from the estimates (3.4.43) and (3.4.45), that

$$|\partial_\rho \tilde{\mathcal{B}}| < \big(3\|\hat{\chi}'\|_{L^\infty(\mathbb{R})}+1\big)\sigma \tag{3.4.54}$$

for all $(t,\rho,x^\Lambda) \in (0,1)\times(-3\rho_0,3\rho_0)\times\mathbb{S}^2$. Appealing again to Taylor's Theorem, it is not difficult to verify from (3.2.21), (3.3.6), (3.4.6), (3.4.26) and (3.4.27) that

$$|tB^i - tB^i_*| \leq C|\rho|^2, \qquad i=1,2,3,$$

and

$$|\partial_\rho(tB^i)| \leq C|\rho|$$

for all $(t,\rho,x^\Lambda) \in (0,1)\times(-\eta,\eta)\times\mathbb{S}^2$ where we can take C to be the same constant as above, see (3.4.39). Using these estimates, the same arguments that lead to the estimates (3.4.44), (3.4.45), (3.4.53) and (3.4.54) can be used to show that

$$|t\tilde{B}^i - tB^i_*| \leq |tB^i - tB^i_*| < \sigma|\rho| < \sigma \tag{3.4.55}$$

and

$$|\partial_\rho(t\tilde{B}^i)| < \big(3\|\hat{\chi}'\|_{L^\infty(\mathbb{R})}+1\big)\sigma \tag{3.4.56}$$

for all $(t,\rho,x^\Lambda) \in (0,1)\times(-3\rho_0,3\rho_0)\times\mathbb{S}^2$ provided that ρ_0 satisfies (3.4.46). Finally, differentiating $\frac{\chi\rho}{m}t\tilde{B}^1$ with respect to ρ gives

$$\partial_\rho\left(t\frac{\chi\rho}{m}\tilde{B}^1\right) = \frac{1}{m}\left[\left(\hat{\chi}'\left(\frac{\rho}{\rho_0}\right)\frac{\rho}{\rho_0}+\chi\right)\big(tB^1_* + (t\tilde{B}^1 - tB^1_*)\big) + \chi\rho t\partial_\rho\tilde{B}^1\right]$$

from which we see, with the help of (3.4.55) and (3.4.56) for $i=1$, that

$$\left|\partial_\rho\left(t\frac{\chi\rho}{m}\tilde{B}^1\right)\right| < \frac{1}{m}\big(3\|\hat{\chi}'\|_{L^\infty(\mathbb{R})}+1\big)\left(\sup_{0<t<1}|tB^1_*(t)| + 2\sigma\right)$$

for all $(t,\rho,x^\Lambda) \in (0,1)\times(-3\rho_0,3\rho_0)\times\mathbb{S}^2$. Choosing $m\geq 2$ large enough so that

$$\frac{1}{m}\sup_{0<t<1}|tB^1_*(t)| < \sigma,$$

the above estimate implies that the inequality

$$\left|\partial_\rho\left(t\frac{\chi\rho}{m}\tilde{B}^1\right)\right| < 3\big(3\|\hat{\chi}'\|_{L^\infty(\mathbb{R})}+1\big)\sigma \tag{3.4.57}$$

also hold for all $(t,\rho,x^\Lambda)\in(0,1)\times(-3\rho_0,3\rho_0)\times\mathbb{S}^2$.

3.5 Global existence

In the following, we choose $\sigma>0$ small enough and $m\in\mathbb{Z}_{\geq 2}$ large enough so that the inequalities (3.4.49), (3.4.52), (3.4.53), (3.4.54), (3.4.56) and (3.4.57) all hold on $(0,1)\times T_{3\rho_0}\times\mathbb{S}^2$ for ρ_0 satisfying (3.4.46) and λ satisfying (3.2.35) (so that $\kappa>0$), and the constant υ is chosen to lie in the interval $(0,\kappa)$. Then, from the Cauchy stability property satisfied by symmetric hyperbolic systems, we deduce, for any given $t_0\in(0,1)$, the existence of a unique solution

$$U\in C^0\big((t_0,1],H^k(T^1_{3\rho_0}\times\mathbb{S}^2)\big)\cap L^\infty\big((t_0,1],H^k(T^1_{3\rho_0}\times\mathbb{S}^2)\big)\cap C^1\big((t_0,1],H^{k-1}(T^1_{3\rho_0}\times\mathbb{S}^2)\big)$$

to (3.4.50) provided that $k\in\mathbb{Z}_{>3/2}$ and the initial data $U|_{t=1}=\mathring{U}\in H^k(T^1_{3\rho_0}\times\mathbb{S}^2)$ is chosen small enough. Moreover, by standard results, this solution will satisfy an energy estimate of the form

$$\|U(t)\|^2_{H^k}+\int_t^1\frac{1}{\tau}\|U(\tau)\|^2_{H^k}\,d\tau\leq C\big(\|U\|_{L^\infty((t_0,0],H^k)}\big)\|\mathring{U}\|^2_{H^k},\qquad t_0<t\leq 1,$$

and the norm $\|U(t_0)\|_{H^k}$ can be made as small as we like by choosing the initial data $\mathring{U}$ at $t=1$ suitably small.

To continue this solution from $t=t_0$ to $t=0$, we now assume that $k\in\mathbb{Z}_{>9/2}$ and choose the initial data $\mathring{U}$ at $t=1$ small enough so that $\|U(t_0)\|_{H^k}$ and t_0 can be taken to be sufficiently small. Then after performing the simple time transformation $t\mapsto -t$, it is not difficult to verify from (3.4.25)-(3.4.28), (3.4.30)-(3.4.33), (3.4.36)-(3.4.38), (3.4.49), (3.4.52) -(3.4.54), (3.4.56) and (3.4.57) that the extended system[2] (3.4.50), which we know is symmetric hyperbolic, satisfies all the assumptions from Section 2.2 with $\mathbb{P}=\mathbb{1}$, where, by choosing $\sigma>0$ sufficiently small and $m\geq 2$ sufficiently large, the constants β_1 and $\mathtt{b}=\tilde{\mathtt{b}}$ from Theorem 2.3.1 can be made as small as we like while the constants λ_1 and α vanish. This allows us to apply

[2]Note that the angular derivatives $\not{\nabla}_\Lambda\tilde{\mathcal{B}}$, $\not{\nabla}_\Lambda\mathscr{C}$, $\not{\nabla}_\Lambda\big(\frac{\rho\chi}{m}\tilde{B}^1\big)$, and $\not{\nabla}_\Lambda\tilde{B}^\ell$, $\ell=0,2,3$, of all the U-independent matrices appearing in (3.4.50) (see also (3.4.29) and (3.4.31)) vanish.

Theorem 2.3.1 to (3.4.50) with initial data given by $U(t_0)$ at $t = t_0$ to obtain the existence of a unique solution

$$U \in C^0\big((0,t_0], H^k(T^1_{3\rho_0}\times\mathbb{S}^2)\big)\cap L^\infty\big((0,t_0], H^k(T^1_{3\rho_0}\times\mathbb{S}^2)\big)\cap C^1\big((0,t_0], H^{k-1}(T^1_{3\rho_0}\times\mathbb{S}^2)\big)$$

that satisfies the energy inequality

$$\|U(t)\|^2_{H^k} + \int_t^1 \frac{1}{\tau}\|U(\tau)\|^2_{H^k}\, d\tau \leq C\big(\|U\|_{L^\infty((1,t_0],H^k)}\big)\|U(t_0)\|^2_{H^k}, \qquad 0 < t \leq t_0,$$

and decay

$$\|U(t)\|_{H^{k-1}} \lesssim t^{\kappa-\upsilon}, \qquad 0 < t \leq t_0,$$

for any given $\upsilon > 0$. This establishes the existence of a unique solution of the IVP (3.4.50)-(3.4.51), which completes the proof of the following theorem.

Theorem 3.5.1. *Suppose $k \in \mathbb{Z}_{>9/2}$, $\upsilon > 0$, $\lambda \in \big(\frac{1}{4}(2+\sqrt{2}), 1\big]$ and $\kappa = \lambda - \frac{1}{4}(2+\sqrt{2})$. Then there exist $\rho_0 \in (0,1)$, $m \in \mathbb{Z}_{\geq 2}$ and $\delta > 0$ such that if $\mathring{U} \in H^k(T^1_{3\rho_0}\times\mathbb{S}^2)$ is chosen so that $\|\mathring{U}\|_{H^k} < \delta$, then there exists a unique solution*

$$U \in C^0\big((0,1], H^k(T^1_{3\rho_0}\times\mathbb{S}^2)\big)\cap L^\infty\big((0,1], H^k(T^1_{3\rho_0}\times\mathbb{S}^2)\big)\cap C^1\big((0,1], H^{k-1}(T^1_{3\rho_0}\times\mathbb{S}^2)\big)$$

of the IVP (3.4.50)-(3.4.51) that satisfies the energy

$$\|U(t)\|^2_{H^k} + \int_t^1 \frac{1}{\tau}\|U(\tau)\|^2_{H^k}\, d\tau \leq C\big(\|U\|_{L^\infty((1,0],H^k)}\big)\|\mathring{U}\|^2_{H^k}$$

and decay

$$\|U(t)\|_{H^{k-1}} \lesssim t^{\kappa-\upsilon}$$

estimates for $0 < t \leq 1$.

The universe gives birth to consciousness, and consciousness gives meaning to the universe.

John Archibald Wheeler

4

A Fuchsian viewpoint on the weak null condition

In this chapter we apply Theorem 2.3.1 to semi-linear wave equations on Minkowski space-time with non-linear quadratic terms that satisfy the *bounded weak null condition* 1.7.2, see Chapter 1. The class of semi-linear wave equations that we consider are of the form

$$\bar{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta\bar{u}^K = \bar{a}^{K\alpha\beta}_{IJ}\bar{\nabla}_\alpha\bar{u}^I\bar{\nabla}_\beta\bar{u}^J, \tag{4.0.1}$$

where the u^I, $1 \leq I \leq N$, are a collection of scalar fields, the $\bar{a}^K_{IJ} = \bar{a}^{K\alpha\beta}_{IJ}\bar{\partial}_\alpha \otimes \bar{\partial}_\beta$, $1 \leq I, J, K \leq N$, are prescribed smooth (2,0)-tensors fields on $\mathbb{R}^4$, and $\bar{\nabla}$ is the Levi-Civita connection of the Minkowski metric $\bar{g} = \bar{g}_{\alpha\beta}d\bar{x}^\alpha \otimes d\bar{x}^\beta$ on $\mathbb{R}^4$. For simplicity we assume that the tensor fields $\bar{a}^K_{IJ}$ are covariantly constant, i.e. $\bar{\nabla}\bar{a}^K_{IJ} = 0$, which is equivalent to the condition that the components of $\bar{a}^K_{IJ}$ are constants in a Cartesian coordinate system $(\hat{x}^\mu)$, that is, $\bar{a}^K_{IJ} = \hat{a}^{K\alpha\beta}_{IJ}\hat{\partial}_\alpha \otimes \hat{\partial}_\beta$ for some set of constant coefficients $\hat{a}^{K\alpha\beta}_{IJ}$.

The way we obtain existence results and decay estimates for equation (4.0.1), follows in essence the same structure that we described in Section 1.4 and that we used in Chapter 3. The main idea is to obtain an extended system (4.3.54) on the bounded manifold $\mathcal{S} = \mathbb{T} \times \mathbb{S}^2$, which is needed to apply the existence theory from [1] . The difference with respect to the previous chapter is that here, we use the flow of the asymptotic equation (1.7.2) to change variables from V_0 to a new variable Y defined by (4.4.5)-(4.4.4) to eliminate the most singular term from the "time" component of the extended system (4.3.54), which results in the evolution equation

(4.4.6). Then we form the Fuchsian system with a new variable that we obtain from the differentiated system (4.3.75), a projection of the extended system given by (4.4.23), and the equations (4.4.5)-(4.4.4). Once we have formed the Fuchsian system (4.4.26), we can apply Theorem 2.3.1, under the flow assumptions 4.4.1, which yields the GIVP result for the Fuchsian system (4.4.26) that is stated in Theorem 4.5.1.

4.0.1 Related works

In [68], J. Keir analysed systems of quasilinear wave equations with quadratic semi-linear terms. His assumptions are more restrictive than the assumptions used in this work since in addition to the boundedness assumption 4.4.1, it is required a stability condition on solutions to the asymptotic equation. Using the boundedness and stability conditions, Keir establishes the global existence of solutions to the future of a truncated outgoing characteristic hypersurface under a suitable small initial data assumption. Keir obtained his results in [68], by using a generalization of the p-weighted energy method of Dafermos and Rodnianski [69] that was developed in [70]. In particular, his results imply that semi-linear systems of wave equations of the form (4.0.1) whose asymptotic equations satisfy his boundedness and stability condition admit solutions on space-time regions of the form $\{\ (\bar{t},\bar{r}) \mid \bar{t} > \max\{0,\bar{r}-\bar{r}_0\}, \bar{r} \geq 0\ \} \times \mathbb{S}^2$, for suitably small initial data that is prescribed on the truncated null-cone $\{\ (\bar{t},\bar{r}) \mid \bar{t} = \max\{0,\bar{r}-\bar{r}_0\}, \bar{r} \geq 0\ \} \times \mathbb{S}^2$, where $(\bar{x}^\mu) = (\bar{t},\bar{r},\bar{\theta},\bar{\phi})$ denote spherical coordinates.

In light of Keir's results, we will restrict our attention to establishing the existence of solutions to (4.0.1) on neighborhoods of spatial infinity of the form

$$\bar{M}_{r_0} = \left\{\ (\bar{t},\bar{r}) \,\middle|\, 0 < \bar{t} < \bar{r} - 1/r_0,\ 1/r_0 < \bar{r} < \infty\ \right\} \times \mathbb{S}^2 \tag{4.0.2}$$

where $r_0 > 0$ is a positive constant and initial data is prescribed on the hypersurface

$$\bar{\Sigma}_{r_0} = \left\{\ (\bar{t},\bar{r}) \,\middle|\, \bar{t} = 0,\ 1/r_0 < \bar{r} < \infty\ \right\} \times \mathbb{S}^2. \tag{4.0.3}$$

This will complement Keir's results, at least in the semi-linear setting, by establishing the existence of solutions on regions not covered by his existence results.

4.0.2 A particular example where our results apply

The following section is a particular example where we can apply the results stated in Theorem 4.5.1, that is, the following discussion is an example of semi-linear wave equations satisfying the *bounded weak null condition*. In [68], Keir showed that systems of wave equations in the form (4.0.1) with

$$\bar{a}^{K\alpha\beta}_{IJ} = \bar{I}^{KL}\bar{C}_{LIJ}\delta^{\alpha}_{0}\delta^{\beta}_{0}, \tag{4.0.4}$$

where $\bar{I}^{KL}$ is a constant, positive definite, symmetric matrix and the $\bar{C}_{LIJ}$ are any constants satisfying

$$\bar{C}_{LIJ} = -\bar{C}_{ILJ}, \tag{4.0.5}$$

have associated asymptotic equations that satisfy the bounded weak null condition 1.7.2. We can verify this since the choice (4.0.4) leads, by (1.7.1)-(1.7.3), to the associated asymptotic equation

$$(2-t)\partial_t\xi^K = -\frac{2}{t}\chi(\rho)\rho^m\bar{I}^{KL}\bar{C}_{LIJ}\xi^I\xi^J. \tag{4.0.6}$$

Introducing the inner-product $(\xi|\eta) = \check{\bar{I}}_{IJ}\xi^I\eta^J$, where $(\check{\bar{I}}_{IJ}) = (\bar{I}^{IJ})^{-1}$, and contracting (4.0.6) with $\check{\bar{I}}_{LK}\xi^L$, we get

$$(2-t)(\xi|\partial_t\xi) = -\frac{2}{t}\chi(\rho)\rho^m\check{\bar{I}}_{LK}\bar{I}^{KM}\bar{C}_{MIJ}\xi^L\xi^I\xi^J = -\frac{2}{t}\chi(\rho)\rho^m\bar{C}_{LIJ}\xi^L\xi^I\xi^J \overset{(4.0.5)}{=} 0. \tag{4.0.7}$$

This equation implies $\partial_t((\xi|\xi)) = 0$, therefore we conclude that any solution of the asymptotic IVP (1.7.4) -(1.7.4) exists for all $t \in (0,1]$ and satisfies $(\xi(t)|\xi(t)) = (\mathring{\xi}|\mathring{\xi})$. Letting $|\cdot|$ denote the Euclidean norm, we then have that $\frac{1}{\sqrt{C}}|\cdot| \leq \sqrt{(\cdot|\cdot)} \leq \sqrt{C}|\cdot|$ for some constant $C > 0$, and consequently, by the above inequality, we arrive at the bound $\sup_{0<t\leq 1}|\xi(t)| \leq C|\mathring{\xi}|$, which verifies that the bounded weak null condition is fulfilled.

The calculation (4.0.7) also shows that this class of semi-linear equations satisfies the structural condition from [71] called *Condition H*. Because of this, the global existence results established in [71] apply and yield the existence of global solutions to (4.0.1) on the region $\bar{t} > 0$ for suitably small initial data with compact support. We further note that due to the compact support of the initial data, the results of [71] can, in fact, be deduced as a special case of the global existence theory developed in [68], but do not apply to the situation we are considering in this chapter because we allow for non-compact initial data in addition to a less restrictive weak null condition.

4.1 The cylinder at spatial infinity

Our first task in the transformation of (4.0.1) into a Fuchsian system is to compactify the space-like region given by (4.0.2) into the *cylinder at infinity*. Since we have already analysed this transformation in the previous chapter, we write here the main results and direct the reader to section Section 3.1 of Chapter 3. We recall the Minkowski space-time

$$\bar{g} = -d\bar{t} \otimes d\bar{t} + d\bar{r} \otimes d\bar{r} + \bar{r}\not{g}, \tag{4.1.1}$$

where $\not{g}$ is the canonic metric on the 2-sphere $\mathbb{S}^2$ and $(\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi})$ are spherical coordinates in $\mathbb{R}^4$. We use below the map

$$\psi \,:\, \bar{M} \longrightarrow M \,:\, (\bar{x}^\mu) = (\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi}) \longmapsto (x^\mu) = \left(1 - \frac{\bar{t}}{\bar{r}}, \frac{\bar{r}}{-\bar{t}^2 + \bar{r}^2}, \bar{\theta}, \bar{\phi}\right), \tag{4.1.2}$$

to push-forward the wave equation (4.0.1). A straightforward calculation shows that the inverse map is given by

$$\psi^{-1} : M \longrightarrow \bar{M} : (x^\mu) = (t, r, \theta, \phi) \longmapsto (\bar{x}^\mu) = \left(\frac{1-t}{rt(2-t)}, \frac{1}{rt(2-t)}, \theta, \phi\right). \tag{4.1.3}$$

We consider a section of the Minkowski space-time defined by

$$\bar{M} = \{\, (\bar{t}, \bar{r}) \in (-\infty, \infty) \times (0, \infty) \mid -\bar{t}^2 + \bar{r}^2 > 0 \,\} \times \mathbb{S}^2, \tag{4.1.4}$$

and with the help of (4.1.2), we map $\bar{M}$ onto the *cylinder at spatial infinity*

$$M = (0, 2) \times (0, \infty) \times \mathbb{S}^2.$$

As discussed in Section 3.1, $\bar{M}$ is the interior of the space-like cone with vertex at the origin in $\mathbb{R}^4$. The diffeomorphism (4.1.2) transforms this region into the manifold M with a boundary composed by

$$\partial M = \mathscr{I}^+ \cup i^0 \cup \mathscr{I}^-$$

where

$$\mathscr{I}^+ = \{0\} \times (0,\infty) \times \mathbb{S}^2, \quad \mathscr{I}^- = \{2\} \times (0,\infty) \times \mathbb{S}^2 \quad \text{and} \quad i^0 = [0,2] \times \{0\} \times \mathbb{S}^2.$$

The compatification defined by (4.1.2) leads to the interpretation of $\mathscr{I}^\pm$ as portions of (+) future and (−) past null-infinity, respectively, and i^0 as spatial infinity. Furthermore, the space-like hypersurface $\{1\} \times (0,\infty) \times \mathbb{S}^2$ in M corresponds to the constant time hypersurface $\bar{t} = 0$ in Minkowski space-time see figures 3.1, 3.2. By straightforward calculation using (4.1.2) and (4.0.2)-(4.0.3) and noting that $\psi(\bar{M}_{r_0}) = M_{r_0}$ and $\psi(\bar{\Sigma}_{r_0}) = \Sigma_{r_0}$ it is not difficult to verify that the region (4.0.2) and the hypersurface (4.0.3) are mapped to

$$M_{r_0} = \{(t,r) \in (1,0) \times (0,r_0) \mid t > 2 - r_0/r\} \times \mathbb{S}^2 \subset M, \qquad r_0 > 0, \tag{4.1.5}$$

where initial data is prescribed on the space-like hypersurface

$$\Sigma_{r_0} = \{1\} \times (0,r_0) \times \mathbb{S}^2 \tag{4.1.6}$$

that forms the "top" of the domain M_{r_0}. By (4.0.2) and (4.0.3), we conclude that any solution of the conformal wave equations on M_{r_0} with initial data prescribed on Σ_{r_0} corresponds uniquely to a solution of the semi-linear wave equations (4.0.1) on $\bar{M}_{r_0}$ with initial data prescribed on $\bar{\Sigma}_{r_0}$.

