# On some problems related to exit times of planar Brownian motion

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**School of Mathematics**

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## Abstract

THE aim of this thesis is to investigate the exit times of planar Brownian motion from planar domains. A planar Brownian motion is simply a $2D$ Brownian motion $Z_t = (X_t, Y_t)$ where $X_t$ and $Y_t$ are two one dimensional independent Brownian motions. This process is often seen as a complex process $Z_t = X_t + Y_t i$ where $i$ is the imaginary unit. This complex representation is very handy and has a lot of nice properties, on top of them the so called *conformal invariance principle*. That is, the action of an analytic or an antiholomorphic function on a planar Brownian motion does not change it into another kind of process. In fact, it transforms it to another planar Brownian motion but running at another speed, i.e a time changed planar Brownian motion. In many cases, people look at planar Brownian motion in a specific domain of the plane, say $U \subsetneq \mathbb{C}$, and then consider the truncated process $Z_{t\wedge\tau_U}$, i.e we stop the motion upon exiting $U$. That is the focus of this thesis.

The thesis is split into 3 chapters. The first one is an introductory section in which we present the theory of planar Brownian motion as well as the key results in the field. The second chapter deals with the problem of finding the starting points in a fixed domain for which the $p^{th}$-moment of the exit time of a planar Brownian motion is maximal. Such starting points are referred to as $p^{th}$ centers. In most cases, we don't know exactly where these $p^{th}$ centers lie even for regular looking domains. The only thing that one expects is that they should not be close to the boundary. We have developed some geometric and analytical methods that exclude parts of the domain from including the $p$-centers, and sometimes these techniques give exactly the set of $p$-centers. Moreover, our methods apply to any domain. Finally, Chapter 3 tackles the planar version of Skorokhod embedding problem. That is, given a real probability centered distribution $\mu$ with finite second moment, is there a simply connected domain so that the real part of the stopped Brownian motion upon leaving it follows the given distribution? Such domains will be called $\mu$-domains. The existence of $\mu$-domains is already known. Our contribution is to provide a uniqueness criterion as well as enlarging the choice of the distributions to cover those of any finite $p^{th}$-moment with $p > 1$. We also came up with a new category of $\mu$-domains, and it turned out that these ones solve entirely an optimization problem related to the heat equation.

## Declaration

This thesis contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

Maher Boudabra

September 11, 2021

# Publications during enrollment

The following papers are almost all joint works with Dr Greg Markowsky who is my supervisor at Monash University, except the paper [4] which is also in collaboration with Pr Dimitrios Betsakos from Aristotle University.

1. A note on the moments of sequences of complex numbers.
   [3 pp, Journal of Indagationes Mathematicae]

2. A new solution to the conformal Skorokhod embedding problem and applications to the Dirichlet eigenvalue problem.
   [14pp, Journal of Mathematical Analysis and Applications]

3. Remarks on the speeds of a class of random walks on the integers
   [6pp, Discrete Mathematics].

4. On the probability of fast exits and long stays of a planar Brownian motion in simply connected domains. [10pp, Journal of Mathematical Analysis and Applications ].

5. Maximizing the $p^{th}$ moment of the exit time of planar Brownian motion from a given domain.
   [15pp, Journal of Applied Probability ].

6. Remarks on Gross' technique for obtaining a conformal Skorohod embedding of planar Brownian motion.
   [14pp, Electronic Communications in Probability ]

7. A variant of Cauchy's argument principle for analytic functions which applies to curves containing zeros.
   [7 pp, Bulletin of Australian Mathematical Society ]

8. On the finiteness of moments of the exit time of planar Brownian motion from comb domains.
   [9pp, To appear in Annales Academiæ Scientiarum Fennicæ, Mathematica ]

## Thesis including published works declaration

I hereby declare that this thesis contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

This thesis includes 3 original papers published in peer reviewed journals. The core theme of the thesis is **the exit times of planar Brownian motion**. The ideas, development and writing up of all the papers in the thesis were the principal responsibility of myself, the student, working within the School of Mathematics at Monash University under the supervision of Dr. *Gregory Markowsky.*

In the case of Chapters 2 and 3, my contribution to the work involved the following:

| Thesis Chapter | Publication Title | Status *(published, in press, accepted or returned for revision, submitted)* | Nature and % of student contribution | Co-author name(s) Nature and % of Co-author's contribution* | Co-author(s), Monash student Y/N* |
|---|---|---|---|---|---|
| *2* | *Maximizing the p-th moment of the exit time of planar Brownian motion from a given domain.* | *Published* | *70% research and writing* | *Greg Markowsky 30%* | *No* |
| *3* | *Remarks on Gross' technique for obtaining a conformal Skorokhod embedding of planar Brownian motion.*<br><br>*A new solution to the conformal Skorokhod embedding problem and applications to the Dirichlet eigenvalue problem.* | *Published*<br><br>*Published* | *70% research and writing*<br><br>*70% research and writing* | *Greg Markowsky 30%*<br><br>*Greg Markowsky 30%* | *No* |

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I hereby certify that the above declaration correctly reflects the nature and extent of the student's and co-authors' contributions to this work. In instances where I am not the responsible author I have consulted with the responsible author to agree on the respective contributions of the authors.

Main Supervisor name : Greg Markowsky

Main Supervisor signature: [redacted] Date: September 11, 2021

## Acknowledgments

First and foremost, I testify there is no God but Allah alone, without any partners, and that Muhammad, peace and blessings be upon him, is His servant and His messenger. I am thankful to almighty Allah the Gracious, the Merciful that by his grace and bounty, I was able to finish this important milestone of my life.

I'm deeply indebted to my dear Mom Tounes, the great mother who has been supporting me during my whole life and especially in this experience as it was the first time I spend several months away from her. I would like to extend my deepest gratitude to my mentor Professor Faiza Maaouia. Pr. Faiza introduced me to Pr. Kais Hamza who, in his turn, accepted to invite me to Monash university. I really appreciate the efforts of Pr. Hamza in supporting me to join the PhD program. It was a great opportunity to develop and improve my knowledge in mathematics, especially in probability theory. Many thanks to John Chan, the administration officer, who assisted me in the enrollment procedures even before coming to Australia. John had been such a great company and a real friend during these years.

Fortunately I was lucky being supervised by Dr. Markowsky. I knew him during the annual summer school that was held at Monash university in January 2018. He took over the second half of the probability and complex analysis course. That was the first time I discovered the realm of the planar Brownian motion theory and was impressed by the deep connection between probability and complex analysis. Dr. Markowsky is an open-hearted person, especially when it comes to discussing my "too many" questions. Moreover, Dr. Markowsky is humble which made me feel very comfortable with him and not hesitate to ask anything. We had really a great time together, thank heaps dear Dr. Markowsky.

## Glossary of notations

Throughout this work, we adopt the following notations :

- $\mathbb{C}$ : The complex plane.
- $\mathbb{D}$ : The unit disc $\{z \mid |z|< 1\}$.
- $S^1$ : The unit circle $\{z \mid |z|= 1\}$.
- $\overline{U}$ : The closure of $U$.
- $\partial U$ : The boundary of $U$.
- $\Re(z)$ : The real part of $z$.
- $\Im(z)$ : The imaginary part of $z$.
- $\mathbb{H}$ : The upper half plane $\{z \mid \Im(z) > 0\}$
- $a.e$ : almost everywhere.
- $\Delta$ : Laplacian operator.

# Contents

## Chapter 1

# Introductory chapter.

This chpater is introductory. It contains an outline of the theory of planar Brownian motion. We give the main definitions and results of the topic. In the next two chapters, we give summaries of the papers mentioned in the Abstract. That is, we present the problems in question and then give the main results and theorems.

**Definition 1.** A Brownian motion $B_t$ is a stochastic process with the following properties :

- $t \mapsto B_t$ is continuous a.e.
- $B_t$ has independent increments.
- $B_s - B_t \sim N(0, |s-t|)$.

The starting point of $B_t$ is $B_0 = x$. When $B_0$ is 0, we say that $B_t$ is a standard Brownian motion. Definition 1 talks about Brownian motion in dimension 1. However we easily extend such a definition to higher dimensions. More precisely, a $d$-dimensional Brownian motion $(B_t^{(1)}, ..., B_t^{(d)})$ is a stochastic process where each $B_t^{(j)}$ is a Brownian motion in the sense of Definition 1 and $B_t^{(j)}$ is independent of $B_t^{(i)}$ when $i \neq j$. If $d = 2$, we use the terminology *planar Brownian motion*. When we don't mention the dimension for the Brownian motion then it is a 1-dimensional process. The $d$-dimensional Brownian motion has the following density

$$p(t,x,y) = \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{2t}}$$

where $t > 0$ and $x, y \in \mathbb{R}^d$.

The Brownian motion has many properties. We refer the reader to [26, 34, 27]. Since the main topic of this thesis is planar Brownian motion, we confine ourselves to focus on this process (properties and mathematical tools). We shall use the shorthand PBM instead of planar Brownian motion.

**Definition 2.** Let $p$ be a positive number. The $p^{th}$ moment of a positive r.v $X$ is $\mathbf{E}(X^p)$. In particular, we say that $X$ has a finite $p^{th}$ moment if that moment is finite, i.e $\mathbf{E}(X^p) < +\infty$.

We write often $X \in L^p$ to say that $\mathbf{E}(X^p) < +\infty$. We define similarly the "$p^{th}$ norm" of $X \in L^p$ by setting $\|X\|_p := \mathbf{E}(X^p)^{\frac{1}{p}}$. The quantity $\|\cdot\|_p$ is a true norm only when $p \geq 1$ [1].

**Proposition 3** (Lyapounov inequality)**.** *If $p < q$ then $\|X\|_p \leq \|X\|_q$. In particular $L^q(\Omega) \subset L^p(\Omega)$.*

The proof of Lyapounov inequality can be found in [22]. One of the powerful tools to study $\mathbf{E}(\tau^p)$ is *the optional stopping theorem.* Here is its statement.

**Theorem 4.** *[26] Let $M_t$ be a martingale and $T$ be a stopping time. Then $\mathbf{E}(M_T) = \mathbf{E}(M_0)$ whenever one of the following conditions holds:*

- $\mathbf{E}(T) < +\infty$.
- *$M_{t\wedge T}$ is bounded a.e.*

[1] For $p < 1$, the triangular inequality does not hold. In particular the unit ball is not convex.

## 1.1 Planar Brownian motion.

PBM is a special case of multi-dimensional Brownian motion. Such a process is highly linked to the theory of complex functions [1, 16]. One of the greatest results illustrating this deep relation is the following theorem of Paul Lévy.

**Theorem 5.** *If $f$ is a non constant analytic function and $Z_t$ is a PBM starting at $a$ then $f(Z_t)$ is a time-changed Brownian motion starting at $f(a)$, where the time change rate is governed by*

$$\sigma(t) := \int_0^t |f'(Z_s)|^2 ds.$$

*In other words, the process $f(Z_{\sigma^{-1}(t)})$ is a PBM.*

An easy case is when $f$ is a rotation; $f(Z_t)$ is a genuine PBM as $|f'| = 1$. Lévy theorem is often referred to as the *conformal invariance principle* of PBM. Note that since the conjugate of a PBM is also a PBM, then Lévy theorem still works perfectly even when we consider non constant antiholomorphic functions [2] instead of analytic ones. The only difference is that when $f$ is antiholomorphic then $\sigma(t)$ becomes $\int_0^t |\frac{df}{d\bar{z}}(Z_s)|^2 ds$. That is, without loss of generality, we state all our results with respect to analytic functions but they are valid for antiholomorphic ones also. The intuition of Lévy theorem can be seen from the scaling property of the Brownian motion. Locally speaking, we have

$$f(z) \approx f(0) + f'(0)z.$$

In particular, if we set $f'(0) = \kappa e^{\vartheta i}$ and $Z_t = X_t + Y_t i$, then for small time $t$ we get the approximation

$$\begin{aligned} f(Z_t) &\approx f(0) + f'(0)Z_t \\ &= f(0) + e^{\vartheta i}(\kappa X_t + \kappa Y_t i). \end{aligned}$$

On the other hand, $e^{\vartheta i}(rX_t + rY_t i)$ is simply a rotated scaled Planar Brownian motion. That is, for small $t$, $f(Z_t)$ is a rotated scaled Brownian motion starting at $f(Z_0)$ and running at the speed $\kappa^2 t = |f'(0)|^2 t$. We can see $(f(Z_t))_t$ as a concatenation of rotated scaled Brownian motions and so, in whole, it is a Brownian path due to Markov property. For the sake of completeness, we give here a clear proof of Theorem 5.

*Proof.* We proceed by showing each the real and imaginary components of $\widehat{Z}_t := f(Z_t)$ are independent one dimensional Brownian motions and running at the same speed. Denote by $R_t$ and $I_t$ the real and the imaginary components of $Z_t$. Since $f$ is analytic, set $f(x, y) := g(x, y) + h(x, y)i$, where $g$ and $h$ are harmonic on $U$. Then by Itô's formula in two dimensions

$$g(Z_t) = g(a) + \int_0^t \frac{\partial g}{\partial x}(Z_s)dR_s + \int_0^t \frac{\partial g}{\partial y}(Z_s)dI_s$$

$$h(Z_t) = h(a) + \int_0^t \frac{\partial h}{\partial x}(Z_s)dR_s + \int_0^t \frac{\partial h}{\partial y}(Z_s)dI_s$$

where the second order terms cancel as a result of $g$ and $h$ being harmonic functions. Both $M_t = g(Z_t)$ and $N_t = h(Z_t)$ are martingales. We need now to compute their quadratic variations using the stochastic calculus machinery [26]. We have

[2] An antiholomorphic function is a function whose conjugate is analytic.

$$\begin{aligned}
[M]_t &= \left[\int_0^t \frac{\partial g}{\partial x}(Z_s)dR_s + \int_0^t \frac{\partial g}{\partial y}(Z_s)dI_s\right]_t \\
&= \left[\int_0^\cdot \frac{\partial g}{\partial x}(Z_s)dR_s, \int_0^\cdot \frac{\partial g}{\partial x}(Z_s)dR_s\right]_t + 2\underbrace{\left[\int_0^t \frac{\partial g}{\partial x}(Z_s)dR_s, \int_0^t \frac{\partial g}{\partial y}(Z_s)dI_s\right]_t}_{=0} \\
&+ \left[\int_0^t \frac{\partial g}{\partial y}(Z_s)dI_s, \int_0^t \frac{\partial g}{\partial y}(Z_s)dI_s\right]_t \\
&= \int_0^t (\frac{\partial g}{\partial x}(Z_s))^2 ds + \int_0^t (\frac{\partial g}{\partial y}(Z_s))^2 ds.
\end{aligned}$$

The same argument for $N_t$ yields

$$[N]_t = \int_0^t (\frac{\partial g}{\partial x}(Z_s))^2 ds + \int_0^t (\frac{\partial g}{\partial y}(Z_s))^2 ds.$$

Then, by the Cauchy-Riemann equations we get

$$[M]_t = [N]_t = \int_0^t |f'(Z_s)|^2 ds = \sigma(t)$$

In other words, $M_t$ and $N_t$ have the same time change. Note that $\sigma$ is strictly increasing as $f$ is a non constant analytic function. Therefore, by the Dambis, Dubins-Schwarz Theorem [26], both $M_{\sigma^{-1}(t)}$ and $N_{\sigma^{-1}(t)}$ are one dimensional Brownian motions. It remains to show that they are independent. We have

$$\begin{aligned}
[M,N]_t &= \left[\int_0^t \frac{\partial g}{\partial x}(Z_s)dR_s + \int_0^t \frac{\partial g}{\partial y}(Z_s)dI_s, \int_0^t \frac{\partial f}{\partial x}(Z_s)dR_s + \int_0^t \frac{\partial f}{\partial y}(Z_s)dI_s\right]_t \\
&= \left[\int_0^t \frac{\partial g}{\partial x}(Z_s)dR_s, \int_0^t \frac{\partial f}{\partial x}(Z_s)dR_s\right]_t + \left[\int_0^t \frac{\partial g}{\partial x}(Z_s)dR_s, \int_0^t \frac{\partial f}{\partial y}(Z_s)dI_s\right]_t \\
&+ \left[\int_0^t \frac{\partial g}{\partial y}(Z_s)dI_s, \int_0^t \frac{\partial f}{\partial x}(Z_s)dR_s\right]_t + \left[\int_0^t \frac{\partial g}{\partial y}(Z_s)dI_s, \int_0^t \frac{\partial f}{\partial y}(Z_s)dI_s\right]_t \\
&= \int_0^t \frac{\partial g}{\partial x}(Z_s)\frac{\partial f}{\partial x}(Z_s) + \frac{\partial g}{\partial y}(Z_s)\frac{\partial f}{\partial y}(Z_s)ds
\end{aligned}$$

and by the Cauchy-Riemann equations $[M,N]_t = 0$. In particular $[M,N]_{\sigma^{-1}(t)} = 0$. Hence, $M_{\sigma^{-1}(t)}$ and $N_{\sigma^{-1}(t)}$ are two uncorrelated Gaussian processes. Then by a classical result, they are independent and so are $M_t$ and $N_t$ . Consequently, $f(Z_t) = M_t + N_t i$ is a time changed Brownian motion running at the speed $\sigma(t) = \int_0^t |f'(Z_s)|^2 ds$. □

*Remark* 6. Theorem 5 says that the action of a non constant analytic function on a PBM transforms it to another PBM but running in another time rate. In particular, a lot of stochastic properties will be preserved. In my opinion, Lévy theorem is the core of the theory of PBM.

**Proposition 7.** *PBM is blind toward a given point. In other words,* $\mathbb{P}_z(\exists t \geq 0 \mid Z_t = a) = 0$ *for all* $z \neq a \in \mathbb{C}$.

*Proof.* Let $Z_t := R_t + I_t i$ be a PBM and $z \neq Z_0$. Without loss of generality we may assume that $z = 0$. The process $W_t = e^{Z_t}$ is a time-changed PBM with time rate

$$\sigma(t) := \int_0^t e^{2R_s} ds.$$

To path of $W_t$ is the same path as the PBM $W_{\sigma^{-1}(t)}$ and so $W_{\sigma^{-1}(t)}$ does not hit $z = 0$. To make this rigorous we need to show that $\sigma(t) \to +\infty$. By Kolmogorov $0-1$ law [34] we get

$$\mathbf{P}(A = \int_0^{+\infty} e^{2R_s} ds < +\infty) = \mathbf{P}(B = \int_0^{+\infty} e^{-2R_s} ds < +\infty) \in \{0, 1\}.$$

Remark that $A \cap B = \{\int_0^{+\infty}(e^{2R_s} + e^{-2R_s}) ds < +\infty\}$. On the other hand, $\int_0^{+\infty}(e^{2R_s} + e^{-2R_s}) ds = +\infty$ since $e^{2x} + e^{-2x} > 1$. Thus $\mathbf{P}(A \cap B) = 0$ and consequently $\mathbf{P}(A) = 0$. □

Another "quick" argument of the blindness toward a given point, say 0, follows from the fact that $\widehat{Z}_t := \frac{1}{Z_t}$ (with $Z_0 = 1$) is also a time changed PBM. Therefore

$$\mathbf{P}_1(\widehat{Z}_t = 0 \text{ for some } t \geq 0) = \mathbf{P}(Z_t = \infty \text{ for some } t \geq 0) = 0.$$

*Remark* 8. Proposition 7 is a bit counterintuitive at first glance as PBM path is a set of points at the end. Rather it says that if we fix a point $z$ in advance, then this path won't hit it almost surely. The proof of Proposition 7 is from [19], but without showing $\sigma(t) \to +\infty$. Later on, we will show that for non constant entire functions, always we have $\sigma(t) \underset{t \to +\infty}{\longrightarrow} +\infty$.

**Definition 9.** The exit time of a PBM $Z_t$ from a domain $U$ is defined by

$$\tau_U := \inf\{t \geq 0 \mid Z_t \notin U\}.$$

The exit time $\tau_U$ is measurable with respect to the canonical Brownian filtration $(\mathcal{F}_{Z_t})_t$ . If $U$ is closed then $\tau_U$ is rather measurable with respect to the augmented Brownian filtration $(\mathcal{F}^+_{Z_t})_t$. In fact, the filtration $(\mathcal{F}^+_{Z_t})_t$ renders $\tau_U$ measurable for any Borel set $U$. An elementary proof can be found in [2] [3]. Unless stated otherwise, the word domain means an open one.

## 1.2 The distribution of $Z_{\tau_U}$.

The stopped PBM $Z_{\tau_U}$ lives on the boundary of $U$ and knowing its law allows us to know which part of the boundary $Z_{\tau_U}$ is more likely to be hit. In case of existence, we denote the p.d.f of $Z_{\tau_U}$ by $\rho_{Z_{\tau_U}}(a, s)$. The measure $\mathbf{P}(Z_\tau \in \cdot)$ is referred to as the *harmonic measure*. The first example to start with is the harmonic measure of the unit disc.

**Lemma 10.** *Starting at the origin, $Z_{\tau_\mathbb{D}}$ is uniformly distributed on the unit circle. In other words* $\arg Z_{\tau_\mathbb{D}} \sim \text{Uni}(-\pi, \pi)$.

*Proof.* Let $\eta \in \mathbb{R}/(2\pi\mathbb{Z})$ [4] and set $\tau := \tau_\mathbb{D}$. The process $W_t := e^{\eta i} Z_t$ is a PBM starting at the origin and it has the same exit time as $Z_t$. Then we have

$$\mathbf{P}_0(\arg Z_\tau \in (\alpha, \beta)) = \mathbf{P}_0(\arg W_\tau \in (\alpha + \eta, \beta + \eta)) = \mathbf{P}_0(\arg Z_\tau \in (\alpha + \eta, \beta + \eta)).$$

[3] An error is corrected in a subsequent paper [3].

[4] This means we don't care about the integer multiples of $2\pi$.

Therefore the measure $\mathbf{P}(\arg Z_\tau \in \cdot)$ is invariant under translation. A classical measure theory argument [17] shows that $\mathbf{P}_0(\arg Z_\tau \in \cdot) = \rho\lambda(\cdot)$ on $\mathcal{B}((-\pi,\pi))$ for some constant $\rho$. Since we are dealing with a probability measure, $\rho$ must be $\frac{1}{2\pi}$. Consequently

$$\mathbf{P}_0(\arg Z_\tau \in d\theta) = \frac{d\theta}{2\pi}.$$

□

*Remark* 11. The behavior of $Z_\tau$ is highly dependent on the starting point. In particular, if $Z_t$ does not start at the origin then $Z_{\tau_{\mathbb{D}}}$ won't be uniformly distributed on the unit circle, as one can see that the boundary points close to the starting point would have more mass.

The distribution of $Z_\tau$ for general domains in not easy. However, in many cases we can overcome this difficulty using the conformal invariance principle.

**Definition 12.** We say that an analytic function $f : U \to W$ is proper if for any compact set $K$, $f^{-1}(K)$ is also compact.

Definition 12 extends to any continuous function between two topological spaces [40]. However, proper functions are presumed to be analytic in this work. The interpretation of properness is the following : If $(z_n)_n$ is a sequence of $U$ that converges to $\partial U$, i.e the complement of any compact set of $U$ contains all but a finite number of $(z_n)_n$, then $(f(z_n))_n$ converges necessarily to $\partial W$. Often, we express that as $f$ maps boundary to boundary. Univalent functions are typical examples of proper ones.

**Lemma 13.** *If $f$ is a proper map from $U$ onto $W$ then*

$$\tau_W = \sigma(\tau_U).$$

*Proof.* The properness assumption yields

$$\begin{aligned}
\tau_U &= \inf\{t \geq 0 \mid Z_t \in \partial U\} \\
&= \inf\{t \geq 0 \mid f(Z_t) \in \partial W\} \\
&= \inf\{\sigma^{-1}(s) \geq 0 \mid f(Z_{\sigma^{-1}(s)}) \in \partial W\} \\
&= \sigma^{-1}(\inf\{s \geq 0 \mid f(Z_{\sigma^{-1}(s)}) \in \partial W\}) \\
&= \sigma^{-1}(\tau_W).
\end{aligned}$$

□

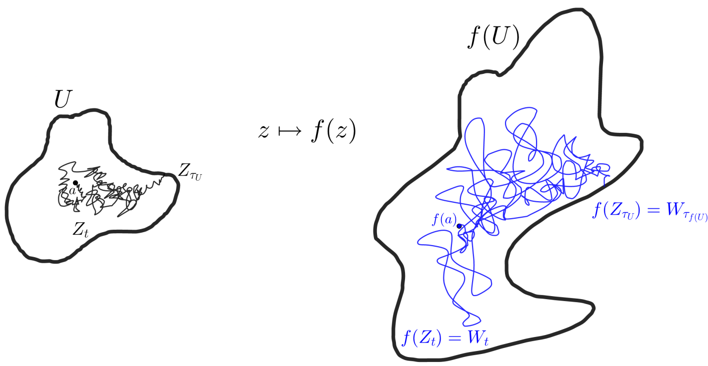


Figure 1.2.1: Lemma 13 says that the projection of the exit time from $U$ under the action of a proper function is the exit time from $f(U)$.

If we drop the properness of $f$ then the projection of the exit time is not necessarily an exit time. This is because an arbitrary analytic function may not map the boundary of $U$ to the boundary of $f(U)$. To see this consider $Z_{t\wedge\tau_{\mathbb{D}}}$ and $f(z) = z(z-1)$. The image of the unit circle is not the boundary of $f(\mathbb{D})$!!! It is rather a self intersecting loop that crosses $f(\mathbb{D})$ as depicted below.

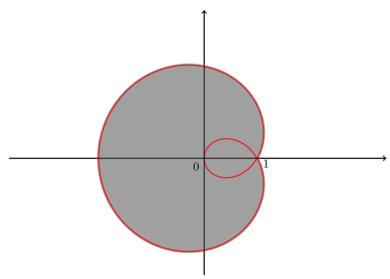


Figure 1.2.2: $f(\mathbb{D})$ is the grey region and the image of unit circle is the red curve. In particular $f(Z_{t\wedge\tau_{\mathbb{D}}})$ can stop while it is still inside $f(\mathbb{D})$.

The following puzzling situation is inspired from [32]. It gives an example when the projection of an exit time is not entirely an exit time. Consider $Z_{t\wedge\tau_U}$ where $U := \mathbb{C} - \{\Re(z) \geq 0, \Im(z) = \frac{\pi}{2}\}$ and $f(z) = e^z$. The image of $U$ is $f(U) = \mathbb{C} - \{0\}$. The projection of $\tau_U$ under the action of $f$, say $\widehat{\tau}_{f(U)}$, is not the exit time from $f(U)$. $f(Z_t)$ will never leave $f(U)$ but it stops when $\arg(Z_{\widehat{\tau}_{f(U)}}) = \frac{\pi}{2}$ and $\Im(Z_{\widehat{\tau}_{f(U)}}) \geq e$ where arg denotes the continuous version of the argument, i.e without taking off the multiples of $2\pi$ [5].

[5] One can think of the continuous version of the argument as $\eta 2\pi$ where $\eta$ is the number of revolutions according to the clock rule.

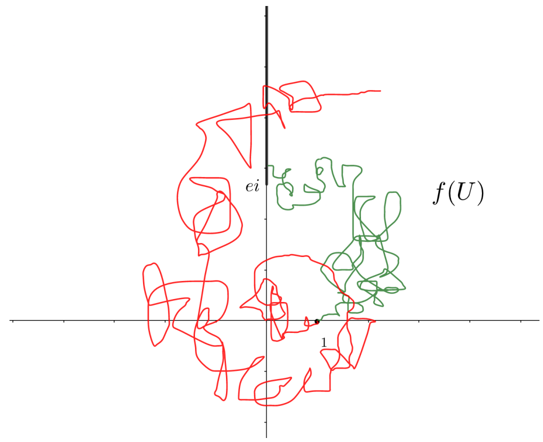


Figure 1.2.3: The green path is stopped while the red one keeps running.

We shall refer the technique of using analytic functions to as *projecting the distribution of PBM at a stopping time by an analytic function.* All subsequent results use univalent functions, but those results are still valid if we relax the univalence to properness.

**Theorem 14.** *[33] If $f$ is a univalent map from $\mathbb{D}$ onto $U$ that extends to be analytic across $\partial\mathbb{D}$ then*

$$\rho_{Z_{\tau_U}}(f(0), s) = \frac{1}{2\pi}|(f^{-1})'(s)|ds. \tag{1.2.1}$$

*Proof.* Set $a := f(0)$. By the conformal invariance principle we have for all $A \subset \partial U$

$$\begin{aligned}
\mathbf{P}_a(Z_{\tau_U} \in A) &= \int_A \mathbf{P}_a(Z_{\tau_U} \in dz) \\
&= \mathbf{P}_0(Z_{\tau(\mathbb{D})} \in f^{-1}(A)) \\
&= \frac{1}{2\pi}\int_{f^{-1}(A)} dz \\
&\overset{w=f(z)}{=} \frac{1}{2\pi}\int_A |f'(f^{-1}(w))|dw \\
&= \frac{1}{2\pi}\int_A |(f^{-1})'(z)|dz
\end{aligned}$$

which ends the proof. □

*Remark* 15. If $f : U \to V$ then almost the same proof of (1.2.1) yields

$$\rho_{Z_{\tau_V}}(f(a), s) = \rho_{Z_{\tau_U}}(a, f^{-1}(s)) \times |(f^{-1})'(s)|. \tag{1.2.2}$$

If $f$ is just analytic then we have the following extension.

**Theorem 16.** *[6] Let $U$ be a domain, and suppose $f$ is a function analytic on $U$. Let $\tau$ be the hitting time of a smooth curve $\gamma \subset U$. Then*

$$\rho_{Z_{\widehat{\tau}}}(f(a), s) = \sum_{x \in f^{-1}(\{s\}) \cap \gamma} \frac{\rho_{Z_{\tau_U}}(a, x)}{|f'(x)|}$$

*where $\widehat{\tau}$ is the projection of $\tau$ under the action of $f$, i.e $\widehat{\tau} = \int_0^\tau |f'(Z_s)|^2 ds$.*

The first application of Theorem 14 is to derive $\rho_{Z_{\tau_{\mathbb{D}}}}(a,s)$ for an arbitrary starting point $a \in \mathbb{D}$. Beforehand, we need the following result about $Aut(\mathbb{D})$, i.e the group of univalent functions from the unit disc onto itself.

