One-point recursions and topological recursion for enumerative problems

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Abstract

This thesis primarily studies one-point invariants and topological recursion for enumerative problems. To make the document reasonably self-contained, we include the background material required to understand our results in the opening chapters. The thesis is the culmination of several research projects, some completed and some ongoing. Each of the following paragraphs provides a brief description of the results obtained from one of these projects.

Harer and Zagier proved a recursion to enumerate gluings of a $2d$-gon that result in an orientable genus g surface, in their work on Euler characteristics of moduli spaces of curves. Analogous results have been discovered for other enumerative problems, so it is natural to pose the following question: How large is the family of problems for which these so-called 1-point recursions exist? We prove the existence of 1-point recursions for a class of enumerative problems that have Schur function expansions. In particular, we recover the Harer–Zagier recursion, but our methodology also applies to the enumeration of dessins d'enfant, Bousquet-Mélou–Schaeffer numbers, monotone Hurwitz numbers, and more.

Do and Norbury initiated the enumeration of lattice points in the Deligne–Mumford compactifications of moduli spaces of curves. They showed that the enumeration may be expressed in terms of polynomials, whose top and bottom degree coefficients store psi-class intersection numbers and orbifold Euler characteristics of $\overline{\mathcal{M}}_{g,n}$, respectively. Furthermore, they ask whether the enumeration is governed by the topological recursion and whether the intermediate coefficients also store algebro-geometric information. We prove that the enumeration does indeed satisfy the topological recursion, although with a modification to the initial spectral curve data. Thus, one can consider this to be one of few known instances of a natural enumerative problem governed by the local topological recursion.

Gromov–Witten theory deals with the enumeration of maps from complex algebraic curves into a complex variety. This theory was motivated by theoretical physics, acting as a mathematical interpretation for certain models of string theory. In the case that the target variety is a non-singular curve, Okounkov and Pandharipande relate Gromov–Witten invariants to classical Hurwitz numbers, giving an explicit way to compute them. We introduce a conjecture that states that certain relative Gromov–Witten invariants of $\mathbb{CP}^1$ are governed by the topological recursion. This conjecture can be seen as a vast generalisation of the Bouchard–Mariño conjecture relating simple Hurwitz numbers with topological recursion. We use the Gromov–Witten/Hurwitz correspondence to deduce a quantum curve for the enumerative problem under consideration. This can be considered strong evidence towards the conjecture.

We conclude the thesis with some results on monotone Hurwitz numbers. First, we give some initial calculations showing how the Kontsevich–Soibelman topological recursion formalism can be used to derive Virasoro constraints for monotone Hurwitz numbers. Such calculations may lead to a better understanding of the relation between the Virasoro algebra and topological recursion more generally. We also prove an identity concerning monotone Hurwitz numbers, which first appeared in the recent work of Cunden, Dahlqvist and O'Connell. Our proof has a very different flavour to their matrix model analysis and instead uses holonomic tools.

Declaration

This thesis is an original work of my research and contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

Anupam Chaudhuri

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Statement of contribution

To make the thesis reasonably self-contained, we include the background material required to understand our results and this appears in Chapters 1 to 3. Among the results in these chapters, only Proposition 3.4.1 and Theorem 3.4.2 are novel and provide an example of deriving quantum curves for enumerative problems.

Chapter 4 is joint work with Norman Do and describes the results contained in our preprint *Generalisations of the Harer–Zagier recursion for 1-point functions* [22]. However, most of Section 4.6 on degree and order bounds for one-point recursions is my own new work and does not appear in the preprint.

Chapter 5 is joint work with Norman Do and Ellena Moskovsky and describes the results contained in our preprint *Local topological recursion governs the enumeration of lattice points in* $\overline{\mathcal{M}}_{g,n}$ [8]. However, Section 5.3 on the asymmetric combinatorial recursion for the enumeration of stable fatgraphs is my own new work and does not appear in the preprint.

Chapter 6 describes work in progress towards the topological recursion for Gromov–Witten invariants of $\mathbb{CP}^1$. We state a new explicit conjecture relating Gromov–Witten invariants of $\mathbb{CP}^1$ to topological recursion, which vastly generalises the Bouchard–Mariño conjecture. The main result of the chapter is Theorem 6.3.2, in which we derive the quantum curve for the enumerative problem. The work contained in this chapter does not yet appear elsewhere in the literature.

Chapter 7 describes some initial calculations towards understanding the relation between Kontsevich–Soibelman topological recursion and Virasoro constraints for enumerative problems, as well as an alternative proof of the Cunden–Dahlqvist–O'Connell identity for monotone Hurwitz numbers via the language of holonomic functions. The work contained in this chapter does not yet appear elsewhere in the literature.

Acknowledgements

First and foremost, I would like to thank my principal supervisor, Norman Do. Under his guidance, I have learned to write mathematics, communicate mathematics and most importantly ask questions in mathematics. Over the years, being in his company has developed my organisational skills and enhanced my perseverance to tackle a problem. I would reiterate my thanks to him for his patience and for tolerating my recursive mistakes.

I would like to express my gratitude to my co-supervisor, Daniel Mathews, as well as my panel members, Heiko Dietrich, Andrew Hammerlindl, Jessica Purcell and Ian Wanless. Without them, my progress would have been slower and perspective narrower.

The topology seminars over the years have been not only fruitful mathematically, but also provided an opportunity to bring peers together. I wouldlike to take this opportunity to thank all of the participants of the seminar for sharing mathematics and for the countless lunches we had together.

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It goes without saying that I would like to thank my parents, who have wholeheartedly and unconditionally supported me in all my endeavours throughout my life. I truly believe that without their unwavering support, I would not be here today.

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Contents

Chapter 1

Problems in enumerative combinatorics

This thesis primarily deals with structures underlying enumerative problems that lie at or near the interface of the combinatorics of permutations and the geometry of surfaces. In this chapter, we discuss various problems in enumerative combinatorics that illuminate these structures, such as the calculation of Bousquet-Mélou–Schaeffer numbers, Hurwitz numbers, monotone Hurwitz numbers and fatgraph counts (Sections 1.1 to 1.4). These problems motivate many of the results in this thesis, particularly those contained in Chapter 4 on one-point recursions. For each case, we define the enumeration in both connected and disconnected forms and relate the latter to a character formula. Such character formulas may be used to derive so-called quantum curves for these enumerations, thus relating them to topological recursion. One of the goals of these opening chapters to the thesis is to motivate the study of topological recursion by enumerative problems. We end this chapter by discussing weighted Hurwitz numbers, which capture a large class of enumerative problems including those discussed above.

1.1 Bousquet-Mélou–Schaeffer numbers

Let m be a positive integer and let σ_0 be a permutation in the symmetric group S_d with cycle type $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$. We want to count the number of ways σ_0 can be written as a product of m permutations. However, we add certain restrictions on these permutations that encode the fact that we are counting degree d maps from a Riemann surface of particular genus to $\mathbb{CP}^1$ with $m+1$ ramification points.

Definition 1.1.1. The *Bousquet-Mélou–Schaeffer number* $B^{(m)}_{g,n}(\mu_1, \mu_2, \ldots, \mu_n)$ is $\frac{1}{d!}$ multiplied by the number of tuples $(\rho_1, \rho_2, \ldots, \rho_m)$ of permutations in S_d such that

- the product $\rho_1 \circ \rho_2 \circ \cdots \circ \rho_m$ has n labelled cycles with lengths $\mu_1, \mu_2, \ldots, \mu_n$;
- $\sum_{i=1}^{m}(d - \ell(\rho_i)) = 2g - 2 + n + d$, where $\ell(\rho_i)$ denotes the number of cycles in ρ_i; and
- the group $\langle \rho_1, \rho_2, \ldots, \rho_m \rangle$ generated by the permutations is a transitive subgroup of S_d.

For brevity, we shall often refer to these as *BMS numbers*.

Example 1.1.2. Let us compute the number $B^{(2)}_{1,1}(3)$. So we count tuples (ρ_1, ρ_2) of permutations in the group S_3 such that $\rho_1 \circ \rho_2$ is a 3-cycle and

$$(3 - \ell(\rho_1)) + (3 - \ell(\rho_2)) = 2 \times 1 - 2 + 1 + 3 \quad \Rightarrow \quad \ell(\rho_1) + \ell(\rho_2) = 2.$$

This equation implies that we have $\ell(\rho_1) = \ell(\rho_2) = 1$. The only possibilities are

$$(\rho_1, \rho_2) = ((1\ 2\ 3), (1\ 2\ 3)) \qquad \text{and} \qquad (\rho_1, \rho_2) = ((1\ 3\ 2), (1\ 3\ 2)).$$

Hence, $B^2_{1,1}(3) = \frac{2}{3!} = \frac{1}{3}$.

By the Riemann existence theorem, one can equivalently consider $B^{(m)}_{g,n}(\mu_1, \ldots, \mu_n)$ to be the weighted count of connected genus g branched covers $f : (\Sigma; p_1, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ such that

- ramification can only occur over ∞ or the mth roots of unity; and
- $f^{-1}(\infty) = \mu_1 p_1 + \cdots + \mu_n p_n$.

Here, the weight of a branched cover is given by $\frac{1}{|\mathrm{Aut}(f)|}$, where $\mathrm{Aut}(f)$ is the group of automorphisms of the branched cover.

An automorphism of a branched cover $f : \Sigma \to \mathbb{CP}^1$ is an automorphism $g : \Sigma \to \Sigma$ such that $f \circ g = f$. Note that the first condition in Definition 1.1.1 corresponds to having ramification profile μ over ∞, the second encodes the genus of the branched cover via the Riemann–Hurwitz formula, and the third guarantees that the branched cover is connected.

We note that there is an alternative pictorial description of BMS numbers via the notion of an m-constellation. Let us recall that a *map* is a 2-cell decomposition of an oriented surface into vertices, edges and faces. The degree of a vertex (or a face) is the number of edges incident to this vertex (or face), counted with multiplicity. Two maps are *isomorphic* if there exists an orientation-preserving homeomorphism of the underlying surfaces that bijectively maps cells of one map to cells of the other, while preserving dimensions of cells and incidences. We shall always consider maps up to isomorphism.

Definition 1.1.3. For an integer $m \geq 2$, a *m-constellation* is a map whose faces are coloured black and white in such a way that

- any face adjacent to a white face is black, and vice versa,
- the degree of any black face is m,
- the degree of any white face is a multiple of m.

The black faces of a constellation are called polygons or *m-gons*. There exists a bijective correspondence between the tuples of permutations in Definition 1.1.1 and m-constellations. Counting planar constellations is carried out in genus zero in [20].

Example 1.1.4. Let us compute the number $B^{(2)}_{1,1}(3)$ from Example 1.1.2 again. In Figure 1.1 below, the genus 1 surface is obtained by gluing together edges carrying the same number of arrows. We label the red vertex 1 and the blue vertex 2. This is the only constellation that contributes to the enumeration and it has three automorphisms given by rotations of multiples of 120° around the central vertex. Therefore, we again see that $B^{(2)}_{1,1}(3) = \frac{1}{3}$.

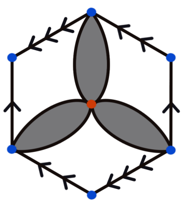

Figure 1.1: The unique 2-constellation contributing to $B^2_{1,1}(3)$.

Definition 1.1.5. Define the *double Bousquet-Mélou–Schaeffer number* $\overline{B}^{(m)}_{g,n}(\mu_1, \ldots, \mu_n)$ to be the weighted count of genus g connected branched covers $f : (\Sigma; p_1, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ such that

- ramification can only occur over 0, ∞ or the mth roots of unity; and
- $f^{-1}(\infty) = \mu_1 p_1 + \cdots + \mu_n p_n$.

Let $q_1, q_2, q_3, \ldots$ be commuting variables and set the weight of a branched cover with ramification profile $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ over 0 to be $\frac{q_{\lambda_1} q_{\lambda_2} \cdots q_{\lambda_\ell}}{|\mathrm{Aut}(f)|}$.

One can get the BMS-number $B^{(m)}_{g,n}(\mu_1, \ldots, \mu_n)$ by choosing the weights of $q_i = 1$.

One can remove the transitivity condition of Definition 1.1.1 to produce the notion of disconnected Bousquet-Mélou–Schaeffer numbers, which we denote by $\overline{B}^{(m)\bullet}_{g,n}(\mu_1, \ldots, \mu_n)$. To define these numbers via branched covers, we introduce the concept of the genus of a disconnected surface. Define the genus of an oriented surface $\Sigma = \sqcup_{i=1}^n \Sigma_i$ with n connected components to be $\sum_{i=1}^n g_i - (n-1)$, where g_i is the usual genus of Σ_i.

Definition 1.1.6. Define $\overline{B}^{(m)\bullet}_{g,n}(\mu_1, \ldots, \mu_n)$ to be the weighted count of possibly disconnected genus g branched covers $f : (\Sigma; p_1, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ such that

- ramification can only occur over $0, \infty$ or the mth roots of unity; and
- $f^{-1}(\infty) = \mu_1 p_1 + \cdots + \mu_n p_n$.

As before, set the weight of a branched cover with ramification profile $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ over 0 to be $\frac{q_{\lambda_1} q_{\lambda_2} \cdots q_{\lambda_\ell}}{|\mathrm{Aut}(f)|}$. Also, observe that the genus can be negative here.

The reason we introduce this disconnected version is that the enumeration is in some sense easier and, in particular, can be expressed using characters of symmetric groups. The connected and disconnected counts are in some sense equivalent, since they can be obtained from each other by inclusion-exclusion. For example, one can express $B^{(m)}_{1,2}(\mu_1, \mu_2)$ in terms of disconnected BMS numbers in the following way.

$$\overline{B}^{(m)}_{1,2}(\mu_1, \mu_2) = \overline{B}^{(m)\bullet}_{1,2}(\mu_1, \mu_2) - \overline{B}^{(m)\bullet}_{1,1}(\mu_1)\overline{B}^{(m)\bullet}_{1,1}(\mu_2) - \overline{B}^{(m)\bullet}_{2,1}(\mu_1)\overline{B}^{(m)\bullet}_{0,1}(\mu_2) - \overline{B}^{(m)\bullet}_{2,1}(\mu_2)\overline{B}^{(m)\bullet}_{0,1}(\mu_1)$$

In the next proposition, we express disconnected BMS numbers using the representation theory of symmetric groups. Calculations such as this form the model for deriving character formulas for other enumerative problems. We assume that the reader is familiar with this material, but refer the reader to the literature for more information [92].

Presently, we set some notation that will be used throughout the thesis. The conjugacy classes of symmetric groups naturally correspond to partitions, which describe the cycle type of permutations in the conjugacy class. Note that the irreducible representations of symmetric groups also naturally correspond to partitions. A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a weakly decreasing sequence of non-negative integers such that its length $\ell(\lambda) := |\{i \geq 1 \mid \lambda_i > 0\}|$ is finite. We often identify λ with its *Young diagram*

$$Y(\lambda) := \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}.$$

If $\Box = (i, j) \in Y(\lambda)$, we say that $\Box$ is a box of λ and write $\Box \in \lambda$ for short. The *content* of $\Box = (i, j) \in Y(\lambda)$ is defined to be $c(\Box) := j - i$. If λ is a partition of d, then we denote this by $\lambda \vdash d$.

Proposition 1.1.7. *Let $P(d)$ denote the set of partitions of d. Let χ^ν_μ denote the character of the symmetric group S_d labelled by $\nu \in P(d)$ evaluated on a permutation of cycle type $\mu \in P(d)$. Let $c(\Box)$ denote the content of the box $\Box$ in a Young diagram and use the notation $[\hbar^k]F(\hbar)$ to denote the coefficient of $\hbar^k$ in the series expansion for $F(\hbar)$. Then for $d = \sum_i \mu_i$, we have*

$$B^{(m)\bullet}_{g,n}(\mu_1, \ldots, \mu_n) = \frac{1}{d! \prod \mu_i}[\hbar^{2g-2+n+d}] \sum_{\nu \in P(d)} \chi^\nu_\mu \dim(\nu) \prod_{\Box \in \nu} (1 + c(\Box)\hbar)^m.$$

Proof. The proof is based on the interplay between two important bases for the centre of the symmetric group algebra $Z\mathbb{C}[S_d]$. The first is the *conjugacy class basis* denoted by $C_\alpha = \sum \sigma$, where the summation is over σ of cycle type α. The second is the *orthogonal idempotent basis* denoted by $\mathcal{E}_\chi = \frac{\chi(1)}{d!} \sum_{g \in S_d} \chi(g) g^{-1}$, where the summation is over χ an irreducible character of S_d.

By definition, the BMS numbers arise as the identity coefficient of a product of conjugacy classes, as follows, where $\mathrm{Aut}(\mu)$ denotes the number of permutations of the tuple μ that leave it invariant.

$$B^{(m)\bullet}_{g,n}(\mu_1, \ldots, \mu_n) = \frac{\mathrm{Aut}(\mu)}{d!} [\mathrm{Id}] \sum_{\substack{\lambda^1, \lambda^2, \ldots, \lambda^m \in P(d) \\ \sum_{i=1}^m (d - \ell(\lambda^i)) = 2g-2+n+d}} C_{\lambda^1} C_{\lambda^2} \cdots C_{\lambda^m} C_\mu$$

Now we change from the conjugacy class basis to the orthogonal idempotent basis via the equation $C_\alpha = \sum_\chi \frac{\chi(C_\alpha)}{\chi(1)} \mathcal{E}_\chi$. Using the fact that $\mathcal{E}_{\chi^1}\mathcal{E}_{\chi^2} = \delta_{\chi^1,\chi^2}\mathcal{E}_{\chi^1}$ and $[\mathrm{Id}]\mathcal{E}_\chi = \frac{\dim(\chi)^2}{d!}$, we have the following.

$$B^{(m)\bullet}_{g,n}(\mu_1, \ldots, \mu_n) = \frac{\mathrm{Aut}(\mu)}{d!} \sum_{\substack{\lambda^1, \lambda^2, \ldots, \lambda^m \in P(d) \\ \sum_{i=1}^m (d - \ell(\lambda^i)) = 2g-2+n+d}} \sum_{\nu \in P(d)} \prod_{i=1}^m \frac{|C_{\lambda^i}|}{\dim(\nu)} \chi^\nu_{\lambda^i} \times \frac{|C_\mu|}{\dim(\nu)} \chi^\nu_\mu \frac{\dim(\nu)^2}{d!}$$

Now interchanging the order of summation, we have the following equation.

$$B^{(m)\bullet}_{g,n}(\mu_1, \ldots, \mu_n) = \frac{\mathrm{Aut}(\mu)}{(d!)^2} |C_\mu| \sum_{\nu \in P(d)} \frac{\chi^\nu_\mu}{\dim(\nu)^{m-1}} \sum_{\substack{\lambda^1, \lambda^2, \ldots, \lambda^m \in P(d) \\ \sum_{i=1}^m (d - \ell(\lambda^i)) = 2g-2+n+d}} \prod_{i=1}^m |C_{\lambda^i}| \chi^\nu_{\lambda^i}$$

Let us write $\ell(\lambda^i) = k_i$ and let $P(d,k)$ denote the set of partitions of d into k parts. We use the fact that $|C_\mu| = \frac{d!}{\mathrm{Aut}(\mu) \prod_{i=1}^n \mu_i}$ to write the above equation as follows.

$$B^{(m)\bullet}_{g,n}(\mu_1, \ldots, \mu_n) = \frac{1}{d! \prod_{i=1}^n \mu_i} \sum_{\nu \in P(d)} \frac{\chi^\nu_\mu}{\dim(\nu)^{m-1}} \sum_{\substack{k_1, k_2, \ldots, k_m \\ \sum_{i=1}^m (d - k_i) = 2g-2+n+d}} \sum_{\substack{\lambda^1 \in P(d,k_1) \\ \vdots \\ \lambda^m \in P(d,k_m)}} \prod_{i=1}^m |C_{\lambda^i}| \chi^\nu_{\lambda^i}$$

Now interchange the product and sum appearing in the above equation.

$$B^{(m)\bullet}_{g,n}(\mu_1, \ldots, \mu_n) = \frac{1}{d! \prod_{i=1}^n \mu_i} \sum_{\nu \in P(d)} \frac{\chi^\nu_\mu}{\dim(\nu)^{m-1}} \sum_{\substack{k_1, k_2, \ldots, k_m \\ \sum_{i=1}^m (d - k_i) = 2g-2+n+d}} \prod_{i=1}^m \sum_{\lambda^i \in P(d,k_i)} |C_{\lambda^i}| \chi^\nu_{\lambda^i}$$

We proceed to use the following fact without proof and refer the interested reader to [98].

$$\sum_{\lambda \in P(d,k)} |C_\lambda| \chi^\nu_\lambda = [\hbar^{d-k}] \dim(\nu) \prod_{\Box \in \nu} (1 + c(\Box)\hbar)$$

Using this fact, we obtain the following.

$$\begin{aligned} B_{g,n}^{(m)\bullet}(\mu_1,\ldots,\mu_n) &= \frac{1}{d!\prod_{i=1}^n \mu_i} \sum_{\nu\in P(d)} \chi_\mu^\nu \dim(\nu) \sum_{\substack{k_1,k_2,\ldots,k_m \\ \sum_{i=1}^m (d-k_i)=2g-2+n+d}} \prod_{i=1}^m [\hbar^{d-k_i}] \prod_{\Box\in\nu}(1+c(\Box)\hbar) \\ &= \frac{1}{d!\prod_{i=1}^n \mu_i}[\hbar^{2g-2+n+d}] \sum_{\nu\in P(d)} \chi_\mu^\nu \dim(\nu) \prod_{\Box\in\nu}(1+c(\Box)\hbar)^m \end{aligned}$$

This concludes the proof. □

1.2 Hurwitz numbers

In this section, we study Hurwitz numbers from a combinatorial viewpoint. We provide some motivation behind considering this enumeration and use monodromy to express them via the representation theory of symmetric groups.

Definition 1.2.1. The *simple Hurwitz number* $H_{g,n}(\mu_1,\mu_2\ldots,\mu_n)$ is the weighted count of genus g connected branched covers $f:(\Sigma;p_1,\ldots,p_n)\to(\mathbb{CP}^1;\infty)$ such that

- $f^{-1}(\infty)=\mu_1p_1+\cdots+\mu_np_n$; and
- the only other ramification is simple and occurs at the mth roots of unity.

The weight of a branched cover f is $\frac{1}{m!|\operatorname{Aut}(f)|}$, where $d=\sum\mu_i$ and we have $m=2g-2+n+d$ from the Riemann–Hurwitz formula.

Given a branched cover $f:\Sigma\to\mathbb{CP}^1$ as in the definition above, let B denote the set of branch points in $\mathbb{CP}^1$ and pick a point y_0 that is not a branch point. We label its preimages $y_1,y_2,\ldots,y_d$ in some way. Then by the general theory of covering spaces, one can naturally define a group homomorphism

$$\phi_f:\pi_1(\mathbb{CP}^1\setminus B,y_0)\to S_d.$$

For each $i=1,2,\ldots,d$, the permutation $\phi_f(\gamma)$ sends i to j, where $\tilde\gamma_i(1)=y_j$ and $\tilde\gamma_i$ is the lift of γ satisfying $\gamma_i(0)=y_i$. This homomorphism is called the *monodromy representation.* A different choice of labelling of the preimage of y_0 corresponds to composing ϕ_f with an inner automorphism of S_d. If $\gamma\in\pi_1(\mathbb{CP}^1\setminus B,y_0)$ is a simple loop winding once around the branch point with ramification profile η, then $\phi_f(\gamma)$ is a permutation of cycle type η.

Given a monodromy representation, the Riemann existence theorem guarantees that we have enough information to recover the branched cover. This lead to the following alternative definition of simple Hurwitz numbers.

Proposition 1.2.2. *The simple Hurwitz number $H_{g,n}(\mu_1,\mu_2,\ldots,\mu_n)$ is $\frac{1}{m!\,d!}$ multiplied by the number of tuples $(\rho_1,\rho_2,\ldots,\rho_m)$ of transpositions in S_d such that*

- *the product $\rho_1\circ\rho_2\circ\cdots\circ\rho_m$ has n labelled cycles with lengths $\mu_1,\mu_2,\ldots\mu_n$;*
- *$m=2g-2+n+d$; and*
- *the group $\langle\rho_1,\rho_2,\ldots,\rho_m\rangle$ generated by the transpositions is a transitive subgroup of S_d.*

If we relax the condition of transitivity in Proposition 1.2.2, we obtain disconnected Hurwitz numbers, which we denote by $H^\bullet_{g,n}(\mu_1,\mu_2\ldots,\mu_n)$. As discussed in Section 1.1, we can express the disconnected Hurwitz numbers using characters of the symmetric group using the following approach.

For many of the enumerative problems we are interested in, we would like to calculate the coefficient of the identity in the product $C_\mu C_\nu B$ for some $B \in \mathbb{C}[S_n]$. In the case of simple Hurwitz numbers, we take $\nu = (1^d)$ and $B = C^m_{(2,1^{d-2})}$. The action of $B \in Z(\mathbb{C}[S_n])$ in the irreducible representation λ is given by multiplication by some eigenvalue $\mathrm{egv}_\lambda(B)$. For the conjugacy class C_α, we have

$$\mathrm{egv}_\lambda(C_\alpha) := \frac{|C_\alpha|\,\chi^\lambda_\alpha}{\dim(\lambda)}.$$

So define $f_2(\lambda) := \frac{|C_{(2,1^{d-2})}|\,\chi^\lambda_{(2,1^{d-2})}}{\dim(\lambda)}$ and let $z(\mu)$ denote the number of elements in the centraliser of the conjugacy class μ. Then we have the following proposition resulting from the above discussion.

Proposition 1.2.3.

$$H^\bullet_{g,n}(\mu_1, \mu_2 \ldots, \mu_n) = \frac{1}{d!\,z(\mu)} \sum_{\lambda \vdash d} \chi^\lambda_\mu \dim(\lambda)\, f_2(\lambda)^m$$

Notice that in the definition of simple Hurwitz numbers, we keep the ramification profile over the point 0 to be $(1, 1, \ldots, 1)$. We can more generally allow arbitrary ramification over 0 and keep track of this ramification with appropriate weights, as in the following definitnion. We call the resulting objects double Hurwitz numbers, using the terminology of Do and Karev [28]. The following definition is a bit different to the definition of double Hurwitz number appearing elsewhere in the literature.