4.2 The conformal wave equation

The next step after the compactification ot he Minkowski space-time is to push-forward the wave equation (4.0.1) using the map (4.1.2). First we let

$$\tilde{g} = \psi_* \bar{g}$$

denote the push-forward of the Minkowski metric (4.1.1) from $\bar{M}$ to M. After a routine calculation it is not difficult to see that

$$\tilde{g} = \Omega^2 g \tag{4.2.1}$$

with

$$\Omega = \frac{1}{r(2-t)t} \tag{4.2.2}$$

and

$$g = -dt \otimes dt + \frac{1-t}{r}(dt \otimes dr + dr \otimes dt) + \frac{(2-t)t}{r^2} dr \otimes dr + \not{g}, \tag{4.2.3}$$

where

$$\not{g} = d\theta \otimes d\theta \sin^2(\theta) d\phi \otimes d\phi. \tag{4.2.4}$$

Using the map (4.1.2) to push-forward the wave equations (4.0.1) yields the system of wave equations

$$\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\tilde{u}^K = \tilde{a}^{K\alpha\beta}_{IJ}\bar{\nabla}_\alpha\tilde{u}^I\tilde{\nabla}_\beta\tilde{u}^J \tag{4.2.5}$$

where $\tilde{\nabla}_\alpha$ is the Levi-Civita connection of the metric $\tilde{g}_{\alpha\beta}$,

$$\tilde{u}^K = \psi_* \bar{u}^K \tag{4.2.6}$$

and

$$\tilde{a}^{K\alpha\beta}_{IJ} = \psi_*(\bar{a}^K_{IJ})^{\alpha\beta}. \tag{4.2.7}$$

Since $M = \psi(\bar{M})$, it is clear the original system of semilinear wave equations (4.0.1) on $\bar{M}$ are completely equivalent to (4.2.5) on M. Next, we observe that the Ricci scalar curvature of $\tilde{g}_{\alpha\beta}$ vanishes by virtue of $\tilde{g}_{\alpha\beta}$ being the push-forward of the Minkowski metric. Furthermore, a straightforward calculation using (4.2.3) shows that the Ricci scalar of the metric $g_{\alpha\beta}$ also vanishes. Consequently, it follows from the formulas (A.3.39)-(A.3.40) and (A.3.44)-(A.3.45), with $n = 4$, from Appendix A.3 that the system of wave equations (4.2.5) transform under the conformal transformation (4.2.1) into

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta u^K = f^K \tag{4.2.8}$$

where ∇ is the Levi-Civita connection of g,

$$\tilde{u}^K = rt(2-t)u^K \tag{4.2.9}$$

and

$$f^K = \tilde{a}^{K\mu\nu}_{IJ}\bigg(\frac{1}{rt(2-t)}\nabla_\mu u^I\nabla_\nu u^J + \frac{1}{(rt(2-t))^2}\big(\nabla_\mu(rt(2-t))u^I\nabla_\nu u^J + \nabla_\mu u^I\nabla_\nu(rt(2-t))u^J\big) + \frac{1}{(rt(2-t))^3}\nabla_\mu\big(rt(2-t)\big)\nabla_\nu(rt(2-t))u^I u^J\bigg).$$

We will refer to the system (4.2.8), (4.2.10) as the *conformal wave equations*. A routine computation involving the metric (4.2.3) then shows that the conformal wave equations (4.2.8) can be expressed as

$$(-2+t)t\partial_t^2 u^K + r^2\partial_r^2 u^K + 2r(1-t)\partial_r\partial_t u^K + \not{g}^{\Lambda\Sigma}\not{\nabla}_\Lambda\not{\nabla}_\Sigma u^K + 2(t-1)\partial_t u^K = f^K \quad (4.2.10)$$

where $\not{\nabla}_\Lambda$ is the Levi-Civita connection of the metric (4.2.4) on $\mathbb{S}^2$.

4.2.1 Expansion formulas for the tensor components $\bar{a}_{IJ}^{K\alpha\beta}$

Before continuing with the transformation of the system (4.2.2) into Fuchsian form, we first derive an expansion formula for the tensor components $\bar{a}_{IJ}^{K\alpha\beta}$. This formula plays an important role in the calculations below. We use the coordinate charts $(\hat{x}^\mu), (\bar{x}^\mu)$ to denote Cartesian and Spherical coordinates respectively. From the coordinate transformation

$$(\hat{x}^\mu) = (\bar{t}, \bar{r}\cos(\phi)\sin(\theta), \bar{r}\sin(\phi)\sin(\theta), \bar{r}\cos(\theta)), \quad (4.2.11)$$

we calculate the Jacobian matrix which is given by

$$(\bar{J}_\mu^\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \\ 0 & \frac{\cos(\theta)\cos(\phi)}{\bar{r}} & \frac{\cos(\theta)\sin(\phi)}{\bar{r}} & -\frac{\sin(\theta)}{\bar{r}} \\ 0 & -\frac{\csc(\theta)\sin(\phi)}{\bar{r}} & \frac{\csc(\theta)\cos(\phi)}{\bar{r}} & 0 \end{pmatrix}. \quad (4.2.12)$$

Using this and the tensorial transformation law

$$\bar{a}_{IJ}^{K\alpha\beta} = \bar{J}_\mu^\alpha \hat{a}_{IJ}^{K\mu\nu}\bar{J}_\nu^\beta, \quad (4.2.13)$$

we can expand the components (4.2.13) in powers of $\bar{r}$ as

$$\bar{a}_{IJ}^{K\alpha\beta} = \frac{1}{\bar{r}^2}\bar{e}_{IJ}^{K\alpha\beta} + \frac{1}{\bar{r}}\bar{d}_{IJ}^{K\alpha\beta} + \bar{c}_{IJ}^{K\alpha\beta}, \quad (4.2.14)$$

where the expansions coefficients can be used to define the following geometric objects (see Appendix A.1 for our indexing conventions) on $\mathbb{S}^2$:

(a) smooth functions $\bar{e}_{IJ}^{Kpq}$, $\bar{d}_{IJ}^{Kpq}$ and $\bar{c}_{IJ}^{Kpq}$,

(b) smooth vector fields $\bar{e}_{IJ}^{Kq\Lambda}$, $\bar{e}_{IJ}^{K\Lambda q}$, $\bar{d}_{IJ}^{Kq\Lambda}$, $\bar{d}_{IJ}^{K\Lambda q}$, $\bar{c}_{IJ}^{Kq\Lambda}$, and $\bar{c}_{IJ}^{K\Lambda q}$,

(c) and smooth (2,0)-tensor fields $\bar{e}_{IJ}^{K\Lambda\Sigma}$, $\bar{d}_{IJ}^{K\Lambda\Sigma}$ and $\bar{c}_{IJ}^{K\Lambda\Sigma}$.

The only terms of the expansion (4.2.14) that we will need to consider in any detail are the $\bar{c}_{IJ}^{K\alpha\beta}$. Now, it can be easily verified that the *non-vanishing* $\bar{c}_{IJ}^{K\alpha\beta}$ are given by

$$\bar{c}_{IJ}^{K00} = \hat{a}_{IJ}^{K00}, \tag{4.2.15}$$
$$\bar{c}_{IJ}^{K01} = \sin(\theta)(\hat{a}_{IJ}^{K01}\cos(\phi) + \hat{a}_{IJ}^{K02}\sin(\phi)) + \hat{a}_{IJ}^{K03}\cos(\theta), \tag{4.2.16}$$
$$\bar{c}_{IJ}^{K10} = \sin(\theta)(\hat{a}_{IJ}^{K10}\cos(\phi) + \hat{a}_{IJ}^{K20}\sin(\phi)) + \hat{a}_{IJ}^{K30}\cos(\theta) \tag{4.2.17}$$

and

$$\begin{aligned}\bar{c}_{IJ}^{K11} = {} & \sin^2(\theta)\left(\hat{a}_{IJ}^{K11}\cos^2(\phi) + (\hat{a}_{IJ}^{K12} + \hat{a}_{IJ}^{K21})\sin(\phi)\cos(\phi) + \hat{a}_{IJ}^{K22}\sin^2(\phi)\right) \\ & + \sin(\theta)\cos(\theta)((\hat{a}_{IJ}^{K13} + \hat{a}_{IJ}^{K31})\cos(\phi) + (\hat{a}_{IJ}^{K23} + \hat{a}_{IJ}^{K32})\sin(\phi)) + \hat{a}_{IJ}^{K33}\cos^2(\theta).\end{aligned} \tag{4.2.18}$$

Furthermore, with the help of (4.2.12) and (4.2.13), we find via a straightforward calculation that the

$$\bar{b}_{IJ}^{K} = \bar{a}_{IJ}^{K00} - \bar{a}_{IJ}^{K01} - \bar{a}_{IJ}^{K10} + \bar{a}_{IJ}^{K11} \tag{4.2.19}$$

which were defined in (1.7.1), can be expressed in terms of Cartesian coordinates as

$$\begin{aligned}\bar{b}_{IJ}^{K} = {} & \hat{a}_{IJ}^{K00} - \sin(\theta)(\hat{a}_{IJ}^{K01}\cos(\phi) + \hat{a}_{IJ}^{K02}\sin(\phi)) - \hat{a}_{IJ}^{K03}\cos(\theta) - \sin(\theta)(\hat{a}_{IJ}^{K10}\cos(\phi) + \hat{a}_{IJ}^{K20}\sin(\phi)) \\ & + \sin^2(\theta)\left(\hat{a}_{IJ}^{K11}\cos^2(\phi) + (\hat{a}_{IJ}^{K12} + \hat{a}_{IJ}^{K21})\sin(\phi)\cos(\phi) + \hat{a}_{IJ}^{K22}\sin^2(\phi)\right) \\ & + \sin(\theta)\cos(\theta)((\hat{a}_{IJ}^{K13} + \hat{a}_{IJ}^{K31})\cos(\phi) + (\hat{a}_{IJ}^{K23} + \hat{a}_{IJ}^{K32})\sin(\phi)) - \hat{a}_{IJ}^{K30}\cos(\theta) + \hat{a}_{IJ}^{K33}\cos^2(\theta).\end{aligned} \tag{4.2.20}$$

4.2.2 Expansion formulas for the tensor components $\tilde{a}_{IJ}^{K\alpha\beta}$

We now turn to deriving expansion formulas for the tensor components $\tilde{a}_{IJ}^{K\alpha\beta}$, defined by (4.2.7), that will determine their behaviour in the limit $t \searrow 0$. These results are essential for writing an explicit expression of the conformal wave equations (4.2.10) as well as the transformation into Fuchsian form, which we carry out in the following section. Furthermore the $\tilde{b}_{IJ}^{K}$ smooth functions on $\mathbb{S}^2$ play a crucial role in the identification of the quadratic terms with the most singular behaviour.

Now, from (4.1.2) and (4.2.7), we find, after a routine calculation, that

$$\tilde{a}_{IJ}^{K00} = 4t^2r^2\tilde{b}_{IJ}^{K} + t^3r^2\tilde{c}_{IJ}^{K00}, \tag{4.2.21}$$
$$\tilde{a}_{IJ}^{K01} = -4tr^3\tilde{b}_{IJ}^{K} + t^2r^3\tilde{c}_{IJ}^{K01}, \tag{4.2.22}$$

$$\tilde{a}_{IJ}^{K10} = -4tr^3\tilde{b}_{IJ}^{K} + t^2r^3\tilde{c}_{IJ}^{K10}, \tag{4.2.23}$$

$$\tilde{a}_{IJ}^{K11} = 4r^4(1-2t)\tilde{b}_{IJ}^{K} + t^2r^4\tilde{c}_{IJ}^{K11}, \tag{4.2.24}$$

$$\tilde{a}_{IJ}^{K0\Lambda} = -2tr(\bar{a}_{IJ}^{K0\Lambda} - \bar{a}_{IJ}^{K1\Lambda})\circ\psi^{-1} + t^2r(\bar{a}_{IJ}^{K0\Lambda} - 3\bar{a}_{IJ}^{K1\Lambda})\circ\psi^{-1} + t^3r\bar{a}_{IJ}^{K1\Lambda}\circ\psi^{-1}, \tag{4.2.25}$$

$$\tilde{a}_{IJ}^{K\Sigma0} = -2tr(\bar{a}_{IJ}^{K\Sigma0} - \bar{a}_{IJ}^{K\Lambda1})\circ\psi^{-1} + t^2r(\bar{a}_{IJ}^{K\Sigma0} - 3\bar{a}_{IJ}^{K\Sigma1})\circ\psi^{-1} + t^3r\bar{a}_{IJ}^{K\Sigma1}\circ\psi^{-1}, \tag{4.2.26}$$

$$\tilde{a}_{IJ}^{K1\Lambda} = 2r^2(\bar{a}_{IJ}^{K0\Lambda} - \bar{a}_{IJ}^{K1\Lambda})\circ\psi^{-1} - 2tr^2(\bar{a}_{IJ}^{K0\Lambda} - \bar{a}_{IJ}^{K1\Lambda})\circ\psi^{-1} - t^2r^2\bar{a}_{IJ}^{K1\Lambda}\circ\psi^{-1}, \tag{4.2.27}$$

$$\tilde{a}_{IJ}^{K\Sigma1} = 2r^2(\bar{a}_{IJ}^{K\Sigma0} - \bar{a}_{IJ}^{K\Sigma1})\circ\psi^{-1} - 2tr^2(\bar{a}_{IJ}^{K\Sigma0} - \bar{a}_{IJ}^{K\Sigma1})\circ\psi^{-1} - t^2r^2\bar{a}_{IJ}^{K\Sigma1}\circ\psi^{-1} \tag{4.2.28}$$

and

$$\tilde{a}_{IJ}^{K\Sigma\Lambda} = \bar{a}_{IJ}^{K\Sigma\Lambda}\circ\psi^{-1}, \tag{4.2.29}$$

where

$$\tilde{b}_{IJ}^{K} = (\bar{a}_{IJ}^{K00} - \bar{a}_{IJ}^{K01} - \bar{a}_{IJ}^{K10} + \bar{a}_{IJ}^{K11})\circ\psi^{-1}, \tag{4.2.30}$$

$$\begin{aligned}\tilde{c}_{IJ}^{K00} =& -4\left((\bar{a}_{IJ}^{K00} - 2\bar{a}_{IJ}^{K01} - 2\bar{a}_{IJ}^{K10} + 3\bar{a}_{IJ}^{K11})\right)\circ\psi^{-1} + t(\bar{a}_{IJ}^{K00} - 5\bar{a}_{IJ}^{K01} - 5\bar{a}_{IJ}^{K10} + 13\bar{a}_{IJ}^{K11})\circ\psi^{-1}\\ &+ t^2(\bar{a}_{IJ}^{K01} + \bar{a}_{IJ}^{K10} - 6\bar{a}_{IJ}^{K11})\circ\psi^{-1} + t^3\bar{a}_{IJ}^{K11}\circ\psi^{-1}\end{aligned}$$

$$\begin{aligned}\tilde{c}_{IJ}^{K01} =& 2(3\bar{a}_{IJ}^{K00} - 3\bar{a}_{IJ}^{K01} - 5\bar{a}_{IJ}^{K10} + 5\bar{a}_{IJ}^{K11})\circ\psi^{-1} - 2t\left((\bar{a}_{IJ}^{K00} - 2\bar{a}_{IJ}^{K01} - 4\bar{a}_{IJ}^{K10} + 5\bar{a}_{IJ}^{K11})\right)\circ\psi^{-1}\\ &- t^2(\bar{a}_{IJ}^{K01} + 2\bar{a}_{IJ}^{K10} - 5\bar{a}_{IJ}^{K11})\circ\psi^{-1} - t^3\bar{a}_{IJ}^{K11}\circ\psi^{-1},\end{aligned}$$

$$\begin{aligned}\tilde{c}_{IJ}^{K10} =& 2(3\bar{a}_{IJ}^{K00} - 5\bar{a}_{IJ}^{K01} - 3\bar{a}_{IJ}^{K10} + 5\bar{a}_{IJ}^{K11})\circ\psi^{-1} - 2t\left((\bar{a}_{IJ}^{K00} - 4\bar{a}_{IJ}^{K01} - 2\bar{a}_{IJ}^{K10} + 5\bar{a}_{IJ}^{K11})\right)\circ\psi^{-1}\\ &- t^2(2\bar{a}_{IJ}^{K01} + \bar{a}_{IJ}^{K10} - 5\bar{a}_{IJ}^{K11})\circ\psi^{-1} - t^3\bar{a}_{IJ}^{K11}\circ\psi^{-1}\end{aligned}$$

and

$$\begin{aligned}\tilde{c}_{IJ}^{K11} =& 2(2\bar{a}_{IJ}^{K00} - 3\bar{a}_{IJ}^{K01} - 3\bar{a}_{IJ}^{K10} + 4\bar{a}_{IJ}^{K11})\circ\psi^{-1}\\ &+ 2t(\bar{a}_{IJ}^{K01} + \bar{a}_{IJ}^{K10} - 2\bar{a}_{IJ}^{K11})\circ\psi^{-1} + t^2\bar{a}_{IJ}^{K11}\circ\psi^{-1}.\end{aligned}$$

We further observe from (4.1.2), (4.1.3), (4.2.14)-(4.2.20) and (4.2.30) that

$$\bar{a}_{IJ}^{Kpq}\circ\psi^{-1} = \bar{c}_{IJ}^{Kpq} + tr(2-t)\bar{d}_{IJ}^{Kpq} + t^2r^2(2-t)^2\bar{e}_{IJ}^{Kpq}, \tag{4.2.31}$$

$$\bar{a}_{IJ}^{K\alpha\Lambda}\circ\psi^{-1} = tr(2-t)\bar{d}_{IJ}^{K\alpha\Lambda} + t^2r^2(2-t)^2\bar{e}_{IJ}^{K\alpha\Lambda}, \tag{4.2.32}$$

$$\bar{a}_{IJ}^{K\Sigma\beta}\circ\psi^{-1} = tr(2-t)\bar{d}_{IJ}^{K\Sigma\beta} + t^2r^2(2-t)^2\bar{e}_{IJ}^{K\Sigma\beta} \tag{4.2.33}$$

and

$$\tilde{b}^K_{IJ} = \bar{b}^K_{IJ}. \tag{4.2.34}$$

4.3 First order transformation into Fuchsian system

Now that we have an explicit expression for the push-forward of the wave equations (4.0.1) as well as the quadratic terms by the formulas (4.2.14)-(4.2.20) and (4.2.21)-(4.2.34) we can continue the transformation process. In the next two sections we proceed in a similar way as we did in the previous chapter, transforming the system into first order form and writing an extended system defined on the space-time $(0,1)\times\mathbb{T}\times\mathbb{S}^2$.

4.3.1 First order variables

We now begin the process of transforming the conformal wave equations (4.2.10) into Fuchsian form. The transformation starts by expressing the wave equation in first order form through the introduction of the variables

$$U^K_0 = t\partial_t u^K, \quad U^K_1 = t^{\frac{1}{2}} r\partial_r u^K, \quad U^K_\Lambda = t^{\frac{1}{2}} \not{\nabla}_\Lambda u^K \quad \text{and} \quad U^K_4 = t^{\frac{1}{2}} u^K. \tag{4.3.1}$$

A short calculation then shows that (4.2.10), when expressed in terms of these variables, becomes

$$(2-t)\partial_t U^K_0 - \frac{2(1-t)}{t} r\partial_r U^K_0 - \frac{1}{t^{\frac{1}{2}}} r\partial_r U^K_1 - \frac{1}{t^{\frac{1}{2}}} \not{g}^{\Lambda\Sigma} \not{\nabla}_\Lambda U^K_\Sigma = -\frac{1}{t^{\frac{1}{2}}} U^K_1 + U^K_0 - f^K, \tag{4.3.2}$$

while the evolution equations for the variables U^K_1, U^K_Λ and U^K_4 are easily computed to be

$$\partial_t U^K_1 = \frac{1}{t^{\frac{1}{2}}} r\partial_r U^K_0 + \frac{1}{2t} U^K_1, \quad \partial_t U^K_\Lambda = \frac{1}{t^{\frac{1}{2}}} \not{\nabla}_\Lambda U^K_0 + \frac{1}{2t} U^K_\Lambda \quad \text{and} \quad \partial_t U^K_4 = \frac{1}{2t} U^K_4 + \frac{1}{t^{\frac{1}{2}}} U^K_0, \tag{4.3.3}$$

respectively. It is worthwhile noting that system (4.3.2)-(4.3.3) is in symmetric hyperbolic form.