**Theorem 17.** *[38] We have*

$$Aut(\mathbb{D}) = \left\{ z \mapsto \xi \frac{a-z}{1-\bar{a}z}, (a,\xi) \in \mathbb{D} \times S^1 \right\}.$$

**Proposition 18.** *We have*

$$\rho_{Z_{\tau_{\mathbb{D}}}}(a,s) = \frac{1}{2\pi} \frac{1-|a|^2}{|1-\bar{a}s|^2}.$$

*Proof.* Consider $f : z \mapsto \frac{a-z}{1-\bar{a}z}$ and apply Theorem 16. □

In case when $U$ and $V$ are simply connected domains, the existence of a univalent map between them is guaranteed by the famous *Riemann mapping theorem*. For further details about it, we refer the reader to [20], [12, Vol I]. Many examples of application of Theorems 16 and 14 can be found in [33] and [6]. In particular

$$\rho_{Z_{\tau_{\mathbb{H}}}}(a,s) = \frac{\Im(a)}{\pi|s-a|^2}$$

where $a$ is the starting point. As noticed, $Z_{\tau_{\mathbb{H}}}$ has Cauchy distribution.

The technique of projecting the distribution of PBM yields sometimes some interesting identities or new proofs of some known ones. We sketch here an example. For a detailed explanation we refer the reader to [33] where the example is taken from. The function $z \mapsto e^{zi}$ maps the upper half plane $i\mathbb{H}$ to the punctured unit disc $\mathbb{D}^* := \mathbb{D} - \{0\}$. Then by Theorem 16 we obtain the density of $Z_{\tau_{\mathbb{D}^*}}$ starting at $a$. That is

$$\rho_{Z_{\tau_{\mathbb{D}^*}}}(a, s = e^{ti}) = \sum_{k\in\mathbb{Z}} \frac{-\ln(|a|)}{\pi(\ln(|a|)^2 + (\arg(a) - t - 2k\pi)^2)}.$$

On the other hand, as PBM is blind toward points (Proposition 7), then

$$\rho_{Z_{\tau_{\mathbb{D}^*}}}(a, s = e^{ti}) = \rho_{Z_{\tau_{\mathbb{D}}}}(a, s = e^{ti}) = \frac{1}{2\pi} \frac{1-|a|^2}{|1-\bar{a}s|^2}.$$

Hence

$$\frac{1}{2\pi} \frac{1-|a|^2}{|1-\bar{a}s|^2} = \sum_{k\in\mathbb{Z}} \frac{-\ln(|a|)}{\pi(\ln(|a|)^2 + (\arg(a) - t - 2k\pi)^2)}. \tag{1.2.3}$$

For simplicity we assume that $a \in (0,1)$. So (1.2.3) becomes

$$\frac{1}{2} \frac{1-a^2}{(1+a^2-2a\cos(t))} = \sum_{k\in\mathbb{Z}} \frac{-\ln(a)}{(\ln(a)^2 + (-t-2k\pi)^2)}$$

and so as long as $t \neq 0$, then by letting $a \to 1$ combined with a standard analysis argument

$$\frac{1}{2(1-\cos(t))} = \sum_{k\in\mathbb{Z}} \frac{1}{(t+2k\pi)^2}. \tag{1.2.4}$$

For $t = 0$ we proceed as follows: from (1.2.4) we subtract $\frac{1}{t^2}$ from both sides. We obtain

$$\frac{1}{2(1-\cos(t))} - \frac{1}{t^2} = \sum_{k\in\mathbb{Z}-\{0\}} \frac{1}{(t+2k\pi)^2}.$$

The LHS tends to $\frac{1}{12}$ and the RHS tends to $\frac{1}{2\pi^2}\sum_{k=1}^{+\infty}\frac{1}{k^2}$. Consequently we recover the famous Basel sum

$$\sum_{k=1}^{+\infty}\frac{1}{k^2}=\frac{\pi^2}{6}.$$

## 1.3 Moments of the exit time.

The exit time from a domain $\tau_U$ measures the time needed for the PBM to quit $U$ starting from some point $z$. A first thing one can think about is the average of $\tau_U$, i.e $\mathbf{E}_z(\tau_U)$. More generally, we are interested in the moments of $\tau_U$, i.e $\mathbf{E}_z(\tau_U^p)$ where $p>0$, in particular the first moment corresponds to the exit time average. It turns out that exit time from a bounded domain $U$ has always finite first moment. This follows directly from the optional stopping Theorem 4 applied to the martingale $|Z_t|^2-2t$. More precisely, we have

$$\mathbf{E}_z(\tau_U)=\frac{1}{2}(\mathbf{E}_z(\underbrace{|Z_{\tau_U}|^2}_{\text{bounded}})-|z|^2)<+\infty$$

In particular when $U=\mathbb{D}$ then $\mathbf{E}_z(\tau_{\mathbb{D}})=\frac{1}{2}(1-|z|^2)$. This approach applies also to $U=\{-b<\Re(z)<a\}$ $(a,b>0)$ even though it is unbounded since what matters here is the real part of the PBM which is moving in the bounded interval $(-b,a)$. In such a case, using the martingale $B_t^2-t$, we get $\mathbf{E}_x(\tau_U)=(a-x)(x+b)$ with $x\in(-b,a)$.

In both examples above, the exit time average $\mathbf{E}_z(\tau_U)$, seen as a function of the starting point, attains its maximum at the center point of $U$. An explicit formula is far from being found for general domains. However, we've developed some techniques about maximizing the function $z\mapsto\mathbf{E}_z(\tau_U^p)$ for a broad range of domains even without knowing the explicit formula. This question will be tackled in a subsequent chapter.

**Example 19.** Consider the case when $U:=\{\Re(z)>0\}$ and $Z_0=x+yi\in(0,+\infty)$. Then the exit time $\tau_U$ is simply the hitting time of $R_t=\Re(Z_t)$ of 0. The density of $\tau_U$ is therefore given by [26]

$$f(t)=\frac{xe^{-\frac{x^2}{2t}}}{\sqrt{2\pi t^3}}.$$

Notably $\mathbf{E}_z(\tau_U^p)<+\infty$ if and only if $p<\frac{1}{2}$ (regardless the starting point $z$). In particular, $Z_t$ spends, on average, an infinite amount of time to get out of $U$.

*Remark* 20. The expectation operator $\mathbf{E}_z$ depends a priori on the starting point of $Z_t$. However, when we want to check whether $\mathbf{E}_z(\tau_U^p)$ is finite or not, the starting point does not matter fortunately [9]. The remark is so useful when we combine it with the monotonicity of domains. In particular, if $U$ contains a half plane then $\mathbf{E}_z(\tau_U^p)=+\infty$ at least for all $p\geq\frac{1}{2}$.

In 1979, D. Burkholder published a paper about PBM and analytic functions [9]. The paper gives a powerful machinery to study such a process. It highlights the strong connection between PBM and analytic functions. One of the result was the characterization the finiteness of $\mathbf{E}_z(\tau_U^p)$ using univalent functions.

Now we give the notion of Hardy norm, which plays a major role when it comes to the problem of finiteness of the moments of the exit times. That is, let $f$ be an analytic function on the unit disc and for any $p>0$ and $0\leq r<1$ set

$$N_{p,r}(f):=\left\{\frac{1}{2\pi}\int_0^{2\pi}|f(re^{\theta i})|^p d\theta\right\}^{\frac{1}{p}}.$$

The quantity $N_{p,r}(f)$ can be interpreted as the $L^p$ norm [6] of the function $f_r : \theta \mapsto f_r(\theta) := f(re^{\theta i})$. It can be shown, using harmonic analysis techniques, that $N_{p,r}(f)$ is non decreasing in terms of $r$ [39]. Hence, we are ready now to define the $p^{th}$-Hardy norm.

**Definition 21.** For any analytic function on the unit disc, the $p^{th}$-Hardy norm of $f$ is defined by

$$H_p(f) := \sup_{0\leq r<1} N_{p,r}(f) = \sup_{0\leq r<1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{\theta i})|^p d\theta \right\}^{\frac{1}{p}}. \tag{1.3.1}$$

The set of analytic functions whose $p^{th}$-Hardy norm is finite is denoted by $\mathcal{H}^p$ and called Hardy space (of index $p$). A crucial result about Hardy norms is that, if $H_p(f)$ is finite then $f$ has a radial extension to the boundary. More precisely $f^*(z) := \lim_{r\to 1} f(rz)$ exists a.e for all $z \in \partial\mathbb{D}$ and it belongs to $L^p$ as well. More details about the topic can be found in [15]. A consequence of Hölder's inequality is the inclusion $\mathcal{H}^q \subseteq \mathcal{H}^p$ whenever $0 < p \leq q$. This leads to the following definition.

**Definition 22.** If $U$ is the range of an analytic function $f$ acting on $\mathbb{D}$ then we define the Hardy number of $U$ as the supremum of all $p > 0$ such that $H_p(f)$ is finite.

In other words, the Hardy number, which we denote by $h_U$, is the largest $p > 0$ such that $f : \mathbb{D} \to U \in \mathcal{H}^p$. That is

$$h_U := \sup\{p > 0 \mid H_p(f) < +\infty\} = \sup\{p > 0 \mid f \in \mathcal{H}^p\}.$$

**Theorem 23.** *[9] If $p > 0$ and $f$ is a univalent map from $\mathbb{D}$ onto $U$ then*

$$\mathbf{E}(\tau_U^{\frac{p}{2}}) < +\infty \iff H_p(f) < +\infty. \tag{1.3.2}$$

The Hardy number as well as the equivalence (1.3.2) seem to depend on the choice of the map $f$, but one expects that the choice of $f$ has no effect as $\mathbf{E}(\tau_U^{\frac{p}{2}})$ is defined once $U$ is given. Indeed, the characterization (1.3.2) is true for any such a map $f$ and this is the subject of Theorem **5.1** of [13]. Another formulation of [9] is the following : The supremum of all $p > 0$ such that $\mathbf{E}(\tau_U^p)$ is finite, is half of the Hardy number of $f$, i.e

$$\sup\{p > 0 \mid \mathbf{E}(\tau_U^p) < +\infty\} = \tfrac{1}{2}h_f. \tag{1.3.3}$$

**Proposition 24.** *If $f$ is a univalent map from $\mathbb{D}$ onto $U$ then*

$$H_p(f)^p = \mathbf{E}_{f(0)}(|Z_{\tau_U}|^p).$$

Consequently, the equivalence (1.3.2) can be extended to

$$\mathbf{E}(\tau_U^{\frac{p}{2}}) < +\infty \iff H_p(f) < +\infty \iff \mathbf{E}_z(|Z_{\tau_U}|^p) < +\infty. \tag{1.3.4}$$

**Theorem 25.** *[9] Let $Z_\tau^* := \sup_{0\leq t\leq\tau} |Z_t|$. For any stopping time $\tau$ and $p > 0$, the following inequalities hold :*

- *$\kappa\mathbf{E}(\tau^{\frac{p}{2}}) \leq \mathbf{E}(Z_\tau^{*p}) \leq k\mathbf{E}(\tau^{\frac{p}{2}})$ for some positive constants $\kappa, k$.*
- *If $\mathbf{E}(\ln(1+\tau)) < +\infty$ then $\mathbf{E}(Z_\tau^{*p}) \leq \gamma\mathbf{E}(|Z_\tau|^p)$ for some positive constant $\gamma$.*

[6] The word norm is an abuse of language as $N_{p,r}(f)$ is not a true norm when $p < 1$.

The following diagram illustrates Theorem 25.

$$
\begin{tikzcd}
\mathbf{E}(\tau^{\frac{p}{2}}) < +\infty \arrow[d, bend left] \\
\mathbf{E}(Z_\tau^{*p}) < +\infty \arrow[d, bend left] \arrow[u, bend left] \\
\mathbf{E}(|Z_\tau|^p) < +\infty \arrow[u, bend left, "\mathbf{E}(\ln(1+\tau))<+\infty"]
\end{tikzcd}
\tag{1.3.5}
$$

Figure 1.3.1: Equivalence between the finiteness of the $p^{th}$ moments of the quantities $\mathbf{E}(Z_\tau^{*p}), \mathbf{E}(|Z_\tau|^p)$ and $\mathbf{E}(\tau^{\frac{p}{2}})$.

In [9], we find the condition $\mathbf{E}(\ln(\tau)) < +\infty$ instead of $\mathbf{E}(\ln(1+\tau)) < +\infty$. However the last one is enough to uphold the same proof. Often it is more likely to have data about $Z_\tau$ rather then the supremum process $Z_\tau^*$, but without the condition $\mathbf{E}(\ln(1+\tau)) < +\infty$ it is not clear that the finiteness of $\mathbf{E}(\tau^{\frac{p}{2}})$ implies that of $\mathbf{E}(|Z_\tau|^p)$ for an arbitrary domain.

**Lemma 26.** *If $\mathbf{E}(\tau^p) < \infty$ for some $p > 0$ then $\mathbf{E}(\ln(1+\tau)) < +\infty$.*

*Remark* 27. The converse of Lemma 26 is wrong since we can have a random variable $X$ such that $\mathbf{E}(\ln(1+X)) < +\infty$ but with no finite $p^{th}$ moment for any $p > 0$. A counterexample is $X$ with p.d.f $f(x) = \frac{\theta}{(1+x)\ln(x)^3}$ where $\theta$ is the normalization constant.

The following theorem will be seen in Chapter 2, but we state it here to show that (1.3.5) (Figure 1.3.1) always holds when $\tau$ is the exit time from a simply connected domain.

**Theorem 28.** *[9, 15] For any simply connected domain $U$ we have $\mathbf{E}(\tau_U^p) < +\infty$ for all $p < \frac{1}{4}$.*

Theorem 28 combined with Theorem 26 imply that $\mathbf{E}(\ln(1+\tau_U))$ is finite for any simply connected domain $U$. Hence the three quantities $\mathbf{E}(\tau^{\frac{p}{2}}), \mathbf{E}(Z_\tau^{*p}), \mathbf{E}(|Z_\tau|^p)$ are all finite or all infinite for positive $p$. A straightforward application is when $U$ is a wedge of aperture $2\theta$, i.e

$$U := \{z \mid -\theta < \arg z < \theta\}$$

with $0 < \theta < \pi$.

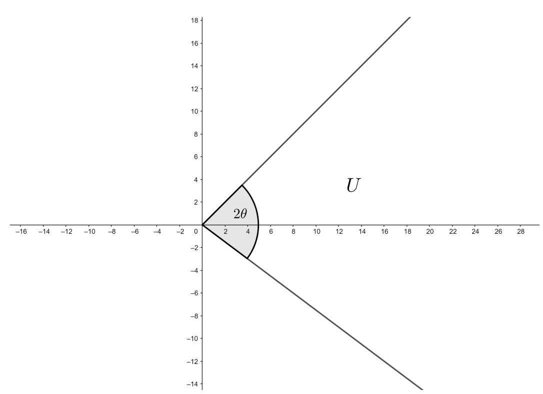


Figure 1.3.2: The wedge of aperture $2\theta$.

**Proposition 29.** *We have*

$$\mathbf{E}(\tau_U^{\frac{p}{2}}) < +\infty \Longleftrightarrow p < \frac{\pi}{2\theta}.$$

*Proof.* Using the map $\varphi : z \longmapsto -z^{\frac{\pi}{2\theta}}$ mixed with the conformal invariance techniques, one can show that the density of the stopped Brownian motion is given by

$$\rho^1_{Z_{\tau_U}}(re^{\pm\theta i}) = \frac{1}{2\theta r(r^{\frac{\pi}{2\theta}} + r^{-\frac{\pi}{2\theta}})}$$

(See [33]). In particular

$$\int_0^{+\infty} r^p \rho^1_{Z_{T(U)}}(re^{\pm\theta i})dr < +\infty \Longleftrightarrow p < \frac{\pi}{2\theta}$$

and since $U$ is a simply connected domain we obtain the criterion for $\mathbf{E}(\tau^{\frac{p}{2}})$:

$$\mathbf{E}(\tau^{\frac{p}{2}}) < +\infty \Longleftrightarrow p < \frac{\pi}{2\theta}.$$

□

The figure (1.3.3) illustrates the regions where we have finiteness/infiniteness of $\mathbf{E}(\tau^{\frac{p}{2}})$ in terms of the half aperture $\theta$.

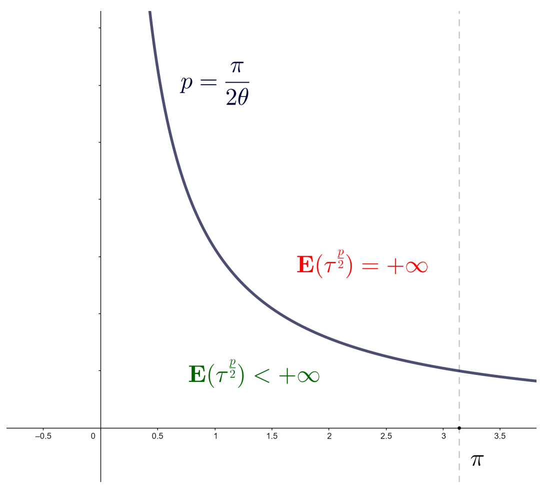


Figure 1.3.3: The curve of the function $p(\theta) = \frac{\pi}{2\theta}$. Below it we have finiteness.

*Remark* 30. Other proofs of Proposition 29 are available in [9]. The asymptotic behavior of the tail of the exit time is provided in [42].

Now we give an example of domain where its exit time has no finite $p^{th}$-moment for any positive $p$.

**Proposition 31.** *The exit time from $\mathbb{C} - \overline{\mathbb{D}}$ has no finite $p^{th}$-moment for any $p > 0$.*

*Proof.* Suppose that $\mathbf{E}(\tau^p) < \infty$ for some $p > 0$, then by Lemma 26 $\mathbf{E}(\ln(1+\tau)) < +\infty$. In this case, $\mathbf{E}(\tau_U^{\frac{p}{2}}) < +\infty \Leftrightarrow \mathbf{E}_z(|Z_\tau|^p) < +\infty$. On the other hand, $\mathbf{E}_z(|Z_\tau|^p) < +\infty$ for all positive $p$ as $Z_\tau$ is of modulus 1 on $\partial(\mathbb{C} - \overline{\mathbb{D}}) = \partial(\mathbb{D})$ . Thus, $\mathbf{E}(\tau_U^{\frac{p}{2}}) < +\infty$ for all positive $p$ as well. However, this can not happen as $\mathbb{C} - \overline{\mathbb{D}}$ contains a half plane. Consequently neither $\mathbf{E}(\ln(1+\tau)) < +\infty$ nor $\tau$ has a finite $p^{th}$-moment. □

## 1.4 Dynkin's formula

One of the major interests in the theory of Brownian motion is to evaluate the expectation of $f(B_t^{\mathbf{x}})$ where $f$ is a smooth real valued function and $B_t^{\mathbf{x}}$ is a multi dimensional Brownian motion starting at $\mathbf{x}$. The following identity, referred to as Itô formula, is the core of this field

$$f(B_t^{\mathbf{x}}) = f(\mathbf{x}) + \int_0^t \nabla f(B_s^{\mathbf{x}}) \cdot dB_s + \frac{1}{2}\int_0^t \Delta f(B_s^{\mathbf{x}}) ds. \tag{1.4.1}$$

We refer the reader to [27] for more detail. In our case, applying the expectation operator to (1.4.1) yields

$$\mathbf{E}_z(f(Z_t)) - f(z) = \frac{1}{2}\mathbf{E}_z\left(\int_0^t \Delta f(Z_s) ds\right).$$

Under some technical assumptions, we can replace $t$ by a stopping time $\tau$, and this what we do often. That is, we get the so called *Dynkin's formula*

$$\mathbf{E}_z(f(Z_\tau)) - f(z) = \frac{1}{2}\mathbf{E}_z\left(\int_0^\tau \Delta f(Z_s) ds\right).$$

In particular

- If $f$ has a constant non zero Laplacian then we obtain the following expression for the average of $\tau$
$$\frac{2}{\Delta f}(\mathbf{E}_z(f(Z_\tau)) - f(z)) = \mathbf{E}_z(\tau). \tag{1.4.2}$$

- If $f$ is harmonic then
$$\mathbf{E}_z(f(Z_\tau)) = f(z).$$

In [9, 30], the two authors provide a method to get $\mathbf{E}_z(\tau_U)$ in case of existence of a suitable function $f$. An application of this result is when $U$ is limited by a conic $\mathcal{H}$ of equation $f(x, y) = \frac{1-ax^2-by^2}{a+b} = 0$ where $a, b$ are two reals such that and $a + b \neq 0$. If the function $f$ satisfies the following properties

$$\begin{cases} f(z) = 0 & z \in \mathcal{H} \\ \Delta f(z) = -2 & z \in U \\ 0 \le f(z) \le \delta(1 + |z|^2) & z \in U \end{cases}$$

then, by plugging it into (1.4.2), we obtain the following expression

$$\mathbf{E}_z(\tau_U) = \frac{1}{a+b}(1 - ax^2 - by^2).$$

When $a + b = 0$, which is the case where $\mathcal{H}$ is an hyperbola with perpendicular asymptotes, then $\mathbf{E}_z(\tau_U) = +\infty$ for any $z \notin \mathcal{H}$. This can be seen either by a monotonicity argument or by Proposition 29 which implies a wedge of 90° angle has an infinite exit time expectation.

**Proposition 32.** *Let $\tau$ be the exit time from the annulus $\{r < |z| < R\}$. Then*

$$\mathbf{P}_z\{|B_\tau| = R\} = \frac{\log|z| - \log r}{\log R - \log r} = 1 - \mathbf{P}_z\{|B_\tau| = r\} \tag{1.4.3}$$

*and*

$$\mathbf{E}_z(\tau) = \frac{1}{2}\left(\frac{R^2 \ln\frac{|z|}{r} - r^2 \ln\frac{|z|}{R}}{\ln\frac{R}{r}} - |z|^2\right). \tag{1.4.4}$$

*Proof.* The idea is to find an harmonic function that takes the value 1 on $\{|z| = R\}$ and 0 on $\{|z| = r\}$ for example. We can check that $f(z) = \frac{\log |z| - \log r}{\log R - \log r}$ does the job. Then we apply Dynkin's formula to $f$. The expectation follows from the martingale $Z_t^2 - 2t$. □

An interesting corollary of (1.4.3) is the following.

**Corollary 33.** *PBM hits any disc of $\mathbb{C}$ a.s.*

*Proof.* By monotonicity we have

$$\lim_{R \to +\infty} \mathbf{P}_z\{|B_\tau| = R\} = 0$$

and so $\mathbf{P}_z\{|B_\tau| = r\} = 1$ where $\tau$ is now the exit time from $\mathbb{C} - \{|z| \leq r\}$. □

Corollary 33 is often rephrased as " PBM is recurrent ". In higher dimension, Brownian motion is transient, i.e the probability it hits a ball $\{\|x\| < r\}$ is less than one. More precisely such a probability is given by $(\frac{r}{\|x\|})^{n-2}$. PBM theory serves also to show some results in complex analysis that have nothing to do with probability at first glance. For example, we can recover the fundamental theorem of Algebra, Liouville theorem, Picard little theorem etc ...

**Theorem 34** (Liouville theorem)**.** *A non constant entire function is unbounded.*

*Proof.* Let $f$ be such a function. Then for some $\delta > 0$ the set $\{z \mid |f'(z)|^2 > \delta\}$ contains a disc $D_\eta := \{|z| < \eta\}$. Consider now the $n^{th}$ exit times $\varpi_n, \varrho_n$ of $Z_t$ from $D_{\frac{\eta}{2}}$ and $D_\eta$. These times are finite a.e by Corollary 33. We obtain then

$$\sigma(+\infty) \geq \sum_n \int_{\varpi_n}^{\varrho_n} |f'(Z_s)|^2 ds \geq \sum_n \delta^2 (\varrho_n - \varpi_n).$$

The r.v's $(\varrho_n - \varpi_n)_n$ are i.i.d nonnegative non degenerate random variables ($\mathbf{E}(\varrho_n - \varpi_n) > 0$). So by Borel-Cantelli lemma, the sum $\sum_{n=0}^{+\infty} \delta^2(\varrho_n - \varpi_n)$ is infinite a.e [7]. Consequently $\sigma(+\infty) = +\infty$ and thus $\{f(Z_t), t \geq 0\}$ is a Brownian path and hence the result follows by Corollary 33. □

*Remark* 35. The proof of Theorem 34 shows that the range of the time change generated by a non-constant entire function is $[0, +\infty)$. In particular, we recover

$$\int_0^{+\infty} e^{2\Re(Z_t)} ds = +\infty$$

which appeared in Proposition 7.

**Proposition 36.** *PBM winds with no bound neither below nor above.*

*Proof.* The statement is equivalent to : $t \mapsto \arg(Z_t)$ is unbounded a.e. Since the property concerns the paths of $Z_t$ then we can show it for $\widetilde{Z}_{\sigma(t)} = e^{Z_t}$ which is a time changed PBM. The argument of

[7] The expectation of $\varrho_n - \varpi_n$ is $\frac{3\eta^2}{16}$ and we have the estimate

$$\sum_{j \leq n} \delta^2(\varrho_n - \varpi_n) \underset{+\infty}{\sim} \delta^2 \frac{3\eta^2}{16} n.$$

by virtue of the strong law of large numbers.

$\widetilde{Z}_{\sigma(t)}$ is $R_t = \Im(Z_t)$ which is a standard Brownian motion. Since $e^z$ is entire then by Remark 35 we obtain

$$\begin{aligned}\{\arg(\widetilde{Z}_t), t \geq 0\} &= \{\arg(\widetilde{Z}_{\sigma(t)}), t \geq 0\} \\ &= \{R_t, t \geq 0\} \\ &= (-\infty, +\infty).\end{aligned}$$

□

## 1.5 Stopped transition probability

It is known that

$$\mathbf{P}_x(Z_t \in A) = \int_A p(t,x,a)da \tag{1.5.1}$$

is valid for any measurable set $A$. However most situations involve stopped PBM upon hitting the boundary of some domain $U$ and so we want to measure $\{Z_t \in A\}$ $(A \subset U)$ but before leaving $U$. More precisely our aim is to find a function $\vartheta(t,x,a)$ (depending on $U$) such that

$$\mathbf{P}_x(Z_t \in A,\, t < \tau_U) = \int_A \vartheta_U(t,x,a)da.$$

In particular if $A = U$ then

$$\mathbf{P}_x(t < \tau_U) = \int_U \vartheta_U(t,x,a)da. \tag{1.5.2}$$

One can think of (1.5.2) as a spatial representation of $\mathbf{P}_x(t < \tau_U)$. In order to derive the density $\vartheta_U$ of $\mathbf{P}_x(Z_t \in A,\, \tau_U \leq t)$, we use the strong Markov property combined with Tonelli's theorem. That is

$$\begin{aligned}\mathbf{P}_x(Z_t \in A,\, \tau_U \leq t) &= \mathbf{P}_x(Z_{(t-\tau_U)+\tau_U} \in A,\, \tau_U \leq t) \\ &= \mathbf{E}_x\left[\mathbf{E}_x(\mathbf{1}_{\{Z_{(t-\tau_U)+\tau_U} \in A\}}\mathbf{1}_{\{\tau_U \leq t\}} \mid \mathcal{F}_\tau)\right] \\ &= \mathbf{E}_x\left[\mathbf{E}_x(\mathbf{1}_{\{Z_{(t-\tau_U)+\tau_U} \in A\}} \mid Z_{\tau_U})\mathbf{1}_{\{\tau_U \leq t\}}\right] \\ &= \mathbf{E}_x\left[\mathbf{E}_{Z_{\tau_U}}(\mathbf{1}_{\{Z_{(t-\tau_U)+\tau_U} \in A\}})\mathbf{1}_{\{\tau_U \leq t\}}\right] \\ &= \mathbf{E}_x\left[\int_A p(t-\tau_U, Z_{\tau_U}, a)\mathbf{1}_{\{\tau_U \leq t\}}da\right] \\ &= \int_A \mathbf{E}_x(p(t-\tau_U, Z_{\tau_U}, a)\mathbf{1}_{\{\tau_U \leq t\}})da.\end{aligned}$$

On the other hand

$$\begin{aligned}\mathbf{P}_x(Z_t \in A,\, t < \tau_U) &= \mathbf{P}_x(Z_t \in A) - \mathbf{P}_x(Z_t \in A,\, \tau_U \leq t) \\ &= \int_A (p(t,x,a) - \mathbf{E}_x(p(t-\tau_U, Z_{\tau_U}, a)\mathbf{1}_{\{\tau_U \leq t\}})da\end{aligned}$$

which holds for all measurable sets $A$. Therefore

$$\vartheta_U(t,x,a) \overset{a.e}{=} p(t,x,a) - \mathbf{E}_x(p(t-\tau_U, Z_{\tau_U}, a)\mathbf{1}_{\{\tau_U \leq t\}}). \tag{1.5.3}$$

We can obtain another representation of $\vartheta(t,x,a)$ by discretizing the set $\{t \leq \tau_U\}$ as the limit of $\{Z_{\frac{kt}{2^n}} \in U,\, k = 0, 1, ..., 2^n\}$. That is

$$\vartheta_U(t,x,a) = \lim_n \int_U \cdots \int_U \prod_{j=0}^{2^n-1} p(\tfrac{kt}{2^n}, w_j, w_{j+1})dw_1 \cdots dw_{2^n-1} \tag{1.5.4}$$

with $z_0 = x$ and $z_{2^n} = a$. For example, (1.5.4) shows immediately that $\vartheta_U(t,\cdot,\cdot)$ is symmetric and i.e $\vartheta_U(t,x,a) = \vartheta_U(t,a,x)$. It can be shown that $\vartheta_U(t,\cdot,\cdot)$ is continuous on $(\mathbb{C} - \partial U) \times (\mathbb{C} - \partial U)$ [11] and hence we can get rid of the almost everywhere statement in (1.5.3).

**Corollary 37.** *We have*

$$\mathbf{E}_x(f(Z_t)\mathbf{1}_{\{t<\tau_U\}}) = \int_{\mathbb{C}} f(z)\vartheta_U(t,x,y)dy. \tag{1.5.5}$$

**Definition 38.** The Green function is defined by

$$G_U(x,y) = \int_0^{+\infty} \vartheta_U(t,x,y)dt.$$

Intuitively, the Green function measures how much time on average the PBM visits at the point $y$ starting from $x$ before leaving $U$ [8]. Using (1.5.5) we get the formula

$$\mathbf{E}_x(\int_0^{\tau_U} f(Z_t)dt) = \int_U f(y)G_U(x,y)dy.$$

## 1.6 PBM and heat equation.

It turns out that PBM is highly connected to the theory of P.D.E. A fundamental example of P.D.E is the so called Dirichlet problem, i.e

$$\begin{cases} \Delta u = 0 & x \in U \\ u = f & x \in \partial U \end{cases}. \tag{1.6.1}$$

The equation $\Delta u = 0$ is the definition of being harmonic for $u$. Another example is the heat equation

$$\begin{cases} (\partial_t - \frac{1}{2}\Delta)u = 0 & x \in U \\ u = f & x \in \partial U \\ u = g & t = 0 \end{cases}. \tag{1.6.2}$$

The transition probability $p(t,x,y)$ of BM (the Brownian density) satisfies the first equation of (1.6.2). In particular, $p(t,x,y)$ has the temporal boundary condition $p(0,x,y) = \delta_x(y)$ where $\delta_x$ is the Dirac function.