Definition 1.2.4. The *double Hurwitz number* $\overline{H}_{g,n}(d_1, d_2, \ldots, d_n)$ is the weighted count of genus g connected branched covers $f : (\Sigma; p_1, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ such that

- $f^{-1}(\infty) = \mu_1 p_1 + \cdots + \mu_n p_n$;
- the ramification profile over 0 is arbitrary; and
- the only other ramification is simple and occurs at the mth roots of unity.

The weight of a branched cover f with ramification profile $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ over 0 is $\frac{q_{\lambda_1} q_{\lambda_2} \cdots q_{\lambda_\ell}}{m!\,|\mathrm{Aut}(f)|}$, where $q_1, q_2, \ldots$ are commuting variables.

1.3 Monotone Hurwitz numbers

Monotone Hurwitz numbers first appeared in a series of papers by Goulden, Guay-Paquet and Novak, in which they arose as coefficients in the large N asymptotic expansions of the Harish-Chandra–Itzykson–Zuber matrix integral over the unitary group $U(N)$ [54, 55, 56]. Their definition resembles that of Hurwitz numbers, but with a monotonicity constraint imposed on the transpositions. The monotonicity condition is rather natural from the standpoint of the Jucys–Murphy elements in the symmetric group algebra $\mathbb{C}[S_d]$.

Definition 1.3.1. The *monotone Hurwitz number* $M_{g,n}(\mu_1, \mu_2 \ldots, \mu_n)$ is $\frac{1}{d!}$ multiplied by the number of tuples $(\tau_1, \tau_2, \ldots, \tau_m)$ of transpositions in S_d such that

- the product $\tau_1 \circ \tau_2 \circ \cdots \circ \tau_m$ has n labelled cycles with lengths $\mu_1, \mu_2, \ldots, \mu_n$;
- $m = 2g - 2 + n + d$;
- the group $\langle \tau_1, \tau_2, \ldots, \tau_m \rangle$ generated by the transpositions is a transitive subgroup of S_d; and
- if $\tau_i = (a_i, b_i)$ with $a_i < b_i$, then $b_1 \le b_2 \le \cdots \le b_m$.

Example 1.3.2. To calculate $M_{0,2}(2,1)$, we first deduce that $m = 3$. Hence, we count all triples of transpositions in S_3 such that the product is a permutation of cycle type $(2,1)$ satisfying the monotonicity condition. There are 27 triples of transpositions, but only 12 of them are monotone.

$$\begin{array}{llll}
(1\ 2)\circ(1\ 2)\circ(1\ 3) & (1\ 2)\circ(1\ 3)\circ(2\ 3) & (1\ 3)\circ(1\ 3)\circ(2\ 3) & (23)\circ(1\ 3)\circ(1\ 3) \\
(1\ 2)\circ(1\ 2)\circ(2\ 3) & (1\ 2)\circ(2\ 3)\circ(1\ 3) & (1\ 3)\circ(2\ 3)\circ(1\ 3) & (23)\circ(1\ 3)\circ(2\ 3) \\
(1\ 2)\circ(1\ 3)\circ(2\ 3) & (1\ 2)\circ(2\ 3)\circ(2\ 3) & (1\ 3)\circ(2\ 3)\circ(2\ 3) & (23)\circ(2\ 3)\circ(1\ 3)
\end{array}$$

So $M_{0,2}(2,1) = \frac{12}{3!} = 2$.

If we relax the condition of transitivity in Definition 1.3.1, we obtain disconnected monotone Hurwitz numbers, which we denote by $M^{\bullet}_{g,n}(\mu_1,\mu_2\dots,\mu_n)$. As discussed in Section 1.2, we can compute these using characters of the symmetric group.

We introduce the Jucys–Murphy elements $\mathcal{J}_k \in \mathbb{C}[S_d]$ for $k = 1,2,\dots,d$, defined as

$$\mathcal{J}_k := (1\ k) + (2\ k) + \cdots + (k-1\ k) \tag{1.1}$$

Jucys studied these elements and showed that they commute [66]. It follows that symmetric polynomials in $\mathcal{J}_1,\mathcal{J}_2,\dots,\mathcal{J}_d$ are central elements in $\mathbb{C}[S_d]$. For a given symmetric function F, it is natural to ask for the expression of $F(\mathcal{J}_1,\mathcal{J}_{,2}\dots,\mathcal{J}_d)$ in terms of the conjugacy class basis of $Z(\mathbb{C}[S_d])$. This would lead us to the character formula for monotone Hurwitz numbers.

Let us introduce the basic families of symmetric functions, which share a close relationship with the representation theory of symmetric groups. We let $x = (x_1,x_2,\dots)$ be an infinite sequence of commuting variables and consider the algebra of symmetric functions with complex coefficients in these variables. Given a partition $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_\ell)$, the monomial symmetric function is defined by

$$m_\lambda(x) = \sum_{\alpha\sim\lambda} x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\cdots,$$

where the summation is over all infinite sequences α of non-negative integers whose non-zero entries form a permutation of λ.

Denote by e_k, h_k, p_k the elementary, complete homogeneous and power-sum symmetric functions, respectively. Namely, for k a positive integer, we define

$$\begin{aligned}
e_k(x) &= m_{(1^k)}(x) = \sum_{i_1<i_2<\cdots<i_k} x_{i_1}x_{i_2}\cdots x_{i_k},\\
h_k(x) &= \sum_{\lambda\vdash k} m_\lambda(x) = \sum_{i_1\le i_2\le\cdots\le i_k} x_{i_1}x_{i_2}\cdots x_{i_k},\\
p_k(x) &= m_{(k)}(x) = x_1^k + x_2^k + x_3^k + \cdots .
\end{aligned}$$

We also set $e_\lambda(x) = \prod_{i=1}^{\ell(\lambda)} e_i(x)$, $h_\lambda(x) = \prod_{i=1}^{\ell(\lambda)} h_i(x)$ and $p_\lambda(x) = \prod_{i=1}^{\ell(\lambda)} p_i(x)$ for a non-empty partition λ and take $m_\emptyset(x) = e_\emptyset(x) = h_\emptyset(x) = 1$.

We are in a position now to write $M^{\bullet}_{g,n}(\mu_1,\mu_2,\dots,\mu_n)$ in terms of the representation theory of symmetric groups. From Definition 1.3.1, we get

$$M^{\bullet}_{g,n}(\mu_1,\mu_2,\dots,\mu_n) = \frac{|\mathrm{Aut}(\mu)|}{d!}[\mathrm{Id}]C_\mu h_m(\mathcal{J}_1,\mathcal{J}_2,\dots,\mathcal{J}_d).$$

For $F(x_1,x_2,\dots,x_d)$ a symmetric polynomial and a partition $\lambda\vdash d$, Jucys obtained the formula [66]

$$\sum_{\mu\vdash d}\chi^\lambda_\mu F(\mathcal{J}_1,\mathcal{J}_2,\dots,\mathcal{J}_d)\,C_\mu = \sum_{\mu\vdash d}\chi^\lambda_\mu F(A_\lambda)C_\mu,$$

where $A_\lambda = \{c(\square) \mid \square \in \lambda\}$ is the multiset of contents of the partition λ. Combining this with orthogonality of characters allows us to deduce that

$$M^\bullet(\mu_1, \mu_2, \ldots, \mu_n) = \frac{1}{d!\, z(\mu)} \sum_\lambda \chi^\lambda_\mu \, h_m(A_\lambda) \dim(\lambda).$$

Again, one can consider the generalisation of this enumerative problem to its double counterpart.

Definition 1.3.3. The *double monotone Hurwitz* number $\overline{M}_{g,n}(\mu_1, \mu_2, \ldots, \mu_n)$ is the weighted count of tuples $(\sigma, \tau_1, \tau_2, \ldots, \tau_m)$ of permutations in S_d such that

- $\tau_1 \tau_2, \ldots, \tau_m$ are transpositions such that, if $\tau_i = (a_i, b_i)$ with $a_i < b_i$, then $b_1 \leq b_2 \leq \cdots \leq b_m$;
- the product $\sigma \circ \tau_1 \circ \tau_2 \circ \cdots \circ \tau_m$ has n labelled cycles with lengths $\mu_1, \mu_2, \ldots, \mu_n$;
- $m = 2g - 2 + n + \ell(\sigma)$; and
- the group $\langle \sigma, \tau_1, \tau_2, \ldots, \tau_m \rangle$ generated by the permutations is a transitive subgroup of S_d.

The weight of such a tuple with σ of cycle type $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is $\frac{q_{\lambda_1} q_{\lambda_2} \cdots q_{\lambda_\ell}}{d!}$.

In Definition 1.3.1, if we strengthen the inequalities to be strict, so that $b_1 < b_2 < \cdots < b_m$, then we obtain *strictly monotone Hurwitz numbers.* The following gives a formal definition.

Definition 1.3.4. The *strictly monotone Hurwitz number* $R_{g,n}(\mu_1, \mu_2 \ldots, \mu_n)$ is $\frac{1}{d!}$ multiplied by the number of tuples $(\tau_1, \tau_2, \ldots, \tau_m)$ of transpositions in S_d such that

- the product $\tau_1 \circ \tau_2 \circ \cdots \circ \tau_m$ has n labelled cycles with lengths $\mu_1, \mu_2, \ldots, \mu_n$;
- $m = 2g - 2 + n + d$;
- the group $\langle \tau_1, \tau_2, \ldots, \tau_m \rangle$ generated by the permutations is a transitive subgroup of S_d; and
- if $\tau_i = (a_i, b_i)$ with $a_i < b_i$, then $b_1 < b_2 < \cdots < b_m$.

Example 1.3.5. To calculate $R_{0,1}(3)$, we first deduce that $m = 2$. Hence, we count all pairs of transpositions in S_3 such that the product is a 3-cycle satisfying the strict monotonicity condition. There are 9 pairs of transpositions, but only 2 of them are strictly monotone: namely, $(1\ 2) \circ (1\ 3)$ and $(1\ 2) \circ (2\ 3)$. So $R_{0,1}(3) = \frac{2}{3!} = \frac{1}{3}$.

If we relax the condition of transitivity in Definition 1.3.4, we obtain disconnected strictly monotone Hurwitz numbers, which we denote by $R^\bullet_{g,n}(\mu_1, \mu_2 \ldots, \mu_n)$. As for the weakly monotone case, we can compute these using characters of the symmetric group as follows.

$$R^\bullet(\mu_1, \mu_2, \ldots, \mu_n) = \frac{1}{d!\, z(\mu)} \sum_\lambda \chi^\lambda_\mu \, e_m(A_\lambda) \dim(\lambda)$$

1.4 Fatgraphs, hypermaps and dessins d'enfant

A fatgraph — also known as a map, embedded graph or ribbon graph — can be thought of as the 1-skeleton of a cell decomposition of an oriented compact surface. Fatgraphs arise in various area of mathematics, including topological graph theory, moduli spaces of Riemann surfaces and matrix models [73].

Definition 1.4.1. A *fatgraph* is a connected graph Γ endowed with a cyclic ordering of the half-edges adjacent to each vertex. A fatgraph is uniquely determined by the triple (X, τ_0, τ_1), where X is the set of half-edges of Γ, $\tau_0 : X \to X$ is the permutation that rotates half-edges anticlockwise about their adjacent vertex, $\tau_1 : X \to X$ is the fixed-point free involution that swaps the two half-edges belonging to the same edge.

Observe that under the permutation model for a fatgraph Γ,

- the set of vertices is in natural bijection with $X_0 = X/\tau_0$,
- the set of edges is in natural bijection with $X_1 = X/\tau_1$, and
- the set of faces is in natural bijection with $X_2 = X/\tau_2$, where τ_2 is defined by $\tau_0\tau_1\tau_2 = \mathrm{id}$.

A fatgraph structure allows one to uniquely thicken the underlying graph to a surface with boundary. In particular, it acquires a type (g, n) where g denotes the genus of the surface and n the number of boundary components.

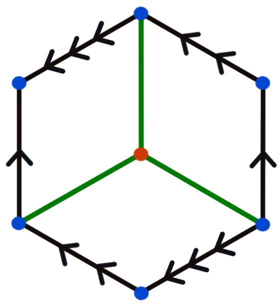

Figure 1.2: By identifying the opposite edges of the hexagon above in pairs, one obtains a genus 1 fatgraph.

An automorphism of a fatgraph Γ is a permutation $\phi : X \to X$ that commutes with τ_0 and τ_1. An automorphism descends to an automorphism of the underlying graph. The group generated by τ_0 and τ_1 acts transitively on X, so an automorphism that fixes an oriented edge is necessarily trivial since $\phi(X) = X$ implies $\phi(\tau_0 X) = \tau_0 X$ and $\phi(\tau_1 X) = \tau_1 X$. A labelled fatgraph is a fatgraph whose boundary components are labelled from 1 up to n. An automorphism of a labelled fatgraph is a permutation $\phi : X \to X$ that commutes with τ_0 and τ_1 and act trivially on X_2.

A natural generalisation of the above definition of *fatgraph* is obtained by changing the cycle type of τ_1 in Definition 1.4.1 to $(m, m, \dots, m)$ for some fixed m. The resulting objects are called m-hypermaps. For more information on these, we refer the reader to [73].

Definition 1.4.2. Define the fatgraph enumeration $F_{g,n}(\mu_1, \mu_2, \dots, \mu_n)$ to be the weighted count of fatgraphs of genus g with n labelled faces of degrees $\mu_1, \mu_2, \dots, \mu_n$. The weight of a fatgraph Γ is $\frac{1}{|\mathrm{Aut}(\Gamma)|}$, where $\mathrm{Aut}(\Gamma)$ denotes the group face-preserving automorphisms. Furthermore, we analogously define $F^m_{g,n}(\mu_1, \mu_2, \dots, \mu_n)$ to be the weighted count of m-hypermaps of genus g with n labelled faces of degrees $\mu_1, \mu_2, \dots, \mu_n$.

We now discuss a variation of fatgraphs known as *dessins d'enfant*. These were originally named by Grothendieck, who introduced them as an approach to the inverse Galois problem via the study of Belyi functions [60].

Definition 1.4.3. A meromorphic function $f : X \to \mathbb{CP}^1$ from a compact Riemann surface that is unramified outside $\{0, 1, \infty\} \subset \mathbb{CP}^1$ is called a *Belyi function*.

Take the segment $[0, 1] \subset \mathbb{CP}^1$. Consider the point 0 to be red (•) and consider the point 1 to be blue (•). The preimage $H = f^{-1}([0, 1]) \subset X$ is a fatgraph in the Riemann surface X, with each edge joining a red vertex and a blue vertex. This representation of a Belyi function is called a *dessin d'enfant*.

Definition 1.4.4. A *dessin d'enfant* is a fatgraph whose vertices are coloured red and blue such that each edge is adjacent to one vertex of each colour. An *isomorphism* between two dessins d'enfant is an isomorphism between their underlying fatgraphs that preserves the vertex colouring.

The set of dessins d'enfant in which every blue vertex has degree two is in natural one-to-one correspondence with the set of fatgraphs. One simply removes the degree two blue vertices and amalgamates the two adjacent edges into a single edge. Similarly, the set of dessins d'enfant in which every blue vertex has degree m is in natural one-to-one correspondence with the set of m-hypermaps.

Definition 1.4.5. Define the dessin d'enfant enumeration $B_{g,n}(\mu_1, \mu_2, \ldots, \mu_n)$ to be the weighted count of dessins d'enfant of genus g with n labelled faces of degrees $2\mu_1, 2\mu_2, \ldots, 2\mu_n$. The weight of a dessin d'enfant Γ is $\frac{1}{|\mathrm{Aut}(\Gamma)|}$, where $\mathrm{Aut}(\Gamma)$ denotes the group of face-preserving automorphisms.

More generally, one can refine the enumeration withs weights that record the degrees of the blue vertices.

Definition 1.4.6. Define the double dessin d'enfant enumeration $\overline{B}_{g,n}(\mu_1, \mu_2, \ldots, \mu_n)$ to be the analogous weighted count of dessins d'enfant, where the weight of a dessin d'enfant Γ with blue vertices of degrees $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ is $\frac{q_{\lambda_1} q_{\lambda_2} \cdots q_{\lambda_\ell}}{|\mathrm{Aut}(\Gamma)|}$.

1.5 Weighted Hurwitz numbers

Our work presented in Chapter 4 is motivated by the Harer–Zagier formula for the enumeration of fatgraphs with one face [62], as well as the analogue for the enumeration of dessins d'enfant with one face [36]. Apart from the obvious combinatorial similarities between these problems, they both also arise from double Schur function expansions. Thus, we propose to study a broad class of "enumerative problems" stored in expansions of the general form

$$Z(\mathbf{p}; \mathbf{q}; \hbar) = \sum_{\lambda \in \mathcal{P}} s_\lambda(p_1, p_2, \ldots)\, s_\lambda(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots)\, F_\lambda(\hbar). \tag{1.2}$$

Here, $\mathcal{P}$ denotes the set of all partitions (including the empty partition), $s_\lambda(p_1, p_2, \ldots)$ denotes the Schur function expressed in terms of power-sum symmetric functions, and $F_\lambda(\hbar)$ is a formal power series in $\hbar$ for each partition λ. We use the shorthand $\mathbf{p} = (p_1, p_2, p_3, \ldots)$ and $\mathbf{q} = (q_1, q_2, q_3, \ldots)$ throughout the section. Following the mathematical physics literature, we will refer to such power series as *partition functions*, although we note that this name does not refer to the integer partitions that appear in the equation above.

One may ask why such double Schur functions are considered and a natural answer is that they encompass a variety of problems, such as enumeration of fatgraphs and dessins d'enfant, as well as Hurwitz numbers of various types. In this section, we give a different perspective coming from integrability, following the work of Alexandrov, Chapuy, Eynard and Harnad on weighted Hurwitz numbers [2]. We briefly discuss how the partition functions defined above arise as KP tau functions when $F_\lambda(\hbar)$ takes on the so-called content-product form.

Recall that a KP tau function $\tau(\mathbf{t}, \mathbf{s})$ is a function of infinite sets of flow variables $\mathbf{t} = (t_1, t_2, \ldots)$ and $\mathbf{s} = (s_1, s_2, \ldots)$ satisfying the infinite system of Hirota bilinear equations, which can be expressed as

$$\operatorname*{Res}_{z=0} \Big(\psi^+(z, \mathbf{t}) \psi^-(z, \mathbf{s}) \Big) = 0.$$

Here the *Baker–Akhiezer function* $\psi^+(z, \mathbf{t})$ and its dual $\psi^-(z, \mathbf{t})$ are defined by the Sato formula

$$\psi^\pm(z, \mathbf{t}) := \exp\left(\sum_{i=1}^\infty t_i z^i \right) \frac{\tau(\mathbf{t} \mp [z^{-1}])}{\tau(\mathbf{t})},$$

where we use the notation $[z^{-1}] = (\frac{1}{z}, \frac{1}{2z^2}, \frac{1}{3z^3}, \ldots)$.

The Hirota bilinear equations can be understood geometrically as equivalent to Plücker relations for a certain infinite-dimensional Grassmannian. This connects KP tau functions with the infinite wedge space, which we will briefly describe. Furthermore, we will require calculations with the infinite wedge space in our work on Gromov–Witten invariants in Chapter 6.

Let V be a vector space with orthonormal basis $\{e_i \mid i \in \mathbb{Z}\}$ and consider it as the direct sum

$$V = V_+ \oplus V_-,$$

where $V_+ = \mathrm{span}\{e_{-i}\}_{i\in\mathbb{N}}$ and $V_- = \mathrm{span}\{e_i\}_{i\in\mathbb{N}^+}$. Introduce the notion of the infinite wedge space $F := \wedge^{\infty/2}V$, which is spanned by elements $|\lambda; N\rangle$, where λ is a partition and $N \in \mathbb{Z}$.

$$|\lambda; N\rangle := \{e_{\ell_1} \wedge e_{\ell_2} \wedge e_{\ell_3} \wedge \cdots \mid \ell_i := \lambda_i - i + N\}$$

Since a partition $\lambda := \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0 \geq 0 \geq \cdots$ is eventually zero, the sequence $\ell_1, \ell_2, \ldots$ contains all but finitely many negative integers. Let us denote the subset of lattice coordinates $\{\ell_1, \ell_2, \ldots\}$ appearing in the basis element $\mid \lambda; N\rangle$ as $S_\lambda(N)$. Note that $F = \bigoplus F_N$ has a natural $\mathbb{Z}$-grading by the integer N, known as the charge.

We are mostly interested in F_0 as other states can be obtained by translating F_0. A special basis element of F_0 called the vacuum $|\emptyset; 0\rangle$ is given by the empty partition and $N = 0$. Let us define the *fermionic operator* ψ_k for $k \in \mathbb{Z}$ by giving its action on the basis element $|\lambda; N\rangle$ as follows.

$$\psi_k(|\lambda; N\rangle) = \begin{cases} e_k \wedge |\lambda; N\rangle, & \text{for } k \notin S_\lambda(N), \\ 0, & \text{for } k \in S_\lambda(N). \end{cases}$$

Similarly, we define the adjoint operator ψ_k^* to be as follows, where $\hat{e}_k$ denotes the removal of the wedge factor.

$$\psi_k^*(|\lambda; N\rangle) = \begin{cases} \hat{e}_k \wedge |\lambda; N\rangle, & \text{for } k \in S_\lambda(N), \\ 0, & \text{for } k \notin S_\lambda(N). \end{cases}$$

These fermionic operators satisfy anti-commutation relations

$$[\psi_i, \psi_j^*]_+ = \delta_{ij}, \qquad [\psi_i, \psi_j]_+ = 0, \qquad \text{and} \qquad [\psi_i^*, \psi_j^*]_+ = 0.$$

Define the normally ordered operator $:\psi_i\psi_j:$ on F_0 to be $\psi_i\psi_j$ if $j > 0$ and $-\psi_j\psi_i$ if $j \leq 0$. We consider the infinite-dimensional Lie algebra $\mathfrak{gl}_\infty$ spanned by the operators $:\psi_i\psi_j:$. Exponentiating elements of $\mathfrak{gl}_\infty$, we obtain elements of the Lie group GL_∞, consisting of invertible endomorphisms having well-defined determinants [94]. A typical element of GL_∞ can be represented by

$$\hat{g} = \exp\left(\sum_{ij} A_{ij} :\psi_i\psi_j:\right).$$

We use the fermionic operators to define the *bosonic operators*

$$\alpha_n := \sum_{k\in\mathbb{Z}+\frac{1}{2}} :\psi_{k-n}\psi_k^*:, \quad \text{for } n \in \mathbb{Z}\setminus\{0\}.$$

The adjoint of α_n is given by

$$\alpha_n^* = \left(\sum :\psi_{k-n}\psi_k^*:\right)^* = \sum :\psi_k\psi_{k-n}^*: = \sum :\psi_{k+n}\psi_k^*: = \alpha_{-n}.$$

The bosonic operators satisfy commutation relations

$$[\alpha_m, \alpha_n] = m\delta_{m+n,0}.$$

There is a natural map $F \to \mathbb{C}[\mathbf{t}]$ given by

$$|\lambda; N\rangle \mapsto \langle\emptyset; N| \exp\left(\sum_{i=1}^{\infty} t_i \alpha_i\right) |\lambda; N\rangle$$

where $\langle\emptyset; N|$ denotes the covacuum, which returns the coefficient of $|\emptyset; N\rangle$. More generally, given an element $\hat{g} \in GL_\infty$, we call the value of $\langle\emptyset; N|\, \hat{g}\, |\emptyset; N\rangle$ the vacuum expectation of the operator $\hat{g}$. For example, we have the result

$$\langle\emptyset; 0| \exp\left(\sum_{i=1}^{\infty} t_i \alpha_i\right) |\lambda; 0\rangle = s_\lambda(\mathbf{t}),$$

where s_λ denotes a Schur function expressed in terms of power-sum symmetric functions $p_1, p_2, \ldots$ and we set $t_i := \frac{p_i}{i}$. Let us define the vertex operators

$$\hat{\gamma}_+(\mathbf{t}) := \exp\left(\sum_{i=1}^{\infty} t_i \alpha_i\right) \qquad \text{and} \qquad \hat{\gamma}_-(\mathbf{t}) := \exp\left(\sum_{i=1}^{\infty} t_i \alpha_{-i}\right).$$

With this notation, we can construct a large family of tau functions of the forms

$$\tau_g(N, \mathbf{t}) := \langle 0; N|\hat{\gamma}_+(\mathbf{t})\hat{g}|0; N\rangle, \tag{1.3}$$

$$\tau_g(N, \mathbf{t}, \mathbf{s}) := \langle 0; N|\hat{\gamma}_+(\mathbf{t})\hat{g}\hat{\gamma}_-(\mathbf{s})|0; N\rangle. \tag{1.4}$$

We consider a special subfamily of tau functions for which the group element $\hat{g}$ is given by

$$\hat{g} := \exp\left(\sum_{i\in\mathbb{Z}} T_i : \psi_i \psi_i^* :\right),$$

whose action is diagonal with respect to the basis $|\lambda; N\rangle$. Such tau functions are called *hypergeometric tau functions* in the literature [89].