To proceed, we use the first order variables (4.3.1) to write $\nabla_\mu u^I$ as

$$\nabla_\mu u^I = t^{-\frac{1}{2}}\big(t^{-\frac{1}{2}} U^I_0 \delta^0_\mu + r^{-1} U^I_1 \delta^1_\mu + U^I_\Lambda \delta^\Lambda_\mu\big).$$

Using this, we then observe that the three main groups of terms from (4.2.10) can be expressed in terms of the first order variables as

$$-\frac{1}{rt(2-t)}\tilde{a}^{K\mu\nu}_{IJ}\nabla_\mu u^I\nabla_\nu u^J = -\frac{1}{2-t}r^{-1}t^{-2}\frac{1}{t}\biggl[\biggl(\tilde{a}^{K00}_{(IJ)}U^I_0U^J_0 + \frac{t^{\frac{1}{2}}}{r}(\tilde{a}^{K01}_{IJ}+\tilde{a}^{K10}_{JI})U^I_0U^J_1 + \frac{t}{r^2}\tilde{a}^{K11}_{(IJ)}U^I_1U^J_1\biggr) + t^{\frac{1}{2}}(\tilde{a}^{K0\Lambda}_{JI}+\tilde{a}^{K\Lambda 0}_{IJ})U^I_\Lambda U^J_0 + \frac{t}{r}(\tilde{a}^{K1\Lambda}_{JI}+\tilde{a}^{K\Lambda 1}_{IJ})U^I_\Lambda U^J_1 + t\tilde{a}^{K\Lambda\Sigma}_{IJ}U^I_\Lambda U^J_\Sigma\biggr],$$

$$-\frac{1}{(rt(2-t))^2}\tilde{a}^{K\mu\nu}_{IJ}(\nabla_\mu(rt(2-t))u^I\nabla_\nu u^J + \nabla_\mu u^I\nabla_\nu(rt(2-t))u^J)$$
$$= -\frac{1}{(2-t)^2}r^{-2}t^{-2}\frac{1}{t}\biggl[\frac{1}{t^{\frac{1}{2}}}\biggl(\bigl(2r(1-t)\tilde{a}^{K00}_{IJ}+t(2-t)\tilde{a}^{K10}_{IJ}\bigr)U^I_4U^J_0 + 2t^{\frac{1}{2}}r(1-t)\tilde{a}^{K0\Sigma}_{IJ}U^I_4U^J_\Sigma + \biggl(\frac{t^{\frac{3}{2}}(2-t)\tilde{a}^{K11}_{IJ}}{r}+2t^{\frac{1}{2}}(1-t)\tilde{a}^{K01}_{IJ}\biggr)U^I_4U^J_1 + t^{\frac{3}{2}}(2-t)\tilde{a}^{K1\Sigma}_{IJ}U^I_4U^J_\Sigma\biggr) + \frac{1}{t^{\frac{1}{2}}}\biggl(\bigl(2r(1-t)\tilde{a}^{K00}_{IJ}+t(2-t)\tilde{a}^{K01}_{IJ}\bigr)U^I_0U^J_4 + 2t^{\frac{1}{2}}r(1-t)\tilde{a}^{K\Lambda 0}_{IJ}U^I_\Lambda U^J_4 + \biggl(\frac{t^{\frac{3}{2}}(2-t)\tilde{a}^{K11}_{IJ}}{r}+2t^{\frac{1}{2}}(1-t)\tilde{a}^{K10}_{IJ}\biggr)U^I_1U^J_4 + t^{\frac{3}{2}}(2-t)\tilde{a}^{K\Lambda 1}_{IJ}U^I_\Lambda U^J_4\biggr)\biggr]$$

and

$$-\frac{1}{(rt(2-t))^3}\tilde{a}^{K\mu\nu}_{IJ}\nabla_\mu(rt(2-t))\nabla_\nu(rt(2-t))u^Iu^J = -\frac{1}{(2-t)^3}r^{-3}t^{-3}\frac{1}{t}\bigl[4r^2(1-t)^2\tilde{a}^{K00}_{(IJ)} + 2tr(1-t)(2-t)(\tilde{a}^{K01}_{(IJ)}+\tilde{a}^{K10}_{(IJ)}) + t^2(2-t)^2\tilde{a}^{K11}_{(IJ)}\bigr]U^I_4U^J_4.$$

With the help of these results, it is then not difficult to verify, using (4.2.20), (4.2.21)-(4.2.29) and (4.2.31)-(4.2.34), that the nonlinear term (4.2.10)becomes

$$-f^K = -\frac{1}{t}2r\bar{b}^K_{IJ}V^I_0V^J_0 + \frac{1}{t}\Bigl[rf^{K00}_{IJ}t^{\frac{1}{2}}U^I_0t^{\frac{1}{2}}U^J_0 + rf^{K01}_{IJ}t^{\frac{1}{2}}U^I_0U^J_1 + rf^{K11}_{IJ}U^I_1U^J_1 + f^{K0\Lambda}_{IJ}t^{\frac{1}{2}}U^I_0U^J_\Lambda + f^{K1\Lambda}_{IJ}U^I_1U^J_\Lambda + f^{K\Sigma\Lambda}_{IJ}U^I_\Sigma U^J_\Lambda + rg^{K0}_{IJ}t^{\frac{1}{2}}U^I_0U^J_4 + rg^{K1}_{IJ}U^I_1U^J_4 + g^{K\Lambda}_{IJ}t^{\frac{1}{2}}U^I_\Lambda U^J_4 + rh^K_{IJ}U^I_4U^J_4 \tag{4.3.4}$$

when written in terms of the first order variables, where $\{f^{Kpq}_{IJ}(t,r), g^{Kp}_{IJ}(t,r), h^K_{IJ}(t,r)\}$, $\{f^{Kp\Lambda}_{IJ}(t,r), g^{K\Lambda}_{IJ}(t,r)\}$ and $\{f^{K\Sigma\Lambda}_{IJ}(t,r)\}$ are collections of smooth scalar, vector, and (2,0)-tensor fields, respectively, on $\mathbb{S}^2$ that depend smoothly on $(t,r)\in\mathbb{R}\times\mathbb{R}$, and we have set

$$V^K_0 = U^K_0 - \frac{1}{t^{\frac{1}{2}}}U^K_1. \tag{4.3.5}$$

The expansion (4.3.4) motivates us to replace the first order variable U^K_0 with V^K_0. Doing so, we see via a routine computation involving (4.3.2) and (4.3.3) that V^K_0 evolves according to

$$(2-t)\partial_tV^K_0 + r\partial_rV^K_0 - \frac{1}{t^{\frac{1}{2}}}g^{\Lambda\Sigma}\nabla_\Lambda U^K_\Sigma = V^K_0 - f^K. \tag{4.3.6}$$

One difficulty with this change of variables is the system of evolution equations (4.3.3) and (4.3.6) for the first order variables V_0^K , U_1^K, U_Λ^K and U_4^K is no longer symmetric hyperbolic. To restore the symmetry, we use the identity $\nabla\!\!\!/_\Lambda U_1 = r\partial_r U_\Lambda$ to write (4.3.3) as

$$\partial_t U_1^K = \frac{1}{t} r\partial_r U_1^K + \frac{1}{t^{\frac{1}{2}}} r\partial_r V_0^K + \frac{1}{2t} U_1^K, \tag{4.3.7}$$

$$\partial_t U_\Lambda^K = -\frac{\mathtt{q}}{t} r\partial_r U_\Lambda^K + \frac{1}{t^{\frac{1}{2}}} \nabla\!\!\!/_\Lambda U_0^K + \frac{\mathtt{q}+1}{t} \nabla\!\!\!/_\Lambda U_1^K + \frac{1}{2t} U_\Lambda^K, \tag{4.3.8}$$

$$\partial_t U_4^K = \frac{1}{2t} U_4 + \frac{1}{t} U_1^K + \frac{1}{t^{\frac{1}{2}}} V_0^K, \tag{4.3.9}$$

where $\mathtt{q}$ is a function of t that we will fix below. Now we propose a change of variable of the form

$$U_1^K = \alpha V_1^K + G(t) V_0^K, \tag{4.3.10}$$

where α is a constant and $G(t)$ is a function of t that we fix below from the symmetric hyperbolic condition and that we simply write as $G(t) = G$. Taking the first time derivative of (4.3.10) and using (4.3.7) we get

$$\begin{aligned}
\alpha\partial_t V_1^K =& - \partial_t G V_0^K - G\partial_t V_0^K + \frac{1}{t^{\frac{1}{2}}} r\partial_r V_0^K + \frac{G}{t} r\partial_r V_0^K + \frac{\alpha}{t} r\partial_r V_1^K + \frac{1}{2t}\left(\alpha V_1^K + G V_0^K\right), \\
(2-t)\alpha\partial_t V_1^K =& (2-t)\left[-G\partial_t V_0^K + \left(\frac{1}{t^{\frac{1}{2}}} + \frac{G}{t}\right) r\partial_r V_0^K + \frac{\alpha}{t} r\partial_r V_1^K + \left(\frac{G}{2t} - \partial_t G\right) V_0^K + \frac{\alpha}{2t} V_1^K\right],
\end{aligned} \tag{4.3.11}$$

multiplying equation (4.3.6) times G we can write

$$-(2-t)G\partial_t V_0^K = G\left[r\partial_r V_0^K - \frac{1}{t^{\frac{1}{2}}} \not{g}^{\Lambda\Sigma} \nabla\!\!\!/_\Lambda U_\Sigma^K - V_0^K + f^K\right], \tag{4.3.12}$$

substituting (4.3.12) into (4.3.11) yields the equation

$$\begin{aligned}
\alpha(2-t)\partial_t V_1^K =& \left(G + \frac{(2-t)}{t^{\frac{1}{2}}} + \frac{(2-t)G}{t}\right) r\partial_r V_0^K + \frac{\alpha(2-t)}{t} r\partial_r V_1^K - \frac{G}{t^{\frac{1}{2}}} \not{g}^{\Lambda\Sigma} \nabla\!\!\!/_\Lambda U_\Sigma^K + \\
& \left(\frac{(2-t)G}{2t} - (2-t)\partial_t G - G\right) V_0^K + \frac{\alpha}{2t} V_1^K + G f^K
\end{aligned} \tag{4.3.13}$$

since we want the system to be symmetric we have to impose the condition

$$\left(G + \frac{(2-t)}{t^{\frac{1}{2}}} + \frac{(2-t)G}{t}\right) = 0, \quad \text{which implies} \quad G(t) = -\frac{t^{\frac{1}{2}}(2-t)}{2}, \tag{4.3.14}$$

substituting (4.3.14) into (4.3.13) yields

$$(2-t)\partial_t V_1^K = \frac{(2-t)}{t} r\partial_r V_1^K + \frac{(2-t)}{2\alpha} \not{g}^{\Lambda\Sigma} \not{\nabla}_\Lambda U_\Sigma^K + \frac{(2-t)}{2t} V_1^K - \frac{(2-t)t^{\frac{1}{2}}}{2\alpha} f^K. \tag{4.3.15}$$

Then, we propose a change of variable of the form

$$U_\Lambda^K = \frac{V_\Lambda^K}{\mathtt{p}}, \tag{4.3.16}$$

the evolution equation for U_Λ^K can be written as

$$\begin{aligned}(2-t)\partial_t V_\Lambda^K =&(2-t)\left[\left(\frac{\partial_t \mathtt{p}}{\mathtt{p}} + \frac{1}{2t}\right) V_\Lambda^K - \frac{\mathtt{q}}{t} r\partial_r V_\Lambda^K + \mathtt{p}\left(\frac{1}{t^{\frac{1}{2}}} - \frac{(\mathtt{q}+1)(2-t)}{2t^{\frac{1}{2}}}\right)\not{\nabla}_\Lambda V_0^K + \right.\\ &\left.\frac{\alpha\mathtt{p}(\mathtt{q}+1)}{t}\not{\nabla}_\Lambda V_1^K\right].\end{aligned} \tag{4.3.17}$$

Similarly to (4.3.14) we want the system to be symmetric, therefore we have to set the conditions

$$(2-t)\mathtt{p}\left(\frac{1}{t^{\frac{1}{2}}} - \frac{(\mathtt{q}+1)(2-t)}{2t^{\frac{1}{2}}}\right) = \frac{1}{t^{\frac{1}{2}}\mathtt{p}} \quad \text{and} \quad \frac{(2-t)}{2\alpha\mathtt{p}} = \frac{\alpha\mathtt{p}(\mathtt{q}+1)(2-t)}{t}, \tag{4.3.18}$$

this implies

$$(2-t)\left(1 - \frac{(\mathtt{q}+1)(2-t)}{2}\right) = \frac{1}{\mathtt{p}^2} \quad \text{and} \quad \frac{1}{\mathtt{p}^2} = \frac{2\alpha^2(\mathtt{q}+1)}{t}, \tag{4.3.19}$$

for simplicity we choose $\alpha = \frac{1}{2}$, equating both expressions in (4.3.19) and solving for $\mathtt{q}$ we get

$$\mathtt{q} = -\frac{-1+2t^2-t^3}{1+4t-4t^2+t^3}, \tag{4.3.20}$$

substituting (4.3.20) into either expression in (4.3.19) we solve for $\mathtt{p}$

$$\mathtt{p} = \sqrt{\frac{1+4t-4t^2+t^3}{2-t}}, \tag{4.3.21}$$

using this value of $\mathtt{p}$ we write the term

$$\frac{(2-t)\partial_t \mathtt{p}}{\mathtt{p}} = \frac{-2t^3+10t^2-16t+9}{2(t^3-4t^2+4t+1)}. \tag{4.3.22}$$

Now, with the help of the functions $\mathtt{p}, \mathtt{q}$ and (4.3.22) we see that the equation (4.3.17) can be written as

$$\begin{aligned}(2-t)\partial_t V^K_\Lambda =&\frac{1}{t^{\frac{1}{2}}\mathtt{p}}\nabla\!\!\!\!/_\Lambda V^K_0+\frac{(2-t)}{\mathtt{p}}\nabla\!\!\!\!/_\Lambda V^K_1-\frac{(2-t)\mathtt{q}}{t}r\partial_r V^K_\Lambda+\\ &\left(\frac{-2t^3+10t^2-16t+9}{2(t^3-4t^2+4t+1)}+\frac{(2-t)}{2t}\right)V^K_\Lambda.\end{aligned} \tag{4.3.23}$$

Setting $U^K_4=V^K_4$ and using the change of variables (4.3.5), (4.3.10) we write the evolution equation (4.3.9) for U^K_4 in the form

$$\partial_t V^K_4=\frac{t^{\frac{1}{2}}}{2}V^K_0+\frac{1}{2t}V^K_1+\frac{1}{2t}V_4. \tag{4.3.24}$$

Summarising, we can write the system (4.3.2), (4.3.3) as

$$(2-t)\partial_t V^K_0+r\partial_r V^K_0-\frac{1}{t^{\frac{1}{2}}p}\not{g}^{\Lambda\Sigma}\nabla\!\!\!\!/_\Lambda V^K_\Sigma=V^K_0-f^K, \tag{4.3.25}$$

$$\begin{aligned}(2-t)\partial_t V^K_1 =&(2-t)\left[\frac{1}{t}r\partial_r V^K_1+\frac{1}{p}\not{g}^{\Lambda\Sigma}\nabla\!\!\!\!/_\Lambda V^K_\Sigma+\frac{1}{2t}V^K_1-t^{\frac{1}{2}}f^K\right]\\ (2-t)\partial_t V_\Lambda =&\frac{1}{t^{\frac{1}{2}}p}\nabla\!\!\!\!/_\Lambda V^K_0+\frac{(2-t)}{p}\nabla\!\!\!\!/_\Lambda V^K_1-\frac{(2-t)q}{t}r\partial_r V^K_\Lambda+\\ &\left(\frac{-2t^3+10t^2-16t+9}{2(t^3-4t^2+4t+1)}+\frac{(2-t)}{2t}\right)V^K_\Lambda\\ \partial_t V^K_4 =&\frac{t^{\frac{1}{2}}}{2}V^K_0+\frac{1}{2t}V^K_1+\frac{1}{2t}V_4.\end{aligned} \tag{4.3.26}$$

Recall that the change of variables that we used are given by

$$V^K_1=2U^K_1+(2-t)t^{\frac{1}{2}}V^K_0,\quad V^K_\Lambda=\mathtt{p}U^K_\Lambda\quad\text{and}\quad V^K_4=U^K_4, \tag{4.3.27}$$

and observe that the evolution equations (4.3.6)-(4.3.9) can be expressed in terms of the variables V^K_0, V^K_1, V^K_Λ and V^K_4 in the following symmetric hyperbolic form:

$$B^0\partial_t V^K+\frac{1}{t}B^1 r\partial_r V^K+\frac{1}{t^{\frac{1}{2}}}B^\Sigma\nabla\!\!\!\!/_\Sigma V^K=\frac{1}{t}\mathcal{B}\mathbb{P}V^K+\mathcal{C}V^K+F^K \tag{4.3.28}$$

where

$$V^K=(V^K_{\mathcal{I}})=\begin{pmatrix}V^K_0 & V^K_1 & V^K_\Lambda & V^K_4\end{pmatrix}^{\mathrm{tr}}, \tag{4.3.29}$$

$$B^0 = \begin{pmatrix} 2-t & 0 & 0 & 0 \\ 0 & 2-t & 0 & 0 \\ 0 & 0 & (2-t)\delta^\Lambda_\Omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.3.30}$$

$$B^1 = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & -(2-t) & 0 & 0 \\ 0 & 0 & (2-t)\mathtt{q}\delta^\Lambda_\Omega & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.3.31}$$

$$B^\Sigma = \begin{pmatrix} 0 & 0 & -\frac{1}{\mathtt{p}}\slashed{g}^{\Sigma\Lambda} & 0 \\ 0 & 0 & -\frac{(2-t)t^{\frac{1}{2}}}{\mathtt{p}}\slashed{g}^{\Sigma\Lambda} & 0 \\ -\frac{1}{\mathtt{p}}\delta^\Sigma_\Omega & -\frac{(2-t)t^{\frac{1}{2}}}{\mathtt{p}}\delta^\Sigma_\Omega & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.3.32}$$

$$\mathcal{B} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & \frac{2-t}{2} & 0 & 0 \\ 0 & 0 & \frac{2-t}{2}\slashed{g}^\Sigma_\Omega & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}, \tag{4.3.33}$$

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{9-16t+10t^2-2t^3}{2(1+4t-4t^2+t^3)}\delta^\Lambda_\Omega & 0 \\ \frac{1}{2}t^{\frac{1}{2}} & 0 & 0 & 0 \end{pmatrix}, \tag{4.3.34}$$

$$\mathbb{P} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \delta^\Lambda_\Sigma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.3.35}$$

and

$$F^K = \begin{pmatrix} -f^K & -(2-t)t^{\frac{1}{2}}f^K & 0 & 0 \end{pmatrix}^{\mathrm{tr}}. \tag{4.3.36}$$

Now, from the definitions (4.3.30), (4.3.31), (4.3.33) and (4.3.35), it is not difficult to verify that $\mathbb{P}$ is a covariantly constant, time-independent, symmetric projection operator that commutes with B^0, B^1 and $\mathcal{B}$, that is,

$$\mathbb{P}^2 = \mathbb{P}, \quad \mathbb{P}^{\mathrm{tr}} = \mathbb{P}, \qquad \partial_t\mathbb{P} = 0, \quad \partial_r\mathbb{P} = 0, \quad \text{and} \quad \slashed{\nabla}_\Lambda\mathbb{P} = 0 \tag{4.3.37}$$

and

$$[B^0, \mathbb{P}] = [B^1, \mathbb{P}] = [\mathcal{B}, \mathbb{P}] = 0, \tag{4.3.38}$$

where the symmetry is with respect to the inner-product

$$h(Y, Z) = \delta^{pq} Y_p Z_q + \not{g}^{\Sigma\Lambda} Y_\Lambda Z_\Sigma + Y_4 Z_4. \tag{4.3.39}$$

Furthermore, it is also not difficult to verify that B^0 and B^1 and $B^\Sigma \eta_\Sigma$ are symmetric with respect to (4.3.39) and that B^0 satisfies

$$h(Y, Y) \le h(Y, B^0 Y) \tag{4.3.40}$$

for all $Y = (Y_\mathcal{I})$ and $0 < t \le 1$, which in particular, implies that the system (4.3.28) is symmetric hyperbolic.

Using (4.3.33) and (4.3.39), we observe, with the help of Young's inequality (i.e. $|ab| \le \frac{\epsilon}{2} a^2 + \frac{1}{2\epsilon} b^2$), that

$$\begin{aligned} h(Y, \mathcal{B}Y) &= 2Y_0^2 + \frac{2-t}{2} Y_1^2 + Y_1 Y_4 + \frac{2-t}{2} \not{g}^{\Lambda\Sigma} Y_\Lambda Y_\Sigma + \frac{1}{2} Y_4^2 \\ &\ge 2Y_0^2 + \frac{2-t-\epsilon}{2} Y_1^2 + \frac{2-t}{2} \not{g}^{\Lambda\Sigma} Y_\Lambda Y_\Sigma + \frac{1}{2}\left(1 - \frac{1}{\epsilon}\right) Y_4^2. \end{aligned}$$

Choosing $\epsilon = \frac{1}{2}\big(1 - t - \sqrt{5 - 2t + t^2}\big)$, we then have

$$\begin{aligned} h(Y, \mathcal{B}Y) &\ge 2Y_0^2 + \frac{1}{4}\big(3 - t + \sqrt{5 - 2t + t^2}\big) Y_1^2 + \frac{2-t}{2} \not{g}^{\Lambda\Sigma} Y_\Lambda Y_\Sigma + \frac{1}{4}\big(3 - t + \sqrt{5 - 2t + t^2}\big) Y_4^2 \\ &\ge 2Y_0^2 + \frac{2-t}{2}\big(Y_1^2 + \not{g}^{\Lambda\Sigma} Y_\Lambda Y_\Sigma + Y_4^2\big) \\ &\ge \frac{1}{2}\Big((2-t)Y_0^2 + (2-t)Y_1^2 + (2-t)\not{g}^{\Lambda\Sigma} Y_\Lambda Y_\Sigma + Y_4^2\Big), \end{aligned}$$

which together with (4.3.30) and (4.3.39) allows us to conclude that

$$h(Y, B^0 Y) \le 2h(Y, \mathcal{B}Y) \tag{4.3.41}$$

for all $Y = (Y_\mathcal{I})$ and $0 < t \le 1$.

Next, from (4.3.5) and (4.3.27), we get

$$t^{\frac{1}{2}} U_0^K = \frac{1}{2}\big(V_1^K + t V_0^K\big), \quad U_1^K = \frac{1}{2}\big(V_1^K - (2-t) t^{\frac{1}{2}} V_0^K\big), \quad U_\Lambda^K = \frac{1}{\mathrm{p}} V_\Lambda^K \quad \text{and} \quad U_4^K = V_4^K. \tag{4.3.42}$$

Using these along with (4.3.4) and (4.3.35) allows us to expand (4.3.36) as

$$F^K = -\frac{2}{t} \bar{b}_{IJ}^K r V_0^I V_0^J \mathbf{e}_0 + G^K \tag{4.3.43}$$

where

$$G^K = G_0^K(t^{\frac{1}{2}}, t, r, V, V) + \frac{1}{t^{\frac{1}{2}}} G_1^K(t^{\frac{1}{2}}, t, r, V, \mathbb{P}V) + \frac{1}{t} G_2^K(t^{\frac{1}{2}}, t, r, \mathbb{P}V, \mathbb{P}V), \quad (4.3.44)$$

$$\mathbf{e}_0 = (\delta_{\mathcal{I}}{}^0) = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{tr}}, \quad (4.3.45)$$

$$V = (V^I) = (V_{\mathcal{I}}^I), \quad (4.3.46)$$

and the $G_a^K(\tau, t, r, Y, Z)$, $a = 0, 1, 2$, are smooth bilinear maps with G_2^K satisfying

$$\mathbb{P}G_2^K = 0. \quad (4.3.47)$$

Remark 4.3.1. Here, we are using the term *smooth bilinear map* to mean a map of the form

$$H^K(\tau, t, r, Y, Z) = H_{IJ}^{Kpq}(\tau, t, r) Y_p^I Z_p^J + H_{IJ}^{Kp\Lambda}(\tau, t, r) Y_p^I Z_\Lambda^J + H_{IJ}^{K\Sigma\Lambda}(\tau, t, r) Y_\Sigma^I Z_\Lambda^J$$

where $H_{IJ}^{Kpq}(\tau, t, r)$, $H_{IJ}^{Kpq}(\tau, t, r)$, and $H_{IJ}^{K\Sigma\Lambda}(\tau, t, r)$ are collections of smooth scalar, vector, and (2,0)-tensor fields on $\mathbb{S}^2$ that depend smoothly on the parameters $(\tau, t, r) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

For the subsequent analysis, it will be advantageous to introduce a change of radial coordinate via

$$r = \rho^m, \quad m \in \mathbb{Z}_{\geq 1}. \quad (4.3.48)$$

Using the transformation law $r\partial_r = r\frac{d\rho}{dr}\partial_\rho = \frac{\rho}{m}\partial_\rho$, we can express the system (4.3.28) as

$$B^0 \partial_t V^K + \frac{1}{t}\frac{\rho}{m} B^1 \partial_\rho V^K + \frac{1}{t^{\frac{1}{2}}} B^\Sigma \not{\nabla}_\Sigma V^K = \frac{1}{t}\mathcal{B}\mathbb{P}V^K + \mathcal{C}V^K + F^K \quad (4.3.49)$$

where now

$$F^K = -\frac{2}{t}\bar{b}_{IJ}^K \rho^m V_0^I V_0^J \mathbf{e}_0 + G^K \quad (4.3.50)$$

and

$$G^K = G_0^K(t^{\frac{1}{2}}, t, \rho^m, V, V) + \frac{1}{t^{\frac{1}{2}}} G_1^K(t^{\frac{1}{2}}, t, \rho^m, V, \mathbb{P}V) + \frac{1}{t} G_2^K(t^{\frac{1}{2}}, t, \rho^m, \mathbb{P}V, \mathbb{P}V). \quad (4.3.51)$$

It is also clear that the neighborhood of infinity M_{r_0} and the initial data hypersurface

Σ_{r_0}, see (4.1.5) and (4.1.6), can be expressed in terms of ρ as

$$M_{r_0} = \left\{(t,\rho) \in (1,0) \times (0,\rho_0) \,\middle|\, t > 2 - \rho_0^m/\rho^m\right\} \times \mathbb{S}^2, \qquad \rho_0 = (r_0)^{\frac{1}{m}}, \tag{4.3.52}$$

and

$$\Sigma_{r_0} = \{1\} \times (0,\rho_0) \times \mathbb{S}^2, \tag{4.3.53}$$

respectively.