In general solving P.D.E is not always easy and requires " nice " domains to work on, especially when it concerns the uniqueness of such a solution. The following theorem highlights how a stopped PBM can serve to give a stochastic representation of the solution for (1.6.1) and (1.6.2).

**Theorem 39.** *Suppose $U$ is a bounded planar domain such that every boundary point satisfies the Poincaré cone condition, and suppose $f$ is a continuous function on $\partial U$.Then the function $u : z \in \overline{U} \mapsto \mathbf{E}_z(f(Z_{\tau_U})$ is the unique continuous function satisfying (1.6.1).*

Regarding the definition of the Poincaré cone condition as well as a $d$-dimensional proof of Theorem 39, we refer the reader to [34].

[8] To see this fact better, recall the Green function of a simple random walk $X_n$ which is defined by

$$G(x,y) := \mathbf{E}_x(\sum_{n=0}^{+\infty} 1_{\{n\leq\tau,\, X_n=y\}}).$$

**Theorem 40.** *If $f$ and $g$ are smooth functions then*

$$u(t,z) = \mathbf{E}_z(g(Z_t)\mathbf{1}_{\{t<\tau_U\}}) + \mathbf{E}_z(f(Z_t)\mathbf{1}_{\{\tau_U \le t\}})$$

*is a continuous solution of (1.6.2) where $\tau_U$ is the exit time of $Z_t$ from $U$.*

In particular, the tail of the exit time $\tau_U$, i.e $\mathbf{P}_z(t < \tau_U)$ satisfies the following P.D.E

$$\begin{cases} (\partial_t - \frac{1}{2}\Delta)u = 0 & x \in U \\ u = 0 & x \in \partial U \\ u = 1 & t = 0 \end{cases} \tag{1.6.3}$$

This duality between P.D.E and stochastic representation is handy. Sometimes it is hard to find an asymptotic estimate of $\mathbf{P}_z(t < \tau_U)$ via straightforward probabilistic techniques, so we try to find the analytic solution for (1.6.3) and then deduce the estimate. Other times, the stochastic representation serves to derive some properties hard to get directly from the analytic solution. The most common method to obtain such a solution is separation of variables, especially when the domain $U$ is " nice " enough (rectangle, disc etc ... ).

**Example 41.** The separation of variables technique yields the following solution for (1.6.3) on the unit disc

$$\mathbf{P}_z(t < \tau_{\mathbb{D}}) = 2\sum_{n=1}^{+\infty} \frac{J_0(\alpha_n|z|)}{\alpha_n J_1(\alpha_n)} e^{-\frac{\alpha_n^2}{2}t}$$

where $\alpha_n$ is the $n^{th}$ positive zero of the Bessel function of the first kind $J_0(x)$ [8]. In particular, we get the estimate

$$\mathbf{P}_z(t < \tau_{\mathbb{D}}) \underset{t\to+\infty}{\sim} 2\frac{J_0(\alpha_1|z|)}{\alpha_1 J_1(\alpha_1)} e^{-\frac{\alpha_1^2}{2}t}. \tag{1.6.4}$$

**Example 42.** The rate of (1.6.3) is defined to be

$$\lambda(U) := \lim_{t\to+\infty} -\frac{\ln \mathbf{P}_z(t < \tau_U)}{t}.$$

It can be shown that $\lambda(U)$ is well defined and finite. The probabilistic way to define $\lambda(U)$ provides a monotonicity property. That is, the rate $\lambda$ decreases when $U$ gets bigger, i.e if $U \subset V$ then $\lambda(V) \le \lambda(U)$.

Now, we are ready to go through the published papers. The next chapter is about the problem of maximizing the $p^{th}$ moments of the exit time. The last chapter is about Planar Skorokhod embedding problem.

# Chapter 2

# Maximizing the $p^{th}$ moments of the exit time.

A high speed particle moving on a surface can be mimicked by a planar Brownian motion. Assume we are interested in maximizing the average of time required by that particle to leave a certain region $U$ in terms of its starting point. Then the question is to find where the function $z \mapsto \mathbf{E}_z(\tau_U)$ would be maximal with $\tau_U$ being the time by which the particle leaves the region. That is, and more generally, fix a domain $U$ and $p > 0$ for which $\mathbf{E}_z(\tau_U^p)$ is finite. Our interest is to find the starting points $z$ for which $\mathbf{E}_z(\tau_U^p)$ is maximal. We shall call such points $p^{th}$ *centers*. In most of the cases, there is no explicit formula for $\mathbf{E}_z(\tau_U^p)$ even when $p = 1$. The case $p = 1$ is referred to as the torsion problem, and was subject to many studies, see for example [41, 23, 28, 36]. The map $z \mapsto \mathbf{E}_z(\tau_U)$ is a solution for Dirichlet problem $\Delta u = -2$, and so P.D.E's techniques are widely used. However, those techniques are devoted especially for convex domains which might represent a kind of limitation to the problem. In this context, we've developed some methods to narrow the search area for the $p$-centers. The contribution of these new methods, compared to existing ones, can be then summarized in the following two main points:

- They deal with any finite $p^{th}$ moment, which in particular covers the torsion problem mentioned above.

- They apply to a larger category of shapes including convex ones.

A straightforward advantage of the provided methods is to save the time of simulation. That is, when simulating the Brownian paths, there is no need to test a large number of starting points in order to maximize the average of the exit time. The central argument used to in both methods is coupling. That is we run two PBM's, say $Z_t$ and $\widetilde{Z}_t$, where $\widetilde{Z}_t$ is related to $Z_t$ in a certain way that enables us to compare their exit times from the domain $U$. In particular, this leads to the comparison between the two tails of the exit times, and therefore we can compare the $p^{th}$-moments in terms of the starting points.

We start by giving some geometric methods to locate $p^{th}$ centers. These ones lean on two concepts, namely partial symmetry and $\Delta$-convexity. The two tools generalize the standard definitions of symmetry and convexity.

**Definition 43.** We say that a line $\Delta : ax + by + c = 0$ is a *partial symmetry axis for $U$* if one of the two sets $U^+ := U \cap \{ax + by + c > 0\}$ or $U^- := U \cap \{ax + by + c < 0\}$ can be folded over $\Delta$ and fits into $U$. In other words, either $\sigma_\Delta(U^+)$ or $\sigma_\Delta(U^-)$ remains inside $U$ where $\sigma_\Delta$ denotes the symmetry over $\Delta$. The smaller side would be called the symmetric side of $U$ over $\Delta$.

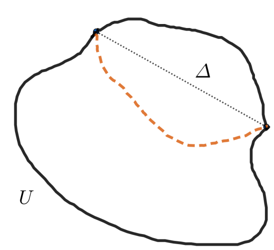


Figure 2.0.1: Partial symmetry axis.

In particular, a symmetry axis is also partial.

**Definition 44.** Let $\Delta$ be a symmetry axis of $U$. We say that $U$ is $\Delta$-convex if

$$\forall\,(z,t) \in U \times [0,1] \;\; tz + (1-t)\sigma_{\Delta}(z) \in U.$$

In other words, for every $z \in V$, the segment joining $z$ and $\sigma_{\Delta}(z)$ remains inside $U$.

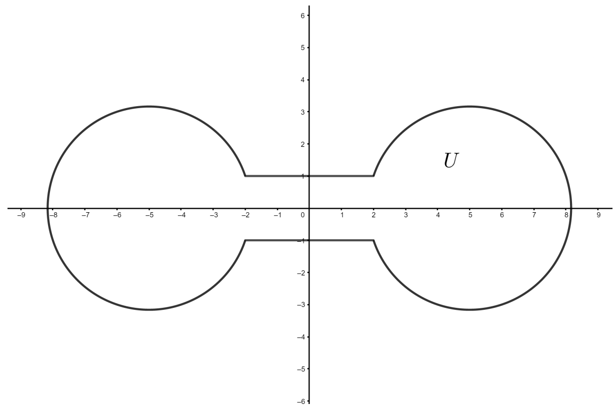


Figure 2.0.2: $U$ is convex with respect to the horizontal axis but not with respect to the vertical axis.

**Theorem 45.** *Let $S$ be the symmetric side of $U$ over some axis $\Delta$. Then all $p^{th}$ centers lie on $U \setminus S$.*

**Proposition 46.** *Suppose $U$ is $\Delta$-convex with respect to a symmetry axis $\Delta$. Then all $p^{th}$ centers of $U$ lie on $\Delta$.*

**Corollary 47.** *If $U$ has two $\Delta$-convexity symmetry axes then it has a unique $p^{th}$ center, precisely its the natural center.*

Now we give the analytical techniques. These one are basically based on the variation of the speed of the image of a planar Brownian motion under the action of an analytic or antiholomorphic function that we refer here as a nice function.

**Theorem 48.** *Let $U$ be a domain with symmetric side $S$ with respect to some line $\Delta$ and $f : U \longrightarrow f(U)$ be a nice map. If*

$$\forall z \in S,\; |f'(z)| \leq |f'(\sigma_{\Delta}(z))|$$

*then the $p^{th}$ centers are in $f(U \setminus S)$.*

**Theorem 49.** *Let $f : U \longrightarrow U$ be an antiholomorphic map and consider the two following sets*

$$\Omega := \{z \in U|\; |f'(z)| < 1\}$$
$$\Lambda := \{z \in U|\; |f'(z)| = 1\}.$$

*If $f(\Omega) \subset U \setminus \Omega$ and $f_{|\Lambda} = id_{\Lambda}$ then $U \setminus \Omega$ contains the $p^{th}$ centers.*

The coupling argument used to show the above results serves also to derive a stochastic domination between exit times at different starting points. More precisely we have the following result.

**Theorem 50.** *Let $U$ be a domain and $\Gamma$ the part of $U$ containing the $p^{th}$ centers according the above techniques. Then for all $z \in U - \overline{\Gamma}$*

$$\mathbf{P}_z(\tau_U > t) \le \mathbf{P}_{f(z)}(\tau_U > t)$$

*where $f$ denotes either a symmetry as in Theorem 45 or a nice map as in Theorem 49. In particular, for any positive function $\Psi$ we have*

$$\mathbf{E}_z(\Psi(\tau_U)) \le \mathbf{E}_{f(z)}(\Psi(\tau_U)).$$

## 2.1 Examples and applications

In this section we provide some examples from [7]. Other ones can be found therein.

**Example 51.** For all $p > 0$, the $p^{th}$ centers of $\mathcal{H} := \{x > 0, x^2 - y^2 < 1\}$ lie on $(\frac{1}{2}, \frac{1}{\sqrt{2}})$.

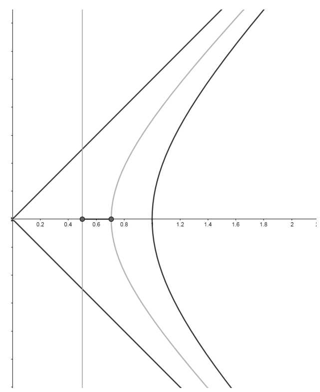

Figure 2.1.1: $p^{th}$ centers lie on the segment $(\frac{1}{2}, \frac{1}{\sqrt{2}})$.

First of all, the fact that all $p^{th}$ moments of $\tau_{\mathcal{H}}$ is a consequence of a result in [31]. The rest follows from these facts :

- $\Re(z) = \frac{1}{2}$ is a partial symmetry axis.
- The function $h : z \mapsto \sqrt{z}$ maps the strip $\{0 < \Re(z) < 1\}$ to $\mathcal{H}$.
- $h'(z) = \frac{1}{2\sqrt{|z|}}$ is decreasing in terms of $|\Im(z)|$. Then we focus on $h((0, 1/2))$.

**Example 52.** Let $\mathscr{C}_{r,R}$ be the crescent-like shape limited by the two circles $C_{(0,\frac{r}{2}),\frac{r}{2}}$ and $C_{(0,\frac{R}{2}),\frac{R}{2}}$. Then for all $p > 0$, the $p^{th}$ centers of $\mathscr{C}_{r,R}$ lie in $(\frac{2Rr}{R+r}, \frac{R+r}{2})$.

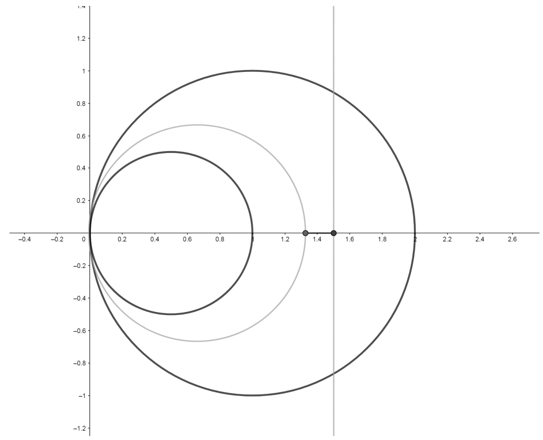

Figure 2.1.2: $p^{th}$ centers lie on the segment $({}^1/_2, {}^1/_{\sqrt{2}})$.

It follows from the following remarks :

- $\Re(z) = \frac{R+r}{2}$ is a partial symmetry axis.
- The function $h : z \mapsto \frac{Rr}{(R-r)z+r}$ maps the strip $\{0 < \Re(z) < 1\}$ to $\mathscr{C}_{r,R}$.
- The derivative growth of $h$ yields the bound $\frac{2Rr}{R+r}$.

**Proposition 53.** *If* $0 < b < a$ *then* $\sqrt{ab} < \frac{a-b}{\ln(\frac{a}{b})} < \frac{\frac{a+b}{2}+\sqrt{ab}}{2}$.

The quantity $\frac{a-b}{\ln({}^a/_b)}$ is called logarithmic mean temperature difference, referred to as **LMTD** ( see [24] ). In literature, the known upper bound is $\frac{a+b}{2}$ which is greater than ours. To see that, we combine two facts :

1. We consider the annulus $\mathcal{A} := \{\sqrt{b} < |z| < \sqrt{a}\}$ with the map $z \mapsto \frac{\sqrt{ab}}{\bar{z}}$. We find that the $p$-centers lie in $\{\sqrt{\sqrt{b}\sqrt{a}} < |z| < \frac{\sqrt{a}+\sqrt{b}}{2}\}$.
2. Dynkin's formula yields

$$\mathbf{E}_z(\tau_{\mathcal{A}}) = \frac{1}{2}\left[\frac{a\ln\frac{|z|}{\sqrt{b}} - b\ln\frac{|z|}{\sqrt{a}}}{\ln\sqrt{\frac{a}{b}}} - |z|^2\right]. \tag{2.1.1}$$

That is, the maximum is attained at $|z| = \sqrt{\frac{a-b}{\ln\frac{a}{b}}}$. A small typo to point out is that the factor $\frac{1}{2}$ is missing from (2.1.1) appeared in the page 10 of [7].

In the study of heat flow, the quantity $\frac{a-b}{\ln(\frac{a}{b})}$ is of great importance and in that context it is known as the logarithmic mean temperature difference, or LMTD (see [24]). We have therefore given a new proof of the fundamental fact that the LMTD lies between the arithmetic and geometric means, and in fact have proved that the upper bound can be lowered to the arithmetic mean of the arithmetic and geometric means.

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# MAXIMIZING THE *p*th MOMENT OF THE EXIT TIME OF PLANAR BROWNIAN MOTION FROM A GIVEN DOMAIN

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**Abstract**

In this paper we address the question of finding the point which maximizes the $p$th moment of the exit time of planar Brownian motion from a given domain. We present a geometrical method for excluding parts of the domain from consideration which makes use of a coupling argument and the conformal invariance of Brownian motion. In many cases the maximizing point can be localized to a relatively small region. Several illustrative examples are presented.

*Keywords:* Planar Brownian motion; exit time

2010 Mathematics Subject Classification: 60J64

## 1. Introduction

Let $Z_t := X_t + iY_t$ be a planar Brownian motion starting at a point $a$ in a domain $U$. We will let $\tau_U = \tau_U(a)$ be the first time that $Z_t$ exits $U$, and we will use the standard notation $\mathbb{E}_a$ to denote expectation conditioned on $Z_0 = a$ a.s. The focus of this paper is the following optimization problem.

> For a given domain in the plane and $0 < p < \infty$, find the point $a$ which maximizes the quantity $\mathbb{E}_a[(\tau_U)^p]$.

We will refer to such a point as a $p$th *center* of $U$; it is not generally unique, as the easy example of an infinite strip shows. For many domains, even simple ones such as an isosceles triangle, it is difficult to find any of the $p$th centers, but we will show how elementary coupling arguments and the conformal invariance of Brownian motion in many cases allows us to locate a small region in $U$ which must contain all $p$th centers. In certain cases in which the domain in question has a high degree of symmetry, it will allow us to locate all $p$th centers.

Before describing our methods, we present a brief overview of some earlier works related to this problem. The case $p = 1$ is commonly referred to as the 'torsion problem' due to its connection with mechanics, and is naturally the most tractable. The function $h(a) = \mathbb{E}_a[\tau_U]$ satisfies $\Delta h = -2$, and therefore PDE techniques can be employed to great effect. Sperb [17, Chapter 6] gives a good account of this problem and methods for attacking it in special cases, such as when the domain in question is convex. Further results along the same lines, focusing in particular on convex domains, can be found in [9], [12], and [17].

Other interesting related problems have been tackled by PDE methods. For example, in the famous paper [2] (see also the related work [3]) eigenvalue techniques are used to demonstrate

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relationships between $\mathbb{E}_a[\tau_U]$ and geometric qualities of the domain, such as the size of the hyperbolic density and the inradius (the radius of the largest disk contained in the domain). The methods developed there have been extended by other authors in a number of different directions. For example, Méndez-Hernández [14] proved a number of related stochastic domination results concerning convex domains in $\mathbb{R}^n$ and various types of symmetrizations. These results allow conclusions to be reached concerning the comparison of $p$th moments of the exit times from these domains. One striking consequence of the eigenvalue methods is the fact that over all domains with a given area, the disk maximizes the $p$th moment of the exit time of Brownian motion for all $p$. The recent work by Kim [11] contains a discussion and refinements of this result. We would also like to mention the interesting and very recent preprint [5], in which questions similar to ours are addressed; the authors there demonstrate bounds on the quantity $\lambda_1^p(U)\sup_{a\in U}\mathbb{E}_a[(\tau_U)^p]$, where $\lambda_1(U)$ denotes the first Dirichlet eigenvalue for the Laplacian in $U$, and prove the existence of extremal domains with regards to this quantity over various classes of convex domains.

Our results differ from those described above in the following ways. We have not employed PDE methods at all, choosing instead to work with an elementary coupling method. Perhaps as a consequence of this, convexity plays little role in our discussion, although a weaker concept called $\Delta$-convexity (defined below) will be important. The type of coupling we will use is not entirely new, and has found a number of uses in related topics, for instance in investigations into the 'hot spots' conjecture such as [1], [4], and [16]. However, we believe that it has not yet been applied directly in the manner we use here. Furthermore, we restrict our attention to two dimensions, which allows conformal mappings to take prominence and to extend the standard notion of coupling. We present several methods for localizing the $p$-centers of a domain, and then consider a number of specific domains, showing in each case how our methods can be used to localize the $p$th centers of the domain. In what follows we assume $p$ is a fixed positive number. However, in order to reduce the qualifications needed to state our results, for any planar domain $U$ for which we are interested in maximizing the $p$th moment, we will assume that $\mathbb{E}_a[(\tau_U)^p] < \infty$ for all points $a \in U$; this would follow if $\mathbb{E}_a[(\tau_U)^p] < \infty$ for any $a \in U$, as is shown in [7].

## 2. Partial symmetry and convexity with respect to a line.

**Definition 2.1.** Let U be a domain of $\mathbb{C}$. We say that a line $\Delta\colon ax + by + c = 0$ is a partial symmetry axis for U if one of the two sets $U^+ := U \cap \{ax + by + c > 0\}$ or $U^- := U \cap \{ax + by + c < 0\}$ can be folded over $\Delta$ and fits into U, more precisely if $\sigma_\Delta(U^+)$ or $\sigma_\Delta(U^-)$ remains inside U, where $\sigma_\Delta$ denotes the symmetry over $\Delta$. The subset among $U^\pm$ that satisfies this property (i.e. the smaller side with respect to the symmetry) is called symmetric side of U over $\Delta$. So, for instance, any line intersecting $\mathbb{D} = \{|z| < 1\}$ is a partial symmetry axis for $\mathbb{D}$ but the line $y = 2x$ is not one for the square $\{|x| < 1, |y| < 1\}$, since the reflection over this line of the point (1,1) is the point (1/5, 7/5), which is not in the closure of the square. Note that both of $U^\pm$ are symmetric sides if and only if $\Delta$ is a symmetry axis for $U$.

**Theorem 2.1.** *Let S be the symmetric side of U over a partial symmetry axis $\Delta$. Then, for any $a \in S$ we can find Brownian motions $Z_t$ starting at a and $\tilde{Z}_t$ starting at $\sigma_\Delta(a)$ defined on the same probability space such that $\tau_U \le \tilde{\tau}_U$ a.s. (where $\tilde{\tau}_U$ is the exit time from U of $\tilde{Z}$). Furthermore, if $\sigma_\Delta(S)$ is strictly contained in $U \setminus (S \cup \Delta)$ then $P(\tau_U < \tilde{\tau}_U) > 0$. In particular, $\mathbb{E}_a[\tau_U^p] \le \mathbb{E}_{\sigma_\Delta(a)}[\tilde{\tau}_U^p]$ (with strict inequality if $\sigma_\Delta(S)$ is strictly contained in $U \setminus (S \cup \Delta)$).*

*Proof.* This follows from a coupling argument. Let $Z_t$ start at $a \in S$, and let $H_\Delta$ be its hitting time of the line $\Delta$. Form the process $\tilde{Z}_t$ by the rule

$$\tilde{Z}_t = \begin{cases} \sigma_\Delta(Z_t) & \text{if } t < H_\Delta, \\ Z_t & \text{if } t \geq H_\Delta. \end{cases}$$

It follows from the strong Markov property and the reflection invariance of Brownian motion that $\tilde{Z}_t$ is a Brownian motion. Clearly $\tau_U = \tilde{\tau}_U$ on the set $\{\tau_U \geq H_\Delta\}$, and our conditions on $S$ imply $\tau_U \leq \tilde{\tau}_U$ on the set $\{\tau_U < H_\Delta\}$. Furthermore, if $\sigma_\Delta(S)$ is strictly contained in $U \setminus (S \cup \Delta)$, then $Z_t$ has some positive probability of leaving $U$ before $\tilde{Z}_t$ does; this is implied for instance by [6, Theorem I.6.6]. The result follows. □

This theorem allows us in essence to exclude the symmetric side of any partial symmetry axis for $U$ in our search for $p$th centers. The only exception to this rule is when $\sigma_\Delta(S) = U \setminus (S \cup \Delta)$, i.e. when $\Delta$ is a symmetry axis of $U$. However, in most cases a symmetry axis will contain all $p$th centers. To see why this is so, we need another definition.

**Definition 2.2.** Let $\Delta$ be a symmetry axis of $U$. We say that $U$ is $\Delta$-convex if

$$tz + (1-t)\sigma_\Delta(z) \in U \quad \text{for all } (z, t) \in U \times [0, 1].$$

In other words, for every $z \in U$, the segment joining $z$ and $\sigma_\Delta(z)$ remains inside $U$.

It is clear that any convex $U$ is $\Delta$-convex for any symmetry axis $\Delta$, and a less trivial example can be given by $\{-f(x) < y < f(x)\}$, where $f$ is a positive continuous function on the real line, which is $\Delta$-convex with $\Delta = \mathbb{R}$. A domain which is not $\Delta$-convex with respect to a symmetry axis can be given by

$$W_\varepsilon = \{-\varepsilon < y < \varepsilon, |x| < 1\} \cup \left\{|z-1| < \frac{1}{2}\right\} \cup \left\{|z+1| < \frac{1}{2}\right\}$$

with $\varepsilon < 1/2$; this has the real and imaginary axes as symmetry axes but is not $\Delta$-convex with respect to the imaginary axis (though it is with respect to the real line). As will be seen below, this domain also shows why $\Delta$-convexity is required in the following proposition.

**Proposition 2.1.** *Suppose $U$ is $\Delta$-convex with respect to a symmetry axis $\Delta$. Then all pth centers of $U$ lie on $\Delta$.*

*Proof.* Let $a \in U \setminus \Delta$, and let

$$\hat{a} = \frac{1}{2}a + \frac{1}{2}\sigma_\Delta(a)$$

be the orthogonal projection of $a$ onto $\Delta$. Let $L$ be the line parallel to $\Delta$ which passes through the point

$$\frac{1}{2}a + \frac{1}{2}\hat{a}.$$

Speaking informally, this is the line halfway between $a$ and $\Delta$. $\Delta$-convexity implies that $L$ is a partial symmetry axis of $U$, and if $S$ is the component of $U \setminus L$ containing $a$ then $\sigma_L(S)$ is strictly contained in $U \setminus (S \cup L)$. It therefore follows from Theorem 2.1 that $\mathbb{E}_a[\tau_U^p] < \mathbb{E}_{\hat{a}}[\tau_U^p]$. The result follows. □

Note that this proposition completely solves our problem in the case that our domain is $\Delta$-convex with respect to two or more non-parallel symmetry axes, since all $p$th centers must lie at their point of intersection, and we have also incidentally proved the purely geometrical fact that all such symmetry axes must coincide at a unique point; more on this in the final section.

Thus, for instance, all $p$-centers of any regular polygon, a circle, an ellipse, a rhombus, and any number of other easily constructed examples must lie at their natural centers. To see an example of a domain with intersecting symmetry axes but where the point of intersection is not a $p$th center, let us return to the domains $W_\varepsilon$ described immediately before this proposition. Proposition 2.1 implies that all $p$th centers lie on the real line, but it is easy to see that if we make $\varepsilon$ sufficiently small then 0, the intersection point of the two symmetry axes, will not be a $p$th center (clearly $\tau_{W_\varepsilon}(1) \geq \tau_{\{|z-1|<1/2\}}(1)$, so that $\mathbb{E}_1[\tau^p_{W_\varepsilon}]$ always remains greater than a positive constant, but $\tau_{W_\varepsilon}(0)$ decreases monotonically to 0 a.s. as $\varepsilon \searrow 0$, so that $\mathbb{E}_1[\tau^p_{W_\varepsilon}] \searrow 0$).

Let us now look at an example that shows the use of the results proved up to this point, but also their limitations. Suppose $U$ is the isosceles right-angled triangle with vertices at $-1$, 1, and $i$. The imaginary axis is an axis of symmetry, and $U$ is $\Delta$-convex with respect to this axis, so all $p$th centers must lie on the imaginary axis. The line $\{y = 1/2\}$ is a partial symmetry axis for $U$, with $U \cap \{y > 1/2\}$ the symmetric side, so all $p$-centers must lie on $\{x = 0, y \leq 1/2\}$. Now common sense tells us that the $p$th centers cannot be too close to the real axis as well, because this is a boundary component, but there is no good partial symmetry axis to apply to conclude that rigorously. The way out of this difficulty is to extend our method of reflection to curves more general than straight lines. For this, we will need to utilize the conformal invariance of Brownian motion, via the following famous theorem of Lévy (see [6] or [15]).

**Theorem 2.2.** *If $f$ is a holomorphic function, then $f(Z_t)$ is a time-changed Brownian motion. More precisely, $f(Z_{\kappa^{-1}(t)})$ is a Brownian motion where*

$$\kappa(t) := \int_0^t |f'(Z_s)|^2 \, \mathrm{d}s \quad \text{for } t \geq 0.$$

This allows us to extend Theorem 2.1 as follows.

**Proposition 2.2.** *Suppose $U$ is a domain with an axis of symmetry $\Delta$, and suppose $f$ is a conformal map defined on $U$ with the property that $|f'(z)| \geq |f'(\sigma_\Delta(z))|$ for all $z \in A$, where $A$ is one component of $U \backslash \Delta$ and $\sigma_\Delta$ is the symmetry over $\Delta$. Then, for any $a \in A$, we can find Brownian motions $Z_t$ starting at f(a) and $\tilde{Z}_t$ starting at $f(\sigma_\Delta(a))$ defined on the same probability space such that $\tau_{f(U)}(f(a)) \geq \tilde{\tau}_{f(U)}(f(\sigma_\Delta(a)))$ a.s. (where $\tilde{\tau}_U$ is the exit time from $U$ of $\tilde{Z}$). In particular,*

$$\mathbb{E}_{f(a)}\big[\tau^p_{f(U)}\big] \geq \mathbb{E}_{f(\sigma_\Delta(a))}\big[\tau^p_{f(U)}\big].$$

*If there is any point in $A$ at which $|f'(z)| > |f'(\sigma_\Delta(z))|$ then*

$$\mathbb{E}_{f(a)}\big[\tau^p_{f(U)}\big] > \mathbb{E}_{f(\sigma_\Delta(a))}\big[\tau^p_{f(U)}\big]$$

*(for this statement we recall the assumption that $\mathbb{E}_w[\tau^p_{f(U)}] < \infty$ for any $w \in f(U)$).*

*Proof.* Let $Z_t$ be a Brownian motion starting at $a$, and let $\tilde{Z}_t$ be defined as in Theorem 2.1. According to Theorem 2.2, the processes $f(Z_t)$ and $f(\tilde{Z}_t)$ are time-changed Brownian motions, and the time changes are given by $\kappa^{-1}(s)$ and $\tilde{\kappa}^{-1}(s)$, respectively, where

$$\left.\begin{aligned} \kappa(t) &= \int_0^t |f'(Z_t)|^2 \, \mathrm{d}t \\ \tilde{\kappa}(t) &= \int_0^t |f'(\tilde{Z}_t)|^2 \, \mathrm{d}t \end{aligned}\right\} \quad t < \tau_U.$$

Now our assumptions imply $|f'(Z_t)| \geq |f'(\tilde{Z}_t)|$ a.s. for all $t < \tau_U$, and thus $\kappa(t) \geq \tilde{\kappa}(t)$ a.s. for all $t < \tau_U$. It follows from this that $\tau \geq \tilde{\tau}$ a.s., where $\tau$ and $\tilde{\tau}$ are the exit times from $f(U)$ of the Brownian motions $f(Z_{\kappa^{-1}(s)})$ and $f(\tilde{Z}_{\tilde{\kappa}^{-1}(s)})$, which begin at $f(a)$ and $f(\sigma_\Delta(a))$, respectively. The result follows. □

We can obtain a corollary that will be useful for the isosceles triangle and in other cases by taking

$$f(z) = z_0 + R\left(\frac{z+i}{z-i}\right) \quad \text{for } z_0 \in \mathbb{C} \text{ and } R > 0,$$

which takes the real axis to the circle $C = \{|z - z_0| = R\}$, the upper half-plane to the outside of $C$, and the lower half-plane to the inside. We have

$$|f'(z)| = \frac{2R}{|z-i|^2},$$

and it is easy to check that $|f'(z)| > |f'(\bar{z})|$ for all $z$ in the upper half-plane. Applying Proposition 2.2 and working through the implications yields the following.