The eigenvalues $r_\lambda(N, g)$ of $\hat{g}$ can be written in the content-product form [89]

$$r_\lambda(N, g) := r_0(N, g) \prod_{(i,j)\in\lambda} r_{N+j-i}(g), \quad r_i(g) := \exp(T_i - T_{i-1}),$$

where

$$r_0(N, g) := \begin{cases} \prod_{i=0}^{N-1} \exp(T_i), & \text{if } N > 0, \\ 1, & \text{if } N = 0, \\ \prod_{i=N}^{-1} \exp(-T_i), & \text{if } N < 0. \end{cases}$$

Then the hypergeometric tau functions of equation (1.4) have double Schur function expansions of the form

$$\tau_g(N, \mathbf{t}, \mathbf{s}) = \sum_\lambda r_\lambda(N, g) s_\lambda(\mathbf{t}) s_\lambda(\mathbf{s}).$$

Let $c_1, c_2, \ldots$ be an infinite sequence of parameters and let

$$G(z) := \prod_{i=1}^{\infty} (1 + c_i z).$$

For now, we are not concerned with $G(z)$ being convergent and treat it formally. Let us recall the notation $\mathcal{E}_\lambda$ for the orthogonal idempotent basis of the centre of the symmetric group algebra and $\mathcal{J}_k$ for the Jucys–Murphy elements. We let $\mathcal{J}$ denote the entire collection of these elements, as defined in Section 1.3. Then

$$G_n(z,\mathcal{J})\mathcal{E}_\lambda := \prod_{i=1}^{n} G(z\mathcal{J}_i)\mathcal{E}_\lambda = \prod_{(i,j)\in\lambda} G(z(j-i))\mathcal{E}_\lambda.$$

Similarly, we can define the dual generating function

$$\tilde{G}(z) := \frac{1}{G(-z)} = \prod_{i=1}^{\infty}(1-c_i z)^{-1}.$$

Now choose T_j depending on $G(z)$ so that

$$T_j^G = \sum_{k=1}^{j} \log G(zk), \quad T_0^{G(z)}(z) = 0, \quad T_{-j}^{G(z)}(z) = -\sum_{k=0}^{j-1} \log G(-kz), \quad \text{for } j > 0.$$

In this case,

$$\hat{g} := \exp\left(\sum_{i\in\mathbb{Z}} T_i^G : \psi_i\psi_i^* :\right),$$

and it follows that

$$r_j(g) := r_j^{G(z)} = G(jz).$$

We have

$$\hat{g}|\lambda; N\rangle = r_\lambda^{G(z)}(N)|\lambda; N\rangle,$$

with eigenvalues

$$r_\lambda^{G(z)}(N) := r_0^{G(z)}(N) \prod_{\square\in\lambda} G(z(N+j-i)),$$

where

$$r_0^{G(z)}(N) = \prod_{j=1}^{N} G((N-j)z)^j, \quad r_0(0) = 1, \quad r_0^{G(z)}(-N) = \prod_{j=1}^{N} G((j-N)z)^{-j}, \quad \text{for } N > 1.$$

Similarly, for the dual generating function $\tilde{G}(z)$, we have

$$r_\lambda^{\tilde{G}(z)}(N) := r_0^{\tilde{G}(z)}(N) \prod_{\square\in\lambda} \tilde{G}(z(N+j-i)).$$

The work of Alexandrov, Chapuy, Eynard and Harnad explains that the coefficients of these particular hypergeometric tau functions in the power-sum symmetric function basis enumerate weighted branched covers or equivalently, weighted paths in the Cayley graph of symmetric groups or weighted constellations [2, 64]. As such, they are termed *weighted Hurwitz numbers*. These give a vast family of enumerative problems that encompass the ones mentioned previously in this chapter, such as enumeration of fatgraphs and hypermaps, BMS numbers, simple and monotone Hurwitz numbers. The class of weighted Hurwitz numbers and their associated tau functions forms the basis for some of our work on one-point recursions in Chapter 4.

Chapter 2

Problems in enumerative geometry

Enumerative geometry is a branch of algebraic geometry concerned with counting geometric objects that satisfy certain constraints, mainly by means of intersection theory. In this chapter, we discuss some problems from enumerative geometry that are relevant to the remainder of the thesis — namely, intersection theory on moduli spaces of curves (Section 2.1), the enumeration of lattice points in moduli spaces of curves (Section 2.2), and Gromov–Witten theory (Section 2.3). These enumerative problems are all closely related to the more general notion of cohomological field theory, whose definition concludes the chapter. Cohomological field theories have a close relation to topological recursion, which is an underlying theme of this thesis.

2.1 Moduli spaces of curves

Consider a genus g smooth complex curve C with $n \geq 0$ distinct points on it. We always suppose that the curve is compact and that the points are labelled $x_1, x_2, \ldots, x_n$. Considered up to isomorphisms preserving the marked points, such a curve determines a point in the moduli space of curves $\mathcal{M}_{g,n}$. We denote this point by $(C; x_1, \ldots, x_n)$. If $2g - 2 + n \geq 0$, then the curve only admits finitely many automorphisms and is called *stable*. For the remaining cases $(g, n) = (0, 0), (0, 1), (0, 2)$ and $(1, 0)$, such an *unstable* curve admits infinitely many automorphisms, in which case there is no good moduli space of curves.

Example 2.1.1. All smooth genus 0 curves with 3 marked points are isomorphic, so $\mathcal{M}_{0,3}$ consists of a single point. A smooth genus 0 curve with 4 marked points is isomorphic to $(\mathbb{CP}^1; 0, 1, \infty, \lambda)$ for some $\lambda \in \mathbb{C} \setminus \{0, 1\}$. So $\mathcal{M}_{0,4}$ can be naturally identified with $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$.

Example 2.1.2. Every elliptic curve is isomorphic to the quotient of $\mathbb{C}$ by a rank 2 lattice, so $\mathcal{M}_{1,1} = \{\text{rank 2 lattices}\}/\mathbb{C}^*$. Here, the single marked point corresponds to the points of the lattice. Consider a basis (z_1, z_2) of a lattice Λ. Multiplying Λ by $\frac{1}{z_1}$ or $\frac{1}{z_2}$, we obtain a lattice with basis $(1, \tau)$, where τ lies in the upper half-plane $\mathbb{H}$. Choosing another basis of the same lattice, we obtain another point $\tau' \in \mathbb{H}$. Thus, the group $SL(2, \mathbb{Z})$ of base changes in a lattice acts on $\mathbb{H}$ via Möbius transformations, given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

Hence, we have $\mathcal{M}_{1,1} = \mathbb{H}/SL(2, \mathbb{Z})$.

For $2g-2+n > 0$, the moduli space $\mathcal{M}_{g,n}$ possesses the structure of a smooth complex $(3g-3+n)$-dimensional orbifold. Alternatively, one can consider it as a smooth $(3g - 3 + n)$-dimensional Deligne–Mumford stack. As we can see from the examples above, $\mathcal{M}_{g,n}$ is not compact in general.

We compactify it by adding points that correspond to nodal curves satisfying a stability condition, thereby arriving at the Deligne–Mumford compactification.

Example 2.1.3. What will happen as $\lambda \to 0$ in the example of $\mathcal{M}_{0,4}$ above? At first sight, we will simply obtain a curve with four marked points, two of which coincide: $x_1 = x_4$. However, such an approach is unfair with respect to the points x_1 and x_4. Indeed, without changing the curve C_λ, we can change its local coordinate via the map $x \to x/\lambda$ and obtain the curve

$$(C, x_1, x_2, x_3, x_4) \simeq (\mathbb{CP}^1, 0, 1/\lambda, \infty, 1).$$

What we see now in the limit is that x_1 and x_4 do not glue together any longer, but this time x_2 and x_3 do tend to the same point. Since there is no reason to prefer one local coordinate to the other, neither of the pictures is better than the other one. The right thing to do is to include both limit curves in the description of the limit.

One component corresponds to the initial local coordinate x, while the other component corresponds to the local coordinate x/λ. To make this more visual, consider the following example. Let $xy = \lambda z^2$ be a family of curves in $\mathbb{CP}^2$ parametrised by λ. On each of these curves, we mark the following points:

$$[x_1, y_1, z_1] = [0 : 1 : 0], \quad [x_2, y_2, z_2] = [1 : \lambda : 1], \quad [x_3, y_3, z_3] = [1 : 0 : 0], \quad [x_2, y_2, z_2] = [\lambda : 1 : 1].$$

Then, for $\lambda \neq 0$, the curve is isomorphic to $\mathbb{CP}^1$ with four marked points, while for $\lambda = 0$ it degenerates into a curve composed of two spheres meeting nodally, with two marked points on each sphere.

The following figure gives a combinatorial model. We have a sphere marked with four points p_1, p_2, p_3, p_4. It degenerates to two spheres glued at the red point having two marked points each. The red point denotes the nodal singularity.

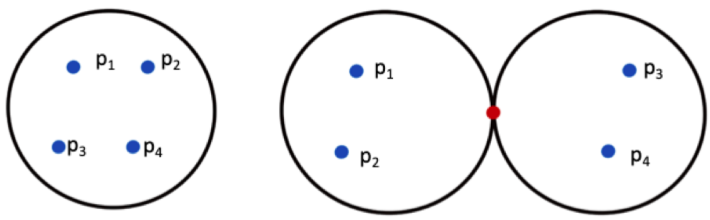

Figure 2.1: On the left, we depict an element of $\mathcal{M}_{0,4}$ and on the right, an element of $\overline{\mathcal{M}}_{0,4} \setminus \mathcal{M}_{0,4}$.

Example 2.1.4. In the case of $\mathcal{M}_{1,1}$ if we let $\tau \to \infty$, we can identify the limit as the torus with a marked point denoted by p_1 and a pinched cycle that creates a nodal point. This can be constructed from a sphere with three marked points, with two of them glued together at a node.

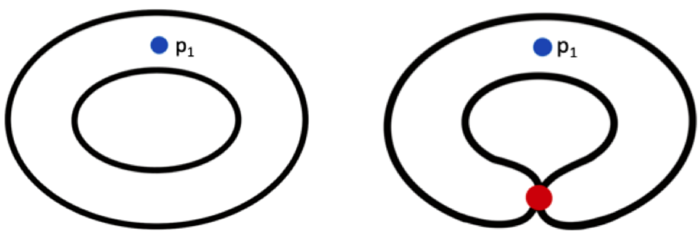

Figure 2.2: On the left, we depict an element of $\mathcal{M}_{1,1}$ and on the right, an element of $\overline{\mathcal{M}}_{1,1} \setminus \mathcal{M}_{1,1}$.

Compactifying the moduli space of curves $\overline{\mathcal{M}}_{g,n}$ requires us to consider non-smooth curves, but with only mild singularities. In particular, we only allow simple nodes, which are those singularities that locally look like the curve given by the equation $xy = 0$ at the origin. The neighbourhood of a node is diffeomorphic to two disks with identified centres. We say that a node is normalised if the two disks with identified centres that form its neighbourhood are unplugged and then replaced by disjoint disks.

Definition 2.1.5. A *stable curve* is a complex algebraic curve C with $n \geq 0$ marked points $x_1, x_2, \ldots, x_n \in C$, satisfying the following conditions.

- The only singularities of C are nodes.
- The marked points are distinct and do not coincide with nodes.
- The curve has finitely many automorphisms that preserve the marked points.

Unless stated otherwise, stable curves are assumed to be connected. The genus of a stable curve C is considered to be the arithmetic genus of the curve. Furthermore, observe that the finite automorphisms condition is satisfied if and only if each irreducible component X satisfies $2g(X) - 2 + n(X) > 0$, where $g(X)$ denotes the genus of the component and $n(X)$ the number of marked points or branches of a node on the component.

As an example, see Figure 2.3, which shows a stable curve of genus 4, with three nodes and one marked point. The middle hole surrounded by the three nodes gives rise to an extra genus beyond that contributed by the irreducible components.

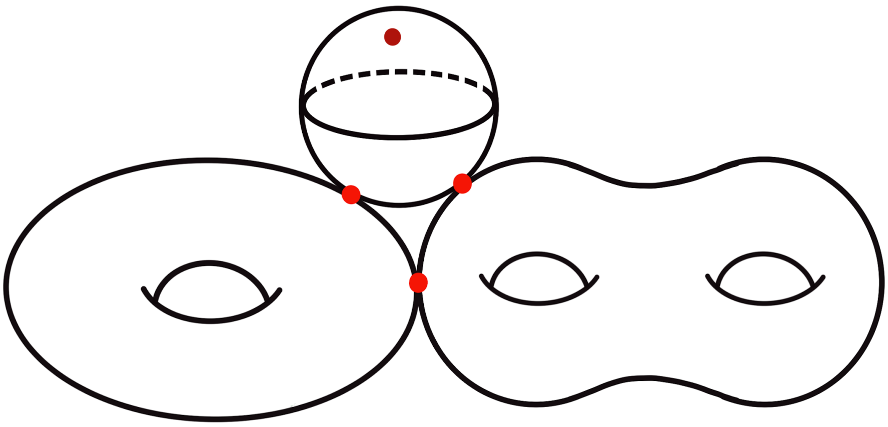

Figure 2.3: A stable curve of genus 4 with one marked point.

Theorem 2.1.6. *There exists a smooth compact complex $(3g-3+n)$-dimensional orbifold $\overline{\mathcal{M}}_{g,n}$, a smooth complex $(3g-2+n)$-dimensional orbifold $\overline{\mathcal{C}}_{g,n}$ and a map $p : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ such that*

- $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ *is an open dense sub-orbifold and $\mathcal{C}_{g,n} \subset \overline{\mathcal{C}}_{g,n}$ its preimage under p;*
- *the fibres of p are stable curves of genus g with n marked points;*
- *each stable curve is isomorphic to exactly one fibre;*
- *the stabiliser of a point $t \in \overline{\mathcal{M}}_{g,n}$ is isomorphic to the automorphism group of the corresponding stable curve C_t.*

Definition 2.1.7. The space $\overline{\mathcal{M}}_{g,n}$ is called the *Deligne–Mumford compactification* of the moduli space of curves $\mathcal{M}_{g,n}$. The family $p : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is called the *universal curve*. The set $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ parametrising singular stable curves is called the *boundary* of $\overline{\mathcal{M}}_{g,n}$.

Example 2.1.8. We have $\overline{\mathcal{M}}_{0,3} = \mathcal{M}_{0,3}$, both of which consist of a single point. Indeed, the unique stable genus 0 curve with 3 marked points is smooth. We have $\overline{\mathcal{M}}_{0,4} = \mathbb{CP}^1$, with the three singular stable genus 0 curves with 4 marked points corresponding to $\lambda \in \{0, 1, \infty\}$, as in Example 2.1.1.

Example 2.1.9. The moduli space $\overline{\mathcal{M}}_{1,1}$ is obtained from $\mathcal{M}_{1,1}$ by adding one point corresponding to the unique singular stable genus 1 curve with 1 marked point.

2.2 Lattice points in moduli spaces of curves

The geometry of moduli spaces of curves $\mathcal{M}_{g,n}$ is proved to be intimately related to the space of graphs embedded in surfaces. Embedded graphs are used to enumerate cells in a certain cell decomposition of decorated moduli spaces. A decorated moduli space is simply a product of the usual moduli space $\mathcal{M}_{g,n}$ by the real positive octant $\mathbb{R}_+^n$. Decorated moduli spaces are no longer complex orbifolds and they carry only real orbifold structures. The moduli space $\mathcal{M}_{g,n}$ has real dimension $6g - 6 + 2n$, so the decorated moduli space has real dimension $6g - 6 + 3n$.

The decorated moduli space has a cell decomposition

$$\mathcal{M}_{g,n} \times \mathbb{R}_+^n \cong \left(\bigsqcup_{\Gamma \in Fat_{g,n}} P_\Gamma \right) / \sim, \tag{2.1}$$

where the indexing set $Fat_{g,n}$ is the set of labelled fatgraphs of genus g with n boundary components, in which each vertex has degree at least three [63]. The notion of a fatgraph was described earlier in Section 1.4. The cell P_Γ is the set of metrics on the fatgraph Γ — in other words, the assignment of a positive real number to each edge. So it naturally satisfies $P_\Gamma \cong \mathbb{R}_+^{e(\Gamma)}$ where $e(\Gamma)$ denotes the number of edges of the fatgraph Γ. The right side of equation (2.1) is known as the *combinatorial moduli space* and we write it as $\mathcal{M}_{g,n}^{\text{comb}}$. For each $(b_1, \ldots, b_n) \in \mathbb{R}_+^n$, there is a natural projection $\pi : \mathcal{M}_{g,n}^{\text{comb}} \to \mathbb{R}_+^n$, which allows us to define the natural homeomorphism

$$\mathcal{M}_{g,n}^{\text{comb}}(b_1, \ldots, b_n) := \pi^{-1}(b_1, \ldots, b_n) \cong \mathcal{M}_{g,n}.$$

Definition 2.2.1. Define the set of all metrics on the labelled fatgraph Γ with fixed boundary lengths $b = (b_1, \ldots, b_n) \in \mathbb{R}_+^n$ to be

$$P_\Gamma(b_1, \ldots, b_n) := P_\Gamma \cap \pi^{-1}(b_1, \ldots, b_n).$$

In particular, we can consider equation (2.1) at the level of fibres in the following way.

$$\mathcal{M}_{g,n}^{\text{comb}}(b_1, \ldots, b_n) = \left(\bigsqcup_{\Gamma \in Fat_{g,n}} P_\Gamma(b_1, \ldots, b_n) \right) / \sim$$

These cell decompositions are fundamental in Kontsevich's proof of Witten's conjecture, which proceeds by calculation of the volume of the moduli space with respect to a particular symplectic structure [69]. This is closely related to Mirzakhani's calculation of the Weil–Petersson volumes of moduli spaces of hyperbolic surfaces and indeed, arises as a particular limit of it [27, 77]. Norbury proposed to discretise the volume calculation, by restricting to positive integer values of $b_1, b_2, \ldots, b_n$ and counting lattice points in the resulting integral polytopes $\mathcal{P}_\Gamma(b_1, b_2, \ldots, b_n)$. These correspond to metric fatgraphs with vertex degrees at least three and integral edge lengths or equivalently, fatgraphs with vertex degrees at least two. Thus, we have the notion of lattice points in $\mathcal{M}_{g,n}$ and the associated enumeration possesses a variety of interesting properties [5, 6, 81, 82].

We now briefly discuss convex polytopes, their volumes and lattice point enumerations. A *convex polytope* $P \subset \mathbb{R}^n$ is a bounded convex set whose closure is the convex hull of a finite set of vertices in $\mathbb{R}^n$. Given a linear map $A : \mathbb{R}^N \to \mathbb{R}^n$ and $b \in \mathbb{R}^n$,

$$P_A(b) := \{x \in \mathbb{R}^N_+ \mid Ax = b\}$$

defines a convex set and it is a convex polytope if A has non-negative entries and non-zero columns. If the matrix A is defined over the integers — that is, $A : \mathbb{Z}^N \to \mathbb{Z}^n$ and A has non-negative entries and non-zero columns — then for $b \in \mathbb{Z}^n$, $P_A(b)$ is a *rational convex polytope*, meaning that its vertices lie in $\mathbb{Q}^n$. One can count integral solutions to $Ax = b$ by defining

$$N_{P_A}(b) := \#\Big\{x \in \mathbb{Z}^N_+ \mid Ax = b\Big\}.$$

The following example appears in [84].

Example 2.2.2. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 0 \end{bmatrix}.$$

By solving for x, we have

$$N_{P_A}(b_1, b_2) = \begin{cases} 0, & b_1 - b_2 \text{ odd}, \\ 0, & b_1 - b_2 \text{ even and } b_1 \le b_2, \\ \frac{1}{2}(b_1 - b_2) - 1, & b_1 - b_2 \text{ even and } b_1 > b_2. \end{cases}$$

Next, we discuss the relationship between convex polytopes and fatgraphs. Given a fatgraph Γ of type (g, n), its incidence matrix A_Γ is defined by

$$\begin{aligned} A_\Gamma : \mathbb{R}^{e(\Gamma)} &\to \mathbb{R}^n \\ \text{edge} &\to \text{incident boundary components.} \end{aligned}$$

Example 2.2.3. Let Γ and Γ' be the genus 0 and genus 1 fatgraphs in the diagram below.

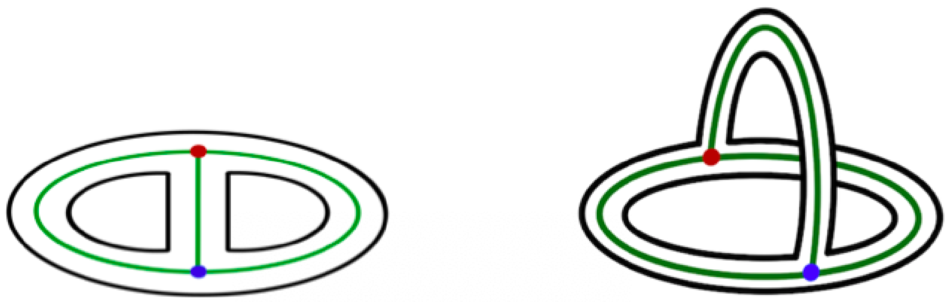

Figure 2.4: On the left, we show the genus 0 fatgraph Γ and on the right, we show the genus 1 fatgraph Γ'.

Then their respective incidence matrices have rows indexed by the faces of the fatgraph and columns indexed by the edges.

$$A_\Gamma = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad\qquad A_{\Gamma'} = \begin{bmatrix} 2 & 2 & 2 \end{bmatrix}$$

For a fatgraph Γ and integers $b_1, \ldots, b_n$, denote the number of lattice points in the rational convex polytope $P_\Gamma(b_1, \ldots, b_n)$ by

$$N_\Gamma(b_1, \ldots, b_n) := N_{P_\Gamma}(b_1, \ldots, b_n) = \#\left\{P_\Gamma(b_1, \ldots, b_n) \cap \mathbb{Z}_+^{e(\Gamma)}\right\}.$$

The number of lattice points in $\mathcal{M}_{g,n}^{\mathrm{comb}}(b_1, \ldots, b_n)$, as defined in [81], is the weighted sum of N_Γ over the finite set of labelled fatgraphs of genus g with n boundary components, in which each vertex has degree at least three.

Definition 2.2.4. For $g \geq 0$, $n \geq 1$ and positive integers $b_1, \ldots, b_n$, define

$$N_{g,n}(b_1, \ldots, b_n) = \sum_{\Gamma \in Fat_{g,n}} \frac{1}{|\mathrm{Aut}(\Gamma)|} N_\Gamma(b_1, \ldots, b_n).$$

In the following, we use the term *quasi-polynomial* to refer to functions $g : \mathbb{Z}_+^n \to \mathbb{R}$ for which there exists a positive integer m and polynomials $p_0, p_1, \ldots, p_{m-1} \in \mathbb{R}[t]$ such that

$$g(t) = p_i(t), \quad \text{for } t \equiv i \pmod{m}.$$

Theorem 2.2.5 (Norbury [81])**.** *The number of lattice points $N_{g,n}(b_1, \ldots, b_n)$ is a symmetric quasi-polynomial of degree $3g - 3 + n$ in $b_1^2, \ldots, b_n^2$ in the sense that it is polynomial on each coset of the sublattice $2\mathbb{Z}^n \subset \mathbb{Z}^n$.*

By symmetry, we can represent the underlying 2^n polynomials corresponding to the quasi-polynomial $N_{g,n}(b_1, \ldots, b_n)$ by the $n + 1$ polynomials $N_{g,n}^{(k)}(b_1, \ldots, b_n)$ for $k = 0, 1, \ldots, n$, corresponding to the first k variables being odd and the remaining variables even. Observe that if k is odd, then $N_{g,n}^{(k)}(b_1, \ldots, b_n) = 0$. The following table gives some examples of these polynomials in low genus.

g	n	k	$N_{g,n}^{(k)}(b_1, b_2, \ldots, b_n)$
0	3	0	1
0	3	2	1
1	1	0	$\frac{1}{48}(b_1^2 - 4)$
0	4	0	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 - 4)$
0	4	2	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 + 2)$
0	4	4	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 + 8)$
1	2	0	$\frac{1}{384}(b_1^2 + b_2^2 - 4)(b_1^2 + b_2^2 - 8)$
1	2	2	$\frac{1}{384}(b_1^4 + b_2^4 + 2b_1^2b_2^2 + 36b_1^2 + 36b_2^2 + 84)$

The lattice point count $N_{g,n}(b_1, \ldots, b_n)$ satisfies a recursion that uniquely determines the polynomials from a finite number of base cases. Moreover, this recursion implies the Witten–Kontsevich theorem when restricted to the top degree terms of the quasi-polynomials.

Theorem 2.2.6 (Norbury [81])**.** *The lattice count $N_{g,n}(b_1, \ldots, b_n)$ satisfies the following recursion relation, which determines the polynomials uniquely from $N_{0,3}(b_1, b_2, b_3)$ and $N_{1,1}(b_1)$. For*

$2g-2+n \geq 2$ and $b_1, b_2, \ldots, b_n > 0$, we have the following equation, where $S = \{1, 2, \ldots, n\}$ and $b_I = (b_{i_1}, b_{i_2}, \ldots, b_{i_k})$ for $I = \{i_1, i_2, \ldots, i_k\}$.

$$\left(\sum_{i=1}^{n} b_i\right) N_{g,n}(\mathbf{b}_S) = \sum_{i<j} \sum_{\substack{p+q=b_i+b_j \\ q \text{ even}}} pq\, N_{g,n-1}(p, \mathbf{b}_{S\setminus\{i,j\}})$$

$$+ \frac{1}{2} \sum_{i} \sum_{\substack{p+q+r=b_i \\ r \text{ even}}} pqr \left[N_{g-1,n+1}(p, q, \mathbf{b}_{S\setminus\{i\}}) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S\setminus\{i\}}}^{\text{stable}} N_{g_1,|I|+1}(p, \mathbf{b}_I)\, N_{g_2,|J|+1}(q, \mathbf{b}_J) \right]$$

The word stable *over the final summation denotes that we exclude all terms with $N_{0,1}$ or $N_{0,2}$.*

Do and Norbury introduced the related count of lattice points in $\overline{\mathcal{M}}_{g,n}$, the Deligne–Mumford compactification of the moduli space of curves [29]. For positive integers $b_1, b_2, \ldots, b_n$, they defined

$$\overline{\mathcal{Z}}_{g,n}(b_1, b_2, \ldots, b_n) \subset \overline{\mathcal{M}}_{g,n}$$

to be the set of stable curves Σ with labelled points $(p_1, p_2, \ldots, p_n)$ such that there exists a morphism $f : \Sigma \to \mathbb{CP}^1$ satisfying the following three conditions.

(C1) The morphism f has degree $b_1 + b_2 + \cdots + b_n$ and is regular over $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$.

(C2) The ramification profile over $1 \in \mathbb{CP}^1$ is of the form $(2, 2, \ldots, 2)$ and the ramification profile over $\infty \in \mathbb{CP}^1$ is of the form $(b_1, b_2, \ldots, b_n)$, with ramification order b_k occurring at the point $p_k \in \Sigma$.

(C3) Each point over $0 \in \mathbb{CP}^1$ has ramification order at least two or is a node of Σ.

The set $\overline{\mathcal{Z}}_{g,n}(b_1, b_2, \ldots, b_n)$ typically comprises a finite set of discrete points in $\overline{\mathcal{M}}_{g,n}$, along with higher-dimensional components that are products of uncompactified moduli spaces of curves. The latter arise from maps $f : \Sigma \to \mathbb{CP}^1$ that have so-called *ghost components* — that is, irreducible components of Σ that map entirely to $0 \in \mathbb{CP}^1$. To properly "count" points in $\overline{\mathcal{Z}}_{g,n}(b_1, b_2, \ldots, b_n)$, one needs to account for the orbifold nature of $\overline{\mathcal{M}}_{g,n}$ and the existence of these ghost components. This can be conveniently expressed via the orbifold Euler characteristic as follows.