4.3.2 The extended system

Rather than solving (4.3.49) on M_{r_0}, we will instead solve an extended version of this system on the extended spacetime $(0,1) \times \mathcal{S}$ where

$$\mathcal{S} = \mathbb{T} \times \mathbb{S}^2$$

and $\mathbb{T}$ is the 1-dimensional torus obtained from identifying the end points of the interval $[-3\rho_0, 3\rho_0]$. Initial data will be prescribed on the hypersurface $\{1\} \times \mathcal{S}$.

To define the extended system, we let $\hat{\chi}(\rho)$ denote a smooth cut-off function satisfying $\hat{\chi} \geq 0$, $\hat{\chi}|_{[-1,1]} = 1$ and $\operatorname{supp}(\hat{\chi}) \subset (-2,2)$, and use it to define the smooth cut-off function

$$\chi(\rho) = \hat{\chi}\big(\rho/\rho_0\big)$$

on $\mathbb{T}$, which is easily seen to satisfy $\chi \geq 0$, $\chi|_{[-\rho_0,\rho_0]} = 1$ and $\operatorname{supp}(\chi) \subset (-2\rho_0, 2\rho_0)$. With the help of this cut-off function, we then define the *extended system* by

$$B^0\partial_t V^K + \frac{1}{t}\frac{\chi\rho}{m}B^1\partial_\rho V^K + \frac{1}{t^{\frac{1}{2}}}B^\Sigma \slashed{\nabla}_\Sigma V^K = \frac{1}{t}\mathcal{B}\mathbb{P}V^K + \mathcal{C}V^K + \mathcal{F}^K \tag{4.3.54}$$

where

$$\mathcal{F}^K = \frac{1}{t}Q^K \mathbf{e}_0 + \mathcal{G}^K, \tag{4.3.55}$$

$$Q^K = -2\bar{b}^K_{IJ}\chi(\rho)\rho^m V_0^I V_0^J, \tag{4.3.56}$$

$$\mathcal{G}^K = \mathcal{G}_0 + \frac{1}{t^{\frac{1}{2}}}\mathcal{G}_1 + \frac{1}{t}\mathcal{G}_2, \tag{4.3.57}$$

$$\mathcal{G}_0^K = G_0^K(t^{\frac{1}{2}}, t, \chi(\rho)\rho^m, V, V), \tag{4.3.58}$$

$$\mathcal{G}_1^K = G_1^K(t^{\frac{1}{2}}, t, \chi(\rho)\rho^m, V, \mathbb{P}V), \tag{4.3.59}$$

$$\mathcal{G}_2^K = G_2^K(t^{\frac{1}{2}}, t, \chi(\rho)\rho^m, \mathbb{P}V, \mathbb{P}V) \tag{4.3.60}$$

and

$$\mathbb{P}\mathcal{G}_2^K = 0. \tag{4.3.61}$$

By definition, see (4.3.29), the fields V^K are time-dependent sections of the vector bundle

$$\mathbb{V} = \bigcup_{y\in\mathcal{S}} \mathbb{V}_y$$

over $\mathcal{S}$ with fibers $\mathbb{V}_y = \mathbb{R}\times\mathbb{R}\times \mathrm{T}^*_{\mathrm{pr}(y)}\mathbb{S}^2\times\mathbb{R}$ where $\mathrm{pr} : \mathcal{S} = \mathbb{T}\times\mathbb{S}^2 \longrightarrow \mathbb{S}^2$ is the canonical projection. We further note that (4.3.39) defines an inner-product on $\mathbb{V}$, and recall that B^0, B^1 and $B^\Sigma\xi_\Sigma$ are symmetric with respect to this inner-product. The symmetry of these operators together with the lower bound (4.3.40) for B^0 imply that the extended system (4.3.54) is symmetric hyperbolic, a fact that will be essential to our arguments below.

Noting from the definition (4.3.52) that the boundary of the region M_{r_0} can be decomposed as

$$\partial M_{r_0} = \Sigma_{r_0}\cup\Sigma^+_{r_0}\cup\Gamma^-\cup\Gamma^+_{r_0}$$

where

$$\Gamma^- = [0,1]\times\{0\}\times\mathbb{S}^2, \quad \Gamma^+_{r_0} = \left\{ (t,r)\in[0,1]\times(0,\rho_0] \,\middle|\, t = 2-\frac{\rho_0^m}{\rho^m} \right\}\times\mathbb{S}^2$$

and

$$\Sigma^+_{r_0} = \{0\}\times\left(0,\frac{\rho_0}{2^{\frac{1}{m}}}\right)\times\mathbb{S}^2, \tag{4.3.62}$$

we find that $n^- = -d\rho$ and $n^+ = -dt + m\frac{\rho_0^m}{\rho^{m+1}}d\rho$ define outward pointing co-normals to Γ^- and $\Gamma^+_{r_0}$, respectively. Furthermore, we have from (4.3.30)-(4.3.32) that

$$\left(n_0^- B^0 + n_1^-\frac{\chi\rho}{m}B^1 + n_\Sigma^- B^\Sigma\right)\Big|_{\Gamma^-} = 0 \tag{4.3.63}$$

and

$$\left(n_0^+ B^0 + n_1^+\frac{\chi\rho}{m}B^1 + n_\Sigma^+ B^\Sigma\right)\Big|_{\Gamma^+_{r_0}} = \begin{pmatrix} -(1-t)(2-t) & 0 & 0 & 0\\ 0 & -(2-t)(3-t) & 0 & 0\\ 0 & 0 & -(2-t)(1-\mathfrak{q}(2-t))\delta^\Lambda_\Omega & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.3.64}$$

where in deriving this we have used the fact that $2-t = \frac{\rho_0^m}{\rho^m}$ on $\Gamma^+_{r_0}$. By (4.3.73), we have that $1-\mathfrak{q}(2-t)$ satisfies $1 < 1-\mathfrak{q}(2-t) < 3$ for $0<t<1$. From this inequality, (4.3.39), (4.3.63) and (4.3.64), we deduce that

$$h\Big(Y, \Big(n_0^- B^0 + n_1^- \frac{\chi\rho}{m} B^1 + n_\Sigma^- B^\Sigma\Big)\Big|_{\Gamma^-} Y\Big) \le 0$$

and

$$h\Big(Y, \Big(n_0^+ B^0 + n_1^+ \frac{\chi\rho}{m} B^1 + n_\Sigma^+ B^\Sigma\Big)\Big|_{\Gamma^+_{r_0}} Y\Big) \le 0$$

for all $Y = (Y_{\mathcal{I}})$. Consequently, by definition, see [62, §4.3], the surfaces Γ^- and $\Gamma^+_{r_0}$ are weakly spacelike, and it follows that any solution of the extended system (4.3.54) on the extended spacetime $(0,1)\times\mathcal{S}$ will yield by restriction a solution of the system (4.3.49) on the region (4.3.52) that is uniquely determined by the restriction of the initial data to (4.3.53). From this property and the above arguments, we conclude that the existence of solutions to the conformal wave equations (4.2.10) on M_{r_0} can be obtained from solving the initial value problem

$$B^0\partial_t V^K + \frac{1}{t}\frac{\chi\rho}{m} B^1 \partial_\rho V^K + \frac{1}{t^{\frac{1}{2}}} B^\Sigma \slashed{\nabla}_\Sigma V^K = \frac{1}{t}\mathcal{B}\mathbb{P}V^K + \mathcal{C}V^K + \mathcal{F}^K \quad \text{in } (0,1)\times\mathcal{S}, \tag{4.3.65}$$

$$V^K = \mathring{V}^K \quad \text{in } \{1\}\times\mathcal{S}, \tag{4.3.66}$$

for initial data $\mathring{V}^K = (\mathring{V}_{\mathcal{I}}{}^K)$ satisfying the constraints

$$\slashed{\nabla}_\Lambda \mathring{V}^K_4 = \frac{1}{\sqrt{2}}\mathring{V}^K_\Lambda \quad \text{and} \quad \frac{\rho}{m}\partial_\rho \mathring{V}^K_4 = \frac{1}{2}\big(\mathring{V}^K_1 - \mathring{V}^K_0\big) \quad \text{in } \Sigma_{r_0}. \tag{4.3.67}$$

Moreover, solutions to (4.2.10) generated this way are independent of the particular form of the initial data $\mathring{V}$ on $(\{1\}\times\mathcal{S})\setminus\Sigma_{r_0}$ and are determined from solutions of the IVP (4.3.65)-(4.3.66) via

$$u^K(t,r,\theta,\phi) = \frac{1}{t^{\frac{1}{2}}} V^K_4(t, r^{\frac{1}{m}}, \theta, \phi). \tag{4.3.68}$$

Finally, solutions to the semilinear wave equations (4.0.1) on $\bar{M}_{r_0}$ can then be obtained from (4.3.68) using (4.2.6) and (4.2.9), which yield the explicit formula

$$\bar{u}^K(\bar{t},\bar{r},\theta,\phi) = \frac{\bar{r}}{\bar{r}^2-\bar{t}^2}\Big(1-\frac{\bar{t}}{\bar{r}}\Big)^{\frac{1}{2}}\Big(1+\frac{\bar{t}}{\bar{r}}\Big)V^K_4\bigg(1-\frac{\bar{t}}{\bar{r}}, \Big(\frac{\bar{r}}{\bar{r}^2-\bar{t}^2}\Big)^{\frac{1}{m}}, \theta, \phi\bigg). \tag{4.3.69}$$

Initial data transformations

The relation between the initial data

$$(\bar{u}^K, \partial_{\bar{t}}\bar{u}^K) = (\bar{v}^K, \bar{w}^K) \quad \text{in } \bar{\Sigma}_{r_0} \tag{4.3.70}$$

for the semilinear wave equations (4.0.1) and the corresponding initial data

$$(u^K, \partial_t u^K) = (v^K, w^K) \quad \text{in } \Sigma_{r_0}$$

for the conformal wave equations (4.2.8) is given by

$$v^K(r,\theta,\phi) = \frac{1}{r}\bar{v}^K\left(\frac{1}{r},\theta,\phi\right) \quad \text{and} \quad w^K(r,\theta,\phi) = -\frac{1}{r^2}\bar{w}^K\left(\frac{1}{r},\theta,\phi\right)$$

as can be readily verified with the help of (4.1.2), (4.2.6) and (4.2.9). The initial data for the conformal wave equations, in turn, determines via (4.3.1), (4.3.5), (4.3.27) and (4.3.48) the following initial data for the system (4.3.49):

$$\tilde{V}(\rho,\theta,\phi) = \begin{pmatrix} \frac{1}{\rho^m}\left[\frac{1}{\rho^m}\partial_r\bar{v}^K\left(\frac{1}{\rho^m},\theta,\phi\right) + \bar{v}^K\left(\frac{1}{\rho^m},\theta,\phi\right) - \frac{1}{\rho^m}\bar{w}^K\left(\frac{1}{\rho^m},\theta,\phi\right)\right] \\ -\frac{1}{\rho^m}\left[\frac{1}{\rho^m}\partial_r\bar{v}^K\left(\frac{1}{\rho^m},\theta,\phi\right) + \bar{v}^K\left(\frac{1}{\rho^m},\theta,\phi\right) + \frac{1}{\rho^m}\bar{w}^K\left(\frac{1}{\rho^m},\theta,\phi\right)\right] \\ \frac{\sqrt{2}}{\rho^m}\partial_\theta\bar{v}^K\left(\frac{1}{\rho^m},\theta,\phi\right) \\ \frac{\sqrt{2}}{\rho^m}\partial_\phi\bar{v}^K\left(\frac{1}{\rho^m},\theta,\phi\right) \\ \frac{1}{\rho^m}\bar{v}^K\left(\frac{1}{\rho^m},\theta,\phi\right) \end{pmatrix}, \tag{4.3.71}$$

which, of course, satisfies the constraint (4.3.67). By the above discussion, we can extend this data in any matter we like to $\mathcal{S}$ to obtain initial data for the extended system (4.3.65), and thus, we can choose any initial $\mathring{V}$ for (4.3.65) on $\mathcal{S}$ satisfying

$$\mathring{V}|_{\Sigma_{r_0}} = \tilde{V} \tag{4.3.72}$$

in order to obtain solutions to (4.0.1) on $\bar{M}_{r_0}$ that are uniquely determined by the initial data (4.3.70).

4.3.3 The differentiated system

While the extended system (4.3.54) is in Fuchsian form, it is not yet in a form that is required in order to apply the Fuchsian GIVP existence theory developed in [1]. To obtain a system that is in the required form, we need to modify (4.3.54) and complement it with a differentiated version. The differentiated version is obtained by applying the Levi-Civita connection $\mathcal{D}_j$ of the Riemannian metric[1]

$$q = q_{ij} dy^i \otimes dy^j := d\rho \otimes d\rho + \displaystyle{\not}g, \quad y = (y^i) := (\rho, \theta, \phi), \tag{4.3.73}$$

on $\mathcal{S}$. Noting that

$$\mathcal{D}_i = \delta_i^1 \partial_\rho + \delta_i^\Lambda \displaystyle{\not}\nabla_\Lambda, \tag{4.3.74}$$

where we recall that $\displaystyle{\not}\nabla_\Lambda$ is the Levi-Civita connection of the metric $\displaystyle{\not}g_{\Lambda\Sigma}$ on $\mathbb{S}^2$, we see after a short calculation that applying $\mathcal{D}_j$ to (4.3.54) and multiplying the result by t^κ, where $\kappa \geq 0$ is a constant to be fixed below, yields

$$B^0 \partial_t W_j^K + \frac{1}{t}\frac{\chi\rho}{m} B^1 \partial_\rho W_j^K + \frac{1}{t^{\frac{1}{2}}} B^\Sigma \displaystyle{\not}\nabla_\Sigma W_j^K = \frac{1}{t}\big(\mathcal{B}\mathbb{P} + \kappa B^0\big) W_j^K + \frac{1}{t}\mathcal{Q}_j^K + \mathcal{H}_j^K \tag{4.3.75}$$

where

$$W_j^K = (W_{j\mathcal{I}}^K) := \big(t^\kappa \mathcal{D}_j V_\mathcal{I}^K\big), \tag{4.3.76}$$

$$\mathcal{Q}_j^K = -t^\kappa 2\chi(\rho)\rho^m \bar{b}_{IJ}^K \mathcal{D}_j (V_0^I V_0^J)\mathbf{e}_0 \tag{4.3.77}$$

and

$$\begin{aligned}\mathcal{H}_j^K = \mathcal{C} W_j^K + t^{\kappa - \frac{1}{2}} B^\Sigma [\displaystyle{\not}\nabla_\Sigma, \mathcal{D}_j] V^K - \frac{1}{t}\partial_\rho\left(\frac{\chi\rho}{m} B^1\right)\delta_j^1 W_1^K \\ + t^\kappa \mathcal{D}_j \mathcal{G} - t^{\kappa-1} 2\mathcal{D}_j(\bar{b}_{IJ}^K \chi\rho^m) V_0^I V_0^J \mathbf{e}_0.\end{aligned} \tag{4.3.78}$$

It is worthwhile pointing out that the term $[\displaystyle{\not}\nabla_\Sigma, \mathcal{D}_j]V^K$ does not involve any differentiation since the commutator can be expressed completely in terms of the curvature of the metric $\displaystyle{\not}g_{\Lambda\Sigma}$.

4.4 The asymptotic equation

The next step in the derivation of a suitable Fuchsian equation involves modifying the V_0^K component of the extended system (4.3.54) given by

$$(2-t)\partial_t V_0^K = -\frac{2}{t}\chi\rho^m \bar{b}_{IJ}^K V_0^I V_0^J + V_0^K - \frac{1}{t^\kappa}\frac{\chi\rho}{m} W_{10}^K + \frac{1}{t^{\frac{1}{2}+\kappa}\mathrm{p}} \displaystyle{\not}g^{\Sigma\Lambda} W_{\Lambda\Sigma}^K + \mathcal{G}_0^K, \tag{4.4.1}$$

[1] See Appendix A.1 for our indexing conventions.

where $\mathcal{G}^K = (\mathcal{G}^K_{\mathcal{I}})$, in order to remove the singular term $\frac{1}{t}Q^K$. We remove this singular term using the flow[2] $\mathscr{F}(t,t_0,y,\xi) = (\mathscr{F}^K(t,t_0,y,\xi))$ of the asymptotic equation (1.7.2), i.e.

$$(2-t)\partial_t \mathscr{F}(t,t_0,y,\xi) = \frac{1}{t}Q\big(\mathscr{F}(t,t_0,y,\xi)\big), \tag{4.4.2}$$

$$\mathscr{F}(t,t_0,y,\xi) = \xi. \tag{4.4.3}$$

Before proceeding, we note that, for fixed (t,t_0,y), the flow $\mathscr{F}(t,t_0,y,\xi)$ maps $\mathbb{R}^N$ to itself, and consequently, the derivative $D_\xi F(t,t_0,y,\xi)$ defines a linear map from $\mathbb{R}^N$ to itself, or equivalently, a $N\times N$-matrix.

Using the asymptotic flow, we define a new set of variables $Y(t,y) = (Y^K(t,y))$ via

$$V_0(t,y) = \mathscr{F}(t,1,y,Y(t,y)) \tag{4.4.4}$$

where

$$V_0 = (V_0^K). \tag{4.4.5}$$

A short calculation involving (4.4.1) and (4.4.2) then shows that Y satisfies

$$(2-t)\partial_t Y = \mathscr{L}\mathscr{G} \tag{4.4.6}$$

where

$$\mathscr{L} = (D_\xi \mathscr{F}(t,1,y,Y))^{-1} \quad \text{and} \quad \mathscr{G} = \left(V_0^K - \frac{1}{t^\kappa}\frac{\chi\rho}{m}W^K_{10} + \frac{1}{t^{\frac{1}{2}+\kappa}\mathtt{p}}\not{g}^{\Sigma\Lambda}W^K_{\Lambda\Sigma} + \mathcal{G}^K_0\right). \tag{4.4.7}$$

4.4.1 Asymptotic flow assumptions

We now assume that the flow $\mathscr{F}(t,t_0,y,\xi) = (\mathscr{F}^K(t,t_0,y,\xi))$ satisfies the following: for any $\mathtt{N}\in\mathbb{Z}_{\geq 0}$, there exist constants $R_0>0$, $\epsilon\in[0,1/10]$ and $C_{k\ell}>0$, where $k,\ell\in\mathbb{Z}_{\geq 0}$ and $0\leq k+\ell\leq \mathtt{N}$, and a function $\omega(R)$ satisfying $\lim_{R\searrow 0}\omega(R)=0$ such that

$$\big|\mathscr{F}(t,1,y,\xi)\big| \leq \omega(R) \tag{4.4.8}$$

[2]Note that the flow depends on $y=(y^i)=(\rho,\theta,\phi)\in\mathcal{S}$ through the coefficients $\chi\rho^m\bar{b}^K_{IJ}$, which are smooth functions on $\mathcal{S}$.

and

$$\left|D_\xi^k\mathcal{D}^\ell\mathscr{F}(t,1,y,\xi)\right|+\left|D_\xi^k\mathcal{D}^\ell\big(D_\xi\mathscr{F}(t,1,y,\xi)\big)^{-1}\right|\leq\frac{1}{t^\epsilon}C_{k\ell} \tag{4.4.9}$$

for all $(t,y,\xi)\in(0,1]\times\mathcal{S}\times B_R(\mathbb{R}^N)$ and $R\in(0,R_0]$. A direct consequence of this assumption is that the maps $\mathtt{F}$ and $\check{\mathtt{F}}$ defined by

$$\mathtt{F}(t,y,\xi)=t^\epsilon\mathscr{F}(t,1,y,\xi)\quad\text{and}\quad=\check{\mathtt{F}}(t,y,\xi)=t^\epsilon\big(D_\xi\mathscr{F}(t,1,y,\xi)\big)^{-1}, \tag{4.4.10}$$

respectively, satisfy $\mathtt{F}\in C^0\big([0,1],C^{\mathtt{N}}(\mathcal{S}\times B_R(\mathbb{R}^N),\mathbb{R})\big)$ and $\check{\mathtt{F}}\in C^0\big([0,1],C^{\mathtt{N}}(\mathcal{S}\times B_R(\mathbb{R}^N),\mathbb{M}_{N\times N})\big)$. Furthermore, since $\xi=0$ obviously solves the asymptotic equation (1.7.2), the flow obviously satisfies $\mathscr{F}(t,t_0,y,0)=0$, which in turn, implies that

$$\mathtt{F}(t,y,0)=0 \tag{4.4.11}$$

for all $(t,y)\in[0,1]\times\mathcal{S}$.

Proposition 4.4.1. *Suppose the bounded weak null condition holds (see Definition 1.7.2). Then there exists a $R_0\in(0,\mathcal{R}_0)$ such that the flow $\mathscr{F}(t,t_0,y,\xi)$ of the asymptotic equation (1.7.2) satisfies the flow assumptions (4.4.8)-(4.4.9) for this choice of R_0 and any choice of $\epsilon\in(0,1/10]$.*

Proof. We begin the proof by first establishing the following lemma that gives an effective bound on solutions of the asymptotic equation.