**Corollary 2.1.** *Let $C = \{|z - z_0| = R\}$ be a circle in $\mathbb{C}$, with inside $\mathcal{I}$ and outside $\mathcal{O}$. If $U$ is a domain such that $\sigma_C(U \cap \mathcal{I}) \subseteq (U \cap \mathcal{O})$, then no pth center of $U$ lies in $\mathcal{I}$.*

Note that here $\sigma_C$ denotes reflection over the circular arc $C$, that is,

$$\sigma_C(z) = z_0 + \frac{R^2}{\overline{z - z_0}}.$$

We remark further that the singularity that $f$ has at $i$ does not cause a problem in this result, because $\mathbb{E}_0(Z_t = i \text{ for some } t \geq 0) = 0$, so a.s. a Brownian motion starting at 0 will not hit the singularity in any case. The compact set $\{B_t \colon 0 \leq t \leq \tau_U\}$ is therefore bounded away from $i$ a.s., and the result goes through.

Let us now apply this corollary to the isosceles right-angled triangle $U$ shown in Figure 1. If $C$ is the circle passing through $-1$ and 1 which intersects the real axis at angles of $\pi/8$, then the reflection of the set $\mathcal{A} = \mathcal{I} \cap U$ will be the region $\mathcal{B}$ in the upper half-plane bounded by $C$ and the circle which passes through $-1$ and 1 and intersects the real axis at angles of $\pi/4$; this can be seen by noting that the transformation $\sigma_C$ preserves angles and also preserves the class of circles on the Riemann sphere (which includes lines, interpreted as circles through $\infty$).

As this region lies within $U$, we conclude that no $p$th centers lie within $\mathcal{A}$. A bit of Euclidean geometry shows that $C$ intersects the imaginary axis at $\csc(\pi/8) - \cot(\pi/8) \approx 0.20$, and coupled with our observations above we see that all $p$th centers must lie on the imaginary axis between the points $0.2i$ and $0.5i$. In fact, the upper bound of $0.5i$ can be improved by using the angle bisector of the angles at 1 or $-1$; see Example 3.2. The reader may also have observed that the reflected circular domain does not do a good job of filling the triangle, and therefore it stands to reason that the lower bound may also be improved; more on this in the final section.

Finally, the following result can be useful when $U$ is mapped to itself by an *antiholomorphic* function $\bar{f}$ (this means that the conjugate of $\bar{f}$, $f(z)$, is holomorphic). We will denote the derivative of this function (with respect to $\bar{z}$) by $\overline{f'(z)}$. An example of this is when $U$ is an annulus, as will be explored in Section 3.

**Proposition 2.3.** *Let $\bar{f} \colon U \longrightarrow U$ be antiholomorphic, and consider the two sets*

$$\Omega := \{z \in U \mid |\bar{f}'(z)| < 1\},$$
$$\Lambda := \{z \in U \mid |\bar{f}'(z)| = 1\}.$$

*If $\bar{f}(U \setminus \Omega) \subset \Omega$ and $\bar{f}_{|\Lambda} = id_\Lambda$, then all pth centers are contained in $\Omega$.*

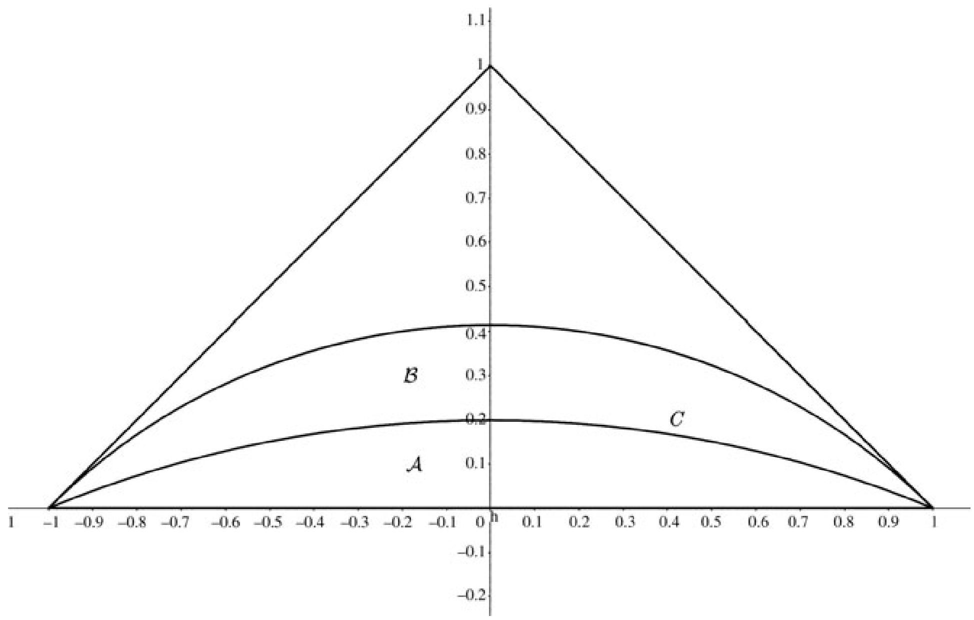


FIGURE 1: Reflection over a circle.

*Proof.* Let $Z_t$ be a Brownian motion starting at $z \in U \setminus \Omega$ and let $H_\Lambda$ be its hitting time of $\Lambda$, and consider $W_t$ the Brownian motion derived from $f(Z_t)$, that is,

$$W_t := f(Z_{\kappa^{-1}(t)}),$$

where

$$\kappa(t) = \int_0^t |\bar{f}'(Z_s)|^2 \, \mathrm{d}s.$$

Now we are going to construct two Brownian motions $\widetilde{Z}_t$ and $\widetilde{W}_t$ starting respectively at $z$ and $w := f(z)$, such that $\widetilde{W}_t$ leaves $U$ before $\widetilde{Z}_t$, as follows.

(1) If $H_\Lambda < \tau_U^Z$, then run an independent Brownian motion, say $B_t$, starting at $Z_{H_\Lambda}$, and set

$$\widetilde{Z}_t := Z_t \mathbf{1}_{\{t \le H_\Lambda\}} + B_{t-H_\Lambda} \mathbf{1}_{\{H_\Lambda < t\}},$$
$$\widetilde{W}_t := W_t \mathbf{1}_{\{t \le \kappa(H_\Lambda)\}} + B_{t-\kappa(H_\Lambda)} \mathbf{1}_{\{\kappa(H_\Lambda) < t\}},$$

where $\widetilde{W}_t$ and $\widetilde{Z}_t$ are well-defined Brownian motions as

$$\widetilde{W}_{\kappa(H_\Lambda)} = W_{\kappa(H_\Lambda)} = f(Z_{H_\Lambda}) = Z_{H_\Lambda} = \widetilde{Z}_{H_\Lambda}.$$

Now note that

$$\tau_U^{\widetilde{Z}} = H_\Lambda + \inf\{t,\ B_t \notin U \mid B_0 = Z_{H_\Lambda}\} \le \kappa(H_\Lambda) + \inf\{t,\ B_t \notin U \mid B_0 = Z_{H_\Lambda}\} = \tau_U^{\widetilde{W}}.$$

(2) If $H_\Lambda \ge \tau_U^Z$, then just set $\widetilde{Z}_t = Z_t$ and $\widetilde{W}_t = W_t$. Therefore

$$\tau_U^{\widetilde{Z}} \le \kappa(\tau_U^{\widetilde{Z}}) = \tau_U^{\widetilde{W}}.$$

In both cases we have

$$\tau_U^{\widetilde{Z}} \overset{a.s}{\le} \tau_U^{\widetilde{W}}.$$

Hence

$$\mathbb{E}_z((\tau_U^{\tilde{Z}})^p) \le \mathbb{E}_w((\tau_U^{\tilde{W}})^p),$$

which ends the proof. □

**Remark 2.1.** Reflection over the circle, obtained above as a corollary of Proposition 2.2, can just as easily be deduced as a corollary of Proposition 2.3.

## 3. Applications

In this section we work through a series of examples that show how our results may be applied.

**Example 3.1.** Let $U$ be the upper half-disk $\{|z| < 1, \mathrm{Im}(z) > 0\}$. The imaginary axis is an axis of symmetry, and $U$ is $\Delta$-convex with respect to this axis, so all $p$th centers lie on the imaginary axis. The line

$$\Delta =: \left\{\mathrm{Im}(z) = \frac{1}{2}\right\}$$

is clearly a partial symmetry axis with symmetric part

$$U \cap \left\{\mathrm{Im}(z) > \frac{1}{2}\right\},$$

and $U$ is $\Delta$-convex as well. Thus all $p$-centers belong to the set

$$\left\{\mathrm{Re}(z) = 0,\ \ 0 \le \mathrm{Im}(z) \le \frac{1}{2}\right\}.$$

Now, if we let $C$ be the circle $\{|z+i| = \sqrt{2}\}$, then $C$ passes through 1 and $-1$, making an angle of $\pi/4$ at each point with the real axis. If we let $\mathcal{A} = U \cap \mathcal{I}$ as before, with $\mathcal{I}$ the inside of $C$, then $\sigma_C(\mathcal{A}) = \mathcal{B}$, where $\mathcal{B} = U \cap \mathcal{O}$ and $\mathcal{O}$ is the outside of $C$. By Corollary 2.1, no $p$th center lies in $\mathcal{A}$. Thus all $p$th centers lie on the line segment

$$\left\{\mathrm{Re}(z) = 0,\ \ \sqrt{2} - 1 \le \mathrm{Im}(z) \le \frac{1}{2}\right\},$$

which is in bold in Figure 2.

**Example 3.2.** Now let $U$ be an isosceles triangle with vertices at $-1$, 1, and $Ni$ with $N > 0$. It will be convenient for us to index $U$ by the angles at 1 and $-1$, so if we let $\theta$ be this angle then $N = \tan\theta$. Proposition 2.1 tells us that all $p$th centers lie on the imaginary axis. We have seen already from the example discussed in connection with Proposition 2.2 that all $p$th centers must lie below $(M/2)i$, but we will now show how this can be improved. Let $B$ be the angle bisector of one of the base angles of $U$. $B$ is a partial symmetry axis of $U$, with symmetric side given by the component of $U \backslash B$ corresponding to the shorter side of the triangle. Thus, if $\theta > \pi/3$, then all $p$th centers must lie above $B$, while if $\theta < \pi/3$, then all $p$th centers must lie below $B$. Now let $M$ be the perpendicular bisector of the edge connecting 1 to $Mi$ (this is often referred to as the *mediator*). This is also a partial symmetry axis of $U$, and the symmetric side is the component of $U \backslash M$ which does not contain $-1$. Thus, if $\theta > \pi/3$, then all $p$th centers must lie below $M$, while if $\theta < \pi/3$, then all $p$th centers must lie above $M$. Thus the intersections of

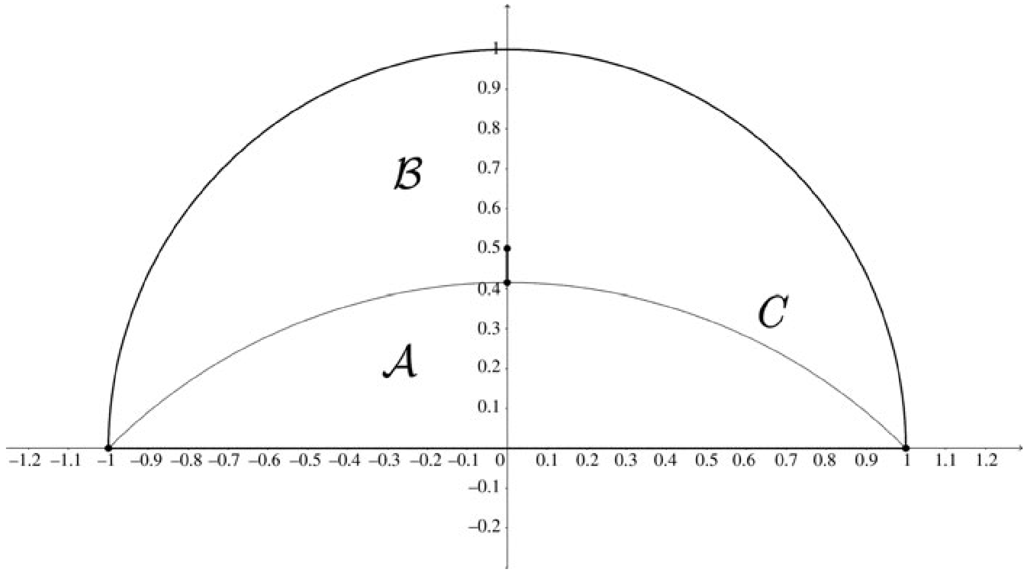


FIGURE 2: Reflection over a circle.

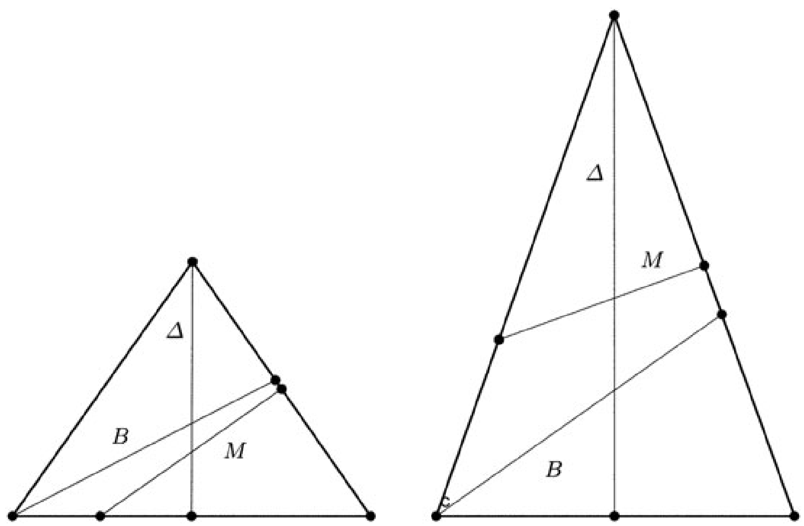


FIGURE 3: The angle bisector and mediator.

$M$ and $B$ with the imaginary axis provide upper and lower bounds for all $p$th centers, although which is the upper bound and which is the lower bound depends on $\theta$. Figure 3 demonstrates this phenomenon ($\Delta$ denotes the imaginary axis).

Naturally they coincide at $\theta = \pi/3$. It can be checked that, regardless of $\theta$, this gives a better upper bound than $N/2$, which was given by reflection over $\{\mathrm{Im}(z) = N/2\}$. A bit of Euclidean geometry shows that the intersection of $B$ with the imaginary axis is at the point $\tan(\theta/2)i$, and the intersection of $M$ with the imaginary axis is at the point $(1/\tan\theta - 1/\sin 2\theta)i$. Furthermore, we always have as a lower bound the intersection of the imaginary axis and the circle passing through 1 and $-1$, making an angle of $\theta/2$ with the real axis; this follows from Corollary 2.1

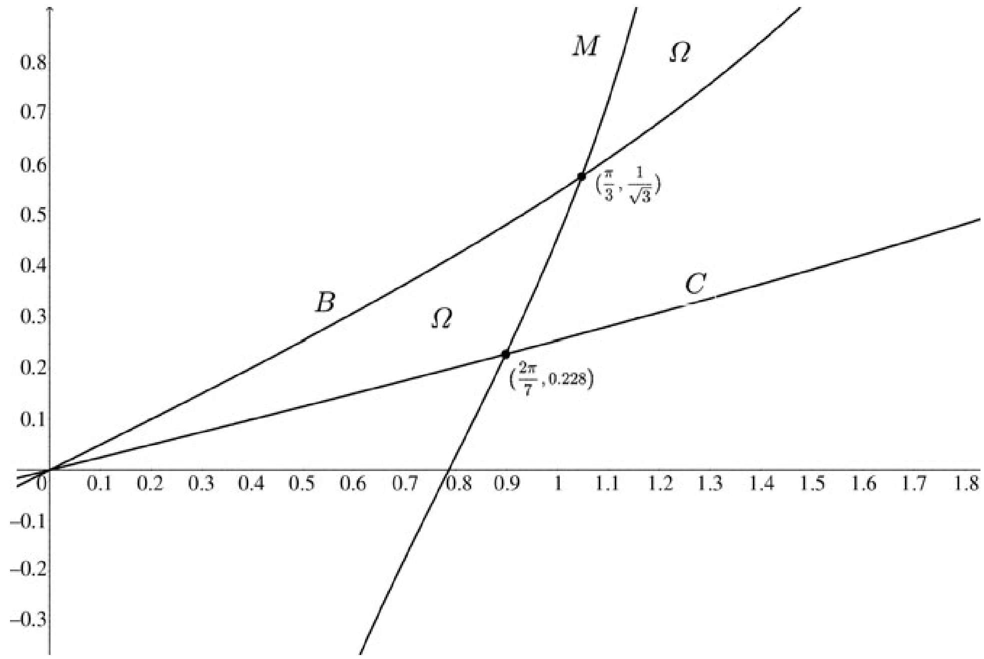


FIGURE 4: Upper and lower bounds for $p$th centers.

as above. This point is

$$\frac{1-\cos(\theta/2)}{\sin(\theta/2)}.$$

Figure 4 shows these upper and lower bounds; all $p$th centers must lie in the regions labeled $\Omega$.

**Remark 3.1.** We believe that a better lower bound can be achieved through numerical conformal mapping; more on this in the final section.

**Example 3.3.** Let $\mathcal{A}_{r,R}$ be the annulus $\{r < |z| < R\}$. Then all $p$-centers lie in $\{\sqrt{rR} < |z| < (R+r)/2\}$.

*Proof.* Consider $f(z) = rR/\bar{z}$, which maps $\mathcal{A}_{r,R}$ to itself. Under the same notation as in Proposition 2.3, we have

$$\Omega := \left\{\sqrt{rR} < |z| < R\right\},$$
$$\Lambda := \{|z| = \sqrt{rR}\}$$

and we can check easily that $f$ satisfies the requirements of Proposition 2.3). Therefore we can eliminate $U \setminus \Omega$ from consideration, and we obtain the lower bound $\sqrt{rR}$. In order to get the upper bound $(R+r)/2$ we can see, as illustrated by Figure 5, that the line $\Delta$ is a partial symmetry axis. The result follows. □

**Remark 3.2.**

- Another way to get the same lower bound as above is to note that Proposition 2.2 extends to non-injective maps in suitable situations. We may use the map

$$f\colon \begin{aligned} \{\ln r < \mathrm{Re}(z) < \ln R\} &\longrightarrow \mathcal{A}_{r,R} \\ z &\longmapsto e^z \end{aligned}$$

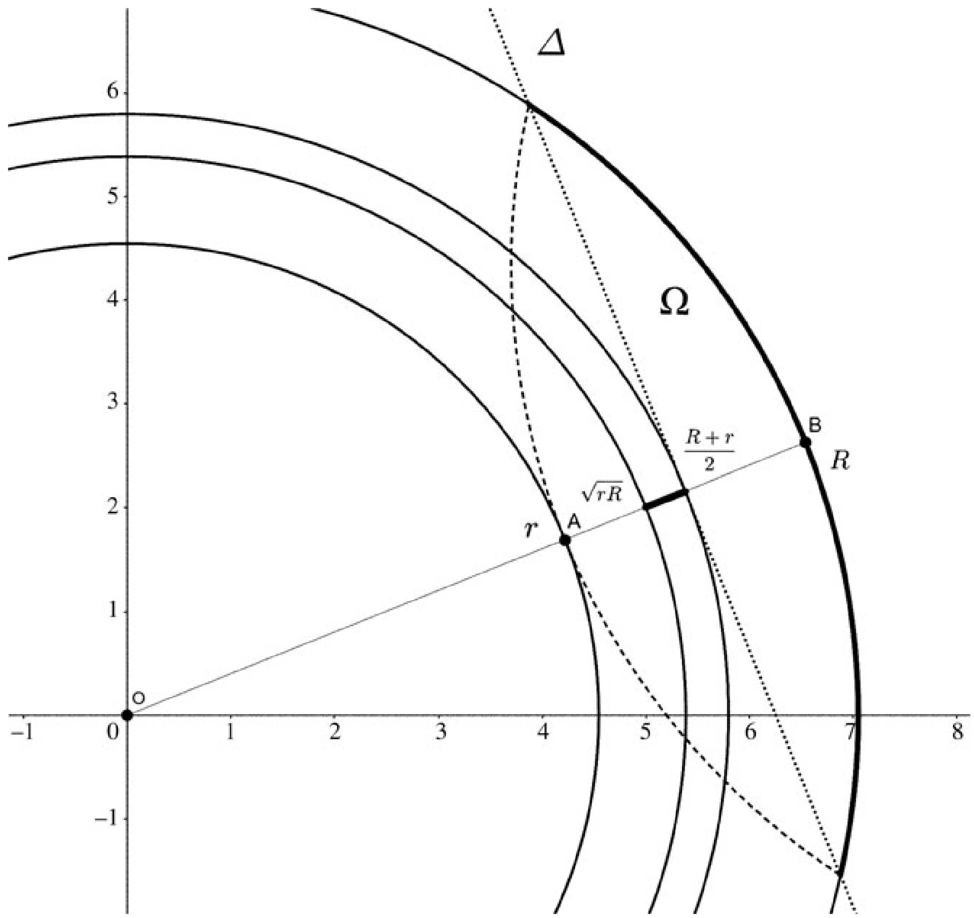


FIGURE 5: Bounds for the annulus.

and apply this extension of Proposition 2.2 with reflection axis $\{\mathrm{Re}(z) = \ln R + \ln r/2\}$ in order to obtain the result.

- It should be mentioned that an explicit formula for the first moment can be obtained by Dynkin's formula, and it is

$$\mathbb{E}_z(\tau_{\mathcal{A}_{r,R}}) = \frac{R^2 \ln(|z|/r) - r^2 \ln(|z|/R)}{\ln(R/r)} - |z|^2.$$

This can be shown to be maximal at

$$|z| = \sqrt{\frac{R^2 - r^2}{2 \ln(R/r)}}.$$

Our estimates are therefore not necessary for the first moments, but as an aside we obtain the inequality

$$\sqrt{Rr} < \sqrt{\frac{R^2 - r^2}{2 \ln(R/r)}} < \frac{R+r}{2}.$$

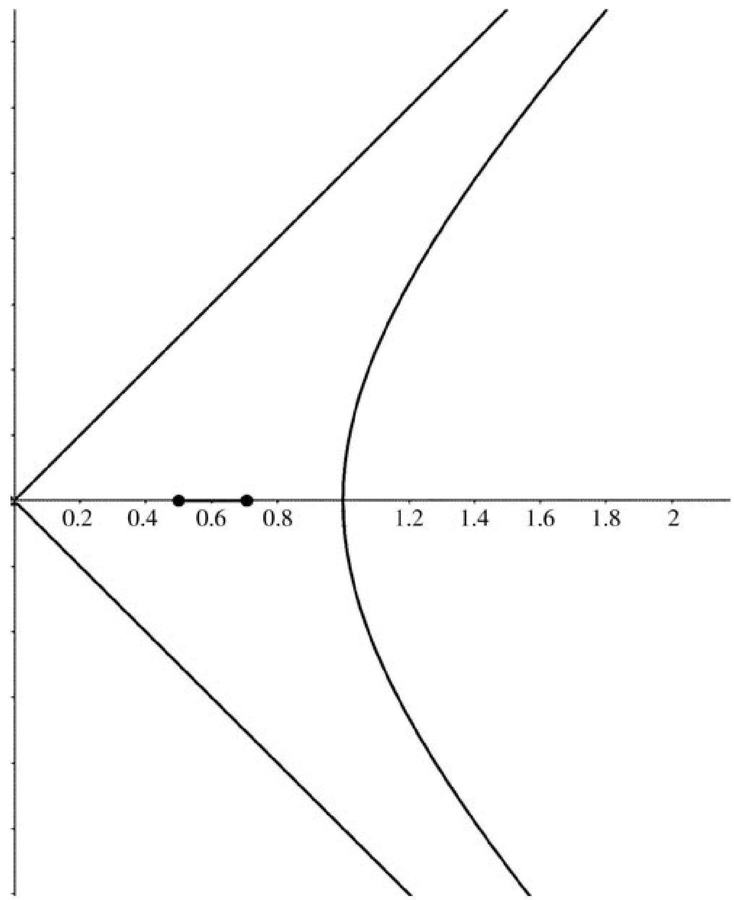


FIGURE 6: Bounds for the hyperbolic region.

Setting $a = R^2$, $b = r^2$, and squaring the inequalities gives the following:

$$\sqrt{ab} < \frac{a-b}{\ln(a/b)} < \frac{a+b+2\sqrt{ab}}{4} \le \frac{a+b}{2}.$$

The quantity

$$\frac{a-b}{\ln(a/b)}$$

is of great importance in the study of heat flow, and in that context it is known as the *logarithmic mean temperature difference*, or LMTD (see [10]). We have therefore given a new proof of the fundamental fact that the LMTD lies between the arithmetic and geometric means, and in fact have proved that the upper bound can be lowered to the arithmetic mean of the arithmetic and geometric means.

**Example 3.4.** Let $\mathscr{H}$ be the region $\{|x| > |y|, x^2 - y^2 < 1\}$; this is the region bounded by the lines $y = \pm x$ and the hyperbola $x^2 - y^2 = 1$; see Figure 6.

It is perhaps not obvious for which $p$ we have $\mathbb{E}_w[\tau^p_{\mathscr{H}}] < \infty$, but we can show that $\mathbb{E}_w[\tau^p_{\mathscr{H}}] < \infty$ for any $p > 0$ and $w \in \mathscr{H}$, as follows. $\mathscr{H}$ is contained in the union of two infinite strips which are orthogonal. Any strip has all moments of its exit time finite: Brownian motion is rotation-invariant, so the moments are the same as for a horizontal strip, and these moments in turn are the same as for a one-dimensional Brownian motion from a bounded interval, since that is what we obtain when we project the Brownian motion onto the imaginary axis; these moments are well known to be finite for all $p$, and in fact they can be calculated explicitly for integer $p$ using the Hermite polynomials. We would like to conclude that the union of these two strips must then have finite $p$th moment, but easy examples show that it is not necessarily the case that the union of two domains with finite $p$th moment must itself have finite $p$th moment. A method does exist for reaching the desired conclusion, however, and it is contained in Theorem 3 and Lemmas 1 and 2 of [13]. It is straightforward to verify that our infinite strips satisfy the required conditions: their intersection is bounded, and boundary arcs intersect at non-zero angles. Therefore the exit time for their union has finite $p$th moments for all $p$, and thus so does $\mathscr{H}$. See [13] for details.

Now let us see how our methods can be used to localize the $p$th centers. The real axis is an axis of symmetry, but the domain is not $\Delta$-convex, so we may not apply Proposition 2.1. However, all $p$-centers lie on the real axis, and we may prove this as follows. The map $f(z)=\sqrt{z}$ maps the strip $\{0<\mathrm{Re}(z)<1\}$ conformally onto $\mathscr{H}$. Any horizontal line can be used in Proposition 2.2, and we note that

$$|f'(z)|=\frac{1}{2\sqrt{|z|}}$$

is monotone decreasing in $|z|$ and therefore in $|\mathrm{Im}(z)|$. Thus all $p$th centers must lie on the image of the real axis under $f$, which is again the real axis. So we need only consider points on $\mathbb{R}$. The line $\{\mathrm{Re}(z)=1/2\}$ is a partial symmetry axis, which gives a lower bound of $1/2$ for all $p$th centers. For an upper bound, note that $\{\mathrm{Re}(z)=1/2\}$ is another axis of symmetry of $\{0<\mathrm{Re}(z)<1\}$, and the monotonicity of the derivative shows again via Proposition 2.2 that we only need to look in the region

$$f\left(\left\{0<\mathrm{Re}(z)<\frac{1}{2}\right\}\right),$$

which is the region $\{x^2-y^2<1/2\}$ inside $U$. This gives an upper bound of $1/\sqrt{2}$ on the real axis. Thus all $p$-centers lie on

$$\left\{\mathrm{Im}(z)=0,\ \frac{1}{2}<\mathrm{Re}(z)<\frac{1}{\sqrt{2}}\right\}.$$

This set is in bold in Figure 6.