Definition 2.2.7. For positive integers $b_1, b_2, \ldots, b_n$, define

$$\overline{N}_{g,n}(b_1, b_2, \ldots, b_n) = \chi\left(\overline{\mathcal{Z}}_{g,n}(b_1, b_2, \ldots, b_n)\right),$$

where $\chi\left(\overline{\mathcal{Z}}_{g,n}(b_1, b_2, \ldots, b_n)\right)$ is the orbifold Euler characteristic of $\overline{\mathcal{Z}}_{g,n}(b_1, b_2, \ldots, b_n)$.

One can associate a combinatorial structure that is called a *stable fatgraph* to any morphism from a stable curve $f : \Sigma \to \mathbb{CP}^1$ satisfying the conditions (C1), (C2) and (C3) above as follows. Let $\Gamma' = f^{-1}[0, 1] \setminus \{\text{nodes, ghost components}\} \subset \Sigma$. Define Γ to be the closure of Γ' in the normalisation of Σ — that is, add vertices to non-compact ends of Γ'. Let $S = \Gamma \setminus \Gamma'$ and define two vertices in S to be equivalent if they coincide in $\Sigma/\sim$, where $\sim$ identifies points on the same ghost component. The genus h of an equivalence class in S is the genus of the corresponding collapsed components or zero if there is no corresponding collapsed component, which corresponds to a node. Thus, we obtain the following definition.

Definition 2.2.8. A *stable fatgraph* is a fatgraph endowed with the extra structure of

- a set S of distinguished vertices;
- an equivalence relation $\sim$ on S;
- a genus function $h : S/\sim \to \mathbb{N}$ such that $h(S_0) > 0$ for any equivalence class $S_0 \subset S$ with $|S_0| = 1$.

We consider a stable fatgraph to be connected if it is connected after identification of vertices by the equivalence relation $\sim$. We define the genus of a connected stable fatgraph Γ to be the sum of the genera of the components, plus the sum of the values of the genus function h, plus the first Betti number of the dual graph of Γ. This is precisely to ensure that the enumeration of stable fatgraphs by degree recovers $\overline{N}_{g,n}(b_1, \ldots, b_n)$ as described below. Isomorphisms between stable fatgraphs are isomorphisms of fatgraphs that respect the extra structure — that is, they leave S invariant and preserve h.

Definition 2.2.9. For $(b_1, b_2, \ldots, b_n) \in \mathbb{Z}_+^n$, define $Fat_{g,n}^{\text{stable}}(b_1, b_2, \ldots, b_n)$ to be the set of isomorphism classes of labelled stable fatgraphs, connected after identification of vertices by $\sim$, of genus g with n boundary components of lengths $(b_1, b_2, \ldots, b_n)$, with all vertices of valence 1 contained in S.

The construction above defines a map

$$\overline{\mathcal{Z}}_{g,n}(b_1, b_2, \ldots, b_n) \to Fat_{g,n}^{\text{stable}}(b_1, b_2, \ldots, b_n),$$

which is no longer one-to-one in general since fibres can be infinite. Nevertheless,

$$\overline{N}_{g,n}(b_1, b_2, \ldots, b_n) = \sum_{\Gamma \in Fat_{g,n}^{\text{stable}}(\mathbf{b}_S)} w(\Gamma), \tag{2.2}$$

for weights $w(\Gamma)$ defined as a product of orbifold Euler characteristics of compactified moduli spaces. More explicitly, we take

$$w(\Gamma) = \frac{1}{\text{Aut}(\Gamma)} \prod_{v \in S/\sim} \chi(\overline{\mathcal{M}}_{h(v),n(v)})$$

where we have defined $n(S_0) = |S_0|$ for any equivalent class $S_0 \subset S$ and we set $\chi(\overline{M}_{0,2}) = 1$ to simplify the notation.

Example 2.2.10. As an example of the enumeration, we consider the value $\overline{N}_{1,1}(2) = \frac{1}{4}$. In the case of the uncompactified lattice point count, we have $N_{1,1}(2) = 0$. The contribution in the compactified count comes from the following figure, which shows the pullback of the interval $[0, 1]$ under a stable map from the pinched torus to $\mathbb{CP}^1$.

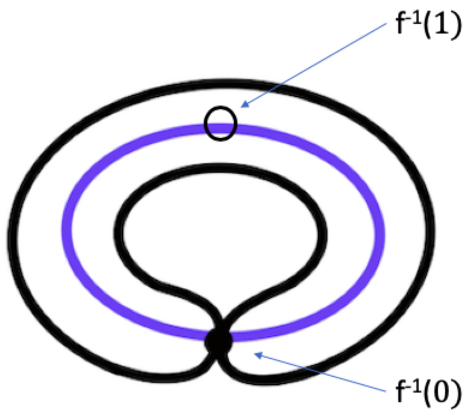

Figure 2.5: The construction of the unique stable fatgraph that contributes to $\overline{N}_{1,1}(2) = \frac{1}{4}$.

The compactified lattice point count $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ has a particularly nice structure, analogous to Theorem 2.2.5.

Theorem 2.2.11 (Do and Norbury [29])**.**

- *The compactified lattice point count* $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ *is a symmetric quasi-polynomial in* $b_1^2, b_1^2, \ldots, b_n^2$ *of degree* $3g - 3 + n$ *in the sense that it is polynomial on each coset of the sublattice* $2\mathbb{Z}^n \subset \mathbb{Z}^n$.
- *For* $\sum_{i=1}^n \alpha_i = 3g - 3 + n$, *the coefficient of* $b_1^{2\alpha_1} b_2^{2\alpha_2} \cdots b_n^{2\alpha_n}$ *in* $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ *is the following intersection number of psi-classes* $\psi_1, \psi_2, \ldots, \psi_n \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$.

$$\frac{1}{2^{5g-6+2n}\alpha_1!\,\alpha_2!\,\cdots\,\alpha_n!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \cdots \psi_n^{\alpha_n}$$

- *The constant coefficient of* $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ *is the orbifold Euler characteristics of* $\overline{\mathcal{M}}_{g,n}$.

$$\overline{N}_{g,n}(0, 0, \ldots, 0) = \chi(\overline{\mathcal{M}}_{g,n})$$

The following recursive formula can be used to effectively compute $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ from the base cases $\overline{N}_{0,3}(b_1, b_2, b_3)$ and $\overline{N}_{1,1}(b_1)$.

Theorem 2.2.12 (Do and Norbury [29])**.** *For* $2g - 2 + n \geq 2$ *and* $b_1, b_2, \ldots, b_n \geq 0$, *we have the following equation, where* $S = \{1, 2, \ldots, n\}$ *and* $\mathbf{b}_I = (b_{i_1}, b_{i_2}, \ldots, b_{i_k})$ *for* $I = \{i_1, i_2, \ldots, i_k\}$.

$$\left(\sum_{i=1}^n b_i\right) \overline{N}_{g,n}(\mathbf{b}_S) = \sum_{i<j} \sum_{\substack{p+q=b_i+b_j \\ q \text{ even}}} [p]q\, \overline{N}_{g,n-1}(p, \mathbf{b}_{S\setminus\{i,j\}})$$
$$+ \frac{1}{2} \sum_i \sum_{\substack{p+q+r=b_i \\ r \text{ even}}} [p][q]r \left[\overline{N}_{g-1,n+1}(p, q, \mathbf{b}_{S\setminus\{i\}}) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=S\setminus\{i\}}}^{\text{stable}} \overline{N}_{g_1,|I|+1}(p, \mathbf{b}_I)\, \overline{N}_{g_2,|J|+1}(q, \mathbf{b}_J)\right]$$

In the summations, p, q, r *vary over all non-negative integers and we use the notation* $[p] = p$ *for* p *positive and* $[0] = 1$. *The word* stable *over the final summation denotes that we exclude all terms with* $\overline{N}_{0,1}$ *or* $\overline{N}_{0,2}$.

2.3 Gromov–Witten theory

Gromov–Witten theory morally counts maps from curves to a target variety satisfying certain conditions. The theory crucially relies on the construction of the moduli space of stable maps and its virtual fundamental class.

Definition 2.3.1. Let X be a complex variety and fix a class $\beta \in H_2(X; \mathbb{Z})$. The moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X, \beta)$ parametrises stable morphisms

$$f : (C; x_1, x_2, \ldots, x_n) \to X,$$

where $(C; x_1, x_2, \ldots, x_n)$ is a nodal curve with marked points and $f_*[C] = \beta$. The morphism is stable if each genus 0 irreducible component of C that maps to a point of X has at least three special points and each genus 1 irreducible component of C that maps to a point of X has at least one special point. A special point is either a marked point or a branch of a node. We consider two such morphisms $f_1 : C_1 \to X$ and $f_2 : C_2 \to X$ to be equivalent if there is an isomorphism $\phi : C_1 \to C_2$ that preserves the marked points and satisfies $f_1 = f_2 \circ \phi$.

Example 2.3.2. One of the simplest non-trivial examples is given by the moduli space of stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{CP}^2, 1)$. Here, we use the identification $H^2(\mathbb{CP}^2; \mathbb{Z}) \cong \mathbb{Z}$ to express the homology class as an integer. This moduli space essentially parametrises maps $\mathbb{CP}^1 \to \mathbb{CP}^2$.

Although the moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is proper, it is often neither smooth nor even equidimensional. The enumerative geometry of stable maps requires integrating over this moduli space, which is made possible due to the construction of the so-called *virtual fundamental class* [87], which we denote by $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}$. It is a homology class whose dimension is given by the formula

$$\dim[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}} = (\dim X - 3)(1-g) + \int_\beta c_1(T_X) + n$$

Example 2.3.3. Let us consider the moduli space $\overline{\mathcal{M}}_{1,0}(\mathbb{CP}^2, 1)$, which demonstrates that the virtual dimension does not match the highest dimensional components of the moduli spaces. This space is the compactification of the moduli space of genus 1 maps to $\mathbb{CP}^2$ of degree 1. Notice that the open locus of smooth curves is empty as all genus 1 curves must have degree at least 3. However, there are non-smooth maps obtained in the following way. Consider a genus 0 curve connected to a genus 1 curve at a node. We can map it to $\mathbb{CP}^2$ by mapping the genus 0 curve onto a line and by collapsing the genus 1 curve to a point. The dimension of this space is 4, coming from 2 for the line, 1 for the point on the line and 1 for the genus 1 curve. However, the virtual dimension given by the above formula is 3.

For $i = 1, 2, \ldots, n$, there is a natural evaluation map

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$$

defined by

$$[f : (C; x_1, x_2, \ldots, x_n) \to X] \mapsto f(x_i).$$

Now suppose that we have subvarieties $W_1, W_2, \ldots, W_n$ of X and wish to enumerate stable maps that send the ith marked point to W_i. The homology class associated to W_i has a Poincaré dual $\gamma_i \in H^*(X)$, so we can consider the cohomology class $\mathrm{ev}_i^*\gamma_i$. The Poincaré dual of this class in some sense represents stable maps that send x_i to a point in W_i. Moreover, since cup product is dual to intersection, the Poincaré dual of the class

$$\mathrm{ev}_1^*\gamma_1 \cup \cdots \cup \mathrm{ev}_n^*\gamma_n$$

in some sense represents stable maps that send x_i to a point in W_i for $i = 1, 2, \ldots, n$. If the set of such maps is finite, at least morally, then the number of them is captured by the integral

$$\int_{\overline{\mathcal{M}}_{g,n}(X,\beta)} \mathrm{ev}_1^*\gamma_1 \cup \cdots \cup \mathrm{ev}_n^*\gamma_n.$$

More generally, let $\mathcal{L}_i$ be the ith cotangent line bundle over $\overline{\mathcal{M}}_{g,n}(X,\beta)$, whose fibre over a point in $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is the cotangent line $T^*_{p_i}C$ at the marked point p_i in the domain curve C. Define $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}(X,\beta);\mathbb{Q})$ to be the first Chern class of $\mathcal{L}_i$. For $i = 1, 2, \ldots, n$, one can consider descendent classes

$$\tau_{b_i}(\gamma) = \psi_i^{b_i}\mathrm{ev}_i^*(\gamma),$$

and these can also be integrated against the virtual fundamental class.

Definition 2.3.4. For $\gamma_1, \gamma_2, \ldots, \gamma_n \in H^*(X;\mathbb{Z})$, we define the *Gromov–Witten invariant*

$$\langle \tau_{b_1}(\gamma_1)\tau_{b_2}(\gamma_2)\cdots\tau_{b_n}(\gamma_n)\rangle_{g,n}^{X,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1)\psi_1^{b_1} \cup \cdots \cup \mathrm{ev}_n^*(\gamma_1)\psi_n^{b_n}.$$

Example 2.3.5. Let pt denote the class of a point in $\mathbb{CP}^2$. Then the Gromov–Witten invariant

$$\langle \tau_0(\mathrm{pt}), \ldots, \tau_0(\mathrm{pt})\rangle_{0,3d-1}^{\mathbb{CP}^2,d}$$

is the number of degree d rational curves in $\mathbb{CP}^2$ passing through $3d-1$ points.

For the remainder of this section, we consider the special case when the algebraic variety is $X = \mathbb{CP}^1$, which is of particular interest to us. Let $\omega \in H^2(\mathbb{CP}^1;\mathbb{Q})$ be the Poincaré dual class of a point and let $1 \in H^0(\mathbb{CP}^1;\mathbb{Q})$ the Poincaré dual of the fundamental class. We consider the connected Gromov–Witten invariants

$$\Big\langle \prod_{i=1}^{\ell} \tau_{b_i}(1) \prod_{i=\ell+1}^{n} \tau_{b_i}(\omega) \Big\rangle_{g,n}^{\mathbb{CP}^1,d}. \tag{2.3}$$

These are defined to be zero unless the dimension condition $\sum b_i = 2g - 2 + 2d + \ell$ is satisfied.

We collect here a few properties of Gromov–Witten invariants of $\mathbb{CP}^1$, as they appear in [87].

- Divisor equation

$$\begin{aligned}\langle \tau_0(\omega)\tau_{b_1}(\gamma_1)\cdots\tau_{b_n}(\gamma_n)\rangle_{g,n+1}^d &= d\,\langle \tau_{b_1}(\gamma_1)\cdots\tau_{b_n}(\gamma_n)\rangle_{g,n}^d \\ &\quad + \sum_{i=1}^{n}\langle \tau_{b_1}(\gamma_1)\cdots\tau_{b_i-1}(\gamma_i\cup\omega)\cdots\tau_{b_n}(\gamma_n)\rangle_{g,n}^d.\end{aligned} \tag{2.4}$$

- String equation

$$\langle \tau_0(1)\tau_{b_1}(\gamma_1)\cdots\tau_{b_n}(\gamma_n)\rangle_{g,n+1} = \sum_{i=1}^{n}\langle \tau_{b_1}(\gamma_1)\cdots\tau_{b_i-1}(\gamma_i)\cdots\tau_{b_n}(\gamma_n)\rangle_{g,n}. \tag{2.5}$$

- Dilaton equation

$$\langle \tau_1(1)\tau_{b_1}(\gamma_1)\cdots\tau_{b_n}(\gamma_n)\rangle_{g,n+1} = (2g-2+n)\,\langle \tau_{b_1}(\gamma_1)\cdots\tau_{b_n}(\gamma_n)\rangle_{g,n} \tag{2.6}$$

- Topological recursion relations. Define the following generating function for descendent classes.

$$F = \exp\left(\sum_{b=0}^{\infty} t_b\tau_b(\omega) + s_b\tau_b(1)\right)$$

  For $\gamma_i \in \{1,\omega\}$, the genus zero topological recursion relation is

$$\begin{aligned}\langle \tau_{b_1}(\gamma_1)\tau_{b_2}(\gamma_2)\tau_{b_3}(\gamma_3)F\rangle_0 &= \langle \tau_0(1)\tau_{b_1-1}(\gamma_1)F\rangle_0\,\langle \tau_0(\omega)\tau_{b_2}(\gamma_2)\tau_{b_3}(\gamma_3)F\rangle_0 \\ &\quad + \langle \tau_0(\omega)\tau_{b_1-1}(\gamma_1)F\rangle_0\,\langle \tau_0(1)\tau_{b_2}(\gamma_2)\tau_{b_3}(\gamma_3)F\rangle_0,\end{aligned}$$

  and the genus one topological recursion relation is

$$\begin{aligned}\langle \tau_{b_1}(\gamma_1)F\rangle_1 &= \langle \tau_0(1)\tau_{b_1-1}(\gamma_1)F\rangle_0\,\langle \tau_0(\omega)F\rangle_1 \\ &\quad + \langle \tau_0(\omega)\tau_{b_1-1}(\gamma_1)F\rangle_0\,\langle \tau_0(1)\tau_0(\omega)\tau_{b_1-1}(\gamma_1)F\rangle_0.\end{aligned}$$

2.4 Cohomological field theories

Kontsevich and Manin [70] introduced the notion of a cohomological field theory, which generalises Gromov–Witten theory by substituting the cohomology of the target space with a vector space equipped with a nonlinear bilinear form, along with other data. As usual, we let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne–Mumford compactification of the moduli space of genus g curves with n marked points.

Definition 2.4.1. A *cohomological field theory* (or CohFT) is a collection of data

$$V, \quad \eta, \quad 1, \quad \{\alpha_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n};\mathbb{Q}) \otimes (V^*)^{\otimes n} \mid 2g-2+n > 0\},$$

where V is a vector space over $\mathbb{Q}$ with a choice of basis $e_1, e_2, \ldots, e_N$, η is a nondegenerate bilinear form, and 1 is an element in V that we refer to as the identity. We require all of the data to satisfy the following conditions, where $\eta_{ij} := \eta(e_i, e_j)$ and η^{ij} denotes the inverse matrix.

- *Permutation of marked points*
  A permutation $\sigma \in S_n$ acts on $\overline{\mathcal{M}}_{g,n}$ by permuting the marked points, which then defines a map $\sigma_m : H^*(\overline{\mathcal{M}}_{g,n};\mathbb{Q}) \to H^*(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$. A permutation $\sigma \in S_n$ also acts on $(V^*)^{\otimes n}$ by permuting the tensor factors, which then defines a map $\sigma_v : (V^*)^{\otimes n} \to (V^*)^{\otimes n}$. We can view each $\alpha_{g,n}$ as a map
  $$V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$$
  and we require this map to form a commutative square with σ_m and σ_v.
- *Gluing maps*
  Let gl_1 denote the gluing map which takes the two marked points labelled $n+1$ and $n+2$ on $\overline{\mathcal{M}}_{g,n+2}$ and glues them together in a node. Then
  $$gl_1 : \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$$
  induces a map $gl_1^* : H^*(\overline{\mathcal{M}}_{g+1,n};\mathbb{Q}) \to H^*(\overline{\mathcal{M}}_{g,n+2};\mathbb{Q})$. We then require for $v_1,\dots,v_n \in V$,
  $$gl_1^*\alpha_{g,n}(v_1 \otimes \cdots \otimes v_n) = \sum_{i,j=1}^{N} \eta^{i,j}\alpha_{g,n+2}(v_1 \otimes \cdots \otimes v_n \otimes e_i \otimes e_j).$$
  Similarly, let denote the gluing map which takes the last marked point on two distinct curves and glues them together in a node. Then
  $$gl_2 : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$$
  induces a map $gl_2^* : H^*(\overline{\mathcal{M}}_{g_1+g_2,n_1+n_2};\mathbb{Q}) \to H^*(\overline{\mathcal{M}}_{g_1,n_1+1};\mathbb{Q}) \otimes H^*(\overline{\mathcal{M}}_{g_2,n_2+1};\mathbb{Q})$. We then require for $v_1,\dots,v_{n_1},w_1,\dots,w_{n_2} \in V$,
  $$\begin{aligned} &gl_2^*\alpha_{g_1+g_2,n_1+n_2}(v_1 \otimes \cdots \otimes v_{n_1} \otimes w_1 \otimes \cdots \otimes w_{n_2}) \\ &\qquad = \sum_{i,j=1}^{N} \eta^{i,j}\alpha_{g_1,n_1+1}(v_1 \otimes \cdots \otimes v_{n_1} \otimes e_i) \otimes \alpha_{g_2,n_2+1}(w_1,\dots w_{n_2} \otimes e_j). \end{aligned}$$
- *Forgetful map*
  Let $p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ denote the map that forgets the last marked point. We require for all $v_1,\dots,v_n \in V$,
  $$p^*\alpha_{g,n}(v_1 \otimes \cdots \otimes v_n) = \alpha_{g,n+1}(v_1 \otimes \cdots \otimes v_n \otimes 1), \tag{2.7}$$
  $$\alpha_{0,3}(v_1 \otimes v_2 \otimes 1) = \eta(v_1,v_2). \tag{2.8}$$

In certain circumstances, it is convenient to remove the conditions of equation (2.7) and equation (2.8), in which case we refer to the resulting structure as a CohFT without unit.

It is known that there is a close relation between CohFTs and the notion of topological recursion. In particular, the work of Dunin-Barkowski, Orantin, Shadrin and Spitz demonstrates that under certain conditions, the correlation differentials that are output by topological recursion store correlators for a CohFT [42]. This allows for one to relate topological recursion to moduli spaces of curves. We will discuss topological recursion in more detail in the next chapter.

Chapter 3

Topological recursion

In this chapter, we give some brief motivation for topological recursion before describing its original formulation, along with some subsequent generalisations. First, we describe the topological recursion, as it was originally discovered by Chekhov, Eynard and Orantin. Next, we describe the local topological recursion, which no longer requires the spectral curve to come from a compact Riemann surface. Finally, we discuss the more recent formulation of topological recursion by Kontsevich and Soibelman, involving the notion of quantum Airy structures. We conclude the chapter with a discussion of the related concept of quantum curves and give an example calculation in the context of BMS numbers.

3.1 Motivation

Topological recursion is a recursive process to compute certain correlation differentials based on the initial data of a spectral curve. The recursive mechanism is based on cutting pairs of pants from a surface with boundary. It was initially discovered by Chekhov, Eynard and Orantin in the context of matrix models [23, 47]. After the work of Mirzakhani [77, 78] computing the volumes of moduli spaces of hyperbolic surfaces with geodesic boundaries, Eynard and Orantin noticed that those same quantities could be obtained through topological recursion. The discovery of the topological recursion in various other settings around that time helped to establish it as a universal theory independent of matrix models.

In this section, we describe some motivation for topological recursion, using Mirzakhani's work as the basis. To initiate our study, we require a brief discussion of Teichmüller theory.

Definition 3.1.1. Let $S_{g,n}$ denote a connected orientable smooth surface of genus g with n labelled boundary components. For $L_1, \ldots, L_n \in \mathbb{R}_+$, the *Teichmüller space* is defined as

$$\mathcal{T}_{g,n}(L_1, \ldots, L_n) = \left\{ (X, f) \;\middle|\; \begin{array}{l} X \text{ is a hyperbolic surface with labelled geodesic} \\ \text{boundary components of lengths } L_1, \ldots, L_n \text{ and} \\ f : S_{g,n} \to X \text{ is a diffeomorphism} \end{array} \right\} / \sim,$$

where the equivalence relation is given by $(X, f) \sim (Y, g)$ if and only if there is an isometry $\phi : X \to Y$ such that $\phi \circ f$ is isotopic to g.

Let $\mathrm{Mod}_{g,n}$ be the mapping class group of $S_{g,n}$ — that is, the group of isotopy classes of orientation-preserving homeomorphisms $S_{g,n} \to S_{g,n}$ leaving boundary components fixed set wise. The mapping class group acts on $\mathcal{T}_{g,n}(L_1, \ldots, L_n)$ by changing the marking f of a pair (X, f). The quotient space

$$\mathcal{M}_{g,n}(L_1, \ldots, L_n) := \mathcal{T}_{g,n}(L_1, \ldots, L_n)/\mathrm{Mod}_{g,n}$$

is the moduli space of hyperbolic surfaces of genus g with n boundary components of lengths $L_1, \ldots, L_n$.

Fix a pair of pants decomposition of $S_{g,n}$ and observe that it must comprise $3g-3+n$ simple closed curves, which we order from 1 up to $3g-3+n$. Then for each element $(X, f) \in \mathcal{T}_{g,n}(L_1, \ldots, L_n)$, one can assign the positive real number lengths $\ell_1(X, f), \ldots, \ell_{3g-3+n}(X, f)$ describing the lengths of the closed curves as well as real number twist parameters $\tau_1(X, f), \ldots, \tau_{3g-3+n}(X, f)$. This defines a homeomorphism

$$\mathcal{M}_{g,n}(L_1, \ldots, L_n) \cong \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n}.$$

These are known as the *Fenchel–Nielsen coordinates* and the interested reader is referred to the literature for more information [90].

The *Weil–Petersson symplectic form* on Teichmüller space is given by $\omega_{g,n} = \sum_{i=1}^{3g-3+n} d\ell_i \wedge d\tau_i$ and it is invariant under the mapping class group [53]. Thus, it descends to a symplectic form on the moduli space $\mathcal{M}_{g,n}(L_1, \ldots, L_n)$. Hence, we may define the volume of the moduli space of hyperbolic surfaces to be

$$V_{g,n}(L_1, \ldots, L_n) = \int_{\mathcal{M}_{g,n}(L_1,\ldots,L_n)} \frac{\omega_{g,n}^{3g-3+n}}{(3g-3+n)!}. \tag{3.1}$$

Since hyperbolic structures on surfaces with geodesic boundary require negative Euler characteristic, we restrict attention to the pairs (g, n) satisfying $2g - 2 + n > 0$. Some examples of $V_{g,n}(L_1, \ldots, L_n)$ are given below.

$$\begin{aligned} V_{0,3}(L_1, L_2, L_3) &= 1 \\ V_{1,1}(L_1) &= \frac{1}{48}\left(L_1^2 + 4\pi^2\right) \\ V_{0,4}(L_1, L_2, L_3, L_4) &= \frac{1}{2}(L_1^2 + L_2^2 + L_3^2 + L_4^2 + 4\pi^2) \end{aligned}$$

Observe that these are polynomials, a fact that was demonstrated in general by Mirzakhani [78]. She furthermore established a recursion for these polynomials and interpreted their coefficients as intersection numbers on Deligne–Mumford compactifications of moduli spaces of curves [77, 78].