Lemma 4.4.2. *For any $R\in(0,\mathcal{R}_0]$, the solutions ξ of the asymptotic IVP (1.7.4)-(1.7.5) exist for $t\in(0,1]$ and satisfies*

$$\sup_{0<t\leq 1}|\xi(t)|\leq\frac{C}{\mathcal{R}_0}R \tag{4.4.12}$$

for any choice of initial data that is bounded by $|\mathring{\xi}|\leq R$.

Proof. Since $Q(\xi)$, see (1.7.3), is independent of t, we can make the asymptotic equation autonomous through the introduction of the new time variable $\tau=-\frac{1}{2}\ln(2-t)+\frac{1}{2}\ln(t)$, which maps the time interval $0<t\leq1$ to $-\infty<\tau\leq0$. In terms this new time variable τ, the asymptotic IVP (1.7.4)-(1.7.5) becomes

$$\partial_\tau\xi=Q(\xi), \tag{4.4.13}$$

$$\xi|_{\tau=0}=\mathring{\xi}. \tag{4.4.14}$$

By the bounded weak null condition, this IVP admits solutions that are defined for $\tau \in (-\infty, 0]$ and satisfy

$$\sup_{-\infty<\tau\leq 0} |\xi(\tau)| \leq C \tag{4.4.15}$$

provided that $|\mathring{\xi}| < \mathcal{R}_0$. Next, we assume that the initial value $\mathring{\xi}$ satisfies $|\mathring{\xi}| < R$ for some $R \in (0, \mathcal{R}_0]$, and we set $\tilde{\xi}(\tau) = \frac{1}{\mathfrak{r}}\xi\big(\frac{\tau}{\mathfrak{r}}\big)$ where $\mathfrak{r} = \frac{R}{\mathcal{R}_0} \in (0, 1]$. Then a quick calculation shows that $\tilde{\xi}$ satisfies asymptotic equation (4.4.13) where that initial value is bounded by $|\tilde{\xi}|_{\tau=0}| = \big|\frac{1}{\mathfrak{r}}\mathring{\xi}\big| < \frac{\mathcal{R}_0}{R}R = \mathcal{R}_0$, and consequently, we deduce from (4.4.15) that $\sup_{-\infty<\tau\leq 0}|\tilde{\xi}(\tau)| \leq C$. But this implies that $\sup_{-\infty<\tau\leq 0}|\xi(\tau)| \leq \frac{C}{\mathcal{R}_0}R$, and the proof of the lemma is complete. □

Implicitly, the solution $\xi = (\xi^K)$ depends on $y \in \mathcal{S}$ and the initial data $\mathring{\xi}$. Fixing $\epsilon > 0$ and differentiating the asymptotic equation (1.7.2) with respect to $y = (y^i)$ shows that

$$\eta_i^K = t^\epsilon \mathcal{D}_i \xi^K \tag{4.4.16}$$

satisfies the differential equation

$$(2-t)\partial_t \eta_i^K = \frac{1}{t}\big((2-t)\epsilon\delta_J^K - 2\chi\rho^m\big(\bar{b}_{JI}^K + \bar{b}_{IJ}^K\big)\xi^I\big)\eta_i^J - \frac{1}{t^{1-\epsilon}}\mathcal{D}_i(2\chi\rho^m\bar{b}_{IJ}^K)\xi^I\xi^J. \tag{4.4.17}$$

Contracting this equation with $\delta_{LK}\delta^{ki}\eta_k^L$ gives

$$\begin{aligned}(2-t)\delta_{LK}\delta^{ki}\eta_k^L\partial_t\eta_i^K = &\frac{1}{t}\delta_{LK}\delta^{ki}\eta_k^L\big((2-t)\epsilon\delta_J^K - 2\chi\rho^m\big(\bar{b}_{JI}^K + \bar{b}_{IJ}^K\big)\xi^I\big)\eta_i^J \\ &- \frac{1}{t^{1-\epsilon}}\delta_{LK}\delta^{ki}\eta_k^L\mathcal{D}_i(2\chi\rho^m\bar{b}_{IJ}^K)\xi^I\xi^J.\end{aligned}$$

Letting $|\eta| = \sqrt{\delta_{KL}\delta^{ij}\eta_i^K\eta_j^L}$, denote the Euclidean norm of $\eta = (\eta_i^K)$, we can write the above equation as

$$\frac{(2-t)}{2}\partial_t|\eta|^2 = \frac{1}{t}\big((2-t)\epsilon|\eta|^2 - 2\chi\rho^m\big(\bar{b}_{JI}^K + \bar{b}_{IJ}^K\big)\delta_{LK}\xi^I\delta^{ki}\eta_k^L\eta_i^J\big) - \frac{1}{t^{1-\epsilon}}\delta_{LK}\delta^{ki}\eta_k^L\mathcal{D}_i(2\chi\rho^m\bar{b}_{IJ}^K)\xi^I\xi^J. \tag{4.4.18}$$

But $\chi\rho^m$ and $\bar{b}_{IJ}^K$ are smooth on $\mathcal{S}$, and consequently, these functions and their derivatives are bounded on $\mathcal{S}$. From this fact and the bound on ξ from Lemma 4.4.2, we deduce from (4.4.18) and the Cauchy Schwartz inequality that for any $\sigma \in (0, \epsilon)$ there exists constants $R_0 \in (0, \mathcal{R}_0]$ and $C > 0$ such that the energy inequality

$$\frac{(2-t)}{2}\partial_t|\eta|^2 \geq \frac{(2-t)}{t}(\epsilon - \sigma)|\eta|^2 - \frac{C}{t^{1-\epsilon}}|\eta|$$

holds for any given $R \in (0, R_0]$ and for all $t \in (0, 1]$. But from this inequality, we see that

$$\partial_t|\eta| \geq \frac{1}{t}(\epsilon - \sigma)|\eta| - \frac{C}{t^{1-\epsilon}}.$$

An application of Grönwall's inequality[3] then yields

$$|\eta(t)| \leq |\eta(1)|t^{\epsilon-\sigma} + t^{\epsilon-\sigma}\int_t^1 \frac{C}{t^{1-\sigma}}\,d\tau = t^{\epsilon-\sigma}|\eta(1)| + \frac{1}{\sigma}t^{\epsilon-\sigma}\big(1-t^{\sigma}\big). \tag{4.4.19}$$

From the definition (4.4.16) and the fact that $\xi(t) = \mathscr{F}(t,1,y,\mathring{\xi})$, we conclude from the above inequality and (4.4.12) that there exist constants $C_0, C_{01} > 0$ such that the flow $\mathscr{F}$ satisfies the bounds

$$|\mathscr{F}(t,1,y,\mathring{\xi})| \leq C_0 R \quad \text{and} \quad |\mathcal{D}\mathscr{F}(t,1,y,\mathring{\xi})| \leq \frac{1}{t^{\sigma}}C_{01}$$

for all $(t,y,\mathring{\xi}) \in (0,1] \times \mathcal{S} \times B_R(\mathbb{R}^N)$, $R \in (0,R_0]$.

Next, differentiating the asymptotic equation (1.7.2) with respect to the initial data $\mathring{\xi}$ shows that the derivative

$$D_{\mathring{\xi}}\xi = \left(\frac{\partial \xi^K}{\partial \mathring{\xi}^L}\right)$$

satisfies the equation

$$(2-t)\partial_t D_{\mathring{\xi}}\xi = \frac{1}{t}LD_{\mathring{\xi}}\xi \tag{4.4.20}$$

where

$$L = (L^K_J) := -2\chi\rho^m\big(\bar{b}^K_{JI} + \bar{b}^K_{IJ}\big)\xi^I\big).$$

Furthermore, multiplying (4.4.20) on the right by $(D_{\mathring{\xi}}\xi)^{-1}$ yields the equation

$$(2-t)\partial_t((D_{\mathring{\xi}}\xi)^{-1})^{\mathrm{tr}} = -\frac{1}{t}L^{\mathrm{tr}}((D_{\mathring{\xi}}\xi)^{-1})^{\mathrm{tr}} \tag{4.4.21}$$

[3]Here, we are using the following form of Grönwall's inequality: if $x(t)$ satisfies $x'(t) \geq a(t)x(t) - h(t)$, $0 < t \leq T_0$, then $x(t) \leq x(T_0)e^{-A(t)} + \int_t^{T_0} e^{-A(t)+A(\tau)}h(\tau)\,d\tau$ where $A(t) = \int_t^{T_0} a(\tau)\,d\tau$. In particular, we observe from this that if, $x(T_0) \geq 0$ and $a(t) = \frac{\lambda}{t} - b(t)$, where $\lambda \in \mathbb{R}$ and $\big|\int_t^{T_0} b(\tau)\,d\tau\big| \leq r$, then

$$x(t) \leq e^r x(T_0)\left(\frac{t}{T_0}\right)^{\lambda} + e^{2r}t^{\lambda}\int_t^{T_0}\frac{|h(\tau)|}{\tau^{\lambda}}\,d\tau$$

for $0 \leq t < T_0$.

for the transpose of $(D_{\mathring{\xi}}\xi)^{-1}$. Multiplying (4.4.20) and (4.4.21) by t^ϵ, we find that

$$(2-t)\partial_t(t^\epsilon D_{\mathring{\xi}}\xi) = \frac{1}{t}\big((2-t)\epsilon + L\big)t^\epsilon D_{\mathring{\xi}}\xi$$

and

$$(2-t)\partial_t(t^\epsilon (D_{\mathring{\xi}}\xi)^{-1})^{\mathrm{tr}} = \frac{1}{t}\big((2-t)\epsilon - L^{\mathrm{tr}}\big)(t^\epsilon (D_{\mathring{\xi}}\xi)^{-1})^{\mathrm{tr}}.$$

Both of the these equations are of the same general form as (4.4.17), and the same arguments used to derive from (4.4.17) the bounds (4.4.19) for $\eta = t^\epsilon \mathcal{D}\xi$ can be used to obtain similar estimates for $t^\epsilon D_{\mathring{\xi}}\xi$ and $(t^\epsilon (D_{\mathring{\xi}}\xi)^{-1})^{\mathrm{tr}}$. Consequently, shrinking R_0 if necessary and arguing as above, we deduce the existence of a constant $C_{10} > 0$ such that the estimate

$$\big|D_{\mathring{\xi}}\xi\big| + \big|(D_{\mathring{\xi}}\xi)^{-1}\big| \leq \frac{1}{t^\sigma}C_{10}$$

holds for $0 < t \leq 1$. From this estimate, we see immediately that

$$\big|D_{\mathring{\xi}}\mathscr{F}(t,1,y,\mathring{\xi})\big| + \big|\big(D_{\mathring{\xi}}\mathscr{F}(t,1,y,\mathring{\xi})\big)^{-1}\big| \leq \frac{1}{t^\sigma}C_{10},$$

for all $(t,y,\mathring{\xi}) \in (0,1] \times \mathcal{S} \times B_R(\mathbb{R}^N)$ and $R \in (0,R_0]$.

Finally, by shrinking R_0 again if necessary, similar arguments as above can be used to derive, for any fixed $\mathtt{N} \in \mathbb{Z}_{\geq 1}$, the bounds

$$\big|D^k_{\mathring{\xi}}\mathcal{D}^\ell\xi\big| + \big|D^k_{\mathring{\xi}}\mathcal{D}^\ell\big(D_{\mathring{\xi}}\xi\big)^{-1}\big| \leq \frac{1}{t^\sigma}C_{kl}$$

on the higher derivatives for $1 \leq k+\ell \leq \mathtt{N}$. It is then clear from this inequality that the flow bounds

$$|D^k_{\mathring{\xi}}\mathcal{D}^\ell\mathscr{F}(t,1,y,\mathring{\xi})| \leq \frac{1}{t^\sigma}C_{\ell k},$$

hold for all $(t,y,\mathring{\xi}) \in (0,1] \times \mathcal{S} \times B_R(\mathbb{R}^N)$, $2 \leq k+\ell \leq \mathtt{N}$, and $R \in (0,R_0]$. This completes the proof of the proposition. $\square$

4.4.2 The complete Fuchsian system

We complete the derivation of the Fuchsian equation by complimenting (4.3.75) and (4.4.6) with a third system obtained from applying the projection operator $\mathbb{P}$ to

(4.3.54), which leads to an equation for the variables

$$X^K = \frac{1}{t^\nu}\mathbb{P}V^K, \tag{4.4.22}$$

where $\nu \geq 0$ is a constant to be fixed below. Now, a straightforward calculation using (4.3.35), (4.3.37)-(4.3.38), (4.3.55), (4.3.57) and (4.3.76) shows that after multiplying (4.3.54) by $t^{-\nu}\mathbb{P}$ that X^K satisfies

$$B^0\partial_t X^K + \frac{1}{t}\frac{\chi\rho}{m}B^1\partial_\rho X^K = \frac{1}{t}(\mathcal{B} - \nu B^0)X^K + \mathcal{K}^K \tag{4.4.23}$$

where

$$\mathcal{K}^K = -\frac{1}{t^{\frac{1}{2}+\kappa+\nu}}\mathbb{P}B^\Sigma W^K_\Sigma + \mathbb{P}\mathcal{C}\left(\frac{1}{t^\nu}\mathbb{P}^\perp V^K + X^K\right) + \frac{1}{t^\nu}\mathbb{P}\mathcal{G}^K_0 + \frac{1}{t^{\frac{1}{2}+\nu}}\mathbb{P}\mathcal{G}^K_1 \tag{4.4.24}$$

and

$$\mathbb{P}^\perp = \mathbb{1} - \mathbb{P} \tag{4.4.25}$$

is the complementary projection oprator. We now complete our derivation of the Fuchsian equation, which will be crucial for our existence proof, by collecting (4.3.75), (4.4.6) and (4.4.23) into the following single system:

$$A^0\partial_t Z + \frac{1}{t}\frac{\chi\rho}{m}A^1\partial_\rho Z + \frac{1}{t^{\frac{1}{2}}}A^\Sigma \nabla\!\!\!\!/_\Sigma Z = \frac{1}{t}\mathcal{A}\Pi Z + \frac{1}{t}\mathcal{Q} + \mathcal{J} \tag{4.4.26}$$

where

$$Z = \begin{pmatrix} W^K_j & X^K & Y \end{pmatrix}^{\mathrm{tr}}, \tag{4.4.27}$$

$$A^0 = \begin{pmatrix} B^0 & 0 & 0 \\ 0 & B^0 & 0 \\ 0 & 0 & (2-t)\mathbb{1} \end{pmatrix}, \tag{4.4.28}$$

$$A^1 = \begin{pmatrix} B^1\delta^j_k\delta^L_K & 0 & 0 \\ 0 & B^1\delta^L_K & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.4.29}$$

$$A^\Sigma = \begin{pmatrix} B^\Sigma & 0 & 0 \\ 0 & B^\Sigma & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.4.30}$$

$$\mathcal{A} = \begin{pmatrix} \mathcal{B}\mathbb{P} + \kappa B^0 & 0 & 0 \\ 0 & \mathcal{B} - \nu B^0 & 0 \\ 0 & 0 & 2\mathbb{1} \end{pmatrix}, \tag{4.4.31}$$

$$\Pi = \begin{pmatrix} \mathbb{1} & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.4.32}$$

$$\mathcal{Q} = \begin{pmatrix} \mathcal{Q}_j^K & 0 & 0 \end{pmatrix}^{\mathrm{tr}} \tag{4.4.33}$$

and

$$\mathcal{J} = \begin{pmatrix} \mathcal{H}_j^K & \mathcal{K}^K & \mathscr{L}\mathscr{G} \end{pmatrix}^{\mathrm{tr}}. \tag{4.4.34}$$

4.4.3 Coefficient properties

We now turn to verifying that the system (4.4.26) satisfies all the assumptions needed to apply the Fuchsian GIVP existence theory from [1].

The projection operator Π and its commutation properties:

By construction, the field Z, defined by (4.4.27), is a time-dependent section of the vector bundle

$$\mathbb{W} = \bigcup_{y\in\mathcal{S}} \mathbb{W}_y$$

over $\mathcal{S}$ with fibers $\mathbb{W}_y = \Big(T_y^*\mathcal{S}\times T_y^*\mathcal{S}\times \big(\mathrm{T}_y^*\mathcal{S}\otimes T^*_{\mathrm{pr}(y)}\mathbb{S}^2\big)\times T_y^*\mathcal{S}\Big)^N \times \mathbb{V}_y^N\times\mathbb{R}^N$ where, as above, $\mathrm{pr} : \mathcal{S} \longrightarrow \mathbb{S}^2$ is the canonical projection and $\mathbb{V}_y = \mathbb{R}\times\mathbb{R}\times\mathrm{T}^*_{\mathrm{pr}(y)}\mathbb{S}^2\times\mathbb{R}$. Letting $\grave{Z} = (\grave{W}_j^K, \grave{X}^K, \grave{Y})$ and Z be as defined above by (4.4.27), we introduce an inner-product on $\mathbb{W}$ via

$$\hbar(Z,\grave{Z}) = \delta_{KL}q^{ij}h(W_i^K,\grave{W}_j^L) + \delta_{KL}h(X^K,\grave{X}^L) + \delta_{KL}Y^K\grave{Y}^L, \tag{4.4.35}$$

where $h(\cdot,\cdot)$ is the inner-product defined previously by (4.3.39). It is then not difficult to verify that this inner-product is compatible, i.e. $\mathcal{D}_j\big(\hbar(Z,\grave{Z})\big) = \hbar(\mathcal{D}_jZ,\grave{Z}) + \hbar(Z,\mathcal{D}_j\grave{Z})$, with the connection $\mathcal{D}_j$ defined above by (4.3.74). We further observe from (4.4.32) that Π defines a projection operator, i.e.

$$\Pi^2 = \Pi, \tag{4.4.36}$$

that is symmetric with respect to the inner-product (4.4.35). It also follows directly from the definitions (4.4.28), (4.4.29) and (4.4.31) that

$$[A^0, \Pi] = [\mathcal{A}, \Pi] = 0, \tag{4.4.37}$$

$$\Pi A^1 = A^1\Pi = A^1, \quad \Pi A^\Sigma \eta_\Sigma = A^\Sigma \eta_\Sigma \Pi = A^\Sigma \eta_\Sigma, \tag{4.4.38}$$

and

$$\Pi^\perp A^1 = A^1\Pi^\perp = \Pi^\perp A^\Sigma \eta_\Sigma = A^\Sigma \eta_\Sigma \Pi^\perp = 0, \tag{4.4.39}$$

where

$$\Pi^\perp = 1\!\!1 - \Pi$$

is the complementary projection operator.

THE OPERATORS A^0, A^1, $A^\Sigma n_\Sigma$ AND $\mathcal{A}$:

Next, we see from (4.3.40), (4.4.28) and (4.4.35) that A^0 satisfies

$$\begin{aligned} \hbar(Z, A^0 Z) &= \delta_{KL} q^{ij} h(W_i^K, B^0 W_j^L) + \delta_{KL} h(X^K, B^0 X^L) + (2-t)\delta_{KL} Y^K Y^L \\ &\geq \delta_{KL} q^{ij} h(W_i^K, W_j^L) + \delta_{KL} h(X^K, X^L) + (2-t)\delta_{KL} Y^K Y^L, \end{aligned}$$

and hence, that

$$\hbar(Z, Z) \leq \hbar(Z, A^0 Z). \tag{4.4.40}$$

Similar calculations using (4.3.37)-(4.3.38), (4.3.41), (4.4.28), (4.4.31) and (4.4.35) show that

$$\kappa \hbar(Z, A^0 Z) \leq \hbar(Z, \mathcal{A} Z) \tag{4.4.41}$$

provided that $\nu, \kappa \geq 0$ and $\kappa + \nu \leq 1/2$. It is also clear from (4.4.28)-(4.4.30) that A^0, A^1 and $A^\Sigma \eta_\Sigma$ are symmetric with respect to the inner-product (4.4.35). Finally, we observe that the inequality

$$\left| \partial_\rho \left(\frac{\chi \rho}{m} B^1 \right) \right| \leq \max_{0 \leq t \leq 1} |B^1(t)| \|\partial_\rho(\chi\rho)\|_{L^\infty(\mathbb{T})} \frac{1}{m}$$

follows easily from (4.3.31) and (4.3.73). With the help of this inequality, we deduce from (4.4.29) that, for any given $\sigma > 0$, there exists an integer $m = m(\sigma) \geq 1$ such

that
$$\left|\partial_\rho\left(\frac{\chi\rho}{m}B^1\right)\right|+\left|\partial_\rho\left(\frac{\chi\rho}{m}A^1\right)\right|<\sigma \quad \text{in } (0,1)\times\mathcal{S}. \tag{4.4.42}$$

THE SOURCE TERM $\mathcal{J}$:

Using (4.3.29), (4.3.35), (4.4.4)-(4.4.5), (4.4.10) and (4.4.22), we can decompose V^K as
$$V^K(t,y)=\mathbb{P}V^K(t,y)+\mathbb{P}^\perp V^K(t,y), \tag{4.4.43}$$
where
$$\mathbb{P}V^K(t,y)=t^\nu X^K(t,y) \tag{4.4.44}$$
and
$$\mathbb{P}^\perp V^K(t,y)=\frac{1}{t^\epsilon}(t^\epsilon V_0^K(t,y))\mathbf{e}_0=\frac{1}{t^\epsilon}\mathtt{F}^K\big(t,y,Y(t,y)\big)\mathbf{e}_0, \tag{4.4.45}$$
while we recall from (4.3.76) that the derivative $\mathcal{D}_jV^K$ is determined by
$$\mathcal{D}_jV^K(t,y)=W_j^K(t,y). \tag{4.4.46}$$
We further observe from (4.4.7) and (4.4.10) that the map $\mathscr{L}$ can be expressed as
$$\mathscr{L}=\frac{1}{t^\epsilon}\check{\mathtt{F}}\big(t,y,Y(t,y)\big). \tag{4.4.47}$$