**Example 3.5.** Let $\mathscr{C}_R$ be the crescent-like shape limited by the two circles

$$\left\{\left|z-\frac{1}{2}\right|=\frac{1}{2}\right\} \quad \text{and} \quad \left\{\left|z-\frac{R}{2}\right|=\frac{R}{2}\right\}$$

(see Figure 7). $\mathscr{C}_R$ is the image of the region

$$\left\{\frac{1}{R}<\mathrm{Re}(z)<1\right\}$$

under the conformal map $f(z)=1/z$. Note that $|f'(z)|=1/|z|^2$ is monotone decreasing in $|z|$, so by the same argument as in Example 3.4, all $p$th centers lie on the real axis. Furthermore,

$$\left\{\mathrm{Re}(z)=\frac{R+1}{2}\right\}$$

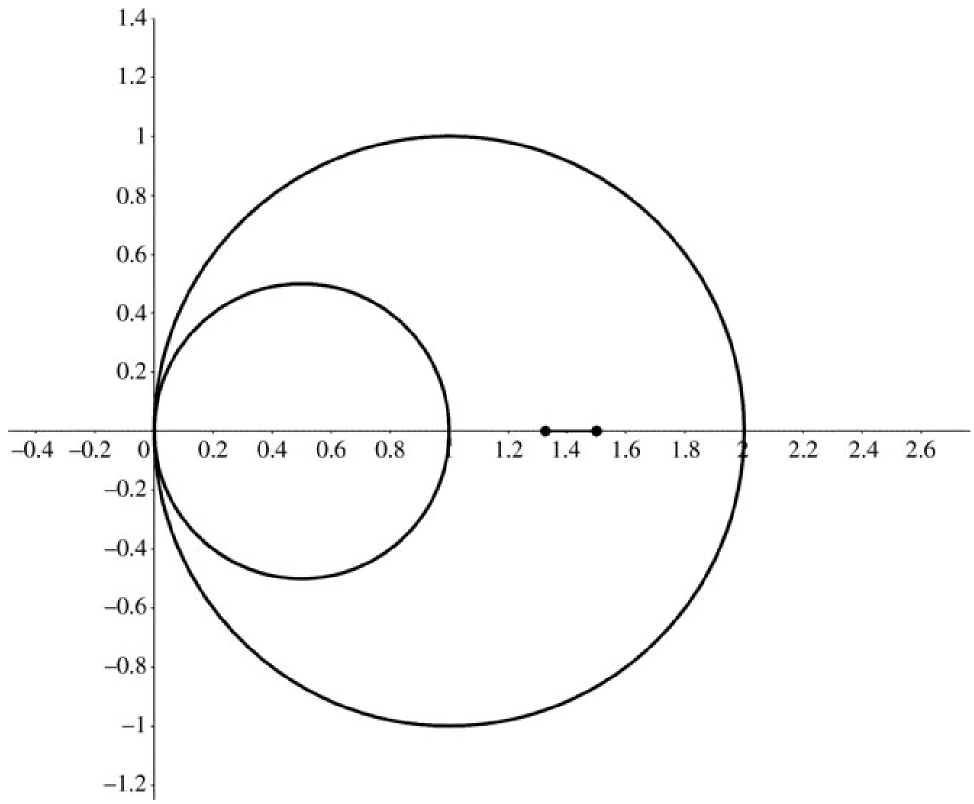


FIGURE 7: Bounds for the crescent region.

is a partial symmetry axis for $\mathscr{C}_R$, and this allows us to eliminate the region

$$\left\{\mathrm{Re}(z) > \frac{R+1}{2}\right\}$$

from consideration. We may also use an axis of symmetry

$$\left\{\mathrm{Re}(z) = \frac{1+1/R}{2}\right\}$$

to conclude via Proposition 2.2 that we can exclude the region

$$f\left(\left\{\frac{1}{R} < \mathrm{Re}(z) < \frac{1+1/R}{2}\right\}\right),$$

which is the region

$$\mathscr{C}_R \cap \left\{\left|z - \frac{R}{R+1}\right| < \frac{R}{R+1}\right\},$$

in the search for $p$th centers. We see that all $p$-centers lie on the interval

$$\left\{\mathrm{Im}(z) = 0,\ \frac{2R}{R+1} < \mathrm{Re}(z) < \frac{R+1}{2}\right\},$$

which is in bold for $R = 2$ in Figure 7.

## 4. Concluding remarks

As remarked earlier, in Figure 1 the regions $\mathcal{A}$ and $\mathcal{B}$ do not fill all of $U$, and it is natural to search for a better bound by finding a conformal map that fills the entire domain. Let us consider the Schwarz–Christoffel transformation sending the unit disk to $U(\theta)$ given by (see [8, Chapter 2])

$$f(z) = A + C \int_0^z (1-w)^{\theta/\pi - 1}(1+w)^{\theta/\pi - 1}(1+\mathrm{i}w)^{-2\theta/\pi}\,\mathrm{d}w$$

for appropriate choices of constants $A$ and $C$; note that this is chosen so that 1 and $-1$ are mapped to the base angles, and $i$ is mapped to the top angle. We have

$$|f'(z)| = |C||1-z|^{\theta/\pi - 1}|1+z|^{\theta/\pi - 1}|1+\mathrm{i}z|^{-2\theta/\pi}.$$

It can then be checked that $|f'(z)| > |f'(\bar{z})|$ whenever $\mathrm{Re}(z) > 0$, so Proposition 2.2 implies that no $p$th centers can be found in the image of $\mathbb{D} \cap \{\mathrm{Re}(z) < 0\}$. From this point a numerical method can be employed, and the resulting bound should improve the one we found, if desired.

As was mentioned in connection with $\Delta$-convexity, there are some purely geometrical consequences of our results. In that context, the following may be proved.

**Proposition 4.1.** *Suppose a domain $U$ is $\Delta$-convex with respect to two parallel symmetry axes. Then we can find $a \in [-\infty, \infty)$ and $b \in (-\infty, \infty]$ so that $U$ is a rotation of the domain $\{a < \mathrm{Re}(z) < b\}$; in other words, $U$ is all of $\mathbb{C}$, is a half-plane, or is an infinite strip.*

As a corollary of this, and of our probabilistic results above, we obtain the following.

**Corollary 4.1.** *Suppose $U$ is a domain which is not all of $\mathbb{C}$, a half-plane, or an infinite strip. Then if there are multiple axes of symmetry to which $U$ is $\Delta$-convex, then they all meet at a unique point.*


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## Chapter 3

# Planar Skorokhod embedding problem.

## 3.1 Problem and first solution.

The Skorokhod embedding problem, as mentioned by its name, was first introduced and solved by the Ukrainian mathematician *Anatoliy Volodymyrovych Skorokhod* in 1961. Here is the original statement: For a given centered probability measure $\mu$ with finite second moment and a standard Brownian motion $B_t$, is it possible to find a stopping time $T$ such that $\mathbf{E}(T) < +\infty$ and the distribution of $B_T$ is $\mu$? This question has motivated many mathematicians since its appearance and many variants and many solutions were given. For a satisfactory survey about the topic, we refer the reader to [35].

In a recent paper [21], the author considered a planar version of Skorokhod problem. That is, given a distribution $\mu$ with zero mean and finite second moment, is there a simply connected domain $U$ (containing the origin) such that if $Z_t = X_t + Y_t i$ is a standard planar Brownian motion, then $X_\tau = \Re(Z_\tau)$ has the distribution $\mu$, where $\tau$ is the exit time from $U$? We shall call such a domain $U$ a $\mu$-domain.

The answer is also affirmative. The construction process adopted by the author generates a domain $U$ with the following properties :

- $U$ is symmetric over the real line.
- $\mathbf{E}(\tau) < \infty$.
- If $\mu(\{x\}) > 0$ then $\partial U$ contains a vertical line segment (possibly infinite).
- If $a, b \in [\inf \mathsf{supp}(\mu), \sup \mathsf{supp}(\mu)]$ such that $(a, b)$ is null set w.r.t $\mu$ [1] then $(a, b) \times (-\infty, +\infty) \subset U$.

The full proof is in [21]. However, we give an outline of the key idea of the proof. The pseudo inverse of $\mu$ (also referred to as the quantile function) is defined by $G_\mu(u) := \inf\{x \mid F(x) \geq u\}$. A well known property of $G_\mu$ is that

$$G_\mu(\mathrm{Uni}(0,1)) \sim \mu$$

which follows immediately from the definition.

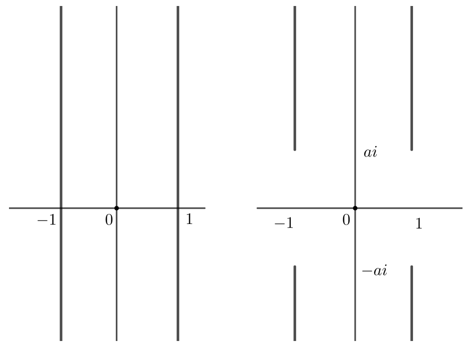
The author considered the scaled even version of $G_\mu$: $\varphi(\theta) := G_\mu(\frac{|\theta|}{\pi})$ for $\theta \in (-\pi, \pi) \setminus \{0\}$. Then $\sum_{n\geq 1} \widehat{\varphi}(n)\cos(n\theta)$ where $\widehat{\varphi}(n)$ is the $n^{th}$ Fourier coefficient of $\varphi$. Then he showed that the function $\psi(z) = \sum_{n\geq 1} \widehat{\varphi}(n) z^n$ is one to one in $\mathbb{D}$. Now it is not hard to see that the domain $U := \psi(\mathbb{D})$ fulfills the requirement. That is, the PBM starting at the origin stopped upon hitting the unit circle is uniformly distributed (Lemma 10) then by conformal invariance the distrubution of $Z_{\tau_U}$ is the same as $\psi(e^{\theta i})$ with $\theta \sim \mathrm{Uni}(-\pi, \pi)$. Therefore $\Re(Z_{\tau_U}) \sim \Re(\psi(e^{\theta i})) = \varphi(\theta) \sim \mu$. One thing to point out concerns the proof given by the author to prove that $\psi$ is one to one. In fact, the variation of the function $\varphi(\theta)$ is the following

| $\theta$ | $(-\pi, 0)$ | $\{0\}$ | $(0, \pi)$ | $\pi$ |
|---|---|---|---|---|
| $\varphi(\theta)$ | $\searrow$ | $-\infty$ | $\nearrow$ | $\ell$ |

where $\ell$ could be infinite depending on $\mu$. Hence, the winding number of $\psi$ must be 1 and then the injectivity follows.

Let us start with the case when $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$ (often called Redmacher distribution). If we turn a blind eye to the constraint $\mathbf{E}(\tau) < +\infty$, there is an infinite number of promising $\mu$-domains that might solve the problem. More precisely, any domain of the form $U_a := \mathbb{C} - (\{\Re(z) = 1, |\Im(z)| \geq a\} \cup \{\Re(z) = -1, |\Im(z)| \geq a\})$ answers the question seemingly.

[1] It means that the c.d.f of $\mu$ is constant on $(a, b)$.




All domains $U_a$ with $a > 0$ have an infinite exit first moment while $U_0$ has a finite one. So as we can see, the condition on $\mathbf{E}(\tau)$ is that what makes the problem more challenging. The first result we give concerns the generalization of Gross construction to a wider category of distributions.

**Theorem 54.** *Given a probability distribution $\mu$ on $\mathbb{R}$ with zero mean and finite nonzero $p^{th}$ moment (with $p \in (1, +\infty)$), we can find a simply connected $\mu$-domain $U$. Furthermore we have $\mathbf{E}(\tau^{\frac{p}{2}}) < \infty$.*

As can be noticed, Gross theorem [21] corresponds to the case $p = 2$. The full proof is contained in [6]. The proof uses the periodic variant of the so called Hilbert transform. We provide here some of its properties.

**Definition 55.** The Hilbert transform of a $2\pi$-periodic function $f$ is defined by

$$H\{f\}(x) := PV\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\cot(\tfrac{t}{2})dt\right\} = \lim_{\eta\to 0}\frac{1}{2\pi}\int_{\eta\le|t|\le\pi} f(x-t)\cot(\tfrac{t}{2})dt$$

where $PV$ denotes the Cauchy principal value, which is required here as the trigonometric function $t \longmapsto \cot(\cdot)$ has a pole at $k\pi$ with $k \in \mathbb{Z}$. The standard Hilbert transform was defined for functions $f$ defined over the whole real line by

$$PV\left\{\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{f(x-t)}{t}dt\right\}.$$

However, replacing $\frac{1}{t}$ by $\cot(\frac{t}{2})$ in the integrator is natural as $\cot(\cdot)$ satisfies the following nice sum identity (See [38] for example):

$$\cot(z) = \frac{1}{z} + 2z\sum_{n=1}^{+\infty}\frac{1}{z^2-n^2} = \frac{\pi}{z} + \pi\sum_{n=1}^{+\infty}\left(\frac{1}{z+\pi n} + \frac{1}{z-\pi n}\right)$$

so $\cot(\cdot)$ could be seen as the periodic (wrapped) version of the function $\frac{1}{t}$.

**Example 56.** For $m \in \mathbb{Z}$ we have

$$H\{\cos(m\cdot)\}(x) = sgn(m)\sin(mx) \tag{3.1.1}$$

In the same context, we define the $p^{th}$ norm of a periodic function $f$ over an interval of length $2\pi$ by

$$||f||_{L^p_{2\pi}} := \left\{\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^p dt\right\}^{\frac{1}{p}}$$

We denote by $L^p_{2\pi}$ the set of periodic functions (of period $2\pi$) of finite $p^{th}$ norm. The Hilbert transform, seen as an operator, has the following nice property.

**Theorem 57.** *[10] If $f$ is in $L^p_{2\pi}$ then $H_f$ does exist a.e and we have*

$$||H\{f\}||_{L^p_{2\pi}} \leq \lambda_p ||f||_{L^p_{2\pi}} \tag{3.1.2}$$

*for some positive constant $\lambda_p$.*

The inequality (3.1.2) holds also for function defined over the whole real line and are referred to as *strong type estimates* [25, Vol I, page 203].

**Proposition 58.** *We have the following functional identities.*

- The Hilbert operator $H$ commutes with positive dilations, i.e if $\Phi_\lambda\{f\}(x) = f(\lambda x)$ then

$$(H \circ \Phi_\lambda)\{f\} = (\Phi_\lambda \circ H)\{f\}. \tag{3.1.3}$$

- The derivative of the Hilbert transform is the Hilbert transform of the derivative. That is

$$H\{f'\} = H\{f\}'. \tag{3.1.4}$$

*Proof.* An immediate application of the definition combined with elementary integration techniques. □

The uniqueness of a $\mu$-domain is not always guaranteed. The author in [21] illustrated the non uniqueness by the following empirical example :

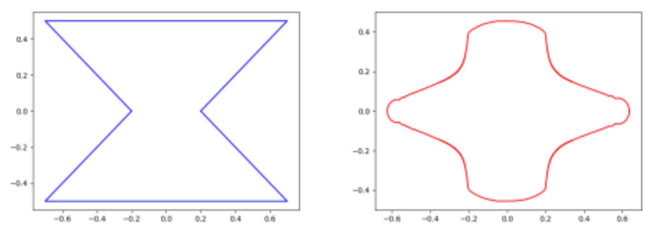



Figure 3.1.1: The right domain is constructed using the distribution of the left domain.

Non uniqueness is easy to conclude. The monotonicity of $\varphi(e^{ti})$ says that if $e^{ti}$ moves clockwise along the unit circle starting from $-1$ to 1, then $\Re(\varphi(e^{ti}))$ moves in the same direction and does not rebound at any point of the boundary. That is, if we apply Gross method to a distribution $\mu$ arising from a domain with dissimilar boundary, we obtain a different domain. Based on this remark, we come up with the following definition.

**Theorem 59.** *For any distribution $\mu$ satisfying the conditions of the previous theorem, there is a unique simply connected domain $U$ such that $\Re(Z_\tau) \sim \mu$ and which is symmetric, $\Delta$-convex with respect to the real axis, and satisfies $\mathbf{E}(\tau^{\frac{p}{2}}) < \infty$.*

The $\Delta$-convexity is the same as defined in Definition 44. The full proof is available in [6], and as above we sketch it here. Consider two potential domains $U, V$ satisfying all requirements and let $f : \mathbb{D} \mapsto U$ and $g : \mathbb{D} \mapsto U$ be two univalent maps fixing 0 and sending reals to reals. The job is then to prove $f \equiv g$ and so $U = V$. The existence of such functions is guaranteed by the Riemann mapping theorem, except the constraint of sending reals to reals which needs a bit of care; it is the subject of the intermediary Lemma 3.1 in [6]. The proof can be summarized in these steps :

1. $f$ and $g$ belong to $L^p$.
2. If $\zeta$ is a r.v uniformly distributed on $\partial\mathbb{D}$ then $Z_{\tau_U} \sim f(\zeta)$ and $Z_{\tau_V} \sim g(\zeta)$.
3. $\Re(f)$ and $\Re(g)$ agree on the unit disc a.e.
4. $f \equiv g$

The following figure illustrates 3 examples. Here $\mathcal{H} := \{x = \frac{y^2}{4} - 1\}$.

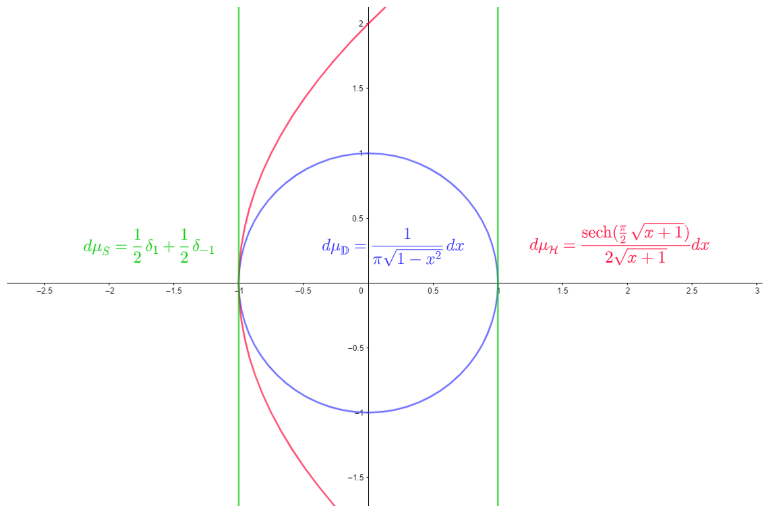



Figure 3.1.2: Three domains generated by three distributions.

The assumptions of the theorem are necessary. The necessity of $\Delta$-convexity could be noticed by the example mentioned by Gross in Fig **3.0.1**.

If $\mathcal{P} := \{y = \frac{x^2}{4} - 1\}$ then we can show that $X_{\tau_\mathcal{P}} = \Re(Z_{\tau_\mathcal{P}})$ has the distribution $d\mu(x) = \frac{\operatorname{sech}(\frac{\pi x}{4})}{4}dx$ which is the same for the strip $\{|\Im(z)| < 2\}$. This demonstrates that symmetry over the real axis is also necessary.

## 3.2 Another solution with application to optimization.

### 3.2.1 Our solution

After scrutinizing closely Gross technique for generating $\mu$-domains, we wondered if we can come up with another method that generates a new "style" of $\mu$-domains different from Gross' ones. Gross considered a "doubled" copy of the pseudo-inverse by taking $G_\mu(\frac{|\theta|}{\pi})$. That is why the $\mu$-domain is symmetric over the real line and in particular the two points $Z_\tau$ and $\overline{Z_\tau}$ to $\Re(Z_\tau)$ contribute to $\Re(Z_\tau)$. A potential explanation of Gross choice is that $G_\mu(\frac{|\theta|}{\pi})$ is continuous at $k\pi$, $k \in \mathbb{Z}$, which is not the case for $G_\mu(\frac{\theta}{2\pi})$. However, we succeeded to overcome such a nonregularity of $G_\mu(\frac{\theta}{2\pi})$. The result is the generation of a new category of $\mu$-domains. It turns out that these new $\mu$-domains solve entirely an optimization problem proposed and partially answered in [29].

**Definition 60.** We will say that a domain is $\Delta^\infty$-*convex* if, given any $z \in U$, the vertical ray $\{w : \Re(w) = \Re(z), \Im(w) \geq \Im(z)\}$ lies entirely in $U$. So, for example, the parabola $y = x^2$ is $\Delta^\infty$-convex, while a horizontal strip is not. More generally, any simply connected domain limited below by a continuous curve $y = f(x)$ is $\Delta^\infty$-convex provided that $f$ goes off to infinity at the ends of its domain. The reason for this name is that it is a variation on the notion of $\Delta$-convexity notion defined in Definition 44.

**Theorem 61.** *If $\mu \in L^p$ for some $p > 1$ then there exists a $\Delta^\infty$-convex $\mu$-domain $U$ containing zero. Furthermore $\mathbf{E}(\tau_U^{\frac{p}{2}}) < \infty$.*

The construction of our domain goes like Gross one, i.e our domain would be the image of the unit disc under the action of a univalent map. As $\varphi_\mu(\theta) := G_\mu(\frac{\theta}{2\pi}) \in L^p$ then it has a Fourier series whose partial sums converge to it in $L^p$, i.e

$$\varphi_\mu(\theta) \overset{L^p}{=} \sum_{n=1}^{+\infty}(a_n \cos(n\theta) + b_n \sin(n\theta)) \tag{3.2.1}$$

where $a_n$ and $b_n$ are the standard Fourier coefficients [2]. In fact, (3.2.1) is also true in the almost everywhere statement, which is the subject of *Carleson-Hunt theorem* [14, 18]. The Hilbert transform of $\varphi_\mu$ is

$$H\{\varphi_\mu\}(\theta) = \sum_{n=1}^{+\infty}(a_n \sin(n\theta) - b_n \cos(n\theta))$$

and it belongs to $L^p$ as well [10]. The power series

$$\widetilde{\varphi}_\mu(z) = \sum_{n=1}^{+\infty}(a_n - b_n i)z^n$$

belongs to $H^p$ since $\Re(\widetilde{\varphi}_\mu(e^{\theta i})) \overset{a.e}{=} \varphi_\mu(\theta)$ and $\Im(\widetilde{\varphi}_\mu(e^{\theta i})) \overset{a.e}{=} H\{\varphi_\mu\}(\theta)$. The map $\widetilde{\varphi}_\mu(z)$ is one to one on the unit disc $\mathbb{D}$ and maps 0 to 0. The domain $U := \widetilde{\varphi}_\mu(\mathbb{D})$ is $\Delta^\infty$-convex since $\varphi_\mu$ is non decreasing a.e on $[0, 2\pi]$. Let $Z_t$ be a planar Brownian motion starting at 0 and stopped at $\tau_{\mathbb{D}}$. Then by conformal invariance $\widetilde{\varphi}_\mu(Z_{\tau_{\mathbb{D}}})$ is a planar Brownian motion starting at $\widetilde{\varphi}_\mu(0) = 0$ and evaluated at $\tau_U$. As $Z_{\tau_{\mathbb{D}}} = e^{\theta i}$ where $\theta := \operatorname{Arg}(Z_{\tau_{\mathbb{D}}}) \sim \mathsf{Uni}(0, 2\pi)$, then $\Re(\widetilde{\varphi}_\mu(Z_{\tau_{\mathbb{D}}})) = \varphi_\mu(\theta)$ has the distribution $\mu$. Finally, the finiteness of $\mathbf{E}(\tau_U^{\frac{p}{2}})$ follows from (1.3.4).

**Theorem 62.** *The $\mu$-domain $U$ given in Theorem 61 is the unique $\mu$-domain which is $\Delta^\infty$-convex and satisfies $\mathbf{E}(\tau_U^{\frac{p}{2}}) < \infty$ for some $p > 1$.*

The proof leans on a lemma that deals with a uniqueness criterion for generalized inverse of a.e nondecreasing functions. That is, if the two generalized inverse of a.e non decreasing functions agree then the original functions coincide as well a.e. All details are covered in paragraph 4 in [5] with some additional comments, namely the necessity of the assumption $\mathbf{E}(\tau_U^{\frac{p}{2}}) < \infty$ with a counter example provided.

We move to practice now by giving some examples of $\mu$-domains generated by our method.

**Example 63.** The uniform distribution on $(-1, 1)$.

We can check that $\varphi_\mu(\theta) = \frac{\theta}{\pi} - 1$ and has the following Fourier series

$$\varphi_\mu(\theta) = -\frac{2}{\pi}\sum_{n=1}^{+\infty}\frac{\sin(n\theta)}{n}.$$

[2] In general there is a constant $\frac{a_0}{2}$ added to the sum, but it is omitted as it equals the average of $\mu$ which is assumed zero.

The power series is

$$\widetilde{\varphi}_\mu(z) = \frac{2i}{\pi}\sum_{n=1}^{+\infty}\frac{z^n}{n} = -\frac{2i}{\pi}\ln(1-z).$$

The $\mu$-domain is called the *catenary* and it is illustrated in Fig 3.2.1. We alert the reader about a small error in the figure of the domain in the paper; it is stretched by $\frac{\pi}{2}$.

Remember that the map $-\frac{2i}{\pi}\ln(1+z)$ produces the same domain and hence the same distribution, but $-\frac{2i}{\pi}\ln(1-z)$ is the one which has a non decreasing real part on $(0, 2\pi)$.

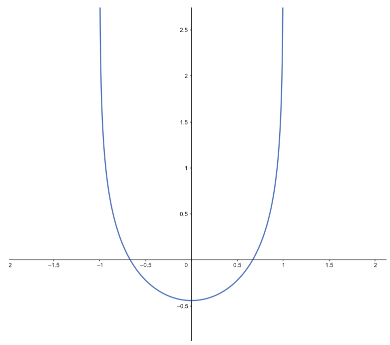


Figure 3.2.1: The catenary, which is our $\mu$-domain obtained from the uniform distribution on $(-1, 1)$.

**Example 64.** The centered and scaled arcsine law on $(-1, 1)$.

We get $\varphi_\mu(\theta) = -\cos(\theta/2)$ and so the power series is

$$\widetilde{\varphi}_\mu(z) = -\frac{8i}{\pi}\sum_{n=1}^{+\infty}\tfrac{n}{1-4n^2}z^n = \tfrac{i}{\pi}\left\{\ln\left(\tfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)(\sqrt{z}+\tfrac{1}{\sqrt{z}})-2\right\}.$$

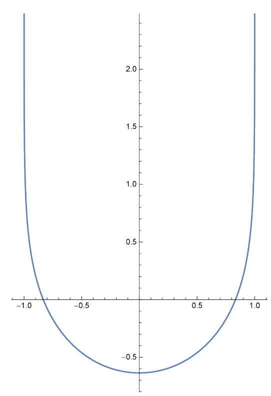


Figure 3.2.2: The extremal lower point of the domain is $-\frac{8i}{\pi}\sum_{n=1}^{+\infty}\frac{(-1)^n n}{1-4n^2} = -2i/\pi \approx -0.636i$

**Example 65.** Let $\mathcal{P}$ be the domain above the parabola of equation

$$2y = x^2 - 1. \tag{3.2.2}$$

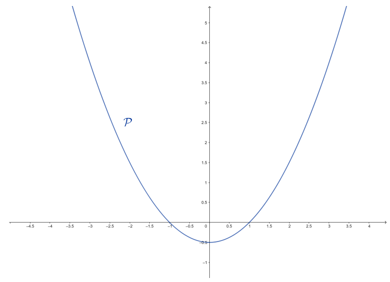



Figure 3.2.3: Parabola of equation $x^2 - 1 = 2y$.

It can be shown that $X_{\tau_{\mathcal{P}}} = \Re(Z_{\tau_{\mathcal{P}}})$ has the density $\frac{\operatorname{sech}(\frac{\sqrt{2}\pi}{2}x)}{\sqrt{2}}$. The map from the unit disc onto $\mathcal{P}$ [30] is given by

$$f(z) = \frac{2i}{\pi^2} \ln \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2 .$$

*Remark* 66. It is worth noting that our method and Gross method generate the same domain if $\mu = p\delta_a + (1-p)\delta_b$ $(a < 0 < b)$, which is the vertical strip $\{a < \Re(z) < b\}$. Moreover, the only $\Delta$-convex and $\Delta^\infty$-convex domain is such a strip.

**Proposition 67.** *If the distribution $\mu$ is of the form*

$$\mu = \sum_{n=1}^{m} x_n \delta_{x_n}$$

*where $x_{n+1} - x_n > 0$ for all $n$, then the domain generated by $\mu$ is the strip $\{x_1 < \Re(z) < x_m\}$ with the vertical slits $(\{x_n\} \times (-\infty, y_n])_{1<n<m}$ removed, where the $y_n$'s are some real numbers.*

It turns out that the boundary of a nice looking $\mu$-domain is explicit in many cases, i.e we can find the Cartesian equation $y = \gamma(x)$ of such a boundary. The method works also for $\mu$-domains generated by Gross method, except that it requires being defined by parts for Gross ones. That is, we have the following result.

**Theorem 68.** *[5] In our $\mu$-domains, every component of the boundary whose real part does not include the atoms of $\mu$ has the equation*

$$y = \gamma(x) = H\{\varphi_\mu\}(\varphi_\mu^{-1}(x)) = H\{F_\mu^{-1}\}(F_\mu(x)) \tag{3.2.3}$$

*where $F_\mu$ is the c.d.f of $\mu$.*

*Remark* 69. The above theorem is also valid for $\mu$-domains obtained by Gross method where $y = \gamma(x)$ is the equation of the lower boundary. Note that the functional $\gamma$ inherits the smothness of $G_\mu$; in particular, $\gamma$ is continuous away from atoms. Another remark is that if we are under the theorem assumptions, the density $\rho$ of the stopped Brownian motion at $z = x + yi$ on the boundary is given by

$$\rho(z) = \frac{F'_\mu(x)}{\sqrt{1+\gamma'(x)^2}}.$$

The same argument applies to a Gross $\mu$-domain, but with a small difference due to the symmetry of those domains, more precisely we have

$$\rho(z = x \pm |y|i) = \frac{F'_\mu(x)}{2\sqrt{1+\gamma'(x)^2}}.$$

| Distribution $\mu$ | Equation of the boundary |
|---|---|
| $\mathsf{Uni}(-1,1)$ | $\gamma(x) = -\frac{2}{\pi}\ln(2\cos(\pi x/2))$ |
| centered and scaled arcsine | $\gamma(x) = -\frac{2}{\pi}(x\ln(\cot(\frac{\arccos(-x)}{4})) + 1)$ |
| $\mathsf{Uni}((-2,-1)\cup(2,1))$ | $\gamma(x) = \frac{2}{\pi}\ln(\frac{\sqrt{2+2\cos(\pi x)}}{4\sin^2(\frac{\pi x}{2})})$ |

Table 3.1: Examples of boundary equations of our $\mu$-domains.