To make connection with topological recursion, we take the Laplace transform of the volume $V_{g,n}(L_1, \ldots, L_n)$ and define

$$W_{g,n}(z_1, \ldots, z_n) := \int_0^\infty \cdots \int_0^\infty L_1 e^{-z_1 L_1} \cdots L_n e^{-z_n L_n}\, V_{g,n}(L_1, \ldots, L_n)\, \mathrm{d}L_1 \cdots \mathrm{d}L_n. \tag{3.2}$$

Some examples of $W_{g,n}(z_1, \ldots, z_n)$ are given below.

$$\begin{aligned} W_{0,3}(z_1, z_2, z_3) &= \frac{1}{z_1^2 z_2^2 z_3^2} \\ W_{1,1}(z_1) &= \frac{1}{24}\left(\frac{3}{z_1^4} + \frac{2\pi^2}{z_1^2}\right) \\ W_{0,4}(z_1, z_2, z_3, z_4) &= \frac{1}{z_1^2 z_2^2 z_3^2 z_4^2}\left(3 \sum_{i=1}^{4} \frac{1}{z_i^2} + 2\pi^2\right) \end{aligned}$$

The following theorem rewrites Mirzakhani's recursion for $V_{g,n}$ in terms of the Laplace transforms $W_{g,n}$ and is due to Eynard and Orantin [47].

Theorem 3.1.2. *Let $W_{g,n}(z_1,\dots,z_n)$ be as above and define $W_{0,2}=\frac{1}{(z_1-z_2)^2}$. Then for $2g-2+n>0$,*

$$W_{g,n}(z_1\dots,z_n)=\operatorname*{Res}_{z=0}\frac{\mathrm{d}z}{z_1^2-z^2}\frac{2\pi}{\sin(2\pi z)}\Bigg[W_{g-1,n+1}(z,-z,z_2,\dots,z_n)+\sum_{\substack{g_1+g_2=g\\ I_1\sqcup I_2=\{z_2,\dots,z_n\}}}W_{g_1,|I_1|+1}(z,I_1)\,W_{g_1,|I_1|+1}(-z,I_2)\Bigg]. \tag{3.3}$$

Example 3.1.3. Let us use the theorem above to compute $W_{1,1}(z_1)$.

$$\begin{aligned}
W_{1,1}(z_1)&=\operatorname*{Res}_{z=0}\frac{\mathrm{d}z}{(z_1^2-z^2)}\frac{\pi}{\sin(2\pi z)}W_{0,2}(z,-z)\\
&=\operatorname*{Res}_{z=0}\frac{\mathrm{d}z}{(z_1^2-z^2)}\frac{\pi}{\sin(2\pi z)}\frac{1}{4z^2}\\
&=\frac{1}{4}\operatorname*{Res}_{z=0}\frac{\mathrm{d}z}{z^2}\left(\frac{1}{z_1^2}+\frac{z^2}{z_1^4}+O(z^4)\right)\left(\frac{1}{2z}+\frac{\pi^2 z}{3}+\frac{7\pi^4z^3}{45}+O\left(z^4\right)\right)\\
&=\frac{1}{8}\operatorname*{Res}_{z=0}\frac{\mathrm{d}z}{z}\left(\frac{1}{z_1^2}+\frac{z^2}{z_1^4}+O(z^4)\right)\left(1+\frac{2\pi^2 z^2}{3}+\frac{14\pi^4z^4}{45}+O\left(z^4\right)\right)\\
&=\frac{1}{24}\left(\frac{3}{z_1^4}+\frac{2\pi^2}{z_1^2}\right)
\end{aligned}$$

Now the question is what can be generalised from formula equation (3.3) that can then be applied to other enumerative problems.

- One would like the calculation to be invariant under change of coordinates $W_{g,n}(z_1,\dots,z_n)$. Furthermore, the recursion involves taking a residue, so it is natural to define

$$\omega_{g,n}(z_1,\dots,z_n):=W_{g,n}(z_1,\dots,z_n)\,\mathrm{d}z_1\otimes\cdots\otimes\mathrm{d}z_n.$$

  This is a symmetric multidifferential on $\mathbb{C}^n$. In this language, equation (3.3) leads to a recursive equation for $\omega_{g,n}$ as follows.

$$\omega_{g,n}=\operatorname*{Res}_{z=0}\frac{\mathrm{d}z_1}{(z_1^2-z^2)\,\mathrm{d}(-z)}\frac{2\pi}{\sin(2\pi z)}\Bigg[\omega_{g,n}(z,-z,z_2,\dots,z_n)+\sum_{\substack{g_1+g_2=g\\ I_1\sqcup I_2=\{z_2,\dots,z_n\}}}\omega_{g_1,I_1}(z,I_1)\,\omega_{g_2,I_2}(-z,I_2)\Bigg]. \tag{3.4}$$

- Now we observe the following and use it to substitute into the right side of the previous equation.

$$\int_{z'=-z}^{z}\omega_{0,2}(z_1,z')=\mathrm{d}z_1\left[\frac{1}{z_1-z}-\frac{1}{z_1+z}\right]=\frac{2z\,\mathrm{d}z_1}{z_1^2-z^2}$$

Hence, we can write

$$\omega_{g,n}=\operatorname*{Res}_{z=0}\frac{\int_{z'=-z}^{z}\omega_{0,2}(z_1,z')}{2z\,\mathrm{d}(-z)}\frac{2\pi}{2z\sin(2\pi z)}\Bigg[\omega_{g,n}(z,-z,z_2,\dots,z_n)+\sum_{\substack{g_1+g_2=g\\ I_1\sqcup I_2=\{z_2,\dots,z_n\}}}\omega_{g_1,I_1}(z,I_1)\,\omega_{g_2,I_2}(-z,I_2)\Bigg]. \tag{3.5}$$

In the next section, we define the spectral curve that forms the input to the topological recursion and allows us to generalise the above equations to other settings. A natural question to keep in mind is how one may guess or derive the spectral curve for a given enumerative geometry problem. We discuss this further in later sections.

3.2 Topological recursion

Original definition

In general, the topological recursion takes as input a spectral curve and produces multidifferentials $\omega_{g,n}$ for integers $g \geq 0$ and $n \geq 1$, which we refered to as *correlation differentials*. If the underlying Riemann surface of the spectral curve is $\mathcal{C}$, then $\omega_{g,n}$ is a symmetric meromorphic section of the line bundle $\pi_1^*(T^*\mathcal{C}) \otimes \pi_2^*(T^*\mathcal{C}) \otimes \cdots \otimes \pi_n^*(T^*\mathcal{C})$ on the Cartesian product $\mathcal{C}^n$, where $\pi_i \colon \mathcal{C}^n \to \mathcal{C}$ denotes projection onto the ith factor. An explicit definition of topological recursion follows.

- **Initial data.** A *spectral curve* is a tuple $(\mathcal{C}, x, y, T)$, where $\mathcal{C}$ is a compact Riemann surface, x and y are meromorphic functions on $\mathcal{C}$, and T is a Torelli marking on $\mathcal{C}$ — that is, a choice of symplectic basis for $H_1(\mathcal{C}; \mathbb{Z})$. We furthermore require the zeroes of $\mathrm{d}x$ to be simple and disjoint from the zeroes and poles of $\mathrm{d}y$.
- **Base cases.** Let $\omega_{0,1}(z_1) = -y(z_1)\,\mathrm{d}x(z_1)$. Let $\omega_{0,2}(z_1, z_2)$ be the unique meromorphic bi-differential on $\mathcal{C}$ that has double poles without residue along the diagonal $z_1 = z_2$, is holomorphic away from the diagonal, and is normalised on the $\mathcal{A}$-cycles of the Torelli marking via the equation

$$\oint_{\mathcal{A}_i} \omega_{0,2}(z_1, z_2) = 0, \qquad \text{for } i = 1, 2, \ldots, \operatorname{genus}(\mathcal{C}).$$

- **Recursion.** For $2g - 2 + n > 0$, the multi-differentials $\omega_{g,n}(z_1, z_2, \ldots, z_n)$ are defined recursively by the following equation, where $S = \{2, 3, \ldots, n\}$ and $\mathbf{z}_I = (z_{i_1}, z_{i_2}, \ldots, z_{i_k})$ for $I = \{i_1, i_2, \ldots, i_k\}$.

$$\omega_{g,n}(z_1, \mathbf{z}_S) = \sum_{\alpha} \operatorname*{Res}_{z=\alpha} K(z_1, z) \Bigg[\omega_{g-1,n+1}\left(z, s(z), \mathbf{z}_S\right) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}}^{\circ} \omega_{g_1,|I|+1}(z, \mathbf{z}_I)\, \omega_{g_2,|J|+1}(s(z), \mathbf{z}_J) \Bigg] \tag{3.6}$$

  The outer summation is over the zeroes α of $\mathrm{d}x$, while the symbol $\circ$ over the inner summation denotes that we exclude all terms with $\omega_{0,1}$. The function $s(z)$ is the unique non-identity holomorphic map defined in a neighbourhood of the simple ramification point $\alpha \in \mathcal{C}$ satisfying $x(s(z)) = x(z)$. Finally, the kernel $K(z_1, z)$ is defined by

$$K(z_1, z) = -\frac{\int_o^z \omega_{0,2}(z_1, \cdot\,)}{[y(z) - y(s(z))]\ \mathrm{d}x(z)},$$

  where o can be taken to be an arbitrary point on the spectral curve.

Notice that the (g', n') appearing in the right side of the above recursion satisfy $2g' - 2 + n' < 2g - 2 + n$. The following schematic diagram demonstrates the point mentioned earlier that this recursive mechanism is based on cutting pairs of pants from a surface with boundary.

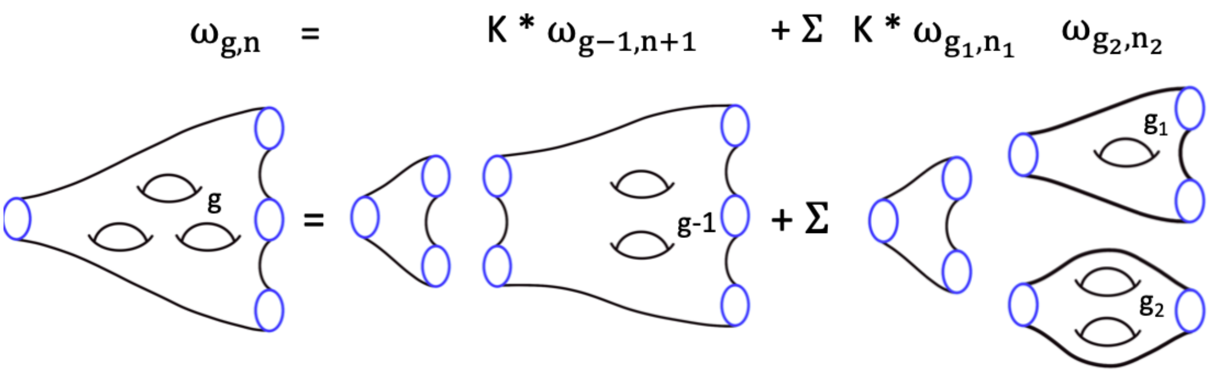

Figure 3.1: Schematic diagram for topological recursion.

As mentioned in the previous section, the topological recursion recovers Mirzakhani's volume polynomials. The spectral curve in that case is

$$\left(\mathbb{CP}^1,\ x(z) = z^2,\ y(z) = \frac{-\sin(2\pi z)}{4\pi},\ \omega_{0,2} = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1 - z_2)^2}\right).$$

For another example, consider the Bouchard–Mariño conjecture [19], which states that simple Hurwitz numbers, as defined in Section 1.2, satisfy the topological recursion formalism applied to the spectral curve

$$\left(\mathbb{CP}^1,\ x(z) = -1 - z + \log(1+z),\ y(z) = z + 1,\ \omega_{0,2} = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1 - z_2)^2}\right).$$

To do the computation we need to analyse the spectral curve. This case is unusual in the sense that $x(z)$ is not meromorphic. The usual topological recursion formulas still apply though, or alternatively, it can be understood by truncating the Taylor series for $x(z)$. The Bouchard–Mariño conjecture was proved by Eynard, Mulase and Safnuk [50]. The diversity of these two enumerative problems mentioned gives some mild indication of the breadth of topological recursion in terms of its applicability.

Properties of topological recursion

The invariants $\omega_{g,n}$ produced by topological recursion share some interesting properties. We briefly give a list of the main ones.

- *Symmetry.* The correlation differentials $\omega_{g,n}(z_1, \ldots, z_n)$ produced by topological recursion are symmetric under permutation of the variables $z_1, \ldots, z_n$. This property is not obvious from the definition of topological recursion [47].
- *Homogeneity.* The correlation differentials $\omega_{g,n}(z_1, \ldots, z_n)$ produced by topological recursion are homogeneous under the transformation $\omega_{0,1} \mapsto \lambda\omega_{0,1}$ for $\lambda \in \mathbb{C}^*$ in the sense that they transform as $\omega_{g,n} \mapsto \lambda^{2-2g-n}\omega_{g,n}$ for $2g - 2 + n > 0$.
- *Pole structure.* For $2g - 2 + n > 0$, the correlation differential $\omega_{g,n}$ is a meromorphic multidifferential on the spectral curve, with poles only at the ramification points and zero residues at those poles.
- *String and dilaton equations.* For $2g - 2 + n > 0$, we have the following string and dilaton equations, where each left side is a summation over the zeroes α of $\mathrm{d}x$, S denotes the set

$\{1, 2, \dots, n\}$, and $\Phi(z)$ is any function satisfying $\mathrm{d}\Phi(z) = y(z)\,\mathrm{d}x(z)$.

$$\sum_{\alpha} \operatorname*{Res}_{z=\alpha} y(z)\,\omega_{g,n+1}(z, \mathbf{z}_S) = -\sum_{k=1}^{n} \mathrm{d}z_k \frac{\partial}{\partial z_k}\left(\frac{\omega_{g,n}(\mathbf{z}_S)}{\mathrm{d}x(z_k)}\right)$$
$$\sum_{\alpha} \operatorname*{Res}_{z=\alpha} \Phi(z)\,\omega_{g,n+1}(z, \mathbf{z}_S) = (2g-2+n)\,\omega_{g,n}(\mathbf{z}_S)$$

Since its discovery, topological recursion is now known or conjectured to govern many different problems, providing a universal framework for their understanding. A few such are listed in the following table, along with their associated spectral curves.

Enumerative problem	**Spectral curve**
Intersection theory on moduli spaces of curves [47]	$x(z) = \frac{1}{2}z^2 \quad y(z) = z$
Enumeration of fatgraphs [85, 38]	$x(z) = z + \frac{1}{z} \quad y(z) = z$
Enumeration of hypermaps [35]	$x(z) = z^{a-1} + \frac{1}{z} \quad y(z) = z$
Weil–Petersson volumes of moduli spaces [51]	$x(z) = z^2 \quad y(z) = \frac{\sin(2\pi z)}{2\pi}$
Simple and orbifold Hurwitz numbers [34]	$x(z) = z\exp(-z^a) \quad y(z) = z^a$
Spin Hurwitz numbers [79]	$x(z) = z\exp(-z^r) \quad y(z) = z$
Stationary Gromov–Witten theory of $\mathbb{CP}^1$ [43]	$x(z) = z + \frac{1}{z} \quad y(z) = \log(z)$
Asymptotics of coloured Jones polynomials of knots [11]	A-polynomials

The spectral curves introduced above have the technical assumption that the zeroes of $\mathrm{d}x$ must be simple and disjoint from the zeroes and poles of $\mathrm{d}y$ [47].There are two possible generalisations of this fact that aim to relax this assumption. Do and Norbury [36] studied spectral curves where zeroes of $\mathrm{d}x$ and poles of $\mathrm{d}y$ intersect. They call such a spectral curve *irregular* and show that in some cases these can be handled by the usual topological recursion formula. Bouchard and Eynard [16] studied spectral curves with non-simple zeroes of $\mathrm{d}x$ but disjoint from zeroes of $\mathrm{d}y$. The computation of correlation differentials $\omega_{g,n}$ becomes more complex in this case. However, since it will not arise in the problems that we study, we do not present this generalised topological recursion here.

Topological recursion and graphs

Given a spectral curve, let $\mathcal{R}$ denote the set of ramification points of $x(z)$ at which $\mathrm{d}x$ vanishes. At such a ramification point $a \in \mathcal{R}$, which is assumed to be simple, we have the natural local coordinate $\zeta_a(z)$ that satisfies $(\zeta_a(z))^2/2 = x(z) - x(a)$. Let the following denote the series expansion of $\omega_{0,1}(z)$ around the ramification point a, with respect to the local coordinate $\zeta_a(z)$.

$$\omega_{0,1}(z) = \sum_k t_{a,k}\zeta_a(z)^k\,\mathrm{d}\zeta_a$$

For ramification points $a_1, a_2 \in \mathcal{R}$ let the following denote the series expansion of $\omega_{0,2}(z_1, z_2)$ with respect to the local coordinates $\zeta_{a_1}(z_1)$ and $\zeta_{a_2}(z_2)$.

$$\omega_{0,2}(z_1, z_2) = \frac{\delta_{a_1,a_2}\,\mathrm{d}\zeta_{a_1}(z_1)\,\mathrm{d}\zeta_{a_2}(z_2)}{(\zeta_{a_1}(z_1) - \zeta_{a_2}(z_2))^2} + \sum_{k_1,k_2} B_{(a_1,k_1),(a_2,k_2)}\zeta_{a_1}(z_1)^{k_1-1}\zeta_{a_2}(z_2)^{k_2-1}\,\mathrm{d}\zeta_{a_1}(z_1)\,\mathrm{d}\zeta_{a_2}(z_2)$$

Let us also introduce the meromorphic 1-forms

$$\xi_{a,\ell}(z) = \operatorname*{Res}_{z'=a} \left(\int_a^{z'} \omega_{0,2}(\,\cdot\,, z) \right) \frac{\mathrm{d}\zeta(z')}{\zeta(z')^{\ell+1}}.$$

The following theorem [47] expresses the correlation differential $\omega_{g,n}(z_1, z_2, \ldots, z_n)$ as a polynomial in the basis $\xi_{a,k}(z)$.

Theorem 3.2.1. *For $2g-2+n>0$, there exists a unique decomposition*

$$\omega_{g,n}(z_1, \ldots, z_n) = 2^{3g-3+n} \sum_{\substack{a_1,a_2,\ldots,a_n \\ d_1,d_2\ldots,d_n}} F_{g,n}((a_1,d_1), \ldots, (a_n,d_n)) \prod_{i=1}^{n} \xi_{a_i,d_i}(z_i),$$

where $F_{g,n}((a_1,d_1), \ldots, (a_n,d_n))$ is a polynomial in the $t_{a,k}$ and the $B_{(a_1,k_1),(a_2,k_2)}$. Moreover, the sum is finite and only terms with $d_1 + \cdots + d_n \leq 3g-3+n$ contribute.

We remark that the coefficient $F_{g,n}((a_1,d_1), \ldots, (a_n,d_n))$ has a nice description as a sum over coloured dual graphs, with contributions coming from intersection numbers on the moduli space of curves [46].

Local topological recursion

One can observe that the topological recursion actually only requires the local information of the meromorphic functions x, y and the bidifferential $\omega_{0,2}$ at the ramification points of the spectral curve, in order to produce the correlation differentials. Thus, one can more generally define topological recursion on spectral curves comprising isolated local germs of x, y and $\omega_{0,2}$, without requiring the existence of a global compact Riemann surface on which this data can be defined. In particular, the local topological recursion requires $\omega_{0,2}$ to become part of the spectral curve data. This viewpoint was promoted by Dunin-Barkowksi, Orantin, Shadrin, and Spitz in their work relating topological recursion to Givental's approach to cohomological field theory [43].

More explicitly, we briefly reproduce the formulas required to apply the local topological recursion. The local spectral curve input consists of

- N families of complex numbers h^i_k for $1 \leq i \leq N$ and $k \in \mathbb{N}$;
- $N \times N$ infinite families of complex numbers $B^{ij}_{k\ell}$ for $1 \leq i, j \leq N$ and $k, \ell \in \mathbb{N}$; and
- N distinct canonical coordinates a_i for $1 \leq i \leq N$.

We consider a small open neighbourhood of $0 \in \mathbb{C}$ and define on it the following analytic functions for $1 \leq i, j \leq N$.

$$x^i(z) = z^2 + a_i, \qquad y^i(z) = \sum_{k=0}^{\infty} h^i_k z^k, \qquad B^{ij}(z,z') = \left[\frac{\delta_{ij}}{(z-z')^2} + \sum_{k,\ell=0}^{\infty} B^{ij}_{k\ell} z^k z'^\ell \right] \mathrm{d}z \mathrm{d}z'$$

The base cases of the recursion are given by

$$\omega^i_{0,1}(z) = 0, \qquad \omega^{ij}_{0,2}(z,z') = B^{ij}(z,z').$$

The recursion formula is then given by

$$\omega_{g,n+1}^{i_0 i_1 \cdots i_n}(z_0, \mathbf{z}_S) = \sum_{j=1}^{N} \operatorname*{Res}_{z=0} \frac{\int_{-z}^{z} B^{i_0 j}(z_0, \cdot)}{2(y^j(z) - y^j(-z))\,\mathrm{d}x^j(z)} \Bigg[\omega_{g-1,n+2}^{jji_1\cdots i_n}(z, -z, \mathbf{z}_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} \omega_{g_1,|I|+1}^{j,i_I}(z, \mathbf{z}_I)\,\omega_{g_2,|J|+1}^{j,i_J}(z, \mathbf{z}_J) \Bigg]. \tag{3.7}$$

It should be clear from these definitions how the local topological recursion agrees with the usual topological recursion by taking x^i and y^i and B^{ij} to be the local expansions of x, y and $\omega_{0,2}$ around the ith and jth ramification points, in the local coordinates under which the involution is given by negation.

3.3 Kontsevich–Soibelman topological recursion

Topological recursion as described in Section 3.2 produces multidifferentials $\omega_{g,n}$ as well as scalars $F_g = \omega_{g,0}$, which enjoy the mysterious property of symplectic invariance in many cases and conjecturally in all cases [49]. Recently, Kontsevich and Soibelman [71] proposed a new point of view that generalises the topological recursion of Chekhov, Eynard and Orantin [23, 47]. The starting point for this so-called *KS topological recursion* is a quantisation of the classical Airy structure, which is a quadratic differential equation in a symplectic vector space. A quantum Airy structure can be described explicitly in terms of four tensors A, B, C, D which must satisfy certain relations, given below in equations (3.12) to (3.16). In this section, we briefly explain the notion of classical and quantum Airy structures, as well as how the tensors A, B, C, D give the topological recursion formalism.

Let V be a vector space of finite or countable dimension over $\mathbb{C}$. Let n denote the dimension of V and define the index set $I = \{1, 2, \ldots, n\}$, where we possibly take n to infinity. Let us choose an ordered basis for V and denote it by $x_1, x_2, \ldots, x_n$. Denote by $y^1, y^2, \ldots, y^n$ the dual basis for V^*. The vector space $W := T^*(V) = V \oplus V^*$ has a standard symplectic structure and the corresponding Possion bracket is given by

$$\{x_i, y^j\} = \delta_{ij}, \qquad \{x_i, x_j\} = 0, \qquad \{y^i, y^j\} = 0.$$

We denote by $\mathrm{Sym}_{\leq 2}(W)$ the Lie algebra $\mathrm{Sym}^0(W) \oplus \mathrm{Sym}^1(W) \oplus \mathrm{Sym}^2(W)$ of polynomial functions on W of degree at most 2, endowed with the natural Lie algebra structure induced by the above Poisson bracket.

Definition 3.3.1. A *classical Airy structure* on V is a collection of $\{L_i\}_{i\in I} \subseteq \mathrm{Sym}_{\leq 2}(W)$ of the form

$$L_i = y^i - \frac{1}{2}\sum_{j,k} A^{ijk} x_j x_k - \sum_{j,k} B_k^{ij} y^k x_j - \frac{1}{2}\sum_{j,k} C_{jk}^i y^j y^k,$$

such that the vector space $\mathrm{span}_{\mathbb{C}}\{L_i\}_{i\in I}$ is closed under the Poisson bracket.

Let us consider some consequences of the definition above. By the closure property, we know that there exist constants $f_{i_1,i_2}^m \in \mathbb{C}$ such that

$$\{L_{i_1}, L_{i_2}\} = \sum_{m=1}^{n} f_{i_1,i_2}^m\, L_m.$$

Using the definition of L_i and the Poisson bracket relation, we get

$$\{L_{i_1}, L_{i_2}\} = \sum_m (B_m^{i_1 i_2} - B_m^{i_2 i_1}) y^m + \cdots,$$

which implies that

$$f^m_{i_1,i_2} = (B^{i_1 i_2}_m - B^{i_2 i_1}_m). \tag{3.8}$$

We define an associative $\mathbb{C}$-algebra $\mathrm{Sym}^{quant}(W)$ generated by $1, \hbar, x_i$ and $\hbar\partial^i$ such that 1 and $\hbar$ are central, and subject to the relation $[\hbar\partial^i, x_j] = \delta_{ij}\hbar$. We define $\mathrm{Sym}^{quant}_{\leq 2}(W)$ as the vector subspace of $\mathrm{Sym}^{quant}(W)$ spanned by

$$\{1, \hbar, x_i, \hbar\partial^i, x_i x_j, \hbar x_i\partial^j + \hbar\partial^j x_i, \hbar^2\partial^i\partial^j \mid i, j \in I\}.$$

Definition 3.3.2. A *quantisation* of a classical Airy structure is a monomorphism $V^* \to \mathrm{Sym}^{quant}_{\leq 2}(W)$ such that modulo $\hbar$, it coincides with the monomorphism $V^* \to \mathrm{Sym}_{\leq 2}(W)$. We call the former monomorphism a *quantum Airy structure*.

We write the quantum Airy structure as

$$\hat{L}_i := \hbar\partial^i - \frac{1}{2}\sum_{j,k} A^{ijk}x_j x_k - \sum_{j,k}\hbar B^{ik}_j \partial^k x_j - \frac{1}{2}\sum_{j,k}\hbar^2 C^i_{jk}\partial^j\partial^k - \hbar D^i, \tag{3.9}$$

for $i \in I$. By the closure property of the classical Airy structure, we have

$$\big[L_i, L_j\big] = \sum_{m=1}^{n} \hbar f^m_{i,j} L_m \tag{3.10}$$

Kontsevich and Soibelman [71] prove that for any quantum Airy structure, there exists a unique solution $\mathcal{Z}^{KS}$ to the collection of constraints $\hat{L}_i \mathcal{Z}^{KS} = 0$ for $i \in I$ of the form

$$\mathcal{Z}^{KS} = \exp\Big(\sum_{2g-2+n>0} \frac{\hbar^{g-1}}{n!} \sum_{k_1,\ldots,k_n \in I} F_{g,n}[k_1, \ldots, k_n]\, x_{k_1} \cdots x_{k_n}\Big). \tag{3.11}$$

Here, the $F_{g,n}[k_1, \ldots, k_n]$ are scalars and are invariant under permutation of $k_1, \ldots, k_n$. It is not in general clear what kind of enumerative invariants these coefficients store; this depends on the choice of the quantum Airy structure. The differential constraints provide a recursive structure to the coefficients $F_{g,n}[k_1, \ldots, k_n]$.