Now, setting
$$X=(X^K),$$
we can use (4.4.43)-(4.4.45) along with (4.3.58)-(4.3.59) to write the source term (4.4.24) as
$$\begin{aligned}\mathcal{K}^K=&-\frac{1}{t^{\frac{1}{2}+\kappa+\nu}}\mathbb{P}B^\Sigma(t,y)W_\Sigma^K(t,y)+\frac{1}{t^{\nu+\epsilon}}\mathtt{F}^K\big(t,y,Y(t,y)\big)\mathbb{P}\mathcal{C}(t)\mathbf{e}_0+\mathbb{P}\mathcal{C}(t)X^K(t,y)\\&+\frac{1}{t^{\nu+2\epsilon}}\mathbb{P}\mathcal{G}_0^K\Big(t^{\frac{1}{2}},t,\chi(\rho)\rho^m,\mathtt{F}\big(t,y,Y(t,y)\big)\mathbf{e}_0,\mathtt{F}\big(t,y,Y(t,y)\big)\mathbf{e}_0\Big)\\&+\sum_{a=0}^{1}\bigg\{\frac{1}{t^{\frac{a}{2}+\epsilon}}\Big[\mathbb{P}\mathcal{G}_a^K\Big(t^{\frac{1}{2}},t,\chi(\rho)\rho^m,\mathtt{F}\big(t,y,Y(t,y)\big)\mathbf{e}_0,X(t,y)\Big)\\&+\mathbb{P}\mathcal{G}_a^K\Big(t^{\frac{1}{2}},t,\chi(\rho)\rho^m,X(t,y),\mathtt{F}\big(t,y,Y(t,y)\big)\mathbf{e}_0\Big)\Big]\end{aligned}$$

$$+\frac{1}{t^{\frac{a}{2}-\nu}}\mathbb{P}\mathcal{G}_a^K\Big(t^{\frac{1}{2}},t,\chi(\rho)\rho^m,X(t,y),X(t,y)\Big)\bigg\}.$$

Using (4.4.43)-(4.4.47) to similarly express the source terms $\mathcal{H}_j^K$ and $\mathscr{L}\mathscr{G}$, see (4.3.78) and (4.4.7), in terms of W_j^K, X^K and Y^K, it is then not difficult, with the help of (4.4.11), (4.4.42) and the assumptions $\epsilon,\kappa,\nu\geq 0$, that we can expand the source term (4.4.34) as

$$\mathcal{J}=\Big(\frac{1}{t^{3\epsilon}}+\frac{1}{t^{\nu+2\epsilon}}+\frac{1}{t^{1-\kappa+2\epsilon}}\Big)\mathcal{J}_0\big(t,y,Z(t,y)\big)+\Big(\frac{1}{t^{\frac{1}{2}+\kappa+\epsilon}}+\frac{1}{t^{\frac{1}{2}+2\epsilon-\nu}}\Big)\mathcal{J}_1\big(t,y,Z(t,y)\big)$$
$$+\frac{1}{t}\big(\sigma+t^{\frac{1}{2}-\kappa-\nu}+t^{\frac{1}{2}-\epsilon}+t^{\frac{1}{2}-\kappa-\epsilon}+t^{2\nu-\epsilon}\big)\mathcal{J}_2\big(t,y,Z(t,y)\big)$$

where $\mathcal{J}_a\in C^0([0,1],C^{\mathtt{N}}(\mathcal{S}\times B_R(\mathbb{W}),\mathbb{W}))$, $a=0,1,2$, for any fixed $\mathtt{N}\in\mathbb{Z}_{\geq 0}$, and these maps satisfy[4]

$$\mathcal{J}_0=\mathrm{O}(Z),\quad \mathcal{J}_1=\mathrm{O}(\Pi Z),\quad \Pi\mathcal{J}_2=\mathrm{O}(\Pi Z)\quad\text{and}\quad \Pi^{\perp}\mathcal{J}_2=\mathrm{O}(\Pi Z\otimes\Pi Z). \tag{4.4.48}$$

To proceed, we choose the constants $\kappa,\nu\in\mathbb{R}_{>0}$ to satisfy the inequalities

$$2\epsilon<\kappa<1-\epsilon,\quad \kappa+\nu<\frac{1}{2}-\epsilon,\quad \epsilon<2\nu\quad\text{and}\quad \kappa\leq\frac{1}{3}, \tag{4.4.49}$$

which is possible since $\epsilon\in[0,1/10]$ by assumption, see Section 4.4.1. For example, if $\epsilon=1/10$, we could choose $\kappa=3/10$ and $\nu=1/15$. Now, it is not difficult to verify that (4.4.49) implies the inequalities

$$3\epsilon\leq 1-\kappa+2\epsilon,\quad \nu+2\epsilon\leq 1-\kappa+2\epsilon,\quad 0<2\nu-\epsilon,\quad 0<\frac{1}{2}-\kappa-\epsilon,\quad 0<\frac{1}{2}-\kappa-\nu,$$
$$\frac{1}{2}+2\epsilon-\nu\leq 1-\frac{\kappa}{2}+\epsilon,\quad \frac{1}{2}+\kappa+\epsilon\leq 1-\frac{\kappa}{2}+\epsilon\quad\text{and}\quad 0<\kappa-2\epsilon\leq 1,$$

and that, with the help of these inequalities, we can, after suitably redefining the maps $\mathcal{J}_a$, rewrite (4.4.48) as

$$\mathcal{J}=\frac{1}{t^{1-\kappa+2\epsilon}}\mathcal{J}_0\big(t,y,Z(t,y)\big)+\frac{1}{t^{1-\frac{\kappa}{2}+\epsilon}}\mathcal{J}_1\big(t,y,Z(t,y)\big)+\frac{1}{t}\big(\sigma+t^{\tilde{\epsilon}}\big)\mathcal{J}_2\big(t,y,Z(t,y)\big) \tag{4.4.50}$$

for some suitably small constant $\tilde{\epsilon}>0$. Here, the constant $\sigma>0$ can be chosen as small as we like, and the redefined maps $\mathcal{J}_a$ have the same smoothness properties as above and satisfy (4.4.48).

[4] Here, we are using the order notation $\mathrm{O}(\cdot)$ from [1, §2.4] where the maps are finitely rather than infinitely differentiable.

Remark 4.4.3. The point of the expansion (4.4.50) is that source term $\mathcal{J}$ satisfies all the assumptions from Section 3.1.(iii) of [1] except for the following:

1. the differentiablity of each of the maps $\mathcal{J}_a$ is finite,
2. and $\mathcal{J}_2$ does not satisfy $\Pi\mathcal{J}_2 = 0$.

Neither of these exceptions pose any difficulties and are easily dealt with. To see why the first exception is not problematic, we observe from arguments of [1] that all of the results of that paper are valid provided that the order of the differentiability of the source term is greater than $n/2 + 3$, where n is the dimension of the spatial manifold. Since the spatial manifold we are considering, i.e. $\mathcal{S}$, is 3-dimensional and we have established above that the maps $\mathcal{J}_a$ are $\mathtt{N}$-times differentiable for any $\mathtt{N} \in \mathbb{Z}_{\geq 0}$, it follows by taking $\mathtt{N} > 3/2 + 3$ that the finite differentiability is no obstruction to applying the results from [1] to the Fuchsian equation (4.4.26). In regards to the second exception, we note, since $\Pi\mathcal{J}_2 = \mathrm{O}(\Pi Z)$, that the term $\frac{1}{t}(\sigma + t^{\tilde{\epsilon}})\Pi\mathcal{J}_2$ can be absorbed into the term $\frac{1}{t}\mathcal{A}\Pi Z$ on the right hand side of the Fuchsian equation (4.4.26) via a redefinition of the operator $\mathcal{A}$. Due to the factor $\sigma + t^{\tilde{\epsilon}}$, we can ensure, for any choice of $\tilde{\kappa} \in (0, \kappa)$, that the redefined matrix $\mathcal{A}$ would satisfy for all $t \in (0, t_0]$ an inequality of the form (4.4.41) with κ replaced by $\tilde{\kappa}$ provided that σ and t_0 are chosen sufficiently small. After doing this, the redefined $\mathcal{J}_2$ would satisfy $\Pi\mathcal{J}_2 = 0$ as required and the source term $\mathcal{J}$ would satisfy all the assumptions needed to apply the existence theory from [1].

THE SOURCE TERM $\mathcal{Q}$:

We now analyze the nonlinear term (4.4.33) (see also (4.3.77)) in more detail. Recalling that the $\chi\rho^m\bar{b}_{IJ}^K$ are smooth functions on $\mathcal{S}$, we can, with the help of the product estimate [53, Ch. 13, Prop. 3.7.] and Hölder's inequality, estimate $\mathcal{Q}$ for any $s \in \mathbb{Z}_{\geq 0}$ by

$$\begin{aligned}\|\mathcal{Q}\|_{H^s(\mathcal{S})} &\lesssim t^\kappa\big(\|\mathcal{D}(V_0V_0)\|_{L^\infty(\mathcal{S})} + \|\mathcal{D}(V_0V_0)\|_{H^s(\mathcal{S})}\big)\\ &\lesssim t^\kappa\big(\|V_0\|_{L^\infty(\mathcal{S})}\|\mathcal{D}V_0\|_{L^\infty(\mathcal{S})} + \|V_0\|_{L^\infty(\mathcal{S})}\|\mathcal{D}V_0\|_{H^s(\mathcal{S})} + \|\mathcal{D}V_0\|_{L^\infty(\mathcal{S})}\|V_0\|_{L^s(\mathcal{S})}\big)\\ &\lesssim \|V_0\|_{L^\infty(\mathcal{S})}\|W\|_{L^\infty(\mathcal{S})} + \|V_0\|_{L^\infty(\mathcal{S})}\|W\|_{H^s(\mathcal{S})} + \|W\|_{L^\infty(\mathcal{S})}\|V_0\|_{L^s(\mathcal{S})}.\end{aligned} \tag{4.4.51}$$

Next, for $k \in \mathbb{Z}_{>3/2}$, we let C_{Sob} denote the constant that appears in the Sobolev inequality [53, Ch. 13, Prop. 2.4.], that is,

$$\|\mathtt{f}\|_{L^\infty(\mathcal{S})} \leq C_{\mathrm{Sob}}\|\mathtt{f}\|_{H^k(\mathcal{S})}. \tag{4.4.52}$$

Then by (4.4.4), the flow bounds (4.4.8)-(4.4.9), and the Sobolev and Hölder inequalities, we see that the inequalities

$$\|V_0\|_{L^\infty(\mathcal{S})} + \|V_0\|_{L^2(\mathcal{S})} \lesssim \omega(R) \tag{4.4.53}$$

and

$$\|V_0\|_{L^s(\mathcal{S})} \lesssim \|V_0\|_{L^2(\mathcal{S})} + \|\mathcal{D}V_0\|_{L^s(\mathcal{S})} \lesssim \omega(R) + \|W\|_{L^s(\mathcal{S})}, \quad s \in \mathbb{Z}_{\geq 1}, \tag{4.4.54}$$

hold for all $t \in (0,1]$ and $\|Y\|_{H^k} \leq R/C_{\mathrm{Sob}}$. Using these estimates, Sobolev's inequality and the estimate $\|W\|_{L^\infty(\mathcal{S})} \lesssim \|W\|_{L^2(\mathcal{S})}$, which follows from Hölder's inequality, we find from setting $s = 0$ and $s = k$ in (4.4.51) that

$$\|\mathcal{Q}\|_{L^2(\mathcal{S})} \lesssim \omega(R)\|W\|_{L^2(\mathcal{S})} \lesssim \omega(R)\|\Pi Z\|_{L^2(\mathcal{S})} \tag{4.4.55}$$

and

$$\|\mathcal{Q}\|_{H^k(\mathcal{S})} \lesssim \big(\omega(R) + \|W\|_{H^k(\mathcal{S})}\big)\|W\|_{H^k(\mathcal{S})} \lesssim \big(\omega(R) + R\big)\|\Pi Z\|_{H^k(\mathcal{S})} \tag{4.4.56}$$

for all $\|Z\|_{H^k(\mathcal{S})} \leq R/C_{\mathrm{Sob}}$. We further observe from (4.4.32) and (4.3.77) that

$$\Pi\mathcal{Q} = \mathcal{Q}. \tag{4.4.57}$$

Remark 4.4.4. The importance of the estimates (4.4.55)-(4.4.56) and the identity (4.4.57) is that, by an obvious modification of the proof of Theorem 3.8. in [1], these results show that terms in the energy estimates for the Fuchsian equation (4.4.26) that arise due to the "bad" singular term $\frac{1}{t}\mathcal{Q}$ can be controlled using the "good" singular $\frac{1}{t}\mathscr{A}\Pi Z$ by choosing $\omega(R)+R$ sufficiently small, which we can do by choosing R suitably small since $\lim_{R\searrow 0}\omega(R) = 0$ by assumption.

Theorem 4.5.1. *Suppose $k \in \mathbb{Z}_{\geq 5}$, $\rho_0 > 0$, the asymptotic flow assumptions (4.4.8)-(4.4.9) are satisfied for constants $\mathtt{N} \in \mathbb{Z}_{\geq k}$, $R_0 > 0$ and $\epsilon \in [0, 1/10]$, the constants $\kappa, \nu \in \mathbb{R}_{>0}$ satisfy the inequalities (4.4.49), and $z \in (0, \kappa)$. Then there exist constants $m \in \mathbb{Z}_{\geq 1}$ and $\delta > 0$ such that for any $\mathring{V} = (\mathring{V}^K) \in H^{k+1}(\mathcal{S}, \mathbb{V}^N)$ satisfying $\|\mathring{V}\|_{H^{k+1}(\mathcal{S})} < \delta$, there exists a unique solution*

$$V = (V^K) \in C^0\big((0,1], H^{k+1}(\mathcal{S}, \mathbb{V}^N)\big) \cap C^1\big((0,1], H^k(\mathcal{S}, \mathbb{V}^N)\big)$$

to the GIVP (4.3.65)-(4.3.66) for the extended system. Moreover, the following hold:

(a) The solution V satisfies the bounds

$$\|V_0(t)\|_{L^\infty(\mathcal{S})} \lesssim 1, \quad \|V_0(t)\|_{H^k(\mathcal{S})} \lesssim \frac{1}{t^\epsilon}, \quad \|\mathbb{P}V(t)\|_{H^k(\mathcal{S})} \lesssim t^\nu,$$
$$\|\mathcal{D}V(t)\|_{H^k(\mathcal{S})} \lesssim \frac{1}{t^\kappa}, \quad \|\mathbb{P}V(t)\|_{H^{k-1}(\mathcal{S})} \lesssim t^{\nu+\kappa-z} \quad \text{and} \quad \|\mathcal{D}V(t)\|_{H^{k-1}(\mathcal{S})} \lesssim \frac{1}{t^z}$$

for $t \in (0,1]$. Additionally, there exists an element $Z^\perp \in H^{k-1}(\mathcal{S}, \mathbb{W})$ satisfying $\mathbb{P}^\perp Z_0^\perp = Z_0^\perp$ such that

$$\|\Pi Z(t)\|_{H^{k-1}(\mathcal{S})} + \|\Pi^\perp Z(t) - Z^\perp\|_{H^{k-1}(\mathcal{S})} \lesssim t^{\kappa-z}$$

for $t \in (0,1]$ where Z is determined from V by (4.4.27).

(b) If, additionally, the initial data $\mathring{V}$ is chosen so that the constraint (4.3.67) is satisfied, then the solution V determines a unique classical solution $\bar{u}^K \in C^2(\bar{M}_{r_0})$, with $r_0 = \rho_0^m$, of the IVP

$$\begin{aligned} \bar{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta\bar{u}^K &= \bar{a}^{K\alpha\beta}_{IJ}\bar{\nabla}_\alpha\bar{u}^I\bar{\nabla}_\beta\bar{u}^J \quad && \text{in } \bar{M}_{r_0}, \\ (\bar{u}^K, \partial_{\bar{t}}\bar{u}^K) &= (\bar{v}^K, \bar{w}^K_1) && \text{in } \bar{\Sigma}_{r_0}, \end{aligned}$$

where $\bar{u}^K$, $\bar{v}^K$ and $\bar{w}^K$ are determined from V by (4.3.69), (4.3.71) and (4.3.72). Furthermore, the $\bar{u}^K$ satisfy the pointwise bounds

$$|\bar{u}^K| \lesssim \frac{\bar{r}}{\bar{r}^2 - \bar{t}^2}\left(1 - \frac{\bar{t}}{\bar{r}}\right)^{\frac{1}{2}+\nu+\kappa-z} \quad \text{in } \bar{M}_{r_0}.$$

Proof.

Existence and uniqueness for the extended system: Having established that the extended system (4.3.65) is symmetric hyperbolic, we can, since $k > 3/2 + 1$ by assumption, appeal to standard local-in-time existence and uniqueness results for symmetric hyperbolic systems, e.g. [53, Ch. 16, Prop. 1.4.], to conclude the existence of a $t^* \in [0,1)$, which we take to be *maximal*, and a unique solution

$$V = (V^K) \in C^0\big((t^*,1], H^{k+1}(\mathcal{S}, \mathbb{V}^N)\big) \cap C^1\big((t^*,1], H^k(\mathcal{S}, \mathbb{V}^N)\big) \tag{4.5.1}$$

to the IVP (4.3.65)-(4.3.66) for given initial data $\mathring{V} = (\mathring{V}^K) \in H^{k+1}(\mathcal{S}, \mathbb{V}^N)$, where the maximal time t^* depends on $\mathring{V}$. Next, by (4.4.4), we have that

$$Y|_{t=1} = \mathring{V}_0 = (\mathring{V}_0^K).$$

From this, (4.3.76), (4.4.22) and (4.4.27), we see, by choosing the initial data to satisfy $\|\mathring{V}\|_{H^{k+1}(\mathcal{S})} < \delta$, that $\|Z(1)\|_{H^k(\mathcal{S})} < \mathring{C}\delta$ for some positive constant $\mathring{C} > 0$ that is independent of δ. We then fix $R \in (0, R_0]$ and choose δ small enough to satisfy

$$\delta < \frac{R}{8\mathring{C}C_{\text{Sob}}} \tag{4.5.2}$$

so that

$$\|Z(1)\|_{H^k(\mathcal{S})} < \mathring{C}\delta < \frac{R}{8C_{\text{Sob}}}. \tag{4.5.3}$$

For Z to be well-defined, it is enough for Z to satisfy

$$\|Z\|_{H^k(\mathcal{S})} \leq \frac{R}{2C_{\text{Sob}}}. \tag{4.5.4}$$

This is because this bound will ensure by Sobolev's inequality (4.4.52) that

$$\|Y\|_{L^\infty} \leq C_{\text{Sob}}\|Y\|_{H^K(\mathcal{S})} \leq C_{\text{Sob}}\|Z\|_{H^k(\mathcal{S})} \leq \frac{R}{2} < R < R_0,$$

which, by the flow assumptions (4.4.8)-(4.4.9), will guarantee that the change of variables (4.4.4) is well-defined and invertible, and hence that Z is well-defined by (4.3.76), (4.4.22) and (4.4.27).

To proceed, we let $t_* \in (t^*, 0)$ denote the first time such that

$$\|Z(t_*)\|_{H^k(\mathcal{S})} = \frac{R}{2C_{\text{Sob}}}, \tag{4.5.5}$$

and if there is no such time, then we set $t_* = t^*$. We note that t_* is well-defined by

(4.5.2) and (4.5.3), and we further note from (4.5.1) and the definition of Z that

$$Z \in C^0\big((t_*, 1], H^k(\mathcal{S}, \mathbb{W})\big) \cap C^1\big((t_*, 1], H^{k-1}(\mathcal{S}, \mathbb{W})\big).$$

Now, since $\mathcal{F}(t, 1, y, 0) = 0$ by virtue of $\xi = 0$ being a solution of the asymptotic equation (1.7.2), it is not difficult to verify that the symmetric hyperbolic equations (4.3.65) and (4.4.26) both admit the trivial solution. Because of (4.5.3), we can therefore appeal to the Cauchy stability property enjoyed by symmetry hyperbolic equations to conclude, by choosing δ small enough, that t_*, where of course $t_* \geq t^*$, can be made to be as small as we like and that the inequality

$$\max_{t_0 \leq t \leq 1} \|Z(t)\|_{H^k(\mathcal{S})} < 2\mathring{C}\delta < \frac{R}{4C_{\text{Sob}}} \tag{4.5.6}$$

is valid for

$$t_0 = \min\{2t_*, 1/2\}.$$

Recalling that we are free to choose the constant $\sigma > 0$, see (4.4.42), as small as we like by choosing the constant $m \in \mathbb{Z}_{\geq 1}$ sufficiently large, we can, for any given $\sigma_* > 0$, arrange, since $\tilde{\epsilon} > 0$ (see (4.4.50)), that

$$\sigma + t^{\tilde{\epsilon}} < \sigma_*, \quad t \in (0, t_0], \tag{4.5.7}$$

by choosing δ small enough to guarantee that t_0 is sufficiently small to ensure that this inequality holds.

In light of Remarks 4.4.3 and 4.4.4, the bounds (4.4.40), (4.4.41), (4.4.42), and (4.5.7), the relations (4.4.36)-(4.4.39), the expansion (4.4.50), and the estimates (4.4.55)-(4.4.56), all taken together, show that if the constants $m \in \mathbb{Z}_{\geq 1}$ and $\delta > 0$ are chosen sufficiently large and small, respectively, and the constants κ, ν are chosen to satisfy (4.4.49), then the Fuchsian system (4.4.26), which Z satisfies, will, after the simple time transformation $t \mapsto -t$, satisfy all the required assumptions needed to apply the time rescaled version, see [1, §3.4.] and the remark below, of Theorem 3.8. from [1].

Remark 4.5.2. From the discussion from Section 3.4. of [1] and Section 4.4.3 of this article, it not difficult to see that the appropriate rescaling power p, see equation (3.106) in [1], in the current context is

$$p = \kappa - 2\epsilon, \tag{4.5.8}$$

which, we note, by (4.4.49), satisfies the required bounds $0 < \kappa - 2\epsilon \leq 1$. We further note from Theorem 3.8. from [1], see also [1, §3.4.], that parameter ζ defined by equation (3.59) of [1], which is involved in determining the decay of solutions, is, in the current context, determined by

$$\zeta = \kappa - z \tag{4.5.9}$$

where $z > 0$ can be made as small as we like by choosing the constant m large enough and the constants R, t_0 small enough to ensure that σ_* and $\|Z\|_{H^\kappa(\mathcal{S})}$ are sufficiently small.