We remarked, at least in most of the mentioned examples, that the boundary equation is easier to find for our $\mu$-domains compared to Gross ones. It is due to the difficulty to extract the Hilbert transform of $\varphi_\mu$ whose construction encodes the main difference difference between our method and Gross one. However, in the case of the centered and scaled arcsine distribution, the generated Gross $\mu$-domain lower boundary equation is retrievable, that is

$$\gamma(x) = -\sqrt{1-x^2}.$$

### 3.2.2 An application to heat equation.

For a planar domain $U$, the function $(t,z) \mapsto \mathbf{P}_z(t < \tau_U)$ satisfies the heat equation

$$\begin{cases} \frac{1}{2}\Delta u - \frac{\partial u}{\partial t} = 0 & z \in U \\ u(t,z) = 0 & z \in \partial U \end{cases} \tag{3.2.4}$$

The equation (3.2.4) is a called a *Dirichlet boundary condition* type [3]. The *rate* of the solution of (3.2.4), that we denote by $\lambda(U)$, is defined to be half of the principal Dirichlet eigenvalue of $U$. The Dirichlet eigenvalue is the minimum of the spectrum of the Laplacian operator on $U$ combined with the boundary condition. A probabilistic characterization of the principal eigenvalue is given by

$$-\frac{\ln \mathbf{P}_z(t < \tau_U)}{t} \underset{t\to+\infty}{\longrightarrow} \lambda. \tag{3.2.5}$$

Formula (3.2.5) shows immediately that the principal eigenvalue decreases if the underlying domain grows. The proof of (3.2.5) can be found in [37]. In [29], the two authors treated the rate of (3.2.4) on domains coming from the *Conformal Skorokhod Embedding Problem.* More precisely, they considered $\mu$-domains (assumed to be standard by letting $0 \in U$). The authors have answered partially the following question: For a specific $\mu$, find extremal $\mu$-domains that attain the highest and lowest possible rate among all $\mu$-domains, i.e seek a $\mu$-domains $U_\mu$ such that

$$\lambda(U_\mu) \leq \lambda(V_\mu)$$

for all $\mu$-domains $V_\mu$. When $\mu$ is the uniform distribution on $(-1,1)$, they proved that the extremal $\mu$-domain is the one above the curve $y = -\frac{2}{\pi}\ln(2\cos(\pi x/2))$ as shown in Example 63. After going through the proof given by the two authors for the uniform distribution on $(-1,1)$, we found that their approach works perfectly for any $\mu$-domain whenever it is $\Delta^\infty$-convex. That is, we have the following result.

**Theorem 70.** *The $\mu$-domain, say $U_\mu$, constructed by our method, has the lowest rate among all $\mu$-domains with finite $\frac{p}{2}^{th}$ moment. In other words, $\lambda(U_\mu) \leq \lambda(V_\mu)$ for all $\mu$-domains $V_\mu$such that $\mathbf{E}(\tau_{V_\mu}^{\frac{p}{2}}) < +\infty$. Furthermore $\lambda(U_\mu) = \frac{\pi^2}{2(\beta-\alpha)^2}$ where $[\alpha,\beta]$ is the smallest interval containing the support of $\mu$.*

[3] We have also other categories, namely Cauchy, Neumann types.

We included the proof in our paper [5]. It leans on the fact that the eigenvalues of rectangular domains (with eventual infinite length or width) are computable, combined with monotonicity with respect to the domain size. In particular, we recover the $\mu$-domain that fits the uniform distribution on $(-1, 1)$.

*Remark* 71. In this context, a straightforward question pops up : What is the domain with the largest rate among all $\mu$-domains? Is Gross' one a potential candidate? The answer seems to be far from being decided.

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**ELECTRONIC COMMUNICATIONS in PROBABILITY**

# Remarks on Gross' technique for obtaining a conformal Skorohod embedding of planar Brownian motion

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**Abstract**

In [7] it was proved that, given a distribution $\mu$ with zero mean and finite second moment, there exists a simply connected domain $\Omega$ such that if $Z_t$ is a standard planar Brownian motion, then $\mathcal{R}e(Z_{\tau_\Omega})$ has the distribution $\mu$, where $\tau_\Omega$ denotes the exit time of $Z_t$ from $\Omega$. In this note, we extend this method to prove that if $\mu$ has a finite $p$-th moment then the first exit time $\tau_\Omega$ from $\Omega$ has a finite moment of order $\frac{p}{2}$. We also prove a uniqueness principle for this construction, and use it to give several examples.

**Keywords:** Planar Brownian motion; conformal invariance.
**AMS MSC 2010:** 60J65; 30C20.
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## 1 Introduction and statement of results

In what follows, $Z_t$ is a standard planar Brownian motion starting at 0, and for any plane domain $\Omega$ containing 0 we let $\tau_\Omega$ denote the first exit time of $Z_t$ from $\Omega$. In the elegant recent paper [7] the following theorem was proved.

**Theorem 1.** Given a probability distribution $\mu$ on $\mathbb{R}$ with zero mean and finite nonzero second moment, we can find a simply connected domain $\Omega$ such that $\mathcal{R}e(Z_{\tau_\Omega})$ has the distribution $\mu$. Furthermore we have $E[\tau_\Omega] < \infty$.

We will prove several new results related to Gross' construction. Our first result is the following generalization.

**Theorem 2.** Given a probability distribution $\mu$ on $\mathbb{R}$ with zero mean and finite nonzero $p$-th moment (with $1 < p < \infty$), we can find a simply connected domain $\Omega$ such that $\mathcal{R}e(Z_{\tau_\Omega})$ has the distribution $\mu$. Furthermore we have $E[(\tau_\Omega)^{p/2}] < \infty$.

The proof of this result depends on a number of known properties of the Hilbert transform and of the exit time $\tau_\Omega$, and is rather short. However the results needed are scattered through a number of different subfields of probability and analysis, and in an attempt to make the paper self-contained we have included a certain amount of exposition on these topics. We will prove the theorem in the next section.

There are several reasons why we feel that our extension is worth noting. The moments of $\tau_\Omega$ have special importance in two dimensions, as they carry a great deal of analytic and geometric information about the domain $\Omega$. The first major work in this

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direction seems to have been by Burkholder in [2], where it was proved among other things that finiteness of the $p$-th Hardy norm of $\Omega$ is equivalent to finiteness of the $\frac{p}{2}$-th moment of $\tau_\Omega$. To be precise, for any simply connected domain $\Omega$ let

$$\mathrm{H}(\Omega) = \sup\{p > 0 : \mathbf{E}[(\tau_\Omega)^p] < \infty\};$$

note that $\mathrm{H}(\Omega)$ is proved in [2, p. 183] to be exactly equal to half of the Hardy number of $\Omega$, as defined in [8], which is

$$\tilde{\mathrm{H}}(\Omega) = \sup\{q > 0 : \lim_{r \nearrow 1} \int_0^{2\pi} |f(re^{i\theta})|^q d\theta < \infty\},$$

where $f$ is a conformal map from the unit disk onto $\Omega$. This equivalence was used in [2, p. 183] to show for instance that $H(W_\alpha) = \frac{\pi}{2\alpha}$, where $W_\alpha = \{0 < Arg(z) < \alpha\}$ is an infinite angular wedge with angle $\alpha$. In fact, coupled with the purely analytic result [8, Thm 4.1] this can be used to determine $\mathrm{H}(\Omega)$ for any starlike domain $\Omega$. If we assume that $V$ is starlike with respect to $0$, then we may define

$$\mathcal{A}_{r,\Omega} = \max\{m(E) : E \text{ is a subarc of } \Omega \cap \{|z| = r\}\}, \tag{1.1}$$

and this quantity is non-increasing in $r$ (here $m$ denotes angular Lebesgue measure on the circle). We may therefore let $\mathcal{A}_\Omega = \lim_{r \nearrow \infty} \mathcal{A}_{r,\Omega}$, and then combining the results in [8] and [2] (see also [12]) we have $\mathrm{H}(\Omega) = \frac{\pi}{2\mathcal{A}_\Omega}$. In this sense, the quantity $\mathrm{H}(\Omega)$ provides us with some sort of measure of the aperture of the domain at $\infty$. Also in [12], a version of the Phragmén-Lindelöf principle was proved that makes use of the quantity $\mathrm{H}(\Omega)$. Furthermore, the quantity $\mathbf{E}[(\tau_\Omega)^p]$ provides us with an estimate for the tail probability $P(\tau_\Omega > \delta)$: by Markov's inequality, $P(\tau_\Omega > \delta) \le \frac{\mathbf{E}[(\tau_\Omega)^p]}{\delta^p}$.

For these reasons, we would argue that Theorem 2 gives a partial answer to the following intriguing question posed by Gross in [7]: given a probability distribution $\mu$ and a corresponding $\Omega$ such that $\mathcal{R}e(Z_{\tau_\Omega})$ has distribution $\mu$, in what sense are properties of $\mu$ reflected in the geometric properties of $\Omega$? We will have more comments on this question in the final section.

Our next result is influenced by Gross' observation that the domain corresponding to a given measure $\mu$ is not unique. Without further conditions this is correct, however we have found that natural conditions can be imposed on the domain so that a uniqueness principle holds. Before stating the result, let us make some definitions. A domain $U$ is *symmetric* if $\bar{z} \in U$ whenever $z \in U$. We will call a domain $U$ $\Delta$*-convex* if, whenever $z_1, z_2 \in U$ with $\mathcal{R}e(z_1) = \mathcal{R}e(z_2)$ then the vertical line segment connecting $z_1$ and $z_2$ lies entirely within $U$. It is straightforward to verify that any $\Delta$-convex domain is automatically simply connected. Furthermore any domain constructed by Gross' technique is both symmetric and $\Delta$-convex (see Section 2), and we have the following result.

**Theorem 3.** For any distribution $\mu$ satisfying the conditions of Theorem 2, there is a unique domain $\Omega$ such that $Re(Z_{\tau_\Omega}) \sim \mu$ and which is symmetric, $\Delta$-convex, and satisfies $E[(\tau_\Omega)^{p/2}] < \infty$.

The importance of this result for our purposes is that it allows us to give certain solutions to the inverse problem of the one solved by Gross. That is, we can give a number of examples of domains generated by Gross' method. To be precise, if $\Omega$ is a domain which is symmetric, $\Delta$-convex, and satisfies $E[(\tau_\Omega)^{p/2}] < \infty$, then it must be the domain generated by Gross' method corresponding to the distribution of $Re(Z_{\tau_\Omega})$. We will exploit this fact in Section 4.

Fourier series $\sum_{n=1}^{+\infty} \widehat{\varphi}(n)\cos(n\theta)$ is still well defined and converges to $\varphi$ in $L^p$ ([6, Thm. 3.5.7]). Parseval's identity is no longer available to us, but the following result allows us to conclude that the Hilbert transform $\sum_{n=1}^{+\infty} \widehat{\varphi}(n)\sin(n\theta)$ of $\varphi$ is also in $L^p$:

**Theorem 4.** [3] If $f$ is in $L^p$ then its periodic Hilbert transform $\mathcal{H}_f$ does exist almost everywhere and we have

$$||\mathcal{H}_f||_{L^p} \leq \lambda_p ||f||_{L^p}, \tag{2.2}$$

for some positive constant $\lambda_p$.

We remark that there are good estimates for the constant $\lambda_p$; see [10, Sec. 4.20, Vol. 1]. From this result we see that, as its real and imaginary parts are in $L^p$, the analytic function $\tilde{\varphi}(z) = \sum_{n=1}^{+\infty} \widehat{\varphi}(n) z^n$ lies in the Hardy space $H^p$, which is the space of all holomorphic maps on the disk with finite Hardy $p$-norm, defined as

$$||f||_{H^q} := \left\{ \lim_{r \nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^q dt \right\}^{\frac{1}{q}}.$$

$\tilde{\varphi}(z)$ is also injective, by the same argument as was used in [7, Prop. 2.2], and therefore $\Omega = \tilde{\varphi}(\mathbb{D})$ is simply connected. By Burkholder's result [2, p. 198] we have that if $f$ is a conformal function on the unit disk then the following equivalence holds:

$$\tau_{f(\mathbb{D})} \in L^{\frac{p}{2}} \iff ||f||_{H^p} < \infty. \tag{2.3}$$

We see therefore that $E[(\tau_\Omega)^{p/2}] < \infty$, and the theorem is proved.

## 3 Proof of Theorem 3

In this section we prove Theorem 3, that the domain $\Omega$ generated by Gross' technique is the unique symmetric, $\Delta$-convex domain with $E[(\tau_\Omega)^{p/2}] < \infty$ such that such that $\mathcal{R}e(Z_{\tau_\Omega})$ has the distribution $\mu$. Before going through the proof, we need the following lemma related to the Riemann mapping theorem.

**Lemma 3.1.** *If $U \subsetneq \mathbb{C}$ is a symmetric simply connected domain containing 0 then there exists a conformal map from $\mathbb{D}$ to $U$ such that $f(0) = 0$ and $f((-1,1)) \subseteq \mathbb{R}$.*

*Proof.* The existence of a conformal map, say $f$, from the unit disc to $U$ and sending zero to itself is guaranteed by the Riemann mapping theorem. It remains to add the constraint that $f((-1,1)) \subseteq \mathbb{R}$. Consideration of the power series shows that the map $\overline{f}(\overline{z})$ is analytic, and as $\mathbb{D}$ and $U$ are symmetric it is a conformal map from $\mathbb{D}$ to $U$. Therefore it is related to $f$ via a rotation acting on the unit disc, that is

$$\overline{f}(\overline{z}) = f(e^{i\theta} z),$$

for some $\theta \in [0, 2\pi)$. The map $\widetilde{f} : z \longmapsto f(e^{i\frac{\theta}{2}} z)$ satisfies the requirement of the lemma since

$$\begin{aligned}
\overline{\widetilde{f}}(\overline{z}) &= \overline{f}(e^{i\frac{\theta}{2}} \overline{z}) \\
&= \overline{f}(\overline{e^{-i\frac{\theta}{2}} z}) \\
&= f(e^{i\theta} e^{-i\frac{\theta}{2}} z) \\
&= f(e^{i\frac{\theta}{2}} z) \\
&= \widetilde{f}(z).
\end{aligned}$$

In particular, if $z$ is real then $\widetilde{f}(z)$ is as well, which ends the proof. □

We proceed now to prove Theorem 3. Suppose $U$ and $V$ are two domains satisfying the conditions of the theorem. Let $f : \mathbb{D} \longrightarrow U$ and $g : \mathbb{D} \longrightarrow V$ be two conformal maps fixing $0$ and sending reals to reals. As $f$ and $g$ are injective, they are monotone on the real line, and we may assume then that they are increasing (if not, consider $f(-z)$ and/or $g(-z)$ instead). The power series $f(z) = \sum_{n=1}^{+\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{+\infty} b_n z^n$ have real coefficients since $a_n = \frac{f^{(n)}(0)}{n!} \in \mathbb{R}$ and $b_n = \frac{g^{(n)}(0)}{n!} \in \mathbb{R}$. The fact that $E[(\tau_U)^{p/2}], E[(\tau_V)^{p/2}] < \infty$ implies that $||f||_{H^p}, ||g||_{H^p} < \infty$ (again, see [2]), and therefore the functions $f$ and $g$ have radial limits defined a.e. on $\{|z| = 1\}$. That is, $f(e^{i\theta}) := \lim_{r \nearrow 1} f(re^{i\theta})$ exists for Lebesque almost every $\theta$ on $[-\pi, \pi]$ (see [15, Thm 17.12] or [3]). We will compare the radial limits of $f$ and $g$ and show that they coincide a.e., but first we need another lemma.

**Lemma 3.2.** *$Z_{\tau_U}$ and $Z_{\tau_V}$ agree in distribution with $f(X)$ and $g(X)$ respectively, where $X$ is a r.v. uniformly distributed on $\{|z| = 1\}$.*

*Proof.* Note that in applying $f$ and $g$ to $X$, we are making use of the radial limits defined above. We will prove the statement for $f$. Let $r_n$ be any sequence in $(0, 1)$ which increases to 1 as $n \to \infty$, and let $\tau_n = \inf\{t > 0 : |Z_t| = r_n\}$. By standard martingale theory (see for instance [16, Ch. 14]), since $f(Z_{\tau_n})$ is a martingale bounded in $L^p$ we are guaranteed the existence of a limiting r.v. $M_\infty$ such that $E[|f(Z_{\tau_n}) - M_\infty|^p] \to 0$. Therefore $f(Z_{\tau_n})$ converges to $M_\infty$ in distribution. On the other hand, $f(Z_{\tau_n})$ is equal in distribution to $f(X_n)$, where $X_n$ is any r.v. uniformly distributed on $\{|z| = r_n\}$. Let us choose $X_n$ and $X$ as follows. Let the probability space in question be the interval $[0, 2\pi)$, with probability measure given by Lebesgue measure divided by $2\pi$. For $\omega$ in the probability space, let $X_n(\omega) = r_n e^{i\omega}$, and similarly $X(\omega) = e^{i\omega}$. By [15, Thm 17.12], we have

$$\lim_{n \to \infty} \int_0^{2\pi} |f(r_n e^{i\theta}) - f(e^{i\theta})| d\theta \to 0. \tag{3.1}$$

Thus, $E[|f(X_n) - f(X)|] \to 0$, which implies that $f(X_n)$ converges to $f(X)$ in distribution. However, $f(X_n)$ and $f(Z_{\tau_n})$ have the same distribution, and therefore $M_\infty$ and $f(X)$ agree in distribution. Now, $f(Z_t)$ is a time-changed Brownian motion, and therefore $f(Z_{\tau_n}) = \hat{Z}_{\sigma(\tau_n)}$, where $\sigma$ denotes the time-change and $\hat{Z}$ is a Brownian motion. By monotone convergence, $\sigma(\tau_n) \nearrow \tau_U$, and thus $f(Z_{\tau_n})$ converges a.s. to $\hat{Z}_{\tau_U}$. It follows that $\hat{Z}_{\tau_U}$ is equal in distribution to $f(X)$. □

$\Delta$-convexity and symmetry imply that $Re(f(e^{i\theta}))$ and $Re(g(e^{i\theta}))$ are a.e. even functions on $[-\pi, \pi]$ and non-increasing on $[0, \pi]$. Since $P(U \in \{e^{i\theta} : -\theta_0 < \theta < \theta_0\}) = \frac{\theta_0}{\pi}$ for $\theta_0 \in (0, \pi]$, it follows that for a.e. $\theta$ we must have $Re(f(e^{i\theta})) = r$, where $r$ is such that $\mu[r, +\infty) = \frac{\theta}{\pi}$, and the same must hold for $Re(g(e^{i\theta}))$. We see that $Re(f)$ and $Re(g)$ agree a.e. on $\{|z| = 1\}$, and since $Im(f), Im(g)$ are obtained from these by the periodic Hilbert transform (see Section 2) we see that $f$ and $g$ agree a.e. on $\{|z| = 1\}$. $f(z)$ and $g(z)$ for $z \in \mathbb{D}$ can be obtained from their boundary values via the Poisson integral formula ([15, Cor. 17.12]), and thus $f$ and $g$ agree. Theorem 3 is proved. □

None of the three conditions in the theorem can be omitted. For example, suppose that $U = \mathbb{C} \backslash \{|Re(z)| \leq 1, |Im(z)| \geq 1\}$. $U$ is symmetric and $\Delta$-convex, but $E[(\tau_\Omega)^{p/2}] = \infty$ for $p \geq 1$. Since $Re(Z_{\tau_\Omega})$ is a measure of bounded support, it will generate by Gross' method a domain $\Omega$ such that $E[(\tau_\Omega)^{p/2}] < \infty$ for all $p$, and this can therefore not be equal to $U$. An example which is symmetric and has finite $p$-th moment for all $p$ but which lacks $\Delta$-convexity is displayed in Figure 2 of [7], and it is pointed out there that uniqueness fails. Finally, as will be shown below in the examples, the parabola and horizontal strip both lead to the same distribution $\mu$. Both domains are $\Delta$-convex and satisfy $E[(\tau_\Omega)^{p/2}] < \infty$ for all $p > 0$, but the parabola is not symmetric. This shows that the condition of symmetry cannot be omitted. On the other hand, it is interesting to note that both of these domains are convex, and that therefore convexity does not seem to be the correct condition for uniqueness.

## 4 Examples

In this section, we consider a series of domains and the corresponding distributions of $Re(Z_\tau)$. In all cases that we consider the boundary of the domain will be well behaved and we will be able to find a p.d.f. of the distribution of $Z_\tau$ on the boundary. By this we mean that we can find a function, $\rho_a^{Z_\tau}(z)$, defined for $z$ on $\partial U$ such that for any interval $I$ on the boundary of $U$, we have $P_a(Z_t \in I) = \int_b^c \rho_a^{Z_\tau}(z(s))ds$, where $z(s)$ is a parameterization of $\partial U$ with $|z'(s)| = 1$ and $z((b,c)) = I$. We will use analytic functions and the conformal invariance of Brownian motion as our primary tool; finding exit distributions in this manner has previously been considered in [13], and following the convention there we will use the notation $\rho_a^{Z_\tau}(z)ds$ to denote this density, with the $ds$ to indicate that the curve $z(s)$ is parameterized by arclength.

If we have found the p.d.f of $Z_\tau$ on $\partial U$, then we can deduce the p.d.f's of $X_\tau$ and $Y_\tau$ provided that the boundary of the domain is smooth enough in the sense that, locally around $z = x + yi$, we have

$$y = \varphi_z(x), \tag{4.1}$$

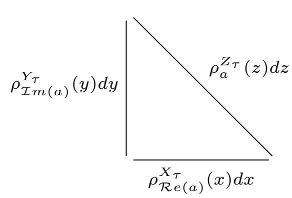
for some differentiable bijective function $\varphi_z$. To see how, let $x$ be an element of $\{\mathcal{R}e(z) | z \in \partial D\}$. Since a positive infinitesimal element $dz \in \partial D$ is expressed as $dz = \sqrt{dx^2 + dy^2}$, then

$$\begin{aligned} \rho_{\mathcal{R}e(a)}^{X_\tau}(x)dx &= \sum_{\mathcal{R}e(z)=x} \rho_a^{Z_\tau}(z)dz \\ &= \sum_{\mathcal{R}e(z)=x} \rho_a^{Z_\tau}(x+yi)\sqrt{dx^2+dy^2} \\ &= \sum_{\mathcal{R}e(z)=x} \rho_a^{Z_\tau}(x+\varphi_z(x)i)\sqrt{1+\varphi_z'(x)^2}dx. \end{aligned} \tag{4.2}$$

Finally we get

$$\begin{aligned} \rho_{\mathcal{R}e(a)}^{X_\tau}(x) &= \sum_{\mathcal{R}e(z)=x} \rho_a^{Z_\tau}(x+\varphi_z(x)i)\sqrt{1+\left\{\frac{d\varphi_z}{dx}(x)\right\}^2} \\ \rho_{\mathcal{I}m(a)}^{Y_\tau}(y) &= \sum_{\mathcal{I}m(z)=y} \rho_a^{Z_\tau}(\varphi_z^{-1}(y)+yi)\sqrt{1+\left\{\frac{d\varphi_z^{-1}}{dy}(y)\right\}^2}. \end{aligned} \tag{4.3}$$

Notice that that both sets $\{z|\mathcal{R}e(z) = x\}$ and $\{z|\mathcal{I}m(z) = y\}$ are countable due to (4.1), which justifies the sum symbols in (4.2). This proves the formula for the distribution of $X_\tau$, and $Y_\tau$ can of course be obtained similarly. The following diagram, which should be viewed at the infinitesimal level, provides the intuitive justification for the formulas.




Before going through examples, we recall a lemma from [13] which we will use to find the exit distribution of Brownian motion from various domains. Let $\gamma$ be any smooth curve parameterized by arclength, $B_t$ a Brownian motion starting at $a$, and $\tau$ a stopping time such that $B_\tau \in \gamma$ a.s. $\rho_\tau^a(w)ds$ will denote the density of $B_\tau$ on $\gamma$, when it exists, with $ds$ denoting the arclength element. The lemma we require is as follows.

**Lemma 4.1.** *[13, Th. 2] Let $U$ be a domain, and suppose $f$ is a function analytic on $U$. Let $B_t$ be a Brownian motion starting at $a$, and $\tau$ a stopping time such that the set of Brownian paths $\{B_t : 0 \le t \le \tau\}$ lie within $U$ a.s. Suppose that $\gamma$ is a smooth curve in $U$ such that $B_\tau \in \gamma$ a.s. Then for any $a \in U$ and $w \in f(\gamma)$ we have*

$$\rho_{\hat{\tau}}^{f(a)}(w)ds = \sum_{z\in f^{-1}(w)\cap\gamma} \frac{\rho_\tau^a(z)}{|f'(z)|}ds. \tag{4.4}$$

In each case below, the stopping time $\tau$ will be the exit time of a domain; note that this does not conflict with the requirement that $\gamma \subseteq U$, since the analytic function $f$ in our examples will always be a function which is analytic on a domain strictly containing the closure of $U$, and the theorem can be applied in this larger domain. We will proceed by applying this lemma to find the exit distribution of various domains, and then projecting these onto the real line using (4.3).

### 4.1 Unit disc

If $Z_t$ starts at zero at stopped at $\tau_{\mathbb{D}}$ then due to the rotational invariance of the Brownian motion $Z_{\tau_{\mathbb{D}}}$ is uniformly distributed on the circle, i.e

$$\rho_0^{Z_{\tau_{\mathbb{D}}}}(e^{\theta i}) = \frac{1}{2\pi}.$$

Using the unit circle equation $x^2 + y^2 = 1$, we extract the distributions of $X_{\tau_{\mathbb{D}}}$ and $Y_{\tau_{\mathbb{D}}}$ on $(-1, 1)$:

$$\begin{aligned}\rho_0^{X_{\tau_{\mathbb{D}}}}(x) &\overset{(4.2)}{=} \sum_{z\in\{x\pm i\sqrt{1-x^2}\}} \rho_a^{Z_\tau}(z) \\ &= \frac{1}{2\pi}\sqrt{1+(\tfrac{x}{1-x^2})^2} + \frac{1}{2\pi}\sqrt{1+(-\tfrac{x}{1-x^2})^2} \\ &= \frac{1}{\pi\sqrt{1-x^2}}.\end{aligned}$$

Similarly for $\rho_0^{Y_{\tau_{\mathbb{D}}}}(y)$. We remark that $X_{\tau_{\mathbb{D}}}$ and $Y_{\tau_{\mathbb{D}}}$ follow the scaled and centered Arc-sine law on $(-1, 1)$ (see [5, p. 49]). If the starting point is $a = u + vi \neq 0$, then the distribution of $Z_{\tau_{\mathbb{D}}}$ is given by

$$\rho_a^{Z_{\tau_{\mathbb{D}}}}(e^{\theta i})d\theta = \frac{1-|a|^2}{2\pi|1-\overline{a}e^{\theta i}|^2}d\theta;$$

see [13, Ex. 1]. Using the coordinates expressions $(x, y) = (\cos\theta, \sin\theta)$, we find the distributions of $X_{\tau_{\mathbb{D}}}$ and $Y_{\tau_{\mathbb{D}}}$:

$$\rho_u^{X_{\tau_{\mathbb{D}}}}(x) = \tfrac{1-|a|^2}{2\pi\sqrt{1-x^2}}\left(\tfrac{1}{|1-\overline{a}(x+\sqrt{1-x^2}i|^2} + \tfrac{1}{|1-\overline{a}(x-\sqrt{1-x^2}i|^2}\right)$$

and

$$\rho_v^{Y_{\tau_{\mathbb{D}}}}(y) = \tfrac{1-|a|^2}{2\pi\sqrt{1-y^2}}\left(\tfrac{1}{|1-\overline{a}(\sqrt{1-y^2}+yi|^2} + \tfrac{1}{|1-\overline{a}(-\sqrt{1-y^2}+yi|^2}\right).$$

In particular we recover $\rho_{\mathcal{I}m(a)}^{Y_{\tau_{\mathbb{D}}}}(y) = \rho_{\mathcal{R}e(-ai)}^{X_{\tau_{\mathbb{D}}}}$.

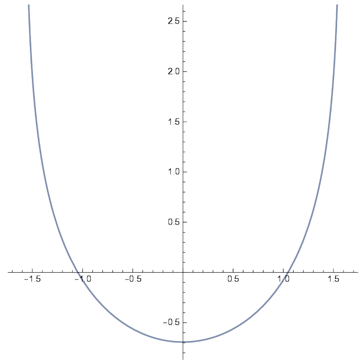



Figure 4.1: $\mathscr{P}$ has two vertical asymptotes, namely at $\pm\frac{\pi}{2}$

## 4.2 Catenary

The following example was brought to our attention by Hugo Panzo and Phanuel Mariano, and is the subject of their interesting preprint [11]. The map $f(z) = -i\ln(1+z)$ maps the unit disc to the domain $\mathscr{P}$ shown in Figure 4.1.

If we set $z = e^{\theta i}$ and $w = f(z)$ then $w = x + yi = \arctan(\frac{\sin\theta}{1+\cos\theta}) - \frac{i}{2}\ln(2 + 2\cos\theta)$. It is not hard to check that $x = \theta/2$, and thus

$$y = -\frac{1}{2}\ln(2 + 2\cos(2x)),$$

which explains the asymptotes at $x = \pm\frac{\pi}{2}$. It is straightforward to verify that $\frac{dx}{dy} = \sqrt{\frac{1}{4e^{2y}-1}}$, and using Lemma 4.1 and (4.2) we get

$$\begin{aligned}
\rho_0^{Y_{\tau_{\mathscr{P}}}}(y) &= \frac{2\mid e^{wi}\mid}{2\pi}\sqrt{1+\left(\sqrt{\frac{1}{4e^{2y}-1}}\right)^2} \\
&= \frac{e^{-y}}{\pi}\sqrt{\frac{4e^{2y}}{4e^{2y}-1}} \\
&= \frac{2}{\pi}\sqrt{\frac{1}{4e^{2y}-1}}.
\end{aligned}$$

Note that the factor 2 in the first equation comes from the fact that each value on the $y$-axis great than $-\ln 2$ has two preimages on the curve. Note that if we rotate $P$ a quarter turn to the left it is $\Delta$-convex and symmetric, so this is (a rotation of) the domain generated by Gross' method for this distribution. On the other hand, this is not the case for the real part, in fact

$$\begin{aligned}
\rho_0^{X_{\tau_{\mathscr{P}}}}(x) &= \frac{\mid e^{wi}\mid}{2\pi}\sqrt{1+\left(\frac{\sin 2x}{1+\cos 2x}\right)^2} \\
&= \frac{e^{-y}}{2\pi}\sqrt{1+\left(\frac{\sin 2x}{1+\cos 2x}\right)^2} \\
&= \frac{\sqrt{2+2\cos 2x}}{2\pi}\sqrt{\frac{2}{1+\cos 2x}} \\
&= \frac{1}{\pi}.
\end{aligned}$$

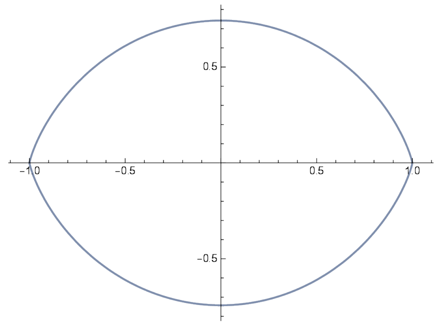



Figure 4.2: The symmetric and $\Delta$-convex domain generated by Uni$(-1, 1)$.