The closure properties of the differential operators give rise to a number of quadratic equation that the tensors A, B, C, D must satisfy and we describe these below. They are expressed in terms of the coefficients of equation (3.9), by forcing them to satisfy equation (3.10) and comparing coefficients of

$$x_c, \hbar x_c\partial^d, \hbar^2\partial^c\partial^c, x_c x_d, \qquad \text{for } c, d \in I.$$

- Equating coefficients of x_c on both sides of equation (3.10), we notice that the following terms are the only ones on the left side that contribute.

  $$[\hbar\partial^i, -\tfrac{1}{2}A^{jab}x_a x_b] + [-\tfrac{1}{2}A^{iab}x_{a'}x_{b'}, \hbar\partial^j]$$

  So the left side contributes $(A^{jic} - A^{ijc})$, while the right side contributes zero, so we obtain

  $$A^{jic} = A^{ijc} \qquad \text{for all } i, j, c.$$

  Similarly, it can be shown that

  $$A^{jic} = A^{jci} \qquad \text{for all } i, j, c. \tag{3.12}$$

  Hence, A^{ijc} is symmetric in all three indices.

- Equating coefficients of $\hbar x_c \partial^d$ on both sides of equation (3.10), we notice that the following terms are the only ones on the left side that contribute.

$$\left[\tfrac{1}{2}A^{iab}x_a x_b\, \tfrac{1}{2}\hbar^2 C^j_{a',b'}\partial^{a'}\partial^{b'}\right] + \left[\tfrac{1}{2}\hbar^2 C^i_{a,b}\partial^a\partial^b, \tfrac{1}{2}A^{ja'b'}x_a x_b\right] + \left[\hbar B^{ia}_b x_a \partial^b, \hbar B^{ja'}_{b'} x_{a'}\partial^{b'}\right]$$

Using equation (3.8), we get

$$\begin{aligned}\frac{1}{4}A^{icb}C^j_{bd} - \frac{1}{4}A^{jcb}C^i_{bd} + B^{ic}_b B^{jb}_d - B^{jc}_b B^{ib}_d &= f^m_{ij}B^{cm}_d \\ \frac{1}{4}A^{icb}C^j_{bd} - \frac{1}{4}A^{jcb}C^i_{bd} + B^{ic}_b B^{jb}_d - B^{jc}_b B^{ib}_d &= -(B^{ij}_m - B^{ji}_m)B^{cm}_d \\ B^{ij}_m B^{cm}_d + \frac{1}{4}A^{icb}C^j_{bd} + B^{ic}_b B^{jb}_d &= B^{ji}_m B^{cm}_d + \frac{1}{4}A^{jcb}C^i_{bd} + B^{jc}_b B^{jb}_d \end{aligned} \tag{3.13}$$

- Equating coefficients of $\hbar^2\partial^c\partial^d$ on both sides of equation (3.10), we notice that the following terms are the only ones on the left side that contribute.

$$\left[\hbar B^{ia}_b x_a \partial^b, \tfrac{1}{2}\hbar^2 C^j_{a',b'}\partial^{a'}\partial^{b'}\right] + \left[\tfrac{1}{2}\hbar^2 C^i_{ab}\partial^{a'}\partial^{b'}, \hbar B^{ja'}_{b'} x_{a'}\partial^{b'}\right]$$

Using equation (3.8), we get

$$\begin{aligned}\tfrac{1}{2}B^{ib}_c C^j_{bd} - \tfrac{1}{2}B^{jb}_c C^i_{bd} + \tfrac{1}{2}C^j_{ca}B^{ia}_d - \tfrac{1}{2}C^i_{ca}B^{ja}_d &= f^m_{ij}C^m_{cd} \\ \tfrac{1}{2}B^{ib}_c C^j_{bd} - \tfrac{1}{2}B^{jb}_c C^i_{bd} + \tfrac{1}{2}C^j_{ca}B^{ia}_d - \tfrac{1}{2}C^i_{ca}B^{ja}_d &= -(B^{ij}_m - B^{ji}_m)C^m_{cd} \\ B^{ij}_m C^m_{cd} + \tfrac{1}{2}B^{ib}_c C^j_{bd} + \tfrac{1}{2}C^j_{ca}B^{ia}_d &= B^{ji}_m C^m_{cd} + \tfrac{1}{2}B^{jb}_c C^i_{bd} + \tfrac{1}{2}C^i_{ca}B^{ja}_d \end{aligned} \tag{3.14}$$

- Equating coefficients of $x_c x_d$ on both sides of equation (3.10), we notice that the following terms are the only ones on the left side that contribute.

$$\left[\hbar B^{ia}_b x_a \partial^b, \tfrac{1}{2}A^{jbd}x_b x_d\right] + \left[\hbar B^{ja}_b x_a \partial^b, \tfrac{1}{2}A^{ibd}x_b x_d\right]$$

Using equation (3.8), we get

$$\begin{aligned}\tfrac{1}{2}B^{ic}_b A^{jbd} - \tfrac{1}{2}B^{jc}_b A^{ibd} + \tfrac{1}{2}B^{ib}_d A^{jcb} - \tfrac{1}{2}B^{jb}_d A^{icb} &= f^m_{ij}A^m_{cd} \\ \tfrac{1}{2}B^{ic}_b A^{jbd} - \tfrac{1}{2}B^{jc}_b A^{ibd} + \tfrac{1}{2}B^{ib}_d A^{jcb} - \tfrac{1}{2}B^{jb}_d A^{icb} &= -(B^{ij}_m - B^{ji}_m)A^{mcd} \\ B^{ij}_m A^{mcd} + \tfrac{1}{2}B^{ic}_b A^{jbd} + \tfrac{1}{2}B^{ib}_d A^{jcb} &= \tfrac{1}{2}B^{jc}_b A^{ibd} + \tfrac{1}{2}B^{jb}_d A^{icb} + B^{ji}_m A^{mcd} \end{aligned} \tag{3.15}$$

- Equating coefficients of $\hbar^2$ on both sides of equation (3.10), we notice that the following terms are the only ones on the left side that contribute.

$$\left[\tfrac{1}{2}A^{iab}x_b x_d, \tfrac{1}{2}\hbar^2 C^j_{a',b'}\partial^{a'}\partial^{b'}\right] + \left[\tfrac{1}{2}\hbar^2 C^i_{a'b'}\partial^{a'}\partial^{b'}, \tfrac{1}{2}A^{ia'b'}x_b x_d\right]$$

Using equation (3.8), we get

$$\begin{aligned}\frac{1}{4}A^{iab}C^j_{ab} - \frac{1}{4}A^{jab}C^i_{ab} &= f^m_{ij}D^m \\ \frac{1}{4}A^{iab}C^j_{ab} - \frac{1}{4}A^{jab}C^i_{ab} &= -(B^{ij}_m - B^{ji}_m)D^m \\ \frac{1}{4}A^{iab}C^j_{ab} + B^{ij}_m D^m &= \frac{1}{4}A^{jab}C^i_{ab} + B^{ji}_m D^m \end{aligned} \tag{3.16}$$

We now discuss the relation between the topological recursion discussed in previous sections and the Kontsevich–Soibelman formulation of topological recursion. Given a spectral curve as defined in Section 3.2, let $z(p)$ denote the local coordinate around the ramification point p. For the sake of this discussion, let us assume that the spectral curve has only one ramification point. Eynard [46] showed that $\omega_{g,n}$ can be written as a polynomial

$$\omega_{g,n}(z_1, z_2, \ldots, z_n) = \sum_{k_1,\ldots,k_n \in I} W_{g,n}[k_1, \ldots, k_n]\, \xi_{k_1}(z_1) \cdots \xi_{k_n}(z_n),$$

in the basis

$$\xi_k(z_0) := \operatorname*{Res}_{z=p} \int^p \omega_{0,2}(z_0, \cdot\,) \frac{(2k+1)\,\mathrm{d}z}{z^{2k+2}}.$$

Let us also define the dual basis

$$\xi_k^*(z_0) = \frac{z^{2k+1}}{2k+1},$$

and the function

$$\theta(p) := \frac{2}{\omega_{0,1}(p) - \omega_{0,1}(s(p))}.$$

It was shown that $W_{g,n}[k_1, \ldots, k_n] = F_{g,n}[k_1, \ldots, k_n]$, where the $F_{g,n}$ arise from a quantum Airy structure [7, 71].Given an enumerative problem governed by topological recursion, one can calculate the associated quantum Airy structure. To demonstrate this procedure, let us carry out an example in the simple case of the Witten–Kontsevich theorem, which is related to topological recursion on the Airy curve.

Kontsevich–Soibelman topological recursion for the Airy curve

One of the landmark results concerning the intersection theory on moduli spaces of curves is the Witten–Kontsevich theorem. In his foundational paper [103], Witten conjectured that a particular generating function for psi-class intersection numbers satisfies the Korteweg–de Vries hierarchy, which is often abbreviated to KdV hierarchy. The proof was subsequently provided by Kontsevich [69], and there are now many proofs in the literature. It is well-known as the prototypical example of an exactly solvable model, whose soliton solutions have attracted tremendous mathematical interest over the past few decades. A thorough analysis of the KdV hierarchy allows Witten's conjecture to be stated in the following alternative way. Let p_i be formal variables and set $\partial_k = \frac{\partial}{\partial p_k}$. Define the sequence of Virasoro operators by

$$L_{-1} = -\frac{1}{\hbar}\partial_1 + \frac{1}{2}p_1^2 + \sum_{i=1}^{\infty}(2i+1)p_{2i+3}\partial_{2i+1}, \tag{3.17}$$

$$L_0 = -\frac{3}{\hbar}\partial_3 + \sum_{i=1}^{\infty}(2i+1)p_{2i+1}\partial_{2i+1} + \frac{1}{8}, \tag{3.18}$$

and for $n \geq 1$,

$$L_n = -\frac{2n+3}{\hbar}\partial_{2n+3} + \sum_{i=0}^{\infty}(2i+2n+1)p_{2i+1}\partial_{2i+2n+1} + \frac{1}{2}\sum_{i=0}^{n-1}(2i-1)(2n-2i-1)\partial_{2i+1}\partial_{2n-2i-1}. \tag{3.19}$$

It is straightforward to check that these operators satisfy the relation $[L_m, L_n] = (m-n)L_{m+n}$ for all $m, n \geq -1$. Thus, they provide a representation of a subalgebra of the Virasoro Lie algebra. One can state the Witten–Kontsevich theorem in terms of these Virasoro operators.

Theorem 3.3.3 (Witten–Kontsevich theorem, Virasoro version)**.** *For every integer $n \geq -1$,*

$$L_n Z = 0,$$

where

$$Z(p_1, p_2, \ldots; \hbar) = \exp\left[\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{2g-2+n}}{n!}\sum_{a_1,\ldots,a_n=0}^{\infty}\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{a_1}\cdots\psi_n^{a_n}\prod_{i=1}^{n}(2a_i+1)!!\,p_{2a_i+1}\right].$$

The Virasoro operators described above give an example of a quantum Airy structure. Furthermore, the partition function $Z(p_1, p_2, \ldots; \hbar)$ is known to arise from the usual topological recursion. We now show how to obtain the tensors A, B, C, D in this particular case. The Airy spectral curve underlying the problem is given by

$$\left(\mathbb{C},\ x(z) = \tfrac{1}{2}z^2,\ y(z) = z,\ \omega_{0,2} = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1 - z_2)^2}\right) \tag{3.20}$$

We need the following series expansion to calculate the tensors.

$$\omega_{0,2}(z, z_2) = \left(\frac{1}{z_2^2} + 2\frac{z}{{z_2}^3} + 3\frac{z^2}{{z_2}^4} + 4\frac{z^3}{{z_2}^5} + 5\frac{z^4}{{z_2}^6} + 6\frac{z^5}{{z_2}^7} + \cdots\right)\mathrm{d}z\,\mathrm{d}z_2 \tag{3.21}$$

The spectral curve has a ramification point at $z = 0$.We use the notation from Section 9 of the paper [7] for the calculation of A, B, C, D. As the spectral curve of equation (3.20) has only one ramification point, we can simplify their notation using $A^{k_1k_2k_3} := A^{(k_1,0),(k_2,0),(k_3,0)}$ and similarly for the other tensors B, C, D. The equations for deriving the tensors are as follows.

$$\begin{aligned}
A^{k_1k_2k_3} &= \operatorname*{Res}_{z=0}\ \xi^*_{k_1}(z)\,\mathrm{d}\xi^*_{k_2}(z)\,\mathrm{d}\xi^*_{k_3}(z)\,\theta(z)\\
&= \operatorname*{Res}_{z=0}\ \frac{z^{2k_1+1}}{2k_1+1}z^{2k_2}\mathrm{d}z\,z^{2k_3}\mathrm{d}z\,\frac{1}{z^2\,\mathrm{d}z}\\
&= \operatorname*{Res}_{z=0}\frac{\mathrm{d}z}{z}\frac{z^{2k_1+2k_2+2k_3}}{(2k_1+1)} && (3.22)\\
B^{k_1k_2}_{k_3} &= \operatorname*{Res}_{z=0}\xi^*_{k_1}(z)\,\mathrm{d}\xi^*_{k_2}(z)\,\xi_{k_3}(z)\,\theta(z)\\
&= \operatorname*{Res}_{z=0}\ \frac{z^{2k_1+1}}{2k_1+1}z^{2k_2}\,\mathrm{d}z\,\frac{(2k_3+1)\,\mathrm{d}z}{z^{(2k_3+2)}}\frac{1}{z^2\,\mathrm{d}z}\\
&= \operatorname*{Res}_{z=0}\frac{\mathrm{d}z}{z}\frac{(2k_3+1)}{(2k_1+1)}z^{2k_1+2k_2-2k_3-2} && (3.23)\\
C^{k_1}_{k_2k_3} &= \operatorname*{Res}_{z=0}\xi^*_{k_1}(z)\ \xi_{k_2}(z)\,\xi_{k_3}(z)\,\theta(z)\\
&= \operatorname*{Res}_{z=0}\frac{z^{2k_1+1}}{2k_1+1}\frac{(2k_2+1)\,\mathrm{d}z}{z^{(2k_2+2)}}\frac{(2k_3+1)\,\mathrm{d}z}{z^{(2k_3+2)}}\frac{1}{z^2\,\mathrm{d}z}\\
&= \operatorname*{Res}_{z=0}\frac{\mathrm{d}z}{z}\frac{(2k_2+1)(2k_3+1)}{(2k_1+1)}z^{2k_1-2k_2-2k_3-4} && (3.24)\\
D^{k_1} &= \delta_{k_1,0}\frac{1}{8} && (3.25)
\end{aligned}$$

The tensors computed above will give us the quantum Airy structure for the Witten–Konstevich intersection numbers and reproduce the Virasoro operators from equations (3.17) to (3.19).

Kontsevich–Soibelman topological recursion for the Bessel curve

We now consider Kontsevich–Soibelman topological recursion for the invariants produced by the Bessel spectral curve [37]. It is given by

$$\left(\mathbb{C},\ x(z) = \tfrac{1}{2}z^2,\ y(z) = \frac{1}{z},\ \omega_{0,2} = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1 - z_2)^2}\right) \tag{3.26}$$

Some examples of correlation differentials produced by the topological recursion applied to the Bessel curve are provided below.

$$\omega_{0,1}(z_1) = -\mathrm{d}z_1 \qquad \omega_{1,2}(z_1, z_2) = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{8{z_2}^2{z_1}^2}$$

$$\omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1 - z_2)^2} \qquad \omega_{1,3}(z_1, z_2, z_3) = -\frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3}{4z_1^2z_2^2z_3^2}$$

$$\omega_{0,3}(z_1, z_2, z_3) = 0 \qquad \omega_{1,4}(z_1, z_2, z_3, z_4) = \frac{3\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4}{4z_1^2z_2^2z_3^2z_4^2}$$

$$\omega_{1,1}(z_1) = -\frac{\mathrm{d}z_1}{8z_1^2} \qquad \omega_{2,1}(z_1) = -\frac{9\,\mathrm{d}z_1}{128{z_1}^4}$$

For $2g - 2 + n > 0$ and positive integers $\mu_1, \ldots, \mu_n$, define the numbers $U_{g,n}(\mu_1, \ldots, \mu_n)$ via the expansion

$$\omega_{g,n}(z_1, z_2, \ldots, z_n) = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} U_{g,n}(\mu_1, \ldots, \mu_n) \prod_{i=1}^{n} \frac{\mu_i}{z_i^{\mu_i} + 1}.$$

Do and Norbury [37] give the following cut-and-join type recursion for these numbers, where $S = \{2, 3, \ldots, n\}$.

$$\begin{aligned}\mu_1 U_{g,n}(\mu_1, \mu_S) = {} & \sum_{k=2}^{n} U_{g,n-1}(\mu_1 + \mu_k - 1, \mu_{S\setminus\{k\}}) \\ & + \frac{1}{2} \sum_{\substack{\alpha+\beta=\mu_1-1 \\ \alpha,\beta \text{ odd}}} \alpha\beta \Big[U_{g-1,n+1}(\alpha, \beta, \mu_S) + \sum_{\substack{g_1+g_2=g \\ I\cup J=S}} U_{g_1}(\alpha, \mu_I)\, U_{g_2}(\beta, \mu_J) \Big]\end{aligned} \tag{3.27}$$

All of the numbers $U_{g,n}(\mu_1, \ldots, \mu_n)$ can be calculated using this recursion from the base cases $U_{0,1}(\mu) = 0$ and $U_{0,2}(\mu_1, \mu_2) = 0$ for all μ_1 and μ_2, and $U_{1,1}(1) = \frac{1}{8}$. The spectral curve has one ramification point at $z = 0$, so the general theory of topological recursion allows us to write

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{k_1+\cdots+k_n \le 3g-3+n} F_{g,n}[k_1, \ldots, k_n]\, \xi_{k_1}(z_1) \cdots \xi_{k_n}(z_n).$$

Here, we take $\xi_k(z) = \frac{(2k+1)\,\mathrm{d}z}{z^{2k+1}}$ and denote the symplectic dual by $\xi_k^*(z) = \frac{z^{2k+1}}{2k+1}$. These enter into the calculations of the tensors A, B, C, D, along with the following expression for $\theta(z)$.

$$\theta(z) = \frac{2}{(y(z) - y(s(z)))\,\mathrm{d}x(z)} = \frac{2}{(\frac{1}{z} + \frac{1}{z})\,z\,\mathrm{d}z} = \frac{1}{\mathrm{d}z} \tag{3.28}$$

Using the same equations as above for the Airy case, we derive the tensors A, B, C, D in the

Bessel case.

$$\begin{aligned}
A^{k_1k_2k_3} &= \operatorname*{Res}_{z=0} \xi^*_{k_1}(z)\, \mathrm{d}\xi^*_{k_2}(z)\, \mathrm{d}\xi^*_{k_3}(z)\, \theta(z) \\
&= \operatorname*{Res}_{z=0} \frac{\mathrm{d}z}{z} z^{2k_1+2k_2+2k_3+2} \\
&= 0, \qquad \text{for all } k_1, k_2, k_3. && (3.29) \\
B^{k_1k_2}_{k_3} &= \operatorname*{Res}_{z=0} \xi^*_{k_1}(z)\, \mathrm{d}\xi^*_{k_2}(z)\, \xi_{k_3}(z)\, \theta(z) \\
&= \operatorname*{Res}_{z=0} \frac{\mathrm{d}z}{z} \frac{(2k_3+1)}{(2k_1+1)} z^{2k_1+2k_2-2k_3} \\
&= \begin{cases} \frac{2k_3+1}{2k_1+1}, & k_1+k_2=k_3, \\ 0, & \text{otherwise.} \end{cases} && (3.30) \\
C^{k_1}_{k_2k_3} &= \operatorname*{Res}_{z=0} \xi^*_{k_1}(z)\, \xi_{k_2}(z)\, \xi_{k_3}(z)\, \theta(z) \\
&= \operatorname*{Res}_{z=0} \frac{\mathrm{d}z}{z} \frac{(2k_3+1)(2k_2+1)}{(2k_1+1)} z^{2k_1-2k_2-2k_3-2} \\
&= \begin{cases} \frac{(2k_2+1)(2k_3+1)}{2k_1+1}, & k_1 = k_2+k_3+1, \\ 0, & \text{otherwise.} \end{cases} && (3.31) \\
D^{k_1} &= \delta_{k_1,0}\frac{1}{8} && (3.32)
\end{aligned}$$

This leads to the differential operators L_n for $n \geq 0$, defined by

$$\begin{aligned}
L_n = -\frac{n+1/2}{\hbar}\partial_{2n+1} + \frac{1}{2}\sum_{i=0}^{\infty}(2i+2n+1)p_{2i+1}\partial_{2i+2n+1} \\
+ \frac{1}{4}\sum_{i=0}^{n-1}(2i-1)(2n-2i-1)\partial_{2i+1}\partial_{2n-2i-1} + \frac{1}{16}\delta_{n,0} \quad (3.33)
\end{aligned}$$

These form a representation of a subalgebra of the Virasoro algebra. They furthermore annihilate the partition function for the Bessel curve, which is a tau function of the KdV hierarchy known in the literature as the Brézin–Gross–Witten tau function.

Konstevich and Soibelman [71] prove the following recursion, where $2g-2+n \geq 2$, $i_1, \ldots, i_n \in I$ and $J = \{i_2, \ldots, i_n\}$.

$$\begin{aligned}
F_{g,n}\big[i_1, \ldots, i_n\big] &= \sum_{m=2}^{n}\sum_{a} B^{i_1 i_m}_{a} F_{g,n-1}\big[a, J \setminus \{i_m\}\big] \\
&+ \frac{1}{2}\sum_{ab} C^{i_1}_{a,b}\bigg(F_{g-1,n+1}\big[a,b,J\big] + \sum_{\substack{g_1+g_2=g \\ K\sqcup L=J}} F_{g_1,|K|+1}\big[a,K\big]\, F_{g_1,|L|+1}\big[b,L\big]\bigg) \quad (3.34)
\end{aligned}$$

First, we notice that the initial conditions for the recursion in equation (3.27) and equation (3.34) agree. That is, $U_{0,1}(i) = F_{0,1}[i] = 0$, $U_{0,2}(i,j) = F_{0,2}(i,j) = 0$ and $F_{0,3}[i,j,k] = A^{ijk} = U_{0,3}(i,j,k) = 0$ for all i,j,k, as well as the fact that $F_{1,1}[0] = D^0 = U_{1,1}(1) = \frac{1}{8}$. We can substitute the calculations of the tensors A, B, C, D above into the equation above to obtain the following explicit recursion.

$$F_{g,n}\big[i_1,\ldots,i_n\big] = \sum_{m=2}^{n} \frac{2(i_1+i_m)+1}{2i_1+1} F_{g,n-1}\big[i_1+i_m, J\setminus\{i_m\}\big]$$
$$+\frac{1}{2}\sum_{a+b=i_1-1}\frac{(2a+1)(2b+1)}{(2i_1+1)}\Bigg(F_{g-1,n+1}\big[a,b,J\big]+\sum_{\substack{g_1+g_2=g\\ J'\cup J''=I}} F_{g_1,1+|J'|}\big[a,J'\big]\,F_{g_1,1+|J'|}\big[b,J''\big]\Bigg) \tag{3.35}$$

At this stage, we simply notice the similarity between the recursions of equations (3.27) and (3.35). The only difference is the transformation of the arguments involved in the recursion. Hence, we have proved by induction the fact that

$$U_{g,n}(2k_1+1, 2k_2+1,\ldots,2k_n+1) = F_{g,n}[k_1,k_2,\ldots,k_n].$$

Our calculations demonstrate how one can derive a recursion of cut-and-join type from the Kontsevich–Soibelman topological recursion. A similar analysis is performed in Section 7.1 for the context of monotone Hurwitz numbers.

3.4 Quantum curves

In this section, we discuss the notion of quantum curves and their relation with topological recursion. A *quantum curve* is an object that we associate to a plane curve

$$C = \big\{(x,y)\in\mathbb{C}^2 \mid P(x,y)=0\big\}$$

and is a Schrödinger-type differential equation

$$\hat{P}(\hat{x},\hat{y})\,\psi(p,\hbar) = 0, \tag{3.36}$$

where $p \in C$, $\hbar$ is a formal parameter, and $\hat{P}(\hat{x},\hat{y})$ is a differential operator-valued non-commutative quantisation of the plane curve with $\hat{x} = x$ and $\hat{y} = \hbar\frac{\partial}{\partial x}$. Observe that these operators satisfy the commutation relation

$$[\hat{x},\hat{y}] = -\hbar. \tag{3.37}$$

For example, a quantisation of $P(x,y) = y^2 - x$ is the operator $\hat{P}(\hat{x},\hat{y}) = \hat{y}^2 - \hat{x} = \hbar^2\frac{\partial^2}{\partial x^2} - x$. equation (3.36) is understood via the WKB method. In other words, we require $\psi(p,\hbar)$ to be of the form

$$\psi(p,\hbar) = \exp\left(\hbar^{-1}F_0(p) + F_1(p) + \hbar F_2(p) + \hbar^2 F_3(p) + \cdots\right), \tag{3.38}$$

and the $F_k(p)$ are calculated recursively via equation (3.36). A simple consequence of equation (3.36) is that the $F_k(p)$ are meromorphic functions of C, where $F_0(p) = \int^p y\,\mathrm{d}x$ may be multi-valued. A fundamental question is whether $F_k(p)$ can be defined directly from the plane curve without using the WKB approximation to produce a natural choice of $\hat{P}(\hat{x},\hat{y})$. A conjectural answer in the case that the plane curve C has genus zero is given by

$$F_k(p) = \sum_{2g-1+n=k} \frac{1}{n!}\int_a^p\int_a^p\cdots\int_a^p \omega_{g,n}(p_1,\ldots,p_n), \tag{3.39}$$

where $\omega_{g,n}(p_1,\ldots,p_n)$ are defined by the topological recursion on the spectral curve given by the plane curve C and a is a base point for the integration, which should be chosen to be a pole of

x [17]. This conjecture is addressed by Gukov and Sułkowski in [61], together with the related issue of constructing $\hat{P}(\hat{x}, \hat{y})$ algorithmically from the wave function.