We therefore conclude from the proof of Theorem 3.8. from [1] that Z, which solves (4.4.26), satisfies an energy estimate of the form

$$\|Z(t)\|^2_{H^k(\mathcal{S})} + \int_t^{t_0} \frac{1}{\tau}\|\Pi Z(\tau)\|^2_{H^k(\mathcal{S})}\, d\tau \leq C_E^2 \|Z(t_0)\|^2 \tag{4.5.10}$$

for all $t \in (t_*, t_0]$. By Grönwall's inequality and (4.5.3), we then have

$$\sup_{t\in(t_*,t_0)} \|Z(t)\|_{H^k(\mathcal{S})} \leq e^{C_E(t_*-t_0)}\|Z(t_0)\|_{H^k(\mathcal{S})} < e^{C_E(t_*-t_0)}\mathring{C}\delta. \tag{4.5.11}$$

Choosing δ now, by shrinking it if necessary, to satisfy $\delta < \frac{R}{3\mathring{C}C_{\text{Sob}}e^{C_E(t_*-t_0)}}$ in addition to (4.5.2), the bounds (4.5.6) and (4.5.11) implies that

$$\sup_{t\in(t_*,1)} \|Z(t)\|_{H^k(\mathcal{S})} < \frac{R}{3C_{\text{Sob}}}. \tag{4.5.12}$$

From this inequality and the definition (4.5.5) for t_*, we conclude that $t_* = t^*$.

Now, from (4.4.10), (4.4.11), Sobolev's inequality, and the Moser estimates (e.g. [53, Ch. 13, Prop. 3.9.]), we see from (4.4.4) and (4.4.27) that V_0 can be bounded by

$$\|V_0(t)\|_{H^k(\mathcal{S})} \leq \frac{1}{t^\epsilon} C(\|Z(t)\|_{H^k(\mathcal{S})})\|Z(t)\|_{H^k(\mathcal{S})} \tag{4.5.13}$$

for Z satisfying (4.5.4), while we see from (4.4.22), (4.4.27) and (4.4.32) that $\mathbb{P}V(t)$ is bounded by

$$\|\mathbb{P}V(t)\|_{H^s(\mathcal{S})} \leq t^\nu \|\Pi Z(t)\|_{H^s(\mathcal{S})}, \quad s \in \mathbb{Z}_{\geq 0}. \tag{4.5.14}$$

Since $t_* = t^*$, the estimates (4.5.12), (4.5.13) and (4.5.14) imply that $\|V(t)\|_{H^k(\mathcal{S})}$ is finite for any $t \in (t^*, 0)$. By the maximality of t^* and the continuation principle for symmetric hyperbolic equations, we conclude that $t^* = 0$, which establishes the existence of solutions to the extended IVP (4.3.65)-(4.3.66) on the spacetime region $(0,1] \times \mathcal{S}$.

Uniform bounds for V: From (4.3.76), (4.4.27), (4.4.53), (4.5.12), (4.5.13) and (4.5.14), we see that the estimates

$$\|V_0(t)\|_{L^\infty(\mathcal{S})} \lesssim \omega(\delta), \quad \|V_0(t)\|_{H^k(\mathcal{S})} \lesssim \frac{1}{t^\epsilon}\delta,$$

and

$$\|\mathbb{P}V(t)\|_{H^k(\mathcal{S})} \lesssim t^\nu \delta, \quad \|\mathcal{D}V(t)\|_{H^k(\mathcal{S})} \lesssim \frac{1}{t^\kappa}\delta$$

hold for $t \in (0,1]$. Furthermore, in view of the Remark 4.5.2, see in particular, (4.5.9), the coefficient properties from Section 4.4.3, and the fact that $\kappa \in (0,1/3]$, we conclude from Theorem 3.8. and Section 3.4. of [1] that, for any fixed $z > 0$, there exists, provided that m and δ are chosen sufficiently large and small respectively, an element $Z^\perp \in H^{k-1}(\mathcal{S}, \mathbb{W})$ satisfying $\mathbb{P}^\perp Z_0^\perp = Z_0^\perp$ such that

$$\|\Pi Z(t)\|_{H^{k-1}(\mathcal{S})} + \|\Pi^\perp Z(t) - Z^\perp\|_{H^{k-1}(\mathcal{S})} \lesssim t^{\kappa - z}$$

for $t \in (0,1]$. With the help of the above inequality, (4.3.76), (4.4.27), (4.4.32) and (4.5.14), we conclude that V also satisfies

$$\|\mathbb{P}V(t)\|_{H^{k-1}(\mathcal{S})} \lesssim t^{\nu+\kappa-z} \quad \text{and} \quad \|\mathcal{D}V(t)\|_{H^{k-1}(\mathcal{S})} \lesssim \frac{1}{t^z} \tag{4.5.15}$$

for $t \in (0,1]$.

Existence for the wave equations (4.0.1): Letting $r_0 = \rho_0^m$, we know from the discussion contained in Section 4.3.2, that if the initial data $\mathring{V}$ is chosen to satisfy the constraints (4.3.67) on the spacelike hypersurface Σ_{r_0}, then the solution $V = (V_0^K, V_1^K, V_\Lambda^K, V_4^K)$ to the extended system (4.3.65) determines a classical solution $\bar{u}^K$ of the semilinear wave equations (4.0.1) on $\bar{M}_{r_0}$ via the formula (4.3.69). Moreover, this solution is uniquely determined by the initial data on Σ_{r_0} that is obtained from the restriction of the initial data $\mathring{V}$ to the initial hypersurface Σ_{r_0} and the transformation formulas (4.3.70) and (4.3.71). To complete the proof, we note from (4.3.68), Sobolev's inequality, the decay estimate (4.5.15), and (4.1.2) that each $\bar{u}^K$ satisfies the pointwise bound

$$|\bar{u}^K| \lesssim \frac{\bar{r}}{\bar{r}^2 - \bar{t}^2}\left(1 - \frac{\bar{t}}{\bar{r}}\right)^{\frac{1}{2}+\nu+\kappa-z} \quad \text{in } \bar{M}_{r_0}.$$

$\square$

Corollary 4.5.3. *Suppose $k \in \mathbb{Z}_{\geq 5}$, $\rho_0 > 0$, $z > 0$ and the bounded weak null condition (see Definition 1.7.2) holds. Then there exist constants $m \in \mathbb{Z}_{\geq 1}$ and $\delta > 0$ such that for any $\mathring{V} = (\mathring{V}^K) \in H^{k+1}(\mathcal{S}, \mathbb{V}^N)$ satisfying $\|\mathring{V}\|_{H^k(\mathcal{S})} < \delta$, there exists a unique solution*

$$V = (V^K) \in C^0\big((0,1], H^{k+1}(\mathcal{S}, \mathbb{V}^N)\big) \cap C^1\big((0,1], H^k(\mathcal{S}, \mathbb{V}^N)\big)$$

to the IVP (4.3.65)-(4.3.66). Moreover, the following hold:

(a) The solution V satisfies the uniform bounds

$$\|V_0(t)\|_{L^\infty(\mathcal{S})} \lesssim 1, \quad \|V_0(t)\|_{H^k(\mathcal{S})} + \|\mathcal{D}V(t)\|_{H^k(\mathcal{S})} \lesssim \frac{1}{t^z}$$

and

$$\|\mathbb{P}V(t)\|_{H^k(\mathcal{S})} \lesssim t^{\frac{1}{2}-z}$$

for $t \in (0,1]$.

(b) If, additionally, the initial data $\mathring{V}$ is chosen so that the constraint (4.3.67) is satisfied, then the solution V determines a unique classical solution $\bar{u}^K \in C^2(\bar{M}_{r_0})$, with $r_0 = \rho_0^m$, of the IVP

$$\begin{aligned} \bar{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta\bar{u}^K &= \bar{a}^{K\alpha\beta}_{IJ}\bar{\nabla}_\alpha\bar{u}^I\bar{\nabla}_\beta\bar{u}^J \quad &&\text{in } \bar{M}_{r_0}, \\ (\bar{u}^K, \partial_{\bar{t}}\bar{u}^K) &= (\bar{v}^K, \bar{w}^K) &&\text{in } \bar{\Sigma}_{r_0}, \end{aligned}$$

where $\bar{u}^K$, $\bar{v}^K$ and $\bar{w}^K$ are determined from V by (4.3.69), (4.3.71) and (4.3.72). Furthermore, the $\bar{u}^K$ satisfy the pointwise bounds

$$|\bar{u}^K| \lesssim \frac{\bar{r}}{\bar{r}^2 - \bar{t}^2}\left(1 - \frac{\bar{t}}{\bar{r}}\right)^{1-z} \quad \text{in } \bar{M}_{r_0}.$$

Proof. By Proposition 4.4.1, we know that the asymptotic flow satisfies the flow assumptions (4.4.8)-(4.4.9) for some $R_0 > 0$ and any $\epsilon \in (0, 1/10]$. Fixing $\epsilon \in (0, 1/11)$, we set $z = \epsilon$, $\nu = \frac{1}{2} - 5z$ and $\kappa = 3z$. It is then not difficult to verify that these choices for z, ν and κ satisfy the inequalities (4.4.49) and $0 < z < \kappa$. The proof now follows directly from Theorem 4.5.1. $\square$

5
Epilogue

In this thesis we showed three applications of the Fuchsian method to a class of semi-linear wave equations which are relevant in the context of General Relativity. The discussion of the asymptotic properties of space-times is an active field of research and in this thesis we provide decay estimates along global existence for wave equations. This is the first step in the analysis of the asymptotic properties of space-times in General Relativity. The main results are listed below:

- Global existence for semi-linear wave equations in Minkowski space-time with non-linear terms satisfying the *null condition*.
- Global existence for semi-linear wave equations in Schwarzschild space-time with non-linear terms satisfying the *null condition*.
- Global existence for semi-linear wave equations in Minkowski space-time whose asymptotic equation satisfies the *bounded weak-null condition*.

The first two results are explained in Chapter 3 of this thesis. These two results are not new since other authors have proved, with different methods, global existence for semi-linear wave equations in Minkowski and Kerr space-time, which is a generalization of Schwarzschild space-time. Global existence results, under a small initial data condition, for systems of wave equation of the form (3.1.1) were given in the pioneering work of Klainerman [40] and Christodoulou [41]. Global existence results, under a small initial data assumption, for solutions to scalar semi-linear wave equations of the form (3.3.1) on Kerr space-time were established in [63].

The last result in this thesis explained in Chapter 4, is new and it complements the work reported in [68]. In this sense, not all the results in this thesis are new. We consider that the relevance of this thesis relies on showing that the Fuchsian method is a viable alternative to study the global existence of solutions to non-linear hyperbolic equations. The Fuchsian method, to the perception of the author, is much *easier* to follow than other methods and it seems to be capable of handling a wide range of different problems. In future works we expect to show more applications of the Fuchsian method for hyperbolic equations with a more complicated structure.

One possible disadvantage of the method is that it is not obvious how to find a suitable transformation of the space-time. In order to apply the Theorem (2.3.1), we need to transform our original space-time into a bounded, non-physical, space-time. This step is the first hindrance that one faces when trying to apply the Fuchsian method to a system of hyperbolic wave equations. The objective of the Fuchsian method is to write the system of wave equations of interest into the Fuchsian form (2.0.1)-(2.0.2), provided that it satisfies the structural conditions and assumptions of the Theorem (2.3.1). Choosing the right compactification of space-time can make this task easy or impossible. One of the questions we still have to answer is how to choose the right transformation and how much freedom do we have to choose between different spaces.

At the moment we do not have a systematic approach for choosing the right transformation. Choosing such a transformation is in a sense, analogous to choosing a *gauge*. The right gauge will allow to transform the original system into a Fuchsian system. Finding the right mapping of the space-time is a puzzling task but we are starting to observe some patterns which might help to identify a *good* class of mappings useful for the method. For example, the map given in (4.1.2) makes evident the structure of the *null condition*. One can see that the null condition is satisfied when the terms $\bar{b}^K_{IJ}$ defined in (4.2.1) vanish. We have observed that when transforming a general second order tensor some maps make the terms $\bar{b}^K_{IJ}$ obvious in some of the components of the tensor. This might be a useful hint to start a classification of transformations for the Fuchsian method. In addition, the map (4.1.2) shows the utility of Friedrich's *cylinder at spatial infinity* for solving non-linear wave equations near spatial infinity on Minkowski and Schwarzschild space-times. The successful approach using this construction leads to the following question, can we use a similar manipulation of space-time near time-like infinity?

Some other problems that we believe can be handled with the Fuchsian method are the same treated in this thesis but in a time-like region of Minkowski space time, that is, a region of the form

$$\bar{M} = \{(\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi}) \in (\infty, \infty) \times (0, \infty) \times \mathbb{S}^2 | \bar{t}^2 - \bar{r}^2 > 0\}. \tag{5.0.1}$$

This could be achieved by finding the analogous *cylinder at time-like infinite*, or a different transformation. Moreover we expect that the Fuchsian method will be capable of handling systems of wave equations on Minkoswki space-time with quasi-linear terms. In essence the Fuchsian method and the analysis carried out in Chapter 3 should apply with some *modifications* to the case of quasi-linear wave equations.

A

A.1 Index conventions

Throughout this thesis we use the following convention for different index quantities:

Alphabet	Examples	Index range	Index quantities
Lowercase Greek	μ, ν, γ	$0, 1, 2, 3$	space-time coordinate components, e.g. $(x^\mu) = (t, r, \theta, \phi)$
Uppercase Greek	Λ, Σ, Ω	$2, 3,$	spherical coordinate components, e.g. $(x^\Lambda) = (\theta, \phi)$
Lowercase Latin	i, j, k	$1, 2, 3$	spatial coordinates components, e.g. $(y^i) = (\rho, \theta, \phi)$
Uppercase Latin	I, J, K	1 to N	wave equation indexing, e.g. u^I
Lowercase Calligraphic	$\mathscr{q}, \mathscr{p}, \mathscr{r}$	0,1	time and radial coordinate components, e.g. $(x^{\mathscr{q}}) = (t, r)$
Uppercase Calligraphic	$\mathcal{I}, \mathcal{J}, \mathcal{K}$	0,1,2,3,4	first order wave formulation indexing, e.g. $V_{\mathcal{I}}^K$

A.2 Calculus Inequalities

In this appendix, we collect, for the convenience of the reader, a number of calculus inequalities that we employ throughout this article. The proof of the following inequalities are well known and may be found, for example, in the books [72], [73] and [53]. As in the introduction, Σ will denote a closed n-dimensional manifold.

Theorem A.2.1. [Hölder's inequality] *If* $0 < p, q, r \le \infty$ *satisfy* $1/p + 1/q = 1/r$, *then*

$$\|uv\|_{L^r} \le \|u\|_{L^p}\|v\|_{L^q}$$

for all $u \in L^p(\Sigma)$ *and* $v \in L^q(\Sigma)$.

Theorem A.2.2. [Sobolev's inequality] *Suppose* $1 \le p < \infty$ *and* $s \in \mathbb{Z}_{>n/p}$. *Then*

$$\|u\|_{L^\infty} \lesssim \|u\|_{W^{s,p}}$$

for all $u \in W^{s,p}(\Sigma)$.

Theorem A.2.3. [Product and commutator estimates]

(i) Suppose $1 \le p_1, p_2, q_1, q_2 \le \infty$, $s \in \mathbb{Z}_{\ge 1}$, *and*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{r}.$$

Then

$$\|\nabla^s(uv)\|_{L^r} \lesssim \|u\|_{W^{s,p_1}}\|v\|_{L^{q_1}} + \|u\|_{L^{p_2}}\|v\|_{W^{s,q_2}}$$

and

$$\|[\nabla^s, u]v\|_{L^r} \lesssim \|\nabla u\|_{L^{p_1}}\|v\|_{W^{s-1,q_1}} + \|\nabla u\|_{W^{s-1,p_2}}\|v\|_{L^{q_2}}$$

for all $u, v \in C^\infty(\Sigma)$.

(ii) Suppose $s_1, s_2, s_3 \in \mathbb{Z}_{\ge 0}$, $s_1, s_2 \ge s_3$, $1 \le p \le \infty$, *and* $s_1 + s_2 - s_3 > n/p$. *Then*

$$\|uv\|_{W^{s_3,p}} \lesssim \|u\|_{W^{s_1,p}}\|v\|_{W^{s_2,p}}$$

for all $u \in W^{s_1,p}(\Sigma)$ *and* $v \in W^{s_2,p}(\Sigma)$.

Theorem A.2.4. [Moser's estimates] *Suppose* $1 \le p \le \infty$, $s \in \mathbb{Z}_{\ge 1}$, $0 \le k \le s$, *and* $f \in C^s(U)$, *where* U *is open and bounded in* $\mathbb{R}$ *and contains* 0, *and* $f(0) = 0$. *Then*

$$\|\nabla^k f(u)\|_{L^p} \le C\big(\|f\|_{C^s(\overline{U})}\big)(1 + \|u\|_{L^\infty}^{s-1})\|u\|_{W^{s,p}}$$

for all $u \in C^0(\Sigma) \cap L^\infty(\Sigma) \cap W^{s,p}(\Sigma)$ *with* $u(x) \in U$ *for all* $x \in \Sigma$.

Lemma A.2.5. [Ehrling's lemma] *Suppose* $1 \leq p < \infty$*,* $s_0, s, s_1 \in \mathbb{Z}_{\geq 0}$*, and* $s_0 < s < s_1$*. Then for any* $\epsilon > 0$ *there exists a constant* $C = C(\epsilon^{-1})$ *such that*

$$\|u\|_{W^{s,p}} \leq \epsilon \|u\|_{W^{s_1,p}} + C(\epsilon^{-1})\|u\|_{W^{s_0,p}}$$

for all $u \in W^{s_1,p}(\Sigma)$*.*

A.3 Conformal Transformations

Conformal maps are of great importance in Riemannian and pseudo Riemannian geometry. In our case we are interested in conformal maps over Lorentzian manifolds. We say that, given two Lorentzian Manifolds $\tilde{M}, M$ equipped with metrics $\tilde{g}$ and g respectively, the map $\psi : \tilde{M} \to M$ is a conformal map if the images under the map ψ of two curves $\tilde{\gamma}_1(\alpha), \tilde{\gamma}_2(\alpha) \in \tilde{M}$ that intersect at the point $p \in \tilde{M}$ forming an angle θ, are curves $\gamma_1(\alpha), \gamma_2(\alpha) \in M$ that intersect forming the same angle θ, see [74], [75]. In other words, a conformal map is said to preserve angles locally. In general relativity they are important since they preserve the causal structure and null cones of space-times. We say that two metrics are conformally equivalent if they are related by a conformal map such that

$$\psi : (\tilde{M}, \tilde{g}) \to (M, g), \tag{A.3.1}$$

and

$$\psi^* g = \Omega^{-2}\tilde{g}, \tag{A.3.2}$$

where

$$\Omega : \tilde{M} \to \mathbb{R}. \tag{A.3.3}$$

In the following calculations, we will simply write

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{A.3.4}$$

to indicate that this two metrics are conformally equivalent and we follow the same conventions as in [76]. A similar relation applies for the inverse metric

$$\tilde{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}. \tag{A.3.5}$$

We start our analysis of geometric quantities from the fact that given two connections $\tilde{\nabla}$ and ∇, which are associated with the metrics $\tilde{g}$ and g respectively, we can define the tensor field

$$C^{\rho}{}_{\mu\nu} w_{\rho} = \nabla_{\mu} w_{\nu} - \tilde{\nabla}_{\mu} w_{\nu}, \tag{A.3.6}$$

where $C^{\rho}{}_{\mu\nu}$ is symmetric in its lower indices by definition, and w_{ν} is a co-vector field. This shows that the difference in the action of ∇ and $\tilde{\nabla}$ is given by

$$\tilde{\nabla}_{\mu} w_{\nu} = \nabla_{\mu} w_{\nu} - C^{\rho}_{\mu\nu} w_{\rho}. \tag{A.3.7}$$

Introducing the vector field t^{ν} and noting that the actions of $\tilde{\nabla}$ and ∇ must agree on scalars, that is,

$$(\nabla - \tilde{\nabla})(w_{\rho} t^{\rho}) = 0, \tag{A.3.8}$$

we can deduce from the Leibniz rule and the previous equation that

$$\tilde{\nabla}_{\mu} t^{\nu} = \nabla_{\mu} t^{\nu} + C^{\nu}_{\mu\rho} t^{\rho}. \tag{A.3.9}$$

By similar calculations, it is not difficult to see that the action of $\tilde{\nabla}$ on a general tensor field can be written in terms of the connection ∇ and the tensor field $C^{\rho}_{\mu\nu}$ as

$$\begin{aligned}\tilde{\nabla}_{\rho} T^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_m} =& \nabla_{\rho} T^{\mu_1 \cdots \nu_n}{}_{\nu_1 \cdots \nu_m} + C^{\mu_1}_{\rho\lambda} T^{\lambda\mu_2 \cdots \mu_n}{}_{\nu_1 \cdots \nu_m} + \cdots + C^{\mu_n}_{\rho\lambda} T^{\mu_1 \cdots \lambda}{}_{\nu_1 \cdots \nu_m} - \\ & C^{\lambda}_{\rho\nu_1} T^{\mu_1 \cdots \mu_n}{}_{\lambda\nu_2 \cdots \nu_m} - \cdots - C^{\lambda}_{\rho\nu_n} T^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \lambda} .\end{aligned} \tag{A.3.10}$$

Note that we can apply this to the metric $g_{\mu\nu}$, and from the assumption that there is a unique derivative operator such that $\tilde{\nabla}_{\mu} \tilde{g}_{\nu\rho} = 0$, we get

$$\tilde{\nabla}_{\mu} \tilde{g}_{\nu\rho} = \nabla_{\mu} \tilde{g}_{\nu\rho} - C^{\sigma}{}_{\mu\nu} \tilde{g}_{\sigma\rho} - C^{\sigma}{}_{\mu\rho} \tilde{g}_{\nu\sigma} = 0, \tag{A.3.11}$$

where we note that

$$C_{\rho\mu\nu} + C_{\nu\mu\rho} = \nabla_{\mu} \tilde{g}_{\nu\rho}, \tag{A.3.12}$$

and from (A.3.12) that

$$C_{\rho\nu\mu} + C_{\mu\nu\rho} = \nabla_{\nu} \tilde{g}_{\mu\rho}, \quad \text{and} \quad C_{\nu\rho\mu} + C_{\mu\rho\nu} = \nabla_{\rho} \tilde{g}_{\mu\nu}. \tag{A.3.13}$$