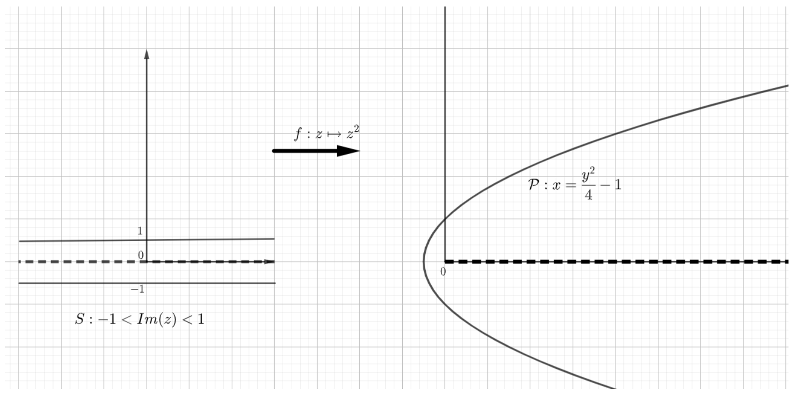



Figure 4.3: Action of $z \longmapsto z^2$ on the strip.

So $X_{\tau_{\mathscr{P}}}$ is uniformly distributed over $(-\pi/2, \pi/2)$. However, this is not the domain generated by Gross' method for the uniform distribution, as it is not symmetric over the real axis. An approximation of that domain is illustrated in Figure 4.2, which also appears in [7] and [11].

Incidentally, [11] contains a great deal more information about this example, as well as another proof that the exit distribution of this domain is uniform when projected onto the real axis.

### 4.3 Parabola

Let $S$ be the horizontal strip $\{z, -1 < \mathrm{Im}(z) < 1\}$ and $\mathcal{P} = f(S)$ where $f : z \longmapsto z^2$.The map $f$ is not conformal as it is 2 to 1, however it maps the boundary $\partial S$ to $\partial \mathcal{P}$. That is

$$\partial \mathcal{P} = \{(u, v) | u = x^2 - 1,\ v = \pm 2x, x \in \mathbb{R}\},$$

so $\mathcal{P}$ is the area limited by the parabola of the equation

$$x = \frac{y^2}{4} - 1. \tag{4.5}$$

The following image may help the reader visualize how the map works. Note that the real axis is mapped to the nonnegative real axis, and each of the strips $\{z, -1 < \mathrm{Im}(z) < 0\}$, $\{z, 0 < \mathrm{Im}(z) < 1\}$ are "bended" into the interior of the parabola minus the nonnegative real axis.

The p.d.f of $Z_{\tau_S}$ starting from the origin is given by

$$\rho_0^{\tau_S}(z = x \pm i) = \frac{\operatorname{sech}(\frac{\pi}{2}x)}{4}, \tag{4.6}$$

where $\operatorname{sech}(z) = \frac{2}{e^z + e^{-z}}$ is the hyperbolic secant function. The distribution of $X_{\tau_\Omega}$ is equally shared between the two horizontal lines of the boundary of the strip because of symmetry, and therefore admits the density $\frac{\operatorname{sech}(\frac{\pi}{2}x)}{2}dx$. (4.6) can be proved by conformal invariance ([13, Ex. 4]) or as a consequence of the optional stopping theorem ([4, Prop. 2]).

The expression of $\rho_0^{Z_{\tau_\mathcal{P}}}$ is

$$\begin{aligned}
\rho_0^{Z_{\tau_\mathcal{P}}}(w = u + vi) &= \sum_{z \in f^{-1}\{w\}} \frac{\rho_0^{Z_{\tau_S}}(z)}{\left|f'(z)\right|} \\
&= \frac{\rho_0^{Z_{\tau_S}}(\frac{v}{2} + i)}{\left|f'(\frac{v}{2} + i)\right|} + \frac{\rho_0^{Z_{\tau_S}}(-\frac{v}{2} + i)}{\left|f'(-\frac{v}{2} + i)\right|} \\
&= 2\frac{\rho_0^{Z_{\tau_S}}(\frac{v}{2} + i)}{\left|f'(\frac{v}{2} + i)\right|} \\
&= \frac{\operatorname{sech}(\frac{\pi}{4}v)}{4\sqrt{\frac{v^2}{4} + 1}},
\end{aligned}$$

where $w \in \partial\mathcal{P}$. Via (4.3), we get for $(u, v) \in (-1, +\infty) \times \mathbb{R}$

$$\begin{aligned}
\rho_0^{X_{\tau_\mathcal{P}}}(u) &= \rho_0^{Z_{\tau_\mathcal{P}}}(u + \sqrt{4u+4}i)\sqrt{1 + \tfrac{4}{4u+4}} + \rho_0^{Z_{\tau_\mathcal{P}}}(u - \sqrt{4u+4}i)\sqrt{1 + \tfrac{4}{4u+4}} \\
&= \frac{\operatorname{sech}(\frac{\pi}{2}\sqrt{u+1})}{2\sqrt{u+1}},
\end{aligned}$$

and

$$\begin{aligned}
\rho_0^{Y_{\tau_\mathcal{P}}}(v) &= \rho_0^{Z_{\tau_\mathcal{P}}}(\tfrac{v^2}{4} - 1 + vi)\sqrt{1 + \tfrac{v^2}{4}} \\
&= \frac{\operatorname{sech}(\frac{\pi}{4}v)}{4}.
\end{aligned}$$

It is a surprising fact that this agrees with the density obtained from the strip $\{z, -2 < \operatorname{Im}(z) < 2\}$, as can be verified by applying Lemma 4.1 with the map $f(z) = 2z$, which takes $\{z, -1 < \operatorname{Im}(z) < 1\}$ to $\{z, -2 < \operatorname{Im}(z) < 2\}$. However, as mentioned in the previous section, this does not contradict Theorem 3 since it is the distribution of $Im(Z_{\tau_\mathcal{P}})$, and $\mathcal{P}$ is not symmetric or $\Delta$-convex with respect to the imaginary axis.

### 4.4 Ellipse of the form $\frac{x^2}{\cosh^2 R} + \frac{y^2}{\sinh^2 R} = 1$

This example leads to a complicated distribution, but is included because it illustrates how our method can be applied to maps which are infinite to one. Let $E$ be the centered ellipse of equation $\frac{x^2}{\cosh^2 R} + \frac{y^2}{\sinh^2 R} = 1$. Although not every ellipse can be expressed in this form, it does capture every possible ratio between major and minor axes, and therefore any ellipse can be expressed simply as a scaling of one of these, with the corresponding $\mu$'s being scalings of each other as well. A conformal map from the disk onto the ellipse is known but is not simple ([9]); however, as is shown in [13, Thm. 2] the map in question does not need to be injective as long as it maps the boundary of its domain of definition onto the boundary of the target domain. It turns out that the holomorphic function $f(z) = \sin(z)$ maps the horizontal strip $S_R := \{z,\, R < \mathrm{Im}(z) < -R\}$ onto $E$. This is how it works: for $z = x + Ri$ we have

$$\begin{aligned}\sin(z) &= \frac{e^{(x+Ri)i} - e^{-(x+Ri)i}}{2i} \\ &= \frac{e^{-R}(\cos x + i\sin x) - e^{R}(\cos x - i\sin x)}{2i} \\ &= \left(\frac{e^R + e^{-R}}{2}\right)\sin x + i\left(\frac{e^R - e^{-R}}{2}\right)\cos x \\ &= \cosh R \sin x + i \sinh R \cos x.\end{aligned}$$

So if we set $\sin z = u + vi$ then $\frac{u^2}{\cosh^2 R} + \frac{v^2}{\sinh^2 R} = 1$. Thus, $f$ maps the lines $\{\mathrm{Im}(z) = R\} and \mathrm{Im}(z) = -R\}$ onto the curve $\frac{x^2}{\cosh^2 R} + \frac{y^2}{\sinh^2 R} = 1$, and it follows that the interior of the strip is mapped onto the interior of the ellipse. Now let $w \in \partial E_{a,b}$ and $\rho_E(w)$ be the p.d.f of $Z_{\tau_E}$, then

$$\begin{aligned}\rho_E(w = u + vi)dw &= \sum_{z \in f^{-1}\{w\}} \frac{\rho_{Z_\tau}(z)}{\left|\cos(z)\right|} dw \\ &= \sum_{z \in f^{-1}\{w\}} \frac{\mathrm{sech}(\frac{\pi x}{2R})}{2R\left|\cos(z)\right|} dw.\end{aligned}$$

If we assume $v \geq 0$, then since $u = \cosh R \sin x$, we may take $x = \arcsin(\frac{u}{\cosh R}) + 2\pi n$ (the other possible values of $x$ for a given $u$ correspond to $v < 0$). Thus,

$$\begin{aligned}\rho_E(w = u + vi)dw &= \sum_{n\in\mathbb{Z}} \frac{\mathrm{sech}(\frac{\pi}{2R}\arcsin(\frac{u}{\cosh R}) + \frac{n\pi^2}{R})}{2R\left|\cos(\arcsin(\frac{u}{\cosh R}) + 2n\pi)\right|} dw \\ &= \frac{1}{2R\left|\cos(\arcsin(\frac{u}{\cosh R}))\right|} \sum_{n\in\mathbb{Z}} \mathrm{sech}(\frac{\pi}{2R}\arcsin(\frac{u}{\cosh R}) + \frac{n\pi^2}{R}) dw \\ &= \frac{\cosh R}{2R\sqrt{\cosh^2 R - u^2}} \sum_{n\in\mathbb{Z}} \mathrm{sech}(\frac{\pi}{2R}\arcsin(\frac{u}{\cosh R}) + \frac{n\pi^2}{R}) dw.\end{aligned}$$

We can now project this density onto the real and imaginary axes as before, using $\frac{dv}{du} = \frac{-u\sinh R}{\cosh R\sqrt{\cosh^2 R - u^2}}$, to get

$$\begin{aligned}\rho_0^{X_{\tau_H}}(u) &= \frac{\cosh R\sqrt{1 + \frac{u^2\sinh^2 R}{\cosh^2 R(\cosh^2 R - u^2)}}}{2R\sqrt{\cosh^2 R - u^2}} \sum_{n\in\mathbb{Z}} \mathrm{sech}(\frac{\pi}{2R}\arcsin(\frac{u}{\cosh R}) + \frac{n\pi^2}{R}) du, \\ \rho_0^{Y_{\tau_H}}(v) &= \frac{\sinh R\sqrt{1 + \frac{v^2\cosh^2 R}{\sinh^2 R(\sinh^2 R - v^2)}}}{2R\sqrt{\sinh^2 R - v^2}} \sum_{n\in\mathbb{Z}} \mathrm{sech}(\frac{\pi}{2R}\arcsin(\frac{v}{\sinh R}) + \frac{n\pi^2}{R}) dv.\end{aligned}$$

### 4.5 Right part of the Hyperbola $x^2 - y^2 = 1$

If $R := \{z|\mathcal{R}e(z) > 1\}$ then the p.d.f of $Z_{\tau_R}$ started at $a = \delta + \eta i \in R$ is given by [13, Ex. 2]

$$\rho_a^{\tau_R}(1 + yi) = \frac{(\delta - 1)}{\pi|1 + iy - a|^2} dy.$$

The square function $s : z \longmapsto z^2$ maps the right part limited by the hyperbola $x^2 - y^2 = 1$, say $H$, to $R$. Therefore for every $z = x + yi \in \partial H$

$$\begin{aligned}\rho_{\sqrt{a}}^{\tau_H}(z)dz &\overset{z^2 = 1+vi}{=} |s'(\sqrt{1 + vi})|\rho_a^{\tau_R}(1 + vi) \\ &= 2(\delta - 1)\frac{\sqrt{x^2 + y^2}}{\pi|1 + vi - a|^2} dz \\ &= 2(\delta - 1)\frac{\sqrt{x^2 + y^2}}{\pi|1 - a + 2xyi|^2} dz.\end{aligned}$$

In particular if $a$ is real, and by using the relation $x^2 - y^2 = 1$, we get the densities of $X_{\tau_H}$ and $Y_{\tau_H}$:

$$\rho_{\sqrt{a}}^{X_{\tau_H}}(x) = \frac{2(\delta-1)}{\pi}\frac{2x^2-1}{\sqrt{x^2-1}}\left\{\frac{1}{|2x\sqrt{x^2-1}i+1-a|^2} + \frac{1}{|2x\sqrt{x^2-1}i-1+a|^2}\right\},$$
$$\rho_{\sqrt{a}}^{Y_{\tau_H}}(y) = \frac{2(\delta-1)}{\pi}\frac{2y^2+1}{\sqrt{1+y^2}|2y\sqrt{y^2+1}i+1-a|^2}.$$

We note that it is known that $E[\tau_H^p] < \infty$ precisely when $p < 1$; see [2, (4.2)]. Also by [2, Thm. 2.1], $E[\tau_H^p] < \infty$ precisely when $E[(Z_{\tau_H})^{2p}] < \infty$, and thus when $p < 1$. This is straightforward to verify from the previous equation, as for example the formula for $\rho_{\sqrt{a}}^{Y_{\tau_H}}$ is asymptotic to $y^{-3}$ at $\infty$.

## 5 Concluding remarks

We do not know whether Theorem 2 holds for $\frac{1}{2} \le p \le 1$. There are many difficulties to proving the result in this range. One is that the analogue of Theorem 4 does not hold, even for $p = 1$; for a counterexample, see [10, p. 212, Vol. 2]. Furthermore $H^p$ and $L^p$ are not as well behaved for $p < 1$; their respective norms are not true norms, for instance, as the triangle inequality fails. In any event, regardless of the veracity of the theorem for $\frac{1}{2} \le p \le 1$, one should certainly exercise extreme caution in attempting to extend it to $p < \frac{1}{2}$. This is because for any simply connected domain $\Omega$ strictly smaller than $\mathbb{C}$ itself we have $E[(\tau_\Omega)^{p/2}] < \infty$ for any $p < \frac{1}{2}$; this is proved in [2]. Thus a measure with infinite $p$-th moment for some $p < \frac{1}{2}$ cannot correspond in this manner to a simply connected domain.

The question posed by Gross in [7] on how properties of $\mu$ are reflected in the geometry of $\Omega$ is, in our opinion, an interesting one. We emphasize in this regard that we have now shown that every $\Delta$-convex, symmetric domain can be obtained uniquely from a probability distribution, so in this context we would hope that geometric conditions will translate directly to probabilistic ones. Gross proposed finding a condition which forced $\Omega$ to be convex; this appears difficult, especially considering that according to Gross' simulations the domain corresponding to a Gaussian is not convex. We would like therefore to suggest several weaker properties that $\Omega$ might have, and propose that finding sufficient conditions on $\mu$ for these might be interesting problems.

- $\Omega$ is starlike with respect to 0.
- $\sup_{z\in\Omega} |\mathcal{I}m(z)| < \infty$. That is, $\Omega$ is contained in an infinite horizontal strip. Note that this would imply that all moments of $\mu$ are finite, because all moments of the exit time of a strip are finite, but that this is not sufficient: if $\Omega$ is the parabolic region $\{x > y^2 - 1\}$, then all moments of $\tau_\Omega$ are finite (proof: $\Omega$ can fit inside a rotated and translated wedge $W_\alpha$ with arbitrarily small aperture $\alpha$, and therefore its exit time is dominated by that of the wedge, which can have finite $p$-th moment for as large $p$ as we like) but $\sup_{z\in\Omega} |\mathcal{I}m(z)| = \infty$.
- $\limsup_{|\mathcal{R}e(z)|\to\infty, z\in\Omega} |\mathcal{I}m(z)| = 0$.

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# A new solution to the conformal Skorokhod embedding problem and applications to the Dirichlet eigenvalue problem

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ABSTRACT

In a recent work by Gross, the following problem was posed and solved: given a measure $\mu$ on $\mathbb{R}$ with finite second moment, find a simply connected domain $U$ in $\mathbb{C}$ such that the real part of the random variable $Z_{\tau_U}$ has the distribution $\mu$, where $Z$ is a planar Brownian motion and $\tau_U$ is the exit time from $U$. The construction developed by Gross yields a domain which is symmetric with respect to the real axis, but it has been noted by other authors that other domains are also possible, in particular there are a number of examples which have the property that a vertical ray starting at a point in the domain lies entirely within the domain. In this paper we give a new solution to the problem posed by Gross, and show that these other cases noted before are special cases of this method. We further show, following a method due to Mariano and Panzo, that the domain generated by this method has the property that it always has the minimal rate (as defined in terms of the spectrum of the Laplacian operator) among all possible domains corresponding to a fixed distribution $\mu$. This gives a partial solution to a question posed by Mariano and Panzo. We show that the domain is unique, provided certain conditions are imposed, and use this to give several examples. We also describe a method for identifying the boundary curve of the domain, and discuss several other related topics.



## 1. Introduction

In what follows, $Z_t$ is a standard planar Brownian motion starting at 0, and for any plane domain $U$ containing 0 we let $\tau_U$ denote the first exit time of $Z_t$ from $U$. In the recent paper [1] the following theorem was proved, which is a direct generalization of the elegant results and methods developed in [11].

**Theorem 1.** *Given a nondegenerate probability distribution $\mu$ on $\mathbb{R}$ with zero mean and finite nonzero p-th moment (with $1 < p < \infty$), we can find a simply connected domain $U$ such that $\Re(Z_{\tau_U})$ has the distribution $\mu$. Furthermore we have $E[(\tau_U)^{\frac{p}{2}}] < \infty$.*

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Note that by *nondegenerate* we mean that $\mu$ is not a point mass at the origin. In fact, this qualification is not present in either [11] or [1], and it seems to have been overlooked. But it is necessary, and we will discuss this more later in the paper.

In what follows, when $\mu$ is given we will refer to $U$ as a *$\mu$-domain*. Therefore Theorem 1 provides the existence of a $\mu$-domain whenever $\mu$ satisfies the moment conditions. In general a $\mu$-domain is not unique, however a uniqueness principle for this construction with additional conditions was proved in [1]. We will say that a domain $U$ is *symmetric* if $\bar{z} \in U$ whenever $z \in U$. We will call a domain $U$ *$\Delta$-convex* if, whenever $z_1, z_2 \in U$ with $\Re(z_1) = \Re(z_2)$ then the vertical line segment connecting $z_1$ and $z_2$ lies entirely within $U$. With these definitions, the uniqueness principle is as follows.

**Theorem 2.** *For any nondegenerate distribution $\mu$ satisfying the conditions of Theorem 1, there is a unique domain $U$ such that $\Re(Z_{\tau_\Omega}) \sim \mu$ and which is symmetric, $\Delta$-convex, and satisfies $E[\tau_\Omega^{\frac{p}{2}}] < \infty$.*

It was pointed out in [1] that this result fails if any of the conditions is omitted, and in particular it was shown that the parabola and horizontal strip are $\mu$-domains when $\mu$ is the hyperbolic secant distribution. Another example of this phenomenon was demonstrated in [14], where it was shown that the catenary (see Fig. 4.2 below) is a $\mu$-domain when $\mu$ is the uniform distribution on $(-1, 1)$, even though it can not be the domain generated by Gross' construction as it is not symmetric. Furthermore, the authors of [14] showed that, among all $\mu$-domains for the uniform distribution, the catenary is the one with the minimal principle Dirichlet eigenvalue, and asked for a characterization of $\mu$-domains which are extremal with respect to the principle Dirichlet eigenvalue.

In this paper, our primary intention is to demonstrate a new method for solving the conformal Skorokhod problem, one which gives the parabola and catenary when applied to the hyperbolic secant and uniform distributions, respectively. This method also has the property that its solution is always the one with the minimal principle Dirichlet eigenvalue among all $\mu$-domains, which therefore gives a partial solution to the problem posed by Mariano and Panzo in [14].

To state our results we need a definition. We will say that a domain is *$\Delta^\infty$-convex* if, given any $z \in U$, the vertical ray $\{w : \Re(w) = \Re(z), \Im(w) \geq \Im(z)\}$ lies entirely in $U$. So, for example, the parabola and catenary described above are $\Delta^\infty$-convex, while a horizontal strip is not. The reason for this name is that it is a variation on the notion of $\Delta$-convexity, as defined above. Our primary results are as follows.

**Theorem 3.** *If $\mu \in L^p$ is nondegenerate for some $p > 1$ then there exists a $\mu$-domain $U$ containing zero which is $\Delta^\infty$-convex. Furthermore $\mathbf{E}(\tau_U^{\frac{p}{2}}) < \infty$.*

**Theorem 4.** *The $\mu$-domain $U$ given in Theorem 3 is the unique $\mu$-domain which is $\Delta^\infty$-convex and satisfies $\mathbf{E}(\tau_U^{\frac{p}{2}}) < \infty$ for some $p > 1$.*

**Theorem 5.** *The $\mu$-domain $U$ constructed in Theorem 3 is always the one with the minimal principle Dirichlet eigenvalue among all $\mu$-domains.*

Sections 2, 3, and 4 are devoted to the proofs of these theorems. The $\mu$-domain generated by our method is bounded below by a boundary curve, and in Section 5 we describe a method of determining this curve from the measure $\mu$. Finally, in Section 6, we present a curiosity, that a formal application of our methods to the Cauchy distribution yields the correct $\Delta^\infty$-convex $\mu$-domain, even though the Cauchy does not satisfy the conditions of our theorems.

## 2. Preliminaries

For a planar domain $U$, the function $(t, z) \mapsto \mathbf{P}_z(t < \tau_U)$ satisfies the heat equation

$$\begin{cases} \frac{1}{2}\Delta u - \frac{\partial u}{\partial t} = 0 \\ u(t, z) = 0 \qquad\quad z \in \partial U. \end{cases} \tag{2.1}$$

The equation (2.1) is commonly referred to as being of *Dirichlet boundary condition* type. The *rate* of the solution of (2.1), which we denote by $\lambda(U)$, is defined to be half of the principal Dirichlet eigenvalue of $U$. This is the minimum of the spectrum of the half of the Laplacian operator on $U$ combined with the boundary condition. Using the expansion of the solution on the Hilbert basis generated by the associated eigenfunctions, we can check that

$$-\frac{\ln \mathbf{P}_z(t < \tau_U)}{t} \underset{t \to +\infty}{\longrightarrow} \lambda. \tag{2.2}$$

In [14], the two authors treated the rate of (2.1) on $\mu$-domains coming from the conformal Skorokhod embedding. More precisely, they proposed the problem of finding, for a given distribution $\mu$, the $\mu$-domains that attain the highest and lowest possible rate. That is, we seek two $\mu$-domains $U_\mu$ and $V_\mu$ such that

$$\lambda(U_\mu) \leq \lambda(\mathscr{D}) \leq \lambda(V_\mu)$$

for all $\mu$-domains $\mathscr{D}$. As mentioned earlier, they partially solved this problem when $\mu$ is the uniform distribution on $(-1, 1)$, showing that the catenary has the minimal rate among all $\mu$-domains. When we prove Theorem 5 we will see that this is a special case of a more general construction, one which always produces the minimal rate solution.

The analytic tools we will need, such as Fourier series, the periodic Hilbert transform, and the Hardy norm, are largely the same as used in [11] and [1]. For the sake of completeness, we recall here two definitions as well as some related results.

A major tool for us is the periodic Hilbert transform.

**Definition 6.** The Hilbert transform of a $2\pi$-periodic function $f$ is defined by

$$H\{f\}(x) := PV\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\cot(\tfrac{t}{2})dt\right\} = \lim_{\eta \to 0}\frac{1}{2\pi}\int_{\eta \leq |t| \leq \pi} f(x-t)\cot(\tfrac{t}{2})dt$$

where $PV$ denotes the *Cauchy principal value*.

The Hilbert transform has some properties of great importance. The Hilbert transform is an automorphism of $L^p$ and it satisfies the *strong type estimate*

$$||H_f||_p \leq \lambda_p ||f||_p \tag{2.3}$$

for some positive constant $\lambda_p$ [12, Vol I, page 203]. Furthermore, under suitable conditions it serves to swap the real and the imaginary parts of the boundary values of holomorphic functions. That is, if $f$ is an analytic function on the disk which extends suitably to the boundary, then the Hilbert transform of $\Re(f)$ is $\Im(f)$ and the Hilbert transform of $\Im(f)$ is $\Re(f)$. Another important property is that the Hilbert operator $H$ commutes with positive dilations. That is, if $\Phi_\lambda\{f\}(x) = f(\lambda x)$ then

$$(H \circ \Phi_\lambda)\{f\} = (\Phi_\lambda \circ H)\{f\}. \tag{2.4}$$

**Definition 7.** For any holomorphic function on the unit disc we define its $p$th-Hardy norm by

$$H_p(f) := \sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}|f(re^{\theta i})|^p d\theta\right\}^{\frac{1}{p}}. \tag{2.5}$$

The set of holomorphic functions whose $p$th-Hardy norm is finite is denoted by $\mathcal{H}^p$ and called the Hardy space (of index $p$).

The Hardy norm of a function $f$ is merely the upper bound of $\{N_r(f)\}_{0<r<1}$ where

$$N_r(f) := \left\{\frac{1}{2\pi}\int_0^{2\pi}|f(re^{\theta i})|^p d\theta\right\}^{\frac{1}{p}} \quad \forall 0<r<1.$$

The quantity $N_r(f)$ is simply the $L^p$ norm of the restriction $\theta \mapsto f(re^{\theta i})$. It can be shown, using harmonic analysis techniques, that $N_r(f)$ is non-decreasing in terms of $r$ [17]. This explains the use of sup in (2.5). Another crucial result about Hardy norms is that, if $H_p(f)$ is finite then $f$ has a radial extension to the boundary. More precisely $f^*(z) := \lim_{r\to 1} f(rz)$ exists for almost every $z \in \partial\mathbb{D}$ (with respect to Lebesgue measure), and this extension belongs to $L^p$ as well. In [2], the author provided a powerful theorem that guarantees the equivalence between the finiteness of the $p$th Hardy norm of $f$ and the finiteness of $\mathbf{E}(\tau_{f(\mathbb{D})}^{\frac{p}{2}})$.

Another important tool for us the following standard result.

**Lemma 8** *(Schwarz' integral formula). If $f \in \mathcal{H}^p$ then for all $z \in \mathbb{D}$*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{ti}+z}{e^{ti}-z}\Re(f(e^{ti}))dt + i\Im(f(0)). \tag{2.6}$$

The formula (2.6) says that, under some assumptions, the boundary behavior of $\Re(f)$ determines entirely the map $f$ inside $\mathbb{D}$. In particular, it implies the Poisson integral formula as $\Re(f(z))$ is harmonic. Schwarz' integral formula is also used in the field of boundary value problem for analytic functions. We refer the reader to [17, Th. 17.26], [15, Ch. 7], or [10, Ch. I] for more details.

## 3. Proof of Theorem 3

The proof builds on the methods used in [11], but with some additional ideas required. Let $G_\mu$ be the quantile function for $\mu$, which is the pseudo-inverse of the c.d.f. $F_\mu$ of $\mu$; that is, $G_\mu(u) = \inf\{x|F_\mu(x) \ge u\}$. Consider the $2\pi$-periodic function

$$\varphi_\mu : \begin{array}{rl} (0,2\pi) & \longrightarrow \mathbb{R} \\ \theta & \longmapsto G_\mu(\frac{\theta}{2\pi}). \end{array}$$

It is a well known fact that $G_\mu(\mathsf{Uni}(0,1))$ has the distribution $\mu$ [6]. Therefore, by scaling, if $\theta$ is uniformly distributed on $(0,2\pi)$ then

$$\varphi_\mu(\theta) \sim \mu \tag{3.1}$$

which is straightforward using the definition of $G_\mu$. As $\varphi_\mu \in L^p$ then it has a Fourier series whose partial sums converge to it in $L^p$, i.e.

$$\varphi_\mu(\theta) \overset{L^p}{=} \sum_{n=1}^{+\infty}(a_n\cos(n\theta)+b_n\sin(n\theta)) \tag{3.2}$$

where $a_n$ and $b_n$ are the standard Fourier coefficients.[1] In fact, (3.2) is also true in the almost everywhere statement, which is the subject of the *Carleson-Hunt theorem* [5,8]. The Hilbert transform of $\varphi_\mu$ is

$$H\{\varphi_\mu\}(\theta) = \sum_{n=1}^{+\infty}(a_n\sin(n\theta)-b_n\cos(n\theta))$$

and it belongs to $L^p$ as well [3]. The power series

$$\widetilde{\varphi}_\mu(z) = \sum_{n=1}^{+\infty}(a_n - b_n i)z^n \tag{3.3}$$

belongs to $H^p$ since $\Re(\widetilde{\varphi}_\mu(e^{\theta i})) \overset{\text{a.e.}}{=} \varphi_\mu(\theta)$ and $\Im(\widetilde{\varphi}_\mu(e^{\theta i})) \overset{\text{a.e.}}{=} H\{\varphi_\mu\}(\theta)$. The map $\widetilde{\varphi}_\mu(z)$ is one to one on the unit disc $\mathbb{D}$ and maps 0 to 0. We give the proof of this fact in a separate lemma. The domain $U := \widetilde{\varphi}_\mu(\mathbb{D})$ is $\Delta^\infty$-convex since $\varphi_\mu$ is non decreasing a.e. on $[0,2\pi]$. Let $Z_t$ be a planar Brownian motion starting at 0 and stopped at $\tau_{\mathbb{D}}$. Then by conformal invariance $\widetilde{\varphi}_\mu(Z_{\tau_{\mathbb{D}}})$ is a planar Brownian motion starting at $\widetilde{\varphi}_\mu(0)=0$ and evaluated at $\tau_U$. As $Z_{\tau_{\mathbb{D}}} = e^{\theta i}$ where $\theta := \operatorname{Arg}(Z_{\tau_{\mathbb{D}}}) \sim \mathsf{Uni}(0,2\pi)$, then $\Re(\widetilde{\varphi}_\mu(Z_{\tau_{\mathbb{D}}})) = \varphi_\mu(\theta)$ has the distribution $\mu$ by (3.1). Since $0 \in U$ and $\theta \mapsto \varphi_\mu(\theta)$ is non-decreasing it follows that it is $\Delta^\infty$-convex. Finally, the finiteness of $\mathbf{E}(\tau_U^{p/2})$ comes from Theorem 4.1 in [2]. $\square$

**Lemma 9.** *The map $\widetilde{\varphi}_\mu(z)$ is one to one on the unit disc.*

**Proof.** Recall that $G$ is a non-decreasing function on $(0,2\pi)$. We may find a sequence of functions $G_n$ on $(0,2\pi)$ which converges to $G$ in $L^1$ such that each $G_n$ has the following properties.