The path from the quantum curve to the plane curve is well-defined. It is a little deeper than simple substitution $\hat{x} \to x$ and $\hat{y} \to y$ into $\hat{P}(\hat{x}, \hat{y})$, since we deduced that equation (3.36) is satisfied only on the plane curve $P(x, y) = 0$. This is achieved via the semi-classical limit $\hbar \to 0$, where the differential operator $\hat{P}(\hat{x}, \hat{y})$ reduces to a multiplication operator that vanishes precisely on the plane curve. The action of $\hbar \frac{\partial}{\partial x}$ on

$$\psi_0(p, \hbar) = \exp\left(\hbar^{-1} \int^p y \, \mathrm{d}x\right)$$

is multiplication by y, so

$$\hat{P}(\hat{x}, \hat{y}) \, \psi(p, \hbar) = [P(x, y) + O(\hbar)] \, \psi(p, \hbar),$$

and in the $\hbar \to 0$ limit, $\hat{y} = \hbar \frac{\partial}{\partial x}$ in $\hat{P}(\hat{x}, \hat{y})$ gets replaced by its symbol y. Higher order corrections in $\hbar$ are required since $(\hbar \frac{\partial}{\partial x})^2 \to y^2 + O(\hbar)$ under its action on $\psi_0(x, \hbar)$.

On the other hand, constructing the quantum curve from the plane curve is far from canonical. The main issues lie in the construction of the wave function $\psi(p, \hbar)$ and the ambiguity in ordering the non-commuting operators $\hat{x}$ and $\hat{y}$ in $\hat{P}$. The conjectural formula of equation (3.39) is one attempt to remedy this. Such a wave function is enough to reconstruct the operator $\hat{P}(\hat{x}, \hat{y})$. Every $\hat{P}(\hat{x}, \hat{y})$ can be expressed as:

$$\hat{P}(\hat{x}, \hat{y}) = P(\hat{x}, \hat{y}) + \hbar P_1(\hat{x}, \hat{y}) + \hbar^2 P_2(\hat{x}, \hat{y}) + \cdots, \tag{3.40}$$

where each $P_k(\hat{x}, \hat{y})$ is a normal ordered operator-valued polynomial. So in $P_k(\hat{x}, \hat{y})$, all $\hat{y}$ terms in a monomial are placed to the right of $\hat{x}$ terms and there is no explicit $\hbar$ dependence. Then these polynomials can be reconstructed recursively from the wave function.

The differential operator $\hat{P}(\hat{x}, \hat{y})$ generates a principal ideal in the algebra $\mathcal{D}$ of differential operators which act on $\mathbb{C}[x]$. The quotient $\mathcal{D}/\langle \hat{P} \rangle$ of the algebra $\mathcal{D}$ by the principal ideal $\langle \hat{P} \rangle = \mathcal{D}\hat{P}$ is a $\mathcal{D}$-module which gives a way to study $\hat{P}(\hat{x}, \hat{y})$ intrinsically. The wave function $\psi(p, \hbar)$ can be retrieved via the $\mathcal{D}$-module homomorphisms it defines.

$$\mathcal{D}/\langle \hat{P} \rangle \to \mathbb{C}[[x^{\pm 1}, h^{\pm 1}]]$$

equation (3.39) relates quantum curves to topological recursion. The plane curve obtained from the quantum curve forms the essential input data for topological recursion. We briefly discuss two properties shared by quantum curves and topological recursion that further demonstrate the relationship between these objects.

- *Invariance under isomorphism*
  Consider the following isomorphism between plane curves

$$(x, y) \to (x, y + \tfrac{\mathrm{d}}{\mathrm{d}x} g(x)), \tag{3.41}$$

  for any polynomial $g(x)$. So their defining polynomials $P(x, y) = 0$ and $Q(x, y) = 0$ are related by $Q(x, y) = P(x, y - \frac{\mathrm{d}}{\mathrm{d}x} g(x))$. Now

$$\hat{P}(\hat{x}, \hat{y}) \, \psi(p, \hbar) = 0 \quad \Rightarrow \quad \hat{Q}(\hat{x}, \hat{y}) \exp(\hbar^{-1} g(x)) \, \psi(p, \hbar) = 0,$$

  for $P(x(p), y(p)) = 0$ and $Q(x(p), y(p) + \frac{\mathrm{d}}{\mathrm{d}x} g(x(p))) = 0$, where $\hat{Q}(\hat{x}, \hat{y})$ has to be defined carefully as follows: replace each operator $\hat{y}$ in $\hat{P}(\hat{x}, \hat{y})$ with the operator $\hat{y} - \frac{\mathrm{d}}{\mathrm{d}x} g(x)$ and do not apply normal ordering.

The isomorphism equation (3.41) preserves the underlying curve, not its embedding, together with the function x defined on the curve. The change in wave function for curves related by such an isomorphism only effects the $\hbar^{-1}$ term in the exponent of $\psi(p,\hbar)$ and all $F_k(p)$ for $p > 0$ are unchanged under the isomorphism. So we see that $F_k(p)$ for $k > 0$ are in some sense intrinsic to the underlying curve equipped with the functions x and y. One can also observe that the $\omega_{g,n}$ generated by the topological recursion are also unchanged under the isomorphism.

- *Local factorisation*
  The quantum curves for fundamental cases like the Airy curve $\frac{1}{2}y^2 - x = 0$ and Bessel curve $xy^2 - \frac{1}{2} = 0$ satisfy equation (3.39). For an arbitrary spectral curve, these give a model for the quantum curve near a simple ramification point. The Airy and Bessel quantum curves then annihilate the part of the wave function related to the Airy and Bessel spectral curves, respectively.

The quantum curve often helps us to predict whether an enumerative problem is guided by topological recursion and, if so, it provides the initial data of the spectral curve. Note that topological recursion produces correlation differentials, which store information by some genus g and tuple $(\mu_1, \ldots, \mu_n)$. On the other hand, the quantum curve controls the wave function, which stores information only by the Euler characteristic $2g - 2 + n$ and the degree $d = \sum \mu_i$.

Below is a brief list of enumerative problems discussed in previous chapters, along with their associated quantum curves. In many cases, the derivation of the quantum curve led to a conjecture at the deeper level of topological recursion.

Enumerative problem	**Quantum curve**
Orbifold Hurwitz numbers [79]	$\hat{y} - \left(\exp(\frac{q-1}{2}\hat{y})\hat{x}\exp(\frac{-(q-1)}{2}\hat{y})\right)^q \exp(q\hat{y})$
Spin Hurwitz numbers [79]	$\hat{y} - x^{\frac{3}{2}} \exp\left(\frac{1}{r+1}\sum_{i=0}^{r} \hat{x}^{-1}\hat{y}^i\hat{x}\hat{y}^{r-i}\right)x^{-\frac{1}{2}}$
Spin orbifold Hurwitz numbers [79]	$\hat{y} - x^{q+\frac{1}{2}} \exp\left(\frac{q}{r+1}\sum_{i=0}^{r} \hat{x}^{-q}\hat{y}^i\hat{x}^q\hat{y}^{r-1}\right)x^{-\frac{1}{2}}$
Monotone orbifold Hurwitz numbers [28]	$\hat{x}^{a-1} + \prod_{j=0}^{a-1}(1 + \hat{x}\hat{y} + j\hbar)\hat{y}$
Enumeration of m-hypermaps [35]	$\hat{y}^m - \hat{x}\hat{y} + 1$

The following diagram shows the relations between the various objects at play, which allows one to calculate quantum curves and conjecture spectral curves for given enumerative problems.

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{spectral}\\ \text{curve}\\ P(x,y)=0\end{array}} \arrow[r] \arrow[d, shift left] & \boxed{\begin{array}{c}\text{correlation}\\ \text{differentials}\\ \omega_{g,n}\end{array}} \arrow[r] & \boxed{\begin{array}{c}\text{free}\\ \text{energies}\\ F_{g,n}\end{array}} \arrow[d] \\
\boxed{\begin{array}{c}\text{quantum}\\ \text{curve}\\ \hat{P}(\hat{x},\hat{y})\end{array}} \arrow[u, shift left] & \boxed{\begin{array}{c}\text{Schrödinger}\\ \text{equation}\\ \hat{P}\psi = 0\end{array}} \arrow[l] & \boxed{\begin{array}{c}\text{partition}\\ \text{function}\\ Z(\mathbf{p};\hbar)\end{array}} \arrow[l]
\end{tikzcd}$$

Figure 3.2: Flowchart describing the quantisation of the spectral curve.

Quantum curve for Bousquet-Mélou–Schaeffer numbers

We now give an example of an enumerative problem for which we derive the associated quantum curve — namely, Bousquet-Mélou–Schaeffer numbers. We start with the partition function for the problem, form the principal specialisation to obtain the wave function, then derive the quantum curve equation, following the path along the bottom of the diagram above. A crucial tool is the character formula obtained in Proposition 1.1.7. We note that this demonstrates a technique that works more generally for similar enumerative problems.

For a fixed positive integer m, the *partition function* for the BMS numbers is defined as follows.

$$Z(p_1, p_2, \ldots; \hbar) := \exp \left[\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g-2+n}}{n!} \sum_{\mu_1, \ldots, \mu_n = 1}^{\infty} B_{g,n}^{(m)}(\mu_1, \ldots, \mu_n) \, p_{\mu_1} \cdots p_{\mu_n} \right]$$

The *wave function* for the BMS numbers is formed by taking the principal specialisation of the partition function — in other words, by replacing p_i with x^{-i} in $Z(p_1, p_2, \ldots; \hbar)$.

$$\psi(x, \hbar) = Z(p_1, p_2, \ldots; \hbar) := \exp \left[\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g-2+n}}{n!} \sum_{\mu_1, \ldots, \mu_n = 1}^{\infty} B_{g,n}^{(m)}(\mu_1, \ldots, \mu_n) \, x^{-(\mu_1 + \cdots + \mu_n)} \right]$$

It will be useful to interpret $p_1, p_2, \ldots$ as the power-sum symmetric functions of some infinite sequence of variables.

Proposition 3.4.1. *The wave function for BMS numbers can be expressed as*

$$\psi(x, \hbar) = \sum_{d=0}^{\infty} \frac{1}{\hbar^d} \frac{1}{d!} \prod_{i=1}^{d-1} (1 + i\hbar)^m x^{-d}.$$

Proof. The exponential appearing in the partition function passes from the connected enumeration to the disconnected enumeration. So we can express the partition function in the following way, noting that the genus can be negative, as discussed in Section 1.1.

$$\begin{aligned} Z(p_1, p_2, \ldots; \hbar) &= 1 + \sum_{g=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g-2+n}}{n!} \sum_{\mu_1, \ldots, \mu_n = 1}^{\infty} B_{g,n}^{(m)\bullet}(\mu_1, \ldots, \mu_n) \, p_{\mu_1} \cdots p_{\mu_n} \\ &= 1 + \sum_{g=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g-2+n}}{n!} \sum_{\mu_1, \ldots, \mu_n = 1}^{\infty} \frac{\prod p_{\mu_i}}{d! \prod \mu_i} [\hbar^{2g-2+n}] \sum_{\nu \vdash d} \frac{\chi_\mu^\nu}{\hbar^d} \dim(\nu) \prod_{\Box \in \nu} (1 + c(\Box)\hbar)^m \end{aligned}$$

To obtain the second line, we have substituted the character formula for BMS numbers of Proposition 1.1.7.

We are extracting a coefficient of $\hbar^{2g-2+n}$ and then replacing it, so one can remove these operations along with the sum over g.

$$Z(p_1, p_2, \ldots; \hbar) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mu_1, \ldots, \mu_n = 1}^{\infty} \frac{\prod p_{\mu_i}}{d! \prod \mu_i} \sum_{\nu \vdash d} \frac{\chi_\mu^\nu}{\hbar^d} \dim(\nu) \prod_{\Box \in \nu} (1 + c(\Box)\hbar)^m$$

Express this now as a sum over degree d and over partitions μ, rather than tuples.

$$\begin{aligned} Z(p_1, p_2, \ldots; \hbar) &= 1 + \sum_{d=1}^{\infty} \frac{1}{\hbar^d} \frac{1}{d!} \sum_{\mu \vdash d} \frac{\prod p_{\mu_i}}{|\mathrm{Aut}(\mu)| \prod \mu_i} \sum_{\nu \vdash d} \chi_\mu^\nu \dim(\nu) \prod_{\Box \in \nu} (1 + c(\Box)\hbar)^m \\ &= 1 + \sum_{d=1}^{\infty} \frac{1}{\hbar^d} \frac{1}{d!} \sum_{\nu \vdash d} \dim(\nu) \prod_{\Box \in \nu} (1 + c(\Box)\hbar)^m \sum_{\mu \vdash d} \frac{\chi_\mu^\nu}{|\mathrm{Aut}(\mu)| \prod \mu_i} \prod p_{\mu_i} \end{aligned}$$

Here, we have interchanged the order of summation over μ and ν. Now we use the change of basis between Schur functions and power-sum symmetric functions $s_\nu = \sum_\mu \frac{\chi^\nu_\mu}{|\mathrm{Aut}(\mu)| \prod \mu_i} \prod p_{\mu_i}$ to write the partition function as follows.

$$Z(p_1, p_2, \ldots; \hbar) = \sum_{d=0}^{\infty} \frac{1}{\hbar^d} \frac{1}{d!} \sum_{\nu \vdash d} s_\nu \dim(\nu) \prod_{\Box \in \nu} (1 + c(\Box)\hbar)^m$$

Now we use the standard fact [76]

$$s_\nu(t^1, t^2, t^3, \ldots) = \begin{cases} t^d, & \text{if } \nu = (d, 0, 0, \ldots) \text{ for some } d, \\ 0, & \text{otherwise.} \end{cases}$$

Substituting this into previous expression for the partition function, we have

$$\begin{aligned} \psi(x, \hbar) &= Z(p_1, p_2, \ldots; \hbar)\big|_{p_i = x^{-i}} \\ &= \sum_{d=0}^{\infty} \frac{1}{\hbar^d} \frac{1}{d!} \sum_{\nu \vdash d} s_\nu(x^{-1}, x^{-2}, x^{-3}, \ldots) \dim(\nu) \prod_{\Box \in \nu} (1 + c(\Box)\hbar)^m \\ &= \sum_{d=0}^{\infty} \frac{1}{\hbar^d} \frac{1}{d!} \prod_{i=1}^{d-1} (1 + i\hbar)^m x^{-d} \end{aligned}$$

□

Theorem 3.4.2 (Quantum curve for BMS numbers). *Let $\overline{\psi}(x, \hbar) = x^{-1/\hbar} \psi(x, \hbar)$. Then for $\hat{x} = x$ and $\hat{y} = \hbar \frac{\partial}{\partial x}$, we have*

$$\left[\hat{y}(\hat{x}\hat{y})^{m-1} + (-1)^m (\hat{x}\hat{y} + 1)\right] \overline{\psi}(x, \hbar) = 0.$$

Proof. In order to interpret the statement of this theorem, we need to make precise what we mean, given that the $\hbar$-expansion of $\overline{\psi}(x, \hbar)$ is not well-defined, due to the $x^{-1/\hbar}$ prefactor. We do this by conjugating the operator to instead prove that

$$x^{1/\hbar} \left[\hat{y}(\hat{x}\hat{y})^{m-1} + (-1)^m (\hat{x}\hat{y} + 1)\right] x^{-1/\hbar} \psi(x, \hbar) = 0.$$

This gives a well-defined differential operator acting on $\psi(x, \hbar) \in \mathbb{Q}[\hbar^{\pm 1}][x^{-1}]$.

Start by writing Proposition 3.4.1 as

$$\psi(x, \hbar) = \sum_{d=0}^{\infty} a_d(\hbar) x^{-d} \qquad \text{with} \qquad a_d(\hbar) = \frac{1}{\hbar^d} \frac{1}{d!} \prod_{i=1}^{d-1} (1 + i\hbar)^m.$$

We compare two consecutive coefficients of $\psi(x, \hbar)$ to obtain

$$\frac{a_{d+1}(\hbar)}{a_d(\hbar)} = \frac{(1 + \hbar d)^m}{\hbar(d+1)}.$$

Cross-multiply this equation, multiply both sides by x^{-d}, and then sum over all non-negative integers d to obtain the following equation.

$$\sum_{d=0}^{\infty} \hbar(d+1) a_{d+1}(\hbar) x^{-d} = \sum_{d=0}^{\infty} (1 + \hbar d)^m a_d(\hbar) x^{-d}$$

This can then be expressed as

$$-\hat{x}^2 \hat{y} \psi(x, \hbar) = (1 - \hat{x}\hat{y})^m \psi(x, \hbar) \qquad \Rightarrow \qquad \left[\hat{x}^2 \hat{y} + (1 - \hat{x}\hat{y})^m\right] \psi(x, \hbar) = 0.$$

To obtain the operator that annihilates $\overline{\psi}(x,\hbar)$, we conjugate the operator appearing in the previous equation using the fact that $x^{-1/\hbar}\hat{x}\hat{y}x^{1/\hbar} = \hat{x}\hat{y}+1$. This gives us

$$[\hat{x}(\hat{x}\hat{y}+1) + (-\hat{x}\hat{y})^m]\,\overline{\psi}(x,\hbar) = 0.$$

By applying $\hat{x}^{-1}$ on the left and rearranging, we obtain the desired result. $\square$

Remark 3.4.3. The modified wave function $\overline{\psi}(x,\hbar) = x^{-1/\hbar}\psi(x,\hbar)$ may seem unusual, but is consistent with other derivations of quantum curves [61]. It in some sense corresponds to the trivial BMS number $B_{0,1}^{(m)}(0) = 1$. We observe that up to a sign convention in the choice of polarisation, the previous theorem specialises in the case $m = 2$ to the known quantum curve for the enumeration of dessins d'enfant [36]. This class of spectral curves falls into the general class of admissible spectral curves studied by Bouchard and Eynard [17], so one can also invoke their result to show that topological recursion applied to the spectral curve does indeed reconstruct the WKB expansion attached to the quantum curve.

Chapter 4

One-point recursions

Harer and Zagier proved a recursion to enumerate gluings of a $2d$-gon that result in an orientable genus g surface, in their work on Euler characteristics of moduli spaces of curves. Analogous results have been discovered for other enumerative problems, so it is natural to pose the following question: How large is the family of problems for which these so-called 1-point recursions exist?

In this chapter, we prove the existence of 1-point recursions for a class of enumerative problems that have Schur function expansions. In particular, we recover the Harer–Zagier recursion, but our methodology also applies to Bousquet-Mélou–Schaeffer numbers, monotone Hurwitz numbers, the enumeration of dessins d'enfant, and more. On the other hand, we prove that there is no 1-point recursion that governs simple Hurwitz numbers. Our results are effective in the sense that one can explicitly compute particular instances of 1-point recursions. We conclude the chapter with a brief discussion of relations between 1-point recursions and the theory of topological recursion.

4.1 Motivation

The connection between map enumeration and matrix integrals was first established by 't Hooft [99]. This technique was later reinvented by Harer and Zagier [62] for their computation of the Euler characteristics of moduli spaces of curves. In their work, they define $a_g(d)$ to be the number of ways to glue the edges of a $2d$-gon in pairs to obtain an orientable genus g surface. We recall that the data of gluing polygons together to make surface is often referred to as a *fatgraph*, as discussed in detail in Section 1.4. One consequence of Harer and Zagier's calculation is the fact that the numbers $a_g(d)$ satisfy the following recursion [62].

$$(d+1)\, a_g(d) = 2(2d-1)\, a_g(d-1) + (2d-1)(d-1)(2d-3)\, a_{g-1}(d-2) \tag{4.1}$$

Despite the simple appearance of this formula, Zagier later stated in [73] that "No combinatorial interpretation of the recursion... is known." The Harer–Zagier recursion has since attracted a great deal of interest, and there now exist several proofs, some of them combinatorial in nature [1, 21, 59, 91].

A similar three-term recursion involving the enumeration of dessins d'enfant was obtained in the work of Do and Norbury [36], as well as the subsequent work of Chekhov [25]. Let $b_g(d)$ denote the number of ways to glue the edges of a $2d$-gon, whose vertices are alternately coloured red and blue, in pairs to obtain an orientable genus g surface. The vertices may only be glued together if they have the same colour. Then the three-term recursion for the enumeration of dessins d'enfant takes the following form.

$$(d+1)\, b_g(d) = 2(2d-1)\, b_g(d-1) + (d-1)^2(d-2)\, b_{g-1}(d-2) \tag{4.2}$$

The more general enumeration of gluings of n polygons with prescribed perimeters to obtain a surface of genus g was studied by Tutte [100] in the case of genus zero and by Walsh and Lehman for arbitrary genus [102]. They produce an effective recursion for the enumeration and the mechanism for this recursion comes from analysing the result of removing an edge from a fatgraph. The recursion necessarily mixes together the enumeration for various values of n, in contrast to the Harer–Zagier recursion, which only involves terms with $n = 1$.

There are various instances of recursions analogous to those expressed in equation (4.1) and equation (4.2), though in other settings. For example, Ledoux gives a recursion for the moments of the Gaussian orthogonal ensemble, which is analogous to the Harer–Zagier recursion [74]. In general, it is not true that these recursions involve three terms and indeed, the recursion of Ledoux requires five terms. In the context of enumerative geometry and mathematical physics, the analogues of $a_g(d)$ and $b_g(d)$ are known as *1-point invariants*, since they arise as the coefficients of 1-point functions. More generally, the enumeration of gluings of n polygons with prescribed perimeters to obtain a surface of genus g produces numbers known as *n-point invariants*. The previous discussion motivates us to make the following definition.

Definition 4.1.1. We say that the collection of numbers $n_g(d) \in \mathbb{C}$ for $g = 0, 1, 2, \ldots$ and $d = 1, 2, 3, \ldots$ satisfies a *1-point recursion* if there exist integers $i_{\max}$ and $j_{\max}$ and polynomials $p_{ij}(z) \in \mathbb{C}[z]$, not all equal to zero, such that

$$\sum_{i=0}^{i_{\max}} \sum_{j=0}^{j_{\max}} p_{ij}(d)\, n_{g-i}(d-j) = 0, \tag{4.3}$$

for all g and d for which all terms in the equation are defined.

The current work is motivated by the following interrelated questions.

- What unified proofs of 1-point recursions exist, which encompass both equations (4.1) and (4.2)?
- How universal is the the notion of a 1-point recursion?

We partially answer these questions by first observing that the enumeration of both fatgraphs and dessins d'enfant can be expressed in terms of Schur functions. This suggests that 1-point recursions may exist more generally for problems that may be defined in an analogous way.

Thus, we consider *double Schur function expansions* of the following form.

$$\begin{aligned} Z(\mathbf{p}; \mathbf{q}; \hbar) &= \sum_{\lambda \in \mathcal{P}} s_\lambda(p_1, p_2, \ldots)\, s_\lambda(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots) \prod_{\Box \in \lambda} G(c(\Box)\hbar) \\ &= \exp\left[\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \sum_{d_1, d_2, \ldots, d_n = 1}^{\infty} N_{g,n}(d_1, d_2, \ldots, d_n)\, \frac{\hbar^{2g-2+n}}{n!}\, p_{d_1} p_{d_2} \cdots p_{d_n} \right] \end{aligned} \tag{4.4}$$

The precise meaning of all terms appearing in the first line of this equation will be discussed in Section 4.2. It currently suffices to observe that the "enumerative problem" is stored in the numbers $N_{g,n}(d_1, d_2, \ldots, d_n)$ appearing in the second line. These numbers have been recently studied in the work of Alexandrov, Chapuy, Eynard and Harnad [2], where they are given a combinatorial interpretation and referred to as *weighted Hurwitz numbers*, as described in Section 1.5.

The primary contribution of this chapter is an approach to proving 1-point recursions for such "enumerative problems". In particular, our main result is the following.

Theorem 4.1.2. *Let $G(z) \in \mathbb{C}(z)$ be a rational function with $G(0) = 1$ and suppose that finitely many terms of the sequence $q_1, q_2, q_3, \ldots$ are non-zero. Then the numbers $n_g(d) = d\, N_{g,1}(d)$ defined by equation (4.4) satisfy a 1-point recursion.*

The proof of this theorem will be taken up in Section 4.5, where we use the theory and language of holonomic sequences and functions. The basic observation is Lemma 4.4.1, which states that a 1-point recursion exists for $n_g(d)$ if and only if the sequence $n_d = \sum_g n_g(d)\, \hbar^{2g-1}$ is holonomic over $\mathbb{C}(\hbar)$.

One of the features of holonomic sequences and functions is that there are readily available algorithms to carry out computations, such as those found in the gfun package for *Maple* [93] or the HolonomicFunctions package for *Mathematica* [72].

After proving the existence of 1-point recursions in Section 4.5, we give bounds on the degree and order of the recursion in Section 4.6, in terms of the degree of the rational function $G(z)$ and the number of non-zero weights $q_1, q_2, \ldots, q_r$. We explain how such bounds on the degree and order can be used to effectively derive 1-point recursions explicitly.

Example 4.1.3. If we take $G(z) = 1 + z$ and $\mathbf{q} = (0, 1, 0, 0, \ldots)$ in equation (4.4), then we recover the enumeration of fatgraphs introduced earlier. In other words, we have $n_g(d) = a_g(d)$, so Theorem 4.1.2 asserts the existence of a 1-point recursion for the numbers $a_g(d)$.

Analogously, if we take $G(z) = (1 + z)^2$ and $\mathbf{q} = (1, 0, 0, \ldots)$ in equation (4.4), then we recover the enumeration of dessins d'enfant introduced earlier. In other words, we have $n_g(d) = b_g(d)$, so Theorem 4.1.2 asserts the existence of a 1-point recursion for the numbers $b_g(d)$.

The topological recursion discussed in Chapter 3 provides a way to calculate n-point invariants, giving a vast generalisation of the Walsh–Lehman recursion for the enumeration of fatgraphs. Of course, any method to calculate n-point invariants in general may also be used to calculate 1-point invariants in particular. However, 1-point recursions appear to be far more efficient from a computational viewpoint and can give direct information regarding the structure of 1-point invariants that is not apparent from the topological recursion.