Then from equations (A.3.12) and (A.3.13), we deduce that

$$C^{\rho}{}_{\mu\nu} = \frac{1}{2} \tilde{g}^{\rho\sigma} \left(\nabla_{\mu} \tilde{g}_{\nu\sigma} + \nabla_{\nu} \tilde{g}_{\mu\sigma} - \nabla_{\sigma} \tilde{g}_{\mu\nu} \right). \tag{A.3.14}$$

In the case that the derivative operator ∇_μ is the ordinary derivative ∂_μ, (A.3.14) becomes the usual Christoffel symbols

$$C^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\mu\nu} = \frac{1}{2}\tilde{g}^{\rho\sigma}\left(\partial_\mu \tilde{g}_{\nu\sigma} + \partial_\nu \tilde{g}_{\mu\sigma} - \partial_\sigma \tilde{g}_{\mu\nu}\right), \tag{A.3.15}$$

and using the relations $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ and $\tilde{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}$, the tensor field (A.3.14) can be written in terms of the conformal factor Ω and the connection ∇ as

$$C^\rho{}_{\mu\nu} = \Omega^{-1}\left(\delta^\rho_\nu \nabla_\mu \Omega + \delta^\rho_\mu \nabla_\nu \Omega - g^{\rho\sigma} g_{\mu\nu} \nabla_\sigma \Omega\right). \tag{A.3.16}$$

Having the relation (A.3.16), it is relatively easy to find the correspondence between different geometric quantities related by a conformal map. Applying twice (A.3.10) to a co-vector field w_ρ gives

$$\begin{aligned}\tilde{\nabla}_\mu \tilde{\nabla}_\nu w_\rho &= \tilde{\nabla}_\mu\left(\nabla_\nu w_\rho - C^\lambda{}_{\nu\rho} w_\lambda\right)\\ &= \nabla_\mu\left(\nabla_\nu w_\rho - C^\lambda{}_{\nu\rho} w_\lambda\right) - C^\lambda{}_{\mu\nu}\left(\nabla_\lambda w_\rho - C^\kappa{}_{\lambda\rho} w_\kappa\right) - C^\lambda{}_{\mu\rho}\left(\nabla_\nu w_\lambda - C^\kappa{}_{\nu\lambda} w_\kappa\right).\end{aligned} \tag{A.3.17}$$

Then, applying (A.3.17) and (A.3.10) to $\tilde{R}_{\mu\nu\rho}{}^\sigma w_\sigma$, we see that the relation of the Riemann tensor between the physical and the unphysical manifolds is given by

$$\begin{aligned}\tilde{R}_{\mu\nu\rho}{}^\sigma w_\sigma =& \tilde{\nabla}_\mu \tilde{\nabla}_\nu w_\rho - \tilde{\nabla}_\nu \tilde{\nabla}_\mu w_\rho\\ =& \nabla_\mu\left(\nabla_\nu w_\rho - C^\lambda{}_{\nu\rho} w_\lambda\right) - C^\lambda{}_{\mu\nu}\left(\nabla_\lambda w_\rho - C^\kappa{}_{\lambda\rho} w_\kappa\right) - C^\lambda{}_{\mu\rho}\left(\nabla_\nu w_\lambda - C^\kappa{}_{\nu\lambda} w_\kappa\right) -\\ &\left[\nabla_\nu\left(\nabla_\mu w_\rho - C^\lambda{}_{\mu\rho} w_\lambda\right) - C^\lambda{}_{\nu\mu}\left(\nabla_\lambda w_\rho - C^\kappa{}_{\lambda\rho} w_\kappa\right) - C^\lambda{}_{\nu\rho}\left(\nabla_\mu w_\lambda - C^\kappa{}_{\mu\lambda} w_\kappa\right)\right].\end{aligned} \tag{A.3.18}$$

Simplifying (A.3.18) yields

$$\tilde{R}_{\mu\nu\rho}{}^\sigma = R_{\mu\nu\rho}{}^\sigma - 2\nabla_{[\mu} C^\sigma{}_{\nu]\rho} + 2C^\lambda{}_{\rho[\mu} C^\sigma{}_{\nu]\lambda}. \tag{A.3.19}$$

With the help of (A.3.16), we can write the second term of (A.3.19) as

$$\begin{aligned}-2\nabla_{[\mu} C^\sigma{}_{\nu]\rho} =& -\nabla_\mu\left(\delta^\sigma_\rho \nabla_\nu \ln\Omega + \delta^\sigma_\nu \nabla_\rho \ln\Omega - g^{\sigma\kappa} g_{\nu\rho} \nabla_\kappa \ln\Omega\right) +\\ &\nabla_\nu\left(\delta^\sigma_\rho \nabla_\mu \ln\Omega + \delta^\sigma_\mu \nabla_\rho \ln\Omega - g^{\sigma\kappa} g_{\mu\rho} \nabla_\kappa \ln\Omega\right)\\ =& \quad 2\delta^\sigma{}_{[\mu} \nabla_{\nu]} \nabla_\rho \ln\Omega - 2g^{\sigma\kappa} g_{\rho[\mu} \nabla_{\nu]} \nabla_\kappa \ln\Omega.\end{aligned} \tag{A.3.20}$$

Similarly, the third term of (A.3.19) can be expanded using (A.3.16) to obtain the following expression

$$
\begin{aligned}
2C^{\lambda}{}_{\rho[\mu}C^{\sigma}{}_{\nu]\lambda} &= \left(C^{\lambda}{}_{\rho\mu}C^{\sigma}{}_{\nu\lambda} - C^{\lambda}{}_{\rho\nu}C^{\sigma}{}_{\mu\lambda}\right) = \\
&\left(\delta^{\lambda}_{\mu}\nabla_{\rho}\ln\Omega + \delta^{\lambda}_{\rho}\nabla_{\mu}\ln\Omega - g^{\lambda\kappa}g_{\rho\mu}\nabla_{\kappa}\ln\Omega\right)\left(\delta^{\sigma}_{\lambda}\nabla_{\nu}\ln\Omega + \delta^{\sigma}_{\nu}\nabla_{\lambda}\ln\Omega - g^{\sigma\kappa}g_{\lambda\nu}\nabla_{\kappa}\ln\Omega\right) - \\
&\left(\delta^{\lambda}_{\nu}\nabla_{\rho}\ln\Omega + \delta^{\lambda}_{\rho}\nabla_{\nu}\ln\Omega - g^{\lambda\kappa}g_{\rho\nu}\nabla_{\kappa}\ln\Omega\right)\left(\delta^{\sigma}_{\lambda}\nabla_{\mu}\ln\Omega + \delta^{\sigma}_{\mu}\nabla_{\lambda}\ln\Omega - g^{\sigma\kappa}g_{\lambda\mu}\nabla_{\kappa}\ln\Omega\right), \\
&= \big(\delta^{\sigma}_{\mu}\nabla_{\rho}\ln\Omega\nabla_{\nu}\ln\Omega + \delta^{\sigma}_{\nu}\nabla_{\rho}\ln\Omega\nabla_{\mu}\ln\Omega - g^{\sigma\kappa}g_{\mu\nu}\nabla_{\rho}\ln\Omega\nabla_{\kappa}\ln\Omega + \\
&\quad \delta^{\sigma}_{\rho}\nabla_{\mu}\ln\Omega\nabla_{\nu}\ln\Omega + \delta^{\sigma}_{\nu}\nabla_{\mu}\ln\Omega\nabla_{\rho}\ln\Omega - g^{\sigma\kappa}g_{\rho\nu}\nabla_{\mu}\ln\Omega\nabla_{\kappa}\ln\Omega - \\
&\quad -g^{\sigma\kappa}g_{\rho\nu}\nabla_{\kappa}\ln\Omega\nabla_{\mu}\ln\Omega - \delta^{\sigma}_{\mu}g^{\lambda\kappa}g_{\rho\nu}\nabla_{\kappa}\ln\Omega\nabla_{\lambda}\ln\Omega + \delta^{\kappa}_{\nu}g^{\sigma\gamma}g_{\rho\mu}\nabla_{\kappa}\ln\Omega\nabla_{\gamma}\ln\Omega\big) - \\
&\big(\delta^{\sigma}_{\nu}\nabla_{\rho}\ln\Omega\nabla_{\mu}\ln\Omega + \delta^{\sigma}_{\mu}\nabla_{\rho}\ln\Omega\nabla_{\nu}\ln\Omega - g^{\sigma\kappa}g_{\nu\mu}\nabla_{\rho}\ln\Omega\nabla_{\kappa}\ln\Omega + \\
&\quad \delta^{\sigma}_{\rho}\nabla_{\nu}\ln\Omega\nabla_{\mu}\ln\Omega + \delta^{\sigma}_{\mu}\nabla_{\nu}\ln\Omega\nabla_{\rho}\ln\Omega - g^{\sigma\kappa}g_{\rho\mu}\nabla_{\nu}\ln\Omega\nabla_{\kappa}\ln\Omega - \\
&\quad -g^{\sigma\kappa}g_{\rho\mu}\nabla_{\kappa}\ln\Omega\nabla_{\nu}\ln\Omega - \delta^{\sigma}_{\nu}g^{\lambda\kappa}g_{\rho\mu}\nabla_{\kappa}\ln\Omega\nabla_{\lambda}\ln\Omega + \delta^{\kappa}_{\nu}g^{\sigma\gamma}g_{\rho\mu}\nabla_{\kappa}\ln\Omega\nabla_{\gamma}\ln\Omega\big),
\end{aligned}
\tag{A.3.21}
$$

which can be reduced to

$$
2C^{\lambda}{}_{\rho[\mu}C^{\sigma}{}_{\nu]\lambda} = 2\delta^{\sigma}{}_{[\nu}\nabla_{\mu]}\ln\Omega\nabla_{\rho}\ln\Omega + 2g^{\sigma\kappa}g_{\rho[\mu}\nabla_{\nu]}\nabla_{\kappa}\ln\Omega + 2\delta^{\sigma}{}_{[\mu}g_{\nu]\rho}g^{\lambda\kappa}\nabla_{\kappa}\ln\Omega\nabla_{\lambda}\ln\Omega.
\tag{A.3.22}
$$

Now, using (A.3.20) and (A.3.22), we observe that the Riemann tensor $\tilde{R}_{\mu\nu\rho}{}^{\sigma}$ is related to the Riemann tensor $R_{\mu\nu\rho}{}^{\sigma}$ by

$$
\begin{aligned}
\tilde{R}_{\mu\nu\rho}{}^{\sigma} =& R_{\mu\nu\rho}{}^{\sigma} + 2\delta^{\sigma}{}_{[\mu}\nabla_{\nu]}\nabla_{\rho}\ln\Omega - 2g^{\lambda\sigma}g_{\rho[\mu}\nabla_{\nu]}\nabla_{\lambda}\ln\Omega + \\
& 2\delta^{\sigma}{}_{[\nu}\nabla_{\mu]}\ln\Omega\nabla_{\rho}\ln\Omega + 2g^{\kappa\sigma}g_{\rho[\mu}\nabla_{\nu]}\nabla_{\kappa}\ln\Omega + \\
& 2\delta^{\sigma}{}_{[\mu}g_{\nu]\rho}g^{\lambda\kappa}\nabla_{\kappa}\ln\Omega\nabla_{\lambda}\ln\Omega.
\end{aligned}
\tag{A.3.23}
$$

From equation (A.3.23), we see, by contracting the indices ν and σ of $\tilde{R}_{\mu\nu\rho}{}^{\sigma}$, that the Ricci tensor $R_{\mu\rho}$ and $\tilde{R}_{\mu\rho}$ are related by

$$
\begin{aligned}
\tilde{R}_{\mu\rho} =& R_{\mu\rho} + (2-n)\nabla_{\mu}\nabla_{\rho}\ln\Omega - g_{\mu\rho}g^{\lambda\sigma}\nabla_{\lambda}\nabla_{\sigma}\ln\Omega + (n-2)\nabla_{\mu}\ln\Omega\nabla_{\rho}\ln\Omega \\
& + (2-n)g_{\mu\rho}g^{\lambda\sigma}\nabla_{\lambda}\ln\Omega\nabla_{\sigma}\ln\Omega.
\end{aligned}
\tag{A.3.24}
$$

Contracting $\tilde{R}_{\mu\rho}$ with $\tilde{g}^{\mu\rho} = \Omega^{-2} g^{\mu\rho}$, we obtain the following relation for the curvature scalar

$$\tilde{R} = \Omega^{-2}\left[R - 2(n-1)g^{\mu\rho}\nabla_\mu\nabla_\rho \ln\Omega + (2-n)(n-1)g^{\mu\rho}\nabla_\mu \ln\Omega \nabla_\rho \ln\Omega\right]. \tag{A.3.25}$$

Now that we have the relationship of the Riemann and Rcci tensors and the scalar of curvature between the manifolds $\tilde{M}$, M, we can proceed transforming a given wave equation on the space time $\tilde{M}$ into the conformal manifold M. We start by considering the wave equation

$$\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{u} - \frac{(n-2)}{4(n-1)}\tilde{R}\tilde{u} = \tilde{f} \tag{A.3.26}$$

where

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}. \tag{A.3.27}$$

Here $\tilde{\nabla}$ is the connection associated to the metric $\tilde{g}$, and we define

$$\tilde{u} = \Omega^s u, \tag{A.3.28}$$

with s a constant. Using (A.3.10), we expand (A.3.26) starting with

$$\begin{aligned}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{u} =& \tilde{g}^{\mu\nu}\left[\nabla_\mu\nabla_\nu(\Omega^s u) - C^\kappa_{\mu\nu}\nabla_\kappa(\Omega^s u)\right] \\ =& \tilde{g}^{\mu\nu}\left[\nabla_\mu\nabla_\nu(\Omega^s u) - \Omega^{-1}\nabla_\mu\Omega\nabla_\nu(\Omega^s u) - \Omega^{-1}\nabla_\nu\Omega\nabla_\mu(\Omega^s u) + \right. \\ & \left. \Omega^{-1}g^{\kappa\lambda}g_{\mu\nu}\nabla_\lambda\Omega\nabla_\kappa(\Omega^s u)\right].\end{aligned} \tag{A.3.29}$$

Noting that

$$\nabla_\nu(\Omega^s u) = s\Omega^{s-1}u\nabla_\nu\Omega + \Omega^s\nabla_\nu u, \tag{A.3.30}$$

we get from applying ∇_μ to (A.3.30) that

$$\begin{aligned}\nabla_\mu\nabla_\nu(\Omega^s u) =& s(s-1)\Omega^{s-2}u\nabla_\mu\Omega\nabla_\nu\Omega + s\Omega^{s-1}\nabla_\mu u\nabla_\nu\Omega + s\Omega^{s-1}u\nabla_\mu\nabla_\nu\Omega + \\ & s\Omega^{s-1}\nabla_\mu\Omega\nabla_\nu u + \Omega^s\nabla_\mu\nabla_\nu u.\end{aligned} \tag{A.3.31}$$

Substituting equation (A.3.30) and (A.3.31) into (A.3.29), we obtain

$$
\begin{aligned}
\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{u} =&\tilde{g}^{\mu\nu}\left[s(s-1)\Omega^{s-2}u\nabla_\mu\Omega\nabla_\nu\Omega + s\Omega^{s-1}\nabla_\mu u\nabla_\nu\Omega + s\Omega^{s-1}u\nabla_\mu\nabla_\nu\Omega +\right.\\
&s\Omega^{s-1}\nabla_\mu\Omega\nabla_\nu u + \Omega^s\nabla_\mu\nabla_\nu u - 2s\Omega^{s-2}u\nabla_{(\mu}\Omega\nabla_{\nu)}\Omega -\\
&\left.2\Omega^{s-1}\nabla_{(\mu}\Omega\nabla_{\nu)}u + s\Omega^{s-2}ug^{\kappa\lambda}g_{\mu\nu}\nabla_\lambda\Omega\nabla_\kappa\Omega + \Omega^{s-1}g^{\kappa\lambda}g_{\mu\nu}\nabla_\lambda\Omega\nabla_\kappa u\right],\\
=&\tilde{g}^{\mu\nu}\left[\Omega^s\nabla_\mu\nabla_\nu u + (s-2)\Omega^{s-1}\nabla_{(\mu}\Omega\nabla_{\nu)}u + \Omega^{s-1}g^{\kappa\lambda}g_{\mu\nu}\nabla_\lambda\Omega\nabla_\kappa u +\right.\\
&s\Omega^{s-1}u\nabla_\mu\nabla_\nu\Omega + s(s-1)\Omega^{s-2}u\nabla_\mu\Omega\nabla_\nu\Omega - 2s\Omega^{s-2}u\nabla_{(\mu}\Omega\nabla_{\nu)}\Omega +\\
&\left.s\Omega^{s-2}ug^{\kappa\lambda}g_{\mu\nu}\nabla_\lambda\Omega\nabla_\kappa\Omega\right].
\end{aligned}
\tag{A.3.32}
$$

Then using $s = 1 - \frac{n}{2}$, and equations (A.3.25), (A.3.32), we write the wave equation (A.3.26) as

$$
\tilde{f} = \tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{u} - \frac{(n-2)}{4(n-1)}\tilde{R}\tilde{u} = \Omega^{-1-\frac{n}{2}}\left(g^{\mu\nu}\nabla_\mu\nabla_\nu u - \frac{n-2}{4(n-1)}Ru\right), \tag{A.3.33}
$$

or equivalently as

$$
g^{\mu\nu}\nabla_\mu\nabla_\nu u - \frac{n-2}{4(n-1)}Ru = f, \tag{A.3.34}
$$

where we have set

$$
f = \Omega^{1+\frac{n}{2}}\tilde{f}. \tag{A.3.35}
$$

The transformation of the source terms $\tilde{f}$ follows a similar set of computations to equations (A.3.29)-(A.3.30). For example, suppose that the source term $\tilde{f}$ is given by

$$
\tilde{f} = \tilde{a}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{u} + \tilde{b}^\mu\tilde{\nabla}_\mu\tilde{u} + \tilde{h}, \tag{A.3.36}
$$

where $\tilde{a}^{\mu\nu}$, $\tilde{b}^\mu$ are tensor fields and $\tilde{h}$ is a constant, then using (A.3.30), (A.3.32) we can write $\tilde{f}$ in terms of the connection ∇ and the conformal factor as

$$
\begin{aligned}
\tilde{f} =&\tilde{a}^{\mu\nu}\left[\Omega^s\nabla_\mu\nabla_\nu u + (s-2)\Omega^{s-1}\nabla_{(\mu}\Omega\nabla_{\nu)}u + \Omega^{s-1}g^{\kappa\lambda}g_{\mu\nu}\nabla_\lambda\Omega\nabla_\kappa u +\right.\\
&s\Omega^{s-1}u\nabla_\mu\nabla_\nu\Omega + s(s-1)\Omega^{s-2}u\nabla_\mu\Omega\nabla_\nu\Omega - 2s\Omega^{s-2}u\nabla_{(\mu}\Omega\nabla_{\nu)}\Omega +\\
&\left.s\Omega^{s-2}ug^{\kappa\lambda}g_{\mu\nu}\nabla_\lambda\Omega\nabla_\kappa\Omega\right] + \tilde{b}^\mu\left(s\Omega^{s-1}u\nabla_\mu\Omega + \Omega^s\nabla_\mu u\right) + \tilde{h}.
\end{aligned}
\tag{A.3.37}
$$

Assuming now that the scalar functions $\tilde{u}^K$ satisfy a system of wave equations

$$\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{u}^K - \frac{(n-2)}{4(n-1)}\tilde{R}\tilde{u}^K = \tilde{f}^K, \tag{A.3.38}$$

we deduce from (A.3.28), with $s = 1 - \frac{n}{2}$, and equations (A.3.33)-(A.3.35) that the scalar functions

$$u^K = \Omega^{\frac{n}{2}-1}\tilde{u}^K, \tag{A.3.39}$$

satisfy a system of wave equations given by

$$g^{\mu\nu}\nabla_\mu\nabla_\nu u^K - \frac{n-2}{4(n-1)}Ru^K = f^K, \tag{A.3.40}$$

where

$$f^K = \Omega^{1+\frac{n}{2}}\tilde{f}^K. \tag{A.3.41}$$

In Chapter 3, we look at the particular case where the non-linear terms are of the form

$$\tilde{f}^K = q^K_{IJ}(\tilde{u}^L)\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu u^I\tilde{\nabla}^J_\nu. \tag{A.3.42}$$

Using (A.3.30), $s = 1 - \frac{n}{2}$, (A.3.37) with $\tilde{a}^{\mu\nu} = 0$, $\tilde{b}^\mu = \tilde{g}^{\mu\nu}\tilde{\nabla}_\nu$, $\tilde{h} = 0$, and (A.3.42), we deduce that the source term (A.3.41), can be expanded as

$$f^K = q^K_{IJ}(\Omega^{1-\frac{n}{2}}u^L)\left(\Omega^{1-\frac{n}{2}}g^{\mu\nu}\nabla_\mu u^I\nabla_\nu u^J + 2\left(\frac{n}{2}-1\right)\Omega^{2-\frac{n}{2}}g^{\mu\nu}\nabla_\mu\Omega^{-1}\nabla_\nu u^{(I}u^{J)} + \left(1-\frac{n}{2}\right)^2\Omega^{3-\frac{n}{2}}g^{\mu\nu}\nabla_\mu\Omega^{-1}\nabla_\nu\Omega^{-1}u^Iu^J\right). \tag{A.3.43}$$

In Chapter 4, we use a slightly different source term given by

$$\tilde{f} = \tilde{a}^{K\mu\nu}_{IJ}\tilde{\nabla}_\mu\tilde{u}^I\tilde{\nabla}_\nu\tilde{u}^J, \tag{A.3.44}$$

using again (A.3.30)-(A.3.37), (A.3.44), and $s = 1-\frac{n}{2}$, we write (A.3.41) in the form

$$f^K = \tilde{a}^{K\mu\nu}_{IJ}\left(\Omega^{3-\frac{n}{2}}\nabla_\mu u^I\nabla_\nu u^J + \left(\frac{n}{2}-1\right)\Omega^{4-\frac{n}{2}}(\nabla_\mu\Omega^{-1}u^I\nabla_\nu u^J + \nabla_\mu u^I\nabla_\nu\Omega^{-1}u^J) + \left(1-\frac{n}{2}\right)^2\Omega^{5-\frac{n}{2}}\nabla_\mu\Omega^{-1}\nabla_\nu\Omega^{-1}u^Iu^J\right). \tag{A.3.45}$$

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