- $G_n$ is $C^\infty$ and non-decreasing on $(0,2\pi)$.
- $\lim_{\theta\to 0+} G_n(\theta)$ and $\lim_{\theta\to 2\pi-} G_n(\theta)$ both exist and are finite.
- $\lim_{\theta\to 0+} G_n^{(k)}(\theta)$ and $\lim_{\theta\to 2\pi-} G_n^{(k)}(\theta)$ both exist and are finite, and furthermore $\lim_{\theta\to 0+} G_n^{(k)}(\theta) = \lim_{\theta\to 2\pi-} G_n^{(k)}(\theta)$, for all $k \geq 1$. In other words, $G_n'$ extends to be a $C^\infty$ function on the circle.

If we now let $\kappa_n = \lim_{\theta\to 2\pi-} G_n(\theta) - \lim_{\theta\to 0+} G_n(\theta)$, then $\tilde{G}_n(\theta) = G_n(\theta) - \frac{\kappa_n}{2\pi}\theta$ extends to be $C^\infty$ on the entire circle (i.e. with 0 and $2\pi$ identified). A standard result in Fourier analysis now states that the Fourier coefficients $a_n$ of $\tilde{G}_n$ satisfy $a_n = O(|n|^{-2})$ ([18, Cor. 2.4]). Form an analytic function $\tilde{f}_n$ with the Fourier coefficients of $\tilde{G}_n(\theta)$ as in (3.3). The decay of the coefficients of $\tilde{f}_n$ means that the power series converges absolutely on the boundary of the disk, and $\tilde{f}_n$ therefore extends to be continuous on the closed unit disk. Let $f_n(z) = \tilde{f}_n(z) - i\frac{\kappa_n}{\pi}\ln(1-z) + \frac{\kappa_n}{2}$, where ln denotes the analytic logarithm function whose imaginary part Arg takes values in $(-\pi,\pi)$. Then $f_n$ is analytic on the disc and continuous on the closure of the disc minus the point 1. Furthermore, the modulus of $f_n$ approaches $\infty$ as $z$ approaches 1. It is therefore a continuous map from the closed unit disk to the Riemann sphere, with 1 being mapped to $\infty$. Furthermore, it can be checked by elementary geometry, or by trigonometric identities, that

$$\Re(-i\frac{\kappa_n}{\pi}\ln(1-e^{i\theta})) = \frac{\kappa_n}{\pi}\operatorname{Arg}(1-e^{i\theta}) = \frac{\kappa_n}{\pi}(-\frac{\pi}{2}+\frac{\theta}{2}) = \frac{-\kappa_n}{2} + \frac{\kappa_n}{2\pi}\theta.$$

[1] In general there is a constant $\frac{a_0}{2}$ added to the sum, but it is omitted as it equals the average of $\mu$ which is assumed zero.

It follows that $\Re(f_n(e^{i\theta})) = \tilde{G}_n(\theta) + \frac{-\kappa_n}{2} + \frac{\kappa_n}{2\pi}\theta + \frac{\kappa_n}{2} = G_n(\theta)$. Therefore $\Re(f_n(e^{i\theta}))$ is strictly increasing on $(0, 2\pi)$; that is, as $\theta$ increases from 0 to $2\pi$, the image $f_n(e^{i\theta})$ "winds once" about the domain in the Riemann sphere. This proves that $f_n$ is injective; see for instance [17, Thm. 10.31].

Using Lemma 8, it is straightforward to verify that the $L^1$ convergence of $G_n$ to $G$ on $(0, 2\pi)$ implies that $f_n$ converges to $\widetilde{\varphi}_\mu$ uniformly on compact sets (note that $\Im(f_n(0)) = \Im(\widetilde{\varphi}_\mu(0)) = 0$ by construction). By Hurwitz's Theorem (see [15]), $\widetilde{\varphi}_\mu$ is either constant or injective. The case when $\widetilde{\varphi}_\mu$ is constant corresponds to the case when $\mu$ is a point mass at the origin, and since we have excluded this case we see that $\tilde{\phi}_\mu$ is injective. $\square$

**Remark.** The primary difference between this proof and that in [11] is essentially that the $G_n$'s have a jump discontinuity at the point 0 ($\equiv 2\pi$) when viewed as a function defined on the circle. This is why the logarithm made an appearance. There may be other solutions available based upon this same method, however the issue of injectivity of the resulting map must be considered. To be precise, we have basically dealt with the quantile function $G_\mu$ directly, while Gross chose to work with a reflected version of it in order to remove the jump discontinuity. It may seem that other transformations of $G_\mu$ are possible that would yield new solutions, however the advantage of Gross' method and ours is that it is clear in both cases that the resulting analytic function wraps the unit circle once around the boundary of the domain, as in the proof of Lemma 9, and is therefore injective. This may not be the case for more complicated transformations of $G_\mu$.

**Remark.** The assumption that $\mu$ is nondegenerate appears when applying Hurwitz's Theorem, and this same issue also applies to the proof given in [11]. Essentially, in this case the $\mu$-domain would degenerate to a point at the origin; it could not for instance be the domain limited by the boundary $\{\Re(z) = 0, \Im(z) \le -1\}$, since this domain contains a half-plane and therefore $E[\tau_U^{p/2}] = \infty$ for $p \ge 1$ (see Section 6 for more on this).

## 4. A uniqueness criterion for $\mu$-domains

Now we are ready to tackle the uniqueness issue of our $\Delta^\infty$-convex $\mu$-domain. Before that, we need some preliminary tools.

**Definition 10.** We say that a function $F$ defined on $[a, b]$ is non-decreasing almost everywhere if there is a non-decreasing function $\widetilde{F}$ defined on all of $[a, b]$ such that $F = \widetilde{F}$ almost everywhere. For such a function we define its generalized inverse function $F^{-1}$ by

$$F^{-1}(x) = \sup\{t \in [a, b] \mid \widetilde{F}(t) \le x\} = \inf\{t \in [a, b] \mid \widetilde{F}(t) > x\}$$

with the convention $F^{-1}(x) = a$ if $\{t \in [a, b] \mid F(t) \le x\}$ is empty. The swap between sup and inf is justified by the monotonicity of $\widetilde{F}$.

**Lemma 11.** *Let $F$ and $G$ be two function defined on $[a, b]$, non-decreasing a.e. If $F^{-1}(x) = G^{-1}(x)$ for a.e. $x$ then $F$ and $G$ agree a.e.*

**Proof.** Let $\Lambda_F$ and $\Lambda_G$ be the subsets of $[a, b]$ of full measure upon which $F = \widetilde{F}$ and $G = \widetilde{G}$, respectively. Set $\Lambda = \Lambda_F \cap \Lambda_G$ and suppose that $\widetilde{F}(c) \ne \widetilde{G}(c)$ for some $c \in \Lambda$, say $\widetilde{F}(c) < \widetilde{G}(c)$. Since the subset of $\Lambda$ where $\widetilde{F}$ or $\widetilde{G}$ are discontinuous is countable ([16]) and therefore of measure 0, we may discard it and assume that $c$ is a continuity point for both of them in $\Lambda$. Choose $\ell_1, \ell_2$ such that $\widetilde{F}(c) < \ell_1 < \ell_2 < \widetilde{G}(c)$. The continuity of $\widetilde{F}$ and $\widetilde{G}$ at $c$ yields

$$\widetilde{F}(t) < \ell_1 < \ell_2 < \widetilde{G}(t)$$

for all $t \in [c-\delta, c+\delta] \cap \Lambda$ for some $\delta > 0$. Now, if $y \in (\ell_1, \ell_2)$ then $G^{-1}(y) \leq c-\delta$ but $F^{-1}(y) \geq c+\delta$. Thus, $F^{-1}$ and $G^{-1}$ disagree on a set of positive measure. This contradiction proves the lemma. $\square$

Now we prove Theorem 4.

**Proof.** Let $U, V$ be two $\mu$-domains which are $\Delta^\infty$-convex and which satisfy $\mathbf{E}(\tau_U^{p/2}), \mathbf{E}(\tau_V^{p/2}) < \infty$ for some $p > 1$. Let $f, g$ be two conformal maps from $\mathbb{D}$ to $U, V$ fixing 0. By $\Delta^\infty$-convexity, the functions $F(t) = \Re(f(e^{ti}))$ and $G(t) = \Re(g(e^{ti}))$ (defined for a.e. $t \in [0, 2\pi]$ in the sense of radial limits, see [17]) are a.e. non-decreasing so they have well defined generalized inverse functions $\widetilde{F}$ and $\widetilde{G}$. As $U$ and $V$ are $\mu$-domain then $\Re(Z_{\tau_U})$ and $\Re(Z_{\tau_V})$ share the same distribution $\mu$. Therefore, by the conformal invariance of Brownian motion, we get

$$\begin{aligned}\mathbf{P}\{\Re(Z_{\tau_U}) \in (-\infty, x]\} &= \mathbf{P}\{\Re(Z_{\tau_V}) \in (-\infty, x]\} \\ &= \frac{F^{-1}(x)}{2\pi} \\ &= \frac{G^{-1}(x)}{2\pi},\end{aligned}$$

for all $x$ where $F^{-1}$ and $G^{-1}$ are the generalized pseudo inverses as in Definition 10. Thus $F \overset{\text{a.e.}}{=} G$ by applying Lemma 11. Consequently $f-g$ is constant via Schwarz integral formula (2.6), and since they send zero to the same point we conclude the equality of $f$ and $g$. $\square$

**Remark.** The condition that $\mathbf{E}(\tau_U^{p/2}) < \infty$ for some $p > 1$ is necessary. To see this, take for instance the domain $U = \mathbb{C} \backslash \{\Im(z) \leq -1, -1 \leq \Re(z) \leq 1\}$. The resulting distribution of $\Re(Z_{\tau_U})$ is supported on $[-1, 1]$, and therefore has all moments. However, $U$ cannot be the $\mu$-domain generated in Theorem 3 as $\mathbf{E}(\tau_U^{p/2}) = \infty$ for $p \geq \frac{1}{2}$.

Theorem 4 implies that, in practice, to obtain such a $\mu$-domain it is enough to find the Fourier expansion of

$$\varphi_\mu : \begin{aligned} (0, 2\pi) &\longrightarrow \mathbb{R} \\ \theta &\longmapsto G_\mu(\tfrac{\theta}{2\pi}) \end{aligned}$$

and consider $\widetilde{\varphi}_\mu(\mathbb{D})$. We will now prove Theorem 5; in fact, we provide a more precise formulation of the result, as follows.

**Theorem 12.** *The $\mu$-domain, say $U_\mu$, constructed in Theorem 4, has the lowest rate among all $\mu$-domains with finite $p/2$-th moment. In other words*

$$\lambda(U_\mu) \leq \lambda(V_\mu)$$

*for all $\mu$-domains $V_\mu$ such that $\mathbf{E}(\tau_{V_\mu}^{p/2}) < \infty$. Furthermore $\lambda(U_\mu) = \frac{\pi^2}{2(\beta-\alpha)^2}$ where $[\alpha, \beta]$ is the smallest interval containing the support of $\mu$.*

The proof of Theorem 12 is exactly the same as for Theorems 2 and 3 in [14]. However we highlight some facts for the sake of completeness of the paper. The monotonicity property of rates follows from (2.2) for example, which shows that if a domain $W$ contains another domain $W'$, then the rate of $W$ is less than or equal to that of $W'$. By using the separation of variables method to solve the heat equation on an infinite strip of width $L$ and on a rectangle of height $a$ and width $b$, we obtain the following well-known eigenvalues:

- Infinite strip: $\lambda_n = \frac{n^2\pi^2}{L^2}$ for $n = 1, 2, ...$
- Rectangle: $\lambda_{n,m} = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}$ for $n, m = 1, 2, ...$

Consequently, the rates are respectively $\frac{\pi^2}{2L^2}$ and $\frac{\pi^2}{2}(\frac{1}{a^2} + \frac{1}{b^2})$. Now, if $\mu$ has unbounded support, then we can fit a rectangle with arbitrarily large height and width inside $U_\mu$, and this implies that the rate of $U_\mu$ is 0, which is clearly the minimum possible by (2.2). On the other hand, let us suppose as above that $\mu$ has support in the finite interval $(\alpha, \beta)$. We claim that any $\mu$-domain $V_\mu$ satisfying $\mathbf{E}(\tau_{V_\mu}^{p/2}) = \infty$ must be contained in the strip $\{\alpha \le \Re(z) \le \beta\}$. To see this, note that the boundary of any such $V_\mu$ must be contained in $\{\alpha \le \Re(z) \le \beta\}$, and therefore if any part of $V_\mu$ were outside this strip then $V_\mu$ would necessarily contain a half-plane. This would contradict our assumption that $\mathbf{E}(\tau_{V_\mu}^{p/2}) < \infty$, since a half-plane has infinite 1/2-moment (see Section 6). This proves by monotonicity that the rate of $V_\mu$ is at least $\frac{\pi^2}{2(\beta-\alpha)^2}$. The same lower bound holds for the rate of $U_\mu$, but inside $U_\mu$ we can fit a rectangle with arbitrarily large height and width arbitrarily close to $\beta - \alpha$. This gives an upper bound of $\frac{\pi^2}{2(\beta-\alpha)^2}$ on the rate of $U_\mu$, and therefore completes the proof of Theorem 12. $\square$

We now give some examples.

**Example 13.** The uniform distribution on $(-1, 1)$.

We can check that $\varphi_\mu(\theta) = \frac{\theta}{\pi} - 1$ and has the Fourier series

$$\varphi_\mu(\theta) = -\frac{2}{\pi}\sum_{n=1}^{+\infty}\frac{\sin(n\theta)}{n}.$$

The power series is

$$\widetilde{\varphi}_\mu(z) = \frac{2i}{\pi}\sum_{n=1}^{+\infty}\frac{z^n}{n} = -\frac{2i}{\pi}\ln(1-z).$$

This function maps the unit disk to the catenary; see Fig. 4.1. This example is the subject of [14]. Note that $-\frac{2i}{\pi}\ln(1+z)$ produces the same distribution, however $-\frac{2i}{\pi}\ln(1-z)$ is the one which has a non decreasing real part on $(0, 2\pi)$.

**Example 14.** The scaled and centered arcsine law on $(-1, 1)$.

We get $\varphi_\mu(\theta) = -\cos(\frac{\theta}{2})$ and so the power series is

$$\widetilde{\varphi}_\mu(z) = -\frac{8i}{\pi}\sum_{n=1}^{+\infty}\tfrac{n}{1-4n^2}z^n = \tfrac{i}{\pi}\left\{\ln\left(\tfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)(\sqrt{z} + \tfrac{1}{\sqrt{z}}) - 2\right\}.$$

It is perhaps not so easy to deduce the image of the unit disk under this map; however, in the next section we present a method for finding the equation of the boundary curve in terms of the distribution. As we will show there, when applied to this law we obtain the domain in Fig. 4.2, limited by the curve $y = -\frac{2}{\pi}(x\ln(\cot(\frac{\arccos(-x)}{4})) + 1)$.

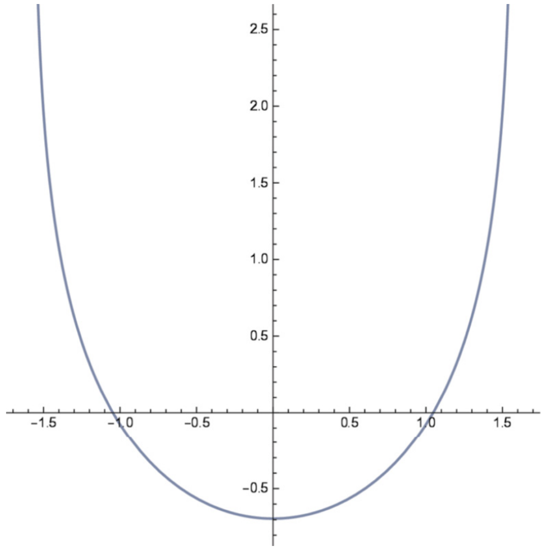


**Fig. 4.1.** The catenary, which is the $\mu$-domain for the uniform distribution.

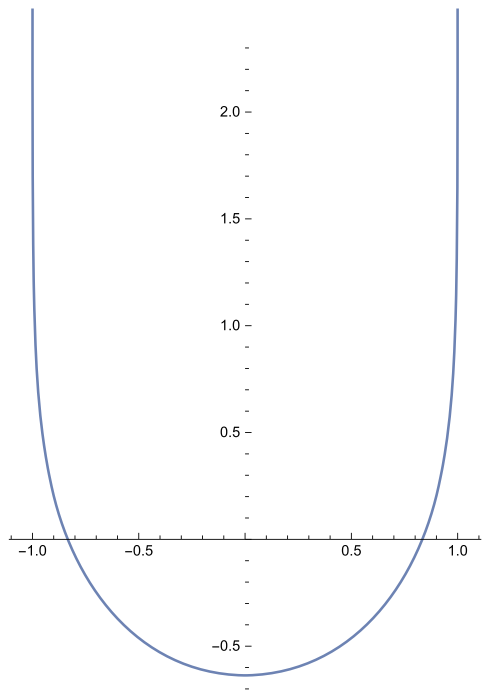


**Fig. 4.2.** The $\mu$-domain for the arcsine distribution. The extremal lower point of the $\mu$-domain is $-\frac{8i}{\pi}\sum_{n=1}^{+\infty}\frac{(-1)^n n}{1-4n^2} = -\frac{2i}{\pi} \approx -0.636i$.

**Example 15.** Consider the density $\frac{\operatorname{sech}(\frac{\sqrt{2}\pi}{2}x)}{\sqrt{2}}$. As is shown in [1], Gross' method applied to this distribution yields a horizontal infinite strip. The method given in Theorem 3 when applied to this distribution yields the function

$$f(z) = \frac{2i}{\pi^2}\Big(\ln\Big(\frac{1+\sqrt{z}}{1-\sqrt{z}}\Big)\Big)^2.$$

This conformal map sends the unit disc onto the parabola limited by the equation (Fig. 4.3)

$$2y = x^2 - 1. \tag{4.1}$$

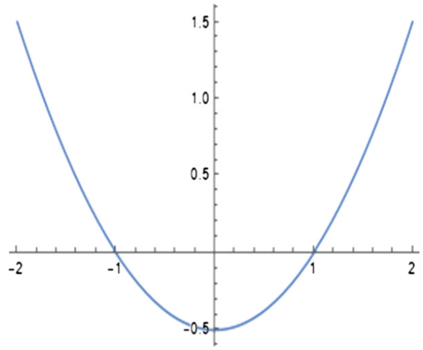


**Fig. 4.3.** Parabola of equation $x^2 = 2y + 1$.

**Example 16.** If the distribution $\mu$ is of the form

$$\mu = \sum_{n=1}^{m} x_n \delta_{x_n}$$

where $x_n > 0$ for all $n$ and $\sum_{n=1}^{m} x_n = 1$, then the $\mu$-domain generated by our method is the strip $\{x_1 < \Re(z) < x_m\}$ with the vertical slits $(\{x_n\} \times (-\infty, y_n])_{1<n<m}$ removed, where the $y_n$'s are some real numbers. In [11] some methods for calculating the values of $y_n$ in the context of Gross' method are presented, and they also work when applied to our method.

## 5. Equation of the boundary

In this section we prove that, in some situations, the boundary of the $\mu$-domain is the graph of a function. This often helps to reduce the computations required to determine the $\mu$-domain. As before, $F_\mu$ stands for the c.d.f. of $\mu$ and $G_\mu$ stands for its pseudo-inverse.

**Theorem 17.** *In our $\mu$-domain, every component of the boundary whose real part does not include the atoms of $\mu$ has the equation*

$$y = \gamma(x) = H\{\varphi_\mu\}(\varphi_\mu^{-1}(x)) = H\{F_\mu^{-1}\}(F_\mu(x)). \tag{5.1}$$

**Remark.** There cannot be boundary components outside the support of $\mu$, so $\gamma$ is only defined for $x$ in the support of $\mu$. Furthermore, $\gamma(x)$ is allowed to be $+\infty$ or $-\infty$ at boundary points of the support, corresponding to vertical asymptotes. A good example of this is Example 21 below, which arises because $\mu$ is a distribution with disjoint support.

**Proof.** Let $U$ be such a $\mu$-domain. Without loss of generality, we may assume $\mu$ with no atoms. Due to the $\Delta^\infty$-convexity of $U$ any vertical line crosses $\partial U$ in at most one point. Hence this boundary is the curve of a function $x \longmapsto \gamma(x)$. The uniqueness of $U$ guaranteed by Theorem 4 yields that $\Re(f(e^{\theta i}))$ is simply $\varphi_\mu(\theta) = G_\mu(\frac{\theta}{2\pi})$. The boundary of $U$ is parameterized by

$$\theta \mapsto (x, y) = (\varphi_\mu(\theta), H\{\varphi_\mu\}(\theta)).$$

Therefore to find $\gamma$ it is enough to express $y$ in terms of $x$. Since $\varphi_\mu$ is increasing then $\theta = \varphi_\mu^{-1}(x)$ and hence

$$\begin{aligned}\gamma(x) &= H\{\varphi_\mu\}(\varphi_\mu^{-1}(x)) \\ &\overset{\varphi_\mu^{-1}=2\pi F_\mu}{=} H\{\Phi_{\frac{1}{2\pi}}\{F_\mu^{-1}\}\}(2\pi F_\mu(x)) \\ &\overset{(2.4)}{=} H\{F_\mu^{-1}\}(F_\mu(x)). \quad \square\end{aligned}$$

The above theorem is also valid for $\mu$-domains obtained by Gross method where $y = \gamma(x)$ is the equation of the lower boundary. We will provide two concrete examples where (5.1) is applied, one for our method and one for Gross method.

**Remark.** The functional $\gamma$ inherits the smoothness of $G_\mu$. In particular, away from the atoms of $\mu$, $\gamma$ is continuous.

**Proposition 18.** *Let $\gamma(x)$ be a continuous and a.e. differentiable function defined over some (finite or infinite) interval $(a, b)$ such that $\gamma(0) < 0$. If*

$$\gamma(x) = H\{F^{-1}\}(F(x))$$

*for some continuous distribution function $F$ then the density of $Z_{\tau_U}$ at $z = x + yi$ is given a.e. by*

$$\rho(x + yi) = \frac{F'(x)}{\sqrt{1 + \gamma'(x)^2}}$$

*where $U$ is the domain above the graph of $\gamma$.*

**Proof.** The proof comes from the formula provided in [1]. $\square$

**Remark.** Suppose $\mu$ has no atoms and $U$ is the $\mu$-domain generated by Gross' method. If $\gamma(x)$ denotes the function determining the lower boundary of $U$, then the same argument shows

$$\rho(x \pm \mid y \mid i) = \frac{F'(x)}{2\sqrt{1 + \gamma'(x)^2}}.$$

**Example 19.** In [14, Thm. 3], the authors gave the following domain

$$\mathbb{U} := \{(x, y) \mid -1 < x < 1,\, y > -\tfrac{2}{\pi} \ln(2\cos(\tfrac{\pi x}{2}))\}$$

as an example of a $\mu$-domain where $\mu = \mathsf{Uni}(-1, 1)$. The $\mu$-domain $\mathbb{U}$ is $\Delta^\infty$-convex so it is unique by our Theorem 4. We show now that the function $\phi : x \mapsto -\frac{2}{\pi} \ln(2\cos(\frac{\pi x}{2}))$ can be deduced from Theorem 17 as expected. The uniform distribution c.d.f. and its pseudo-inverse function are given by

$$\begin{aligned}F_\mu(x) &= \tfrac{x+1}{2} \\ F_\mu^{-1}(u) &= 2u - 1.\end{aligned}$$

The (periodic) Hilbert transform of $F_\mu^{-1}$ is

$$H\{F_\mu^{-1}\}(x) = -\tfrac{2}{\pi} \ln(2\sin(\pi x)),$$

and therefore we get

$$\begin{aligned}\gamma(x) &= -\tfrac{2}{\pi}\ln(2\sin(\pi(\tfrac{x+1}{2}))) \\ &= -\tfrac{2}{\pi}\ln(2\cos(\tfrac{\pi x}{2})) \\ &= \phi(x)\end{aligned}$$

**Example 20.** We mention that the Gross $\mu$-domain generated by the centered and scaled arcsine distribution mentioned in Example 14 is simply the unit disc. That is, after performing the necessary computations and applying Theorem 17, we find the equation of the lower boundary

$$\begin{aligned}\gamma(x) &= -\sin(\arcsin(x) + \tfrac{\pi}{2}) \\ &= -\sqrt{1-x^2}.\end{aligned}$$

Consequently, the generated Gross $\mu$-domain is limited by the union of the graphs of $x \mapsto \pm\sqrt{1-x^2}$. This is the unit disc. The same technique shows that, for the same distribution, the boundary equation of our $\mu$-domain is

$$\gamma(x) = -\frac{2}{\pi}(x\ln(\cot(\tfrac{\arccos(-x)}{4})) + 1).$$

**Example 21.** Here we provide an example with disjoint support. Let $\mu$ be the uniform distribution on $(-2,-1) \cup (1,2)$. The c.d.f. of $\mu$ is given by

$$F_\mu(x) = \begin{cases} 0 & (x \le -2) \\ \frac{x+2}{2} & (-2 < x \le -1) \\ \frac{1}{2} & (-1 \le x < 1) \\ \frac{x}{2} & (1 \le x < 2) \\ 1 & (2 \le x) \end{cases}$$

The Fourier series of $\varphi_\mu(\theta)$ is

$$\varphi_\mu(\theta) = \sum_{n=1}^{\infty} 2\frac{((-1)^n - 2)}{n\pi}\sin(n\theta)$$

and therefore we get the conformal map

$$\widetilde{\varphi}_\mu(z) = -2i\sum_{n=1}^{\infty}\frac{((-1)^n - 2)}{n\pi}z^n = \frac{2i}{\pi}\ln(\frac{z+1}{(1-z)^2}).$$

We have

$$\Im(\widetilde{\varphi}_\mu(e^{\theta i})) = \frac{2}{\pi}\ln(\tfrac{\sqrt{2+2\cos(\theta)}}{4\sin(\frac{\theta}{2})^2})$$

whence the equation of the boundary curve on $(-2,-1)$ for example is

$$\gamma(x) = \frac{2}{\pi}\ln(\tfrac{\sqrt{2+2\cos(\pi x)}}{4\sin(\frac{\pi x}{2})^2}).$$

By symmetry, we get the part over $(1,2)$. An image of the $\mu$-domain obtained is presented in Fig. 5.1.

By contrast, the $\mu$-domain generated by Gross' method is illustrated in Fig. 5.2.

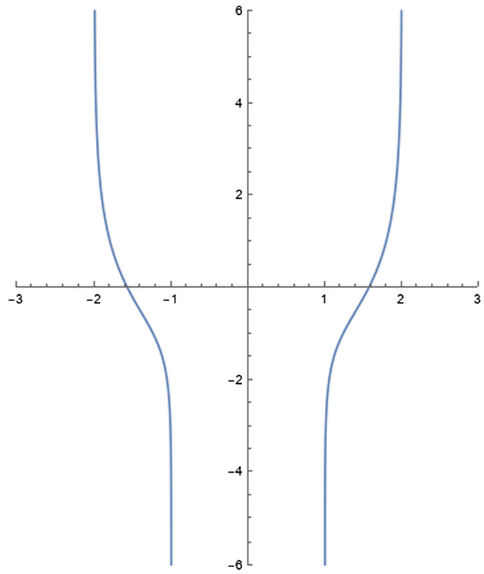


**Fig. 5.1.** The case of disjoint support via our method.

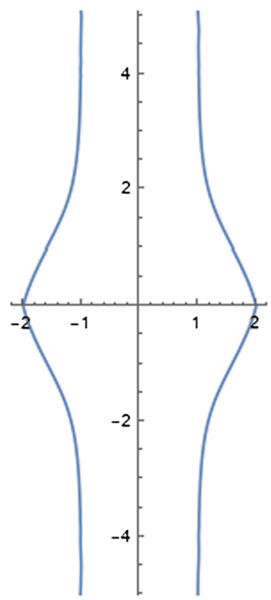


**Fig. 5.2.** The case of disjoint support via Gross' method.

## 6. A surprising pseudo-example and possible extension

The density of the standard Cauchy distribution $\mu$ is given by

$$\varrho_\mu(x) = \frac{1}{\pi(1+x^2)}$$

and its quantile function is $G_\mu(u) = -\cot(\pi u)$ for all $u \in (0,1)$. This distribution does not have a mean as it is not integrable. Therefore, we can not apply the same techniques as before to generate a corresponding $\mu$-domain. However, let us simply ignore this issue and apply the method formally. It can be shown that $\Re(Z_{\tau_{\mathscr{U}}})$ has the density $\varrho_\mu$ where $\mathscr{U}$ is the upper half plane limited by $\{z \mid \Im(z) = -1\}$; a recent proof of this using the optional stopping theorem appears in [4], but it can also be deduced by a direct calculation, using the Poisson kernel, or by properties of stable distributions; see for instance [7, Sec. 1.9] or [9, Ch. VI.2]. We can check that

$$2i\tfrac{e^{\theta i}}{1-e^{\theta i}} = -\cot(\tfrac{\theta}{2}) - i = G_\mu(\tfrac{\theta}{2\pi}) - i. \tag{6.1}$$

Note that the Fourier coefficients technically do not exist here, because the function is not in $L^1$ (however, the sine integrals against $-\cot(\frac{\theta}{2})$ do converge, and if we set the cosine terms all to 0 by the oddness of $-\cot(\frac{\theta}{2})$ then we obtain (6.1)). Nevertheless, we have

$$\vartheta(z) = 2i\sum_{n=1}^{\infty} z^n = 2i\frac{z}{1-z}.$$

This is the Möbius transformation taking the disk to the half-plane $\{z \mid \Im(z) > -1\}$, so that the correct conclusion does hold in this case. It is interesting to note that here also $\mathscr{U}$ satisfies $E[(\tau_{\mathscr{U}})^1/2] = \infty$; this can be seen by realizing $\tau_{\mathscr{U}}$ as the time of $-1$ by the one-dimensional Brownian motion $\Im(Z_t)$, and using standard results on hitting times (see for instance [13]).

As an anonymous referee pointed out, this example suggests that the important requirement for Theorem 3 to hold is that the Fourier series for $G$ converges a.e. to $G$. Our condition that $\mu$ has a finite $p$-th moment for some $p > 1$ is sufficient, but not necessary, for this to hold. It appears likely that our results would admit a significant generalization along these lines, but we refrain from making a precise statement as it would be a bit removed from our main aims in this paper. We thank the referee for this observation.

It is also interesting to note that the type of formal calculations given in this section do not seem to apply to Gross' method.

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