4.2 One-point enumerative problems

Our work is primarily motivated by the Harer–Zagier recursion for the enumeration of fatgraphs with one face [62], as well as the Do–Norbury recursion for the enumeration of dessins d'enfant with one face [36], appearing in equations (4.1) and (4.2). Apart from the obvious similarities between these two problems, they also both arise from double Schur function expansions. So we propose to study the broad class of "enumerative problems" stored in double Schur function expansions of the general form

$$Z(\mathbf{p}; \mathbf{q}; \hbar) = \sum_{\lambda \in \mathcal{P}} s_\lambda(p_1, p_2, \ldots)\, s_\lambda(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots)\, F_\lambda(\hbar).$$

Here, $\mathcal{P}$ denotes the set of all partitions (including the empty partition), $s_\lambda(p_1, p_2, \ldots)$ denotes the Schur function expressed in terms of power sum symmetric functions, and $F_\lambda(\hbar)$ is a formal power series in $\hbar$ for each partition λ. We use the shorthand $\mathbf{p} = (p_1, p_2, p_3, \ldots)$ and $\mathbf{q} = (q_1, q_2, q_3, \ldots)$ throughout the chapter. Following the mathematical physics literature, we will refer to such power series as *partition functions*, although we note that this name does not refer to the integer partitions that appear in the equation above.

For our applications, we will take $F_\lambda(\hbar)$ to have the so-called *content product form*

$$F_\lambda(\hbar) = \prod_{\Box \in \lambda} G(c(\Box)\hbar).$$

Here, the product is over the boxes in the Young diagram for λ, $G(z) \in \mathbb{C}[[z]]$ is a formal power series normalised to have constant term 1, and $c(\Box)$ denotes the content of the box. Recall that the *content* of a box in row i and column j of a Young diagram is the integer $i - j$.

The partition function can be expressed as

$$Z(\mathbf{p};\mathbf{q};\hbar) = \exp\left[\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\sum_{d_1,d_2,\ldots,d_n=1}^{\infty} N_{g,n}(d_1,d_2,\ldots,d_n)\,\frac{\hbar^{2g-2+n}}{n!}\,p_{d_1}p_{d_2}\cdots p_{d_n}\right], \tag{4.5}$$

where $N_{g,n}(d_1,d_2,\ldots,d_n) \in \mathbb{C}[q_1,q_2,\ldots]$. For various natural choices of the formal power series $G(z)$ and the weights $q_1,q_2,q_3,\ldots$, the quantity $N_{g,n}(d_1,d_2,\ldots,d_n)$ enumerates objects of combinatorial interest. We will be primarily concerned with the 1-point invariants that arise when $n = 1$. In particular, we consider the numbers

$$n_g(d) = d\,N_{g,1}(d),$$

with the goal of determining whether or not there exists a 1-point recursion governing these numbers.

We now proceed to examine four classes of combinatorial problems that arise from double Schur function expansions. These were all defined previously in Chapter 1, although we now focus our attention on their expression via Schur functions and their corresponding 1-point enumerative problems.

Fatgraphs and dessins d'enfant

We discussed the enumeration of fatgraphs and dessins d'enfant in Section 1.4 and now proceed to define the associated 1-point enumerations by $a_g(d) := dA_{g,1}(d)$, $\bar{a}_g(d) := d\bar{A}_{g,1}(d)$, $b_g(d) := dB_{g,1}(d)$ and $\bar{b}_g(d) := d\bar{B}_{g,1}(d)$.

Lemma 4.2.1. *The double dessin d'enfant numbers arise from taking $\mathbf{q} = (q_1,q_2,q_3,\ldots)$ and $G(z) = 1 + z$ in equation (4.4). In other words, we have*

$$\begin{aligned}
Z(\mathbf{p};\mathbf{q};\hbar) &= \sum_{\lambda\in\mathcal{P}} s_\lambda(p_1,p_2,\ldots)\,s_\lambda(\tfrac{q_1}{\hbar},\tfrac{q_2}{\hbar},\ldots)\prod_{\Box\in\lambda}(1+c(\Box)\hbar)\\
&= \exp\left[\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\sum_{d_1,d_2,\ldots,d_n=1}^{\infty} \overline{B}_{g,n}(d_1,d_2,\ldots,d_n)\,\frac{\hbar^{2g-2+n}}{n!}\,p_{d_1}p_{d_2}\cdots p_{d_n}\right].
\end{aligned}$$

One obtains the usual dessin d'enfant enumeration by setting $\mathbf{q} = (1,1,1,\ldots)$ in the double dessin d'enfant enumeration.

$$\begin{aligned}
Z(\mathbf{p};\mathbf{q};\hbar) &= \exp\left[\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\sum_{d_1,d_2,\ldots,d_n=1}^{\infty} B_{g,n}(d_1,d_2,\ldots,d_n)\,\frac{\hbar^{2g-2+n}}{n!}\,p_{d_1}p_{d_2}\cdots p_{d_n}\right]\\
&= \sum_{\lambda\in\mathcal{P}} s_\lambda(p_1,p_2,\ldots)\,s_\lambda(\tfrac{1}{\hbar},\tfrac{1}{\hbar},\tfrac{1}{\hbar},\ldots)\prod_{\Box\in\lambda}(1+c(\Box)\hbar)\\
&= \sum_{\lambda\in\mathcal{P}} s_\lambda(p_1,p_2,\ldots)\,s_\lambda(\tfrac{1}{\hbar},0,0,\ldots)\prod_{\Box\in\lambda}(1+c(\Box)\hbar)^2
\end{aligned}$$

The second equality here relies on the fact that $s_\lambda(\frac{1}{\hbar}, \frac{1}{\hbar}, \frac{1}{\hbar}, \ldots) = s_\lambda(\frac{1}{\hbar}, 0, 0, \ldots) \prod (1 + c(\Box)\hbar)$, which is a direct corollary of the hook-length and the hook-content formulas — see equation (4.7).

d	g	$a_g(d)$	$\bar{b}_g(d)$
1	0	1	q_1
2	0	2	$q_2 + q_1^2$
2	1	1	0
3	0	5	$q_3 + 3q_2q_1 + q_1^3$
3	1	10	q_3
4	0	14	$q_4 + 4q_3q_1 + 2q_2^2 + 6q_2q_1^2 + q_1^4$
4	1	70	$5q_4 + 4q_3q_1 + q_2^2$
4	2	21	0
5	0	42	$q_5 + 5q_4q_1 + 5q_3q_2 + 10q_3q_1^2 + 10q_2^2q_1 + 10q_2q_1^3 + q_1^5$
5	1	420	$15q_5 + 25q_4q_1 + 15q_3q_2 + 10q_3q_1^2 + 5q_2^2q_1$
5	2	483	$8q_5$
6	0	132	$q_6 + 6q_5q_1 + 6q_4q_2 + 15q_4q_1^2 + 3q_3^2 + 30q_3q_2q_1 + 20q_3q_1^3 + 5q_2^3 + 30q_2^2q_1^2 + 15q_2q_1^4 + q_1^6$
6	1	2310	$35q_6 + 90q_5q_1 + 60q_4q_2 + 75q_4q_1^2 + 25q_3^2 + 90q_3q_2q_1 + 20q_3q_1^3 + 10q_2^3 + 15q_2^2q_1^2$
6	2	6468	$84q_6 + 48q_5q_1 + 24q_4q_2 + 12q_3^2$
6	3	1485	0

Bousquet-Mélou–Schaeffer numbers

We defined Bousquet-Mélou–Schaeffer numbers in Section 1.1 and now define the associated 1-point enumerations via $b_g^{(m)}(d) := dB_{g,1}^{(m)}(d)$ and $\bar{b}_g^{(m)}(d) := d\bar{B}_{g,1}^{(m)}(d)$.

Lemma 4.2.2. *The m-BMS numbers arise from taking $\mathbf{q} = (1, 0, 0, \ldots)$ and $G(z) = (1 + z)^m$ in equation (4.4). In other words, we have*

$$
\begin{aligned}
Z(\mathbf{p}; \mathbf{q}; \hbar) &= \sum_{\lambda \in \mathcal{P}} s_\lambda(p_1, p_2, \ldots)\, s_\lambda(\tfrac{1}{\hbar}, 0, 0, \ldots) \prod_{\Box \in \lambda} (1 + c(\Box)\hbar)^m \\
&= \exp \left[\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \sum_{d_1, d_2, \ldots, d_n = 1}^{\infty} B_{g,n}^{(m)}(d_1, d_2, \ldots, d_n)\, \frac{\hbar^{2g-2+n}}{n!}\, p_{d_1} p_{d_2} \cdots p_{d_n} \right].
\end{aligned}
$$

The double Bousquet-Mélou–Schaeffer numbers arises taking $\mathbf{q} = (q_1, q_2, q_3, \ldots)$ and $G(z) = (1 + z)^{m-1}$ in equation (4.4).

d	g	$\bar{b}_g^{(3)}(d)$
1	0	q_1
2	0	$q_2 + 2q_1^2$
2	1	q_2
3	0	$q_3 + 6q_2q_1 + 5q_1^3$
3	1	$8q_3 + 12q_2q_1 + q_1^3$
3	2	$3q_3$
4	0	$q_4 + 8q_3q_1 + 4q_2^2 + 28q_2q_1^2 + 14q_1^4$
4	1	$30q_4 + 96q_3q_1 + 34q_2^2 + 100q_2q_1^2 + 10q_1^4$
4	2	$93q_4 + 88q_3q_1 + 34q_2^2 + 16q_2q_1^2$
4	3	$20q_4$
5	0	$q_5 + 10q_4q_1 + 10q_3q_2 + 45q_3q_1^2 + 45q_2^2q_1 + 120q_2q_1^3 + 42q_1^5$
5	1	$80q_5 + 400q_4q_1 + 280q_3q_2 + 770q_3q_1^2 + 560q_2^2q_1 + 700q_2q_1^3 + 70q_1^5$
5	2	$901q_5 + 1990q_4q_1 + 1290q_3q_2 + 1405q_3q_1^2 + 1055q_2^2q_1 + 380q_2q_1^3 + 8q_1^5$
5	3	$1650q_5 + 1200q_4q_1 + 820q_3q_2 + 180q_3q_1^2 + 140q_2^2q_1$
5	4	$248q_5$

Simple Hurwitz numbers

We defined Hurwitz numbers in Section 1.2 and now define the associated one-point enumerations by $h_g(d) := dH_{g,1}(d)$ and $\bar{h}_g(d) := d\bar{H}_{g,1}(d)$.

Lemma 4.2.3. *The simple Hurwitz numbers arise from taking* $\mathbf{q} = (1, 0, 0, \ldots)$ *and* $G(z) = \exp(z)$ *in equation (4.4). In other words, we have*

$$\begin{aligned} Z(\mathbf{p};\mathbf{q};\hbar) &= \sum_{\lambda\in\mathcal{P}} s_\lambda(p_1,p_2,\ldots)\, s_\lambda(\tfrac{1}{\hbar},0,0,\ldots) \prod_{\Box\in\lambda} \exp(c(\Box)\hbar) \\ &= \exp\left[\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\sum_{d_1,d_2,\ldots,d_n=1}^{\infty} H_{g,n}(d_1,d_2,\ldots,d_n)\,\frac{\hbar^{2g-2+n}}{n!}\, p_{d_1}p_{d_2}\cdots p_{d_n}\right]. \end{aligned}$$

Lemma 4.2.4. *The double Hurwitz numbers arise from taking* $\mathbf{q} = (q_1, q_2, q_3, \ldots)$ *and* $G(z) = \exp(z)$ *in equation (4.4). In other words, we have*

$$\begin{aligned} Z(\mathbf{p};\mathbf{q};\hbar) &= \sum_{\lambda\in\mathcal{P}} s_\lambda(p_1,p_2,\ldots)\, s_\lambda(\tfrac{q_1}{\hbar},\tfrac{q_2}{\hbar},\ldots) \prod_{\Box\in\lambda} \exp(c(\Box)\hbar) \\ &= \exp\left[\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\sum_{d_1,d_2,\ldots,d_n=1}^{\infty} \overline{H}_{g,n}(d_1,d_2,\ldots,d_n)\,\frac{\hbar^{2g-2+n}}{n!}\, p_{d_1}p_{d_2}\cdots p_{d_n}\right]. \end{aligned}$$

d	g	$h_g(d)$	$\overline{h}_g(d)$
1	0	1	q_1
1	1	0	0
1	2	0	0
2	0	1	$q_2 + q_1^2$
2	1	$\frac{1}{6}$	$\frac{1}{2}q_2 + \frac{1}{6}q_1^2$
2	2	$\frac{1}{120}$	$\frac{1}{24}q_2 + \frac{1}{120}q_1^2$
3	0	$\frac{3}{2}$	$q_3 + 3q_2q_1 + \frac{3}{2}q_1^3$
3	1	$\frac{9}{8}$	$3q_3 + \frac{9}{2}q_2q_1 + \frac{9}{8}q_1^3$
3	2	$\frac{27}{80}$	$\frac{9}{4}q_3 + \frac{81}{40}q_2q_1 + \frac{27}{80}q_1^3$
4	0	$\frac{8}{3}$	$q_4 + 4q_3q_1 + 2q_2^2 + 8q_2q_1^2 + \frac{8}{3}q_1^4$
4	1	$\frac{16}{3}$	$10q_4 + 24q_3q_1 + \frac{28}{3}q_2^2 + \frac{80}{3}q_2q_1^2 + \frac{16}{3}q_1^4$
4	2	$\frac{208}{45}$	$\frac{82}{3}q_4 + \frac{216}{5}q_3q_1 + \frac{244}{15}q_2^2 + \frac{1456}{45}q_2q_1^2 + \frac{208}{45}q_1^4$
5	0	$\frac{125}{24}$	$q_5 + 5q_4q_1 + 5q_3q_2 + \frac{25}{2}q_3q_1^2 + \frac{25}{2}q_2^2q_1 + \frac{125}{6}q_2q_1^3 + \frac{125}{24}q_1^5$
5	1	$\frac{3125}{144}$	$25q_5 + \frac{250}{3}q_4q_1 + \frac{125}{2}q_3q_2 + \frac{3125}{24}q_3q_1^2 + \frac{625}{6}q_2^2q_1 + \frac{3125}{24}q_2q_1^3 + \frac{3125}{144}q_1^5$
5	2	$\frac{15625}{384}$	$\frac{2125}{12}q_5 + \frac{1250}{3}q_4q_1 + \frac{6875}{24}q_3q_2 + \frac{21875}{48}q_3q_1^2 + \frac{3125}{9}q_2^2q_1 + \frac{15625}{48}q_2q_1^3 + \frac{15625}{384}q_1^5$

Monotone Hurwitz numbers

We defined monotone Hurwitz numbers in Definition 1.3.1 and now define the associated one-point enumerations by $m_g(d) := dM_{g,1}(d)$ and $\bar{m}_g(d) := d\bar{M}_{g,1}(d)$.

Lemma 4.2.5. *The monotone Hurwitz numbers arise from taking* $\mathbf{q} = (1, 0, 0, \ldots)$ *and* $G(z) = \frac{1}{1-z}$ *in equation* (4.4)*. In other words, we have*

$$Z(\mathbf{p}; \mathbf{q}; \hbar) = \sum_{\lambda \in \mathcal{P}} s_\lambda(p_1, p_2, \ldots)\, s_\lambda(\tfrac{1}{\hbar}, 0, 0, \ldots) \prod_{\Box \in \lambda} \frac{1}{1 - c(\Box)\hbar}$$
$$= \exp\left[\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\sum_{d_1,d_2,\ldots,d_n=1}^{\infty} M_{g,n}(d_1, d_2, \ldots, d_n)\, \frac{\hbar^{2g-2+n}}{n!}\, p_{d_1}p_{d_2}\cdots p_{d_n}\right].$$

Lemma 4.2.6. *The double monotone Hurwitz numbers arise from taking* $\mathbf{q} = (q_1, q_2, q_3, \ldots)$ *and* $G(z) = \frac{1}{1-z}$ *in equation* (4.4)*. In other words, we have*

$$Z(\mathbf{p}; \mathbf{q}; \hbar) = \sum_{\lambda \in \mathcal{P}} s_\lambda(p_1, p_2, \ldots)\, s_\lambda(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots) \prod_{\Box \in \lambda} \frac{1}{1 - c(\Box)\hbar}$$
$$= \exp\left[\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\sum_{d_1,d_2,\ldots,d_n=1}^{\infty} \overline{M}_{g,n}(d_1, d_2, \ldots, d_n)\, \frac{\hbar^{2g-2+n}}{n!}\, p_{d_1}p_{d_2}\cdots p_{d_n}\right].$$

d	g	$m_g(d)$	$\overline{m}_g(d)$
1	0	1	q_1
1	1	1	0
1	2	1	0
2	0	1	$q_2 + q_1^2$
2	1	1	$q_2 + q_1^2$
2	2	1	$q_2 + q_1^2$
3	0	2	$q_3 + 3q_2q_1 + 2q_1^3$
3	1	10	$5q_3 + 15q_2q_1 + 10q_1^3$
3	2	42	$21q_3 + 63q_2q_1 + 42q_1^3$
4	0	5	$q_4 + 4q_3q_1 + 2q_2^2 + 10q_2q_1^2 + 5q_1^4$
4	1	70	$15q_4 + 60q_3q_1 + 25q_2^2 + 140q_2q_1^2 + 70q_1^4$
4	2	735	$161q_4 + 644q_3q_1 + 252q_2^2 + 1470q_2q_1^2 + 735q_1^4$
5	0	14	$q_5 + 5q_4q_1 + 5q_3q_2 + 15q_3q_1^2 + 15q_2^2q_1 + 35q_2q_1^3 + 14q_1^5$
5	1	420	$35q_5 + 175q_4q_1 + 140q_3q_2 + 490q_3q_1^2 + 420q_2^2q_1 + 1050q_2q_1^3 + 420q_1^5$
5	2	8778	$777q_5 + 3885q_4q_1 + 2835q_3q_2 + 10605q_3q_1^2 + 8505q_2^2q_1 + 21945q_2q_1^3 + 8778q_1^5$

4.3 Schur function evaluations

In the previous section, we discussed certain enumerative problems of geometric interest that are stored in the following partition function for different choices of the power series $G(z)$ and the parameters $q_1, q_2, \ldots$.

$$\begin{aligned} Z(\mathbf{p};\mathbf{q};\hbar) &= \sum_{\lambda\in\mathcal{P}} s_\lambda(p_1,p_2,\ldots)\, s_\lambda(\tfrac{q_1}{\hbar},\tfrac{q_2}{\hbar},\ldots) \prod_{\Box\in\lambda} G(c(\Box)\hbar) \\ &= \exp\left[\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\sum_{d_1,d_2,\ldots,d_n=1}^{\infty} N_{g,n}(d_1,d_2,\ldots,d_n)\,\frac{\hbar^{2g-2+n}}{n!}\,p_{d_1}p_{d_2}\cdots p_{d_n}\right] \end{aligned}$$

The coefficients $N_{g,n}(d_1,d_2,\ldots,d_n) \in \mathbb{C}[q_1,q_2,\ldots]$ are essentially the weighted Hurwitz numbers appearing in the work of Alexandrov, Chapuy, Eynard and Harnad [2].

We consider in particular the 1-point invariants

$$n_g(d) = d\, N_{g,1}(d)$$

stored in the partition function.[1] In order to obtain information about these numbers, we deform the partition function via a parameter s that keeps track of the unweighted degree in $p_1, p_2, p_3, \ldots$

[1]The extra factor of d in the definition of $n_g(d)$ will have little bearing on our results, but is introduced here for consistency with the original Harer–Zagier recursion and other results in the literature. We remark that the 1-point recursions are generally simpler with this normalisation, as can be witnessed from equations (4.1) and (4.2).

and then extract the 1-point invariants by differentiation.

$$\begin{aligned}\left[\frac{\partial}{\partial s} Z(s\mathbf{p};\mathbf{q};\hbar)\right]_{s=0} &= \sum_{\lambda\in\mathcal{P}} \left[\frac{\partial}{\partial s} s_\lambda(sp_1, sp_2, \ldots)\right]_{s=0} s_\lambda(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots) \prod_{\Box\in\lambda} G(c(\Box)\hbar) \\ &= \sum_{g=0}^{\infty}\sum_{d=1}^{\infty} N_{g,1}(d)\, \hbar^{2g-1}\, p_d\end{aligned}$$

At this stage, it is natural to introduce the so-called *principal specialisation* $p_d = x^d$ to record the degree via the single variable x.

$$\begin{aligned}\left[\frac{\partial}{\partial s} Z(sx, sx^2, sx^3, \ldots;\mathbf{q};\hbar)\right]_{s=0} &= \sum_{\lambda\in\mathcal{P}} \left[\frac{\partial}{\partial s} s_\lambda(sx, sx^2, sx^3, \ldots)\right]_{s=0} s_\lambda(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots) \prod_{\Box\in\lambda} G(c(\Box)\hbar) \\ &= \sum_{g=0}^{\infty}\sum_{d=1}^{\infty} N_{g,1}(d)\, \hbar^{2g-1}\, x^d \end{aligned} \tag{4.6}$$

In this section, we deduce some facts about Schur functions that will be required at a later stage. We begin with the crucial observation that the evaluation of the Schur function appearing in equation (4.6) is zero unless λ is a hook partition. Here, and throughout the chapter, a *hook partition* refers to a partition of the form $(k, 1^{d-k})$, where $1 \leq k \leq d$.

Lemma 4.3.1.

$$\left[\frac{\partial}{\partial s} s_\lambda(sx, sx^2, sx^3, \ldots)\right]_{s=0} = \begin{cases} (-1)^{d-k}\, \frac{x^d}{d}, & \text{if } \lambda = (k, 1^{d-k}) \text{ is a hook partition,} \\ 0, & \text{otherwise.}\end{cases}$$

Proof. The lemma follows from the hook-content formula [76], which states that

$$s_\lambda(s, s, s, \ldots) = \prod_{\Box\in\Lambda} \frac{s + c(\Box)}{h(\Box)}, \tag{4.7}$$

where $c(\Box)$ and $h(\Box)$ denote the content and hook-length of a box in the Young diagram for λ, respectively. Recall that the *hook-length* of a box in a Young diagram is the number of boxes that lie to the right in the same row, or lie below in the same column, and including the box itself.

If λ is a non-empty partition that is not a hook, then its Young diagram contains at least two boxes with content 0. So the hook-content formula implies that $s_\lambda(s, s, s, \ldots)$ is a polynomial divisible by s^2 and it follows that

$$\left[\frac{\partial}{\partial s} s_\lambda(sx, sx^2, sx^3, \ldots)\right]_{s=0} = 0.$$

If $\lambda = (k, 1^{d-k})$ is a hook partition, then its hook-lengths are given by the multiset

$$\{1, 2, \ldots, k-1\} \cup \{1, 2, \ldots, d-k\} \cup \{d\},$$

while its contents are given by

$$\{1, 2, \ldots, k-1\} \cup \{-1, -2, \ldots, -(d-k)\} \cup \{0\}.$$

Thus, we obtain

$$s_\lambda(s, s, s, \ldots) = (-1)^{d-k} \frac{(s+k-1)(s+k-2)\cdots(s+k-d)}{d(k-1)!(d-k)!}.$$

By directly differentiating with respect to s and evaluating at $s = 0$, we obtain

$$\left[\frac{\partial}{\partial s} s_\lambda(s, s, \ldots)\right]_{s=0} = \frac{(-1)^{d-k}}{d}.$$

The powers of x appearing in the statement of the lemma can be reinstated, using the fact that Schur functions are weighted homogeneous. □

Now use Lemma 4.3.1 in equation (4.6) to obtain the following.

$$\begin{aligned}\left[\frac{\partial}{\partial s} Z(sx, sx^2, sx^3, \ldots; \mathbf{q}; \hbar)\right]_{s=0} &= \sum_{g=0}^{\infty}\sum_{d=1}^{\infty} N_{g,1}(d)\hbar^{2g-1}x^d \\ &= \sum_{d=1}^{\infty}\sum_{k=1}^{d}(-1)^{d-k}\frac{x^d}{d}\, s_{(k,1^{d-k})}(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots)\prod_{i=1}^{d} G((k-i)\hbar)\end{aligned}$$

Extracting the x^d coefficient yields the following result.

Lemma 4.3.2. *The 1-point invariants $n_g(d) = d\,N_{g,1}(d)$ defined by equation (4.5) satisfy*

$$\sum_{g=0}^{\infty} n_g(d)\,\hbar^{2g-1} = \sum_{k=1}^{d}(-1)^{d-k}\, s_{(k,1^{d-k})}(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots)\prod_{i=1}^{d} G((k-i)\hbar),$$

for every positive integer d.

We will later be interested in setting the parameter $q_i = 0$ for i sufficiently large. In this case, we write $s_\lambda(\frac{q_1}{\hbar}, \frac{q_2}{\hbar}, \ldots, \frac{q_r}{\hbar})$ to mean the Schur function $s_\lambda(\frac{q_1}{\hbar}, \frac{q_2}{\hbar}, \ldots)$ evaluated at $q_{r+1} = q_{r+2} = \cdots = 0$.

We end this section with the following lemma, which will be used in the following sections [76].

Lemma 4.3.3. *The Schur function indexed by the hook $(k, 1^{d-k})$ can be written as*

$$s_{k,1^{d-k}}(\mathbf{q}) = \sum_{j=1}^{k}(-1)^{j+1} h_{k-j}(\mathbf{q})\, e_{d-k+j}(\mathbf{q}),$$

where h_n and e_n are the complete homogeneous and elementary symmetric functions, respectively. These can be expressed in terms of power-sum symmetric functions as

$$\sum_{n=0}^{\infty} h_n(\mathbf{q})x^n = \exp\left[\sum_{k=1}^{\infty}\frac{q_k}{k}x^k\right] \qquad \text{and} \qquad \sum_{n=0}^{\infty} e_n(\mathbf{q})x^n = \exp\left[\sum_{k=1}^{\infty}(-1)^{k-1}\frac{q_k}{k}x^k\right].$$

In the case $\mathbf{q} = (q, 0, 0, \ldots)$, the above expression evaluates to the following.

Corollary 4.3.4.

$$s_{(k,1^{d-k})}(q, 0, 0, \ldots) = \binom{d-1}{k-1}\frac{q^d}{d!}$$

4.4 Holonomic sequences and functions

Recall that a sequence $\{a_i\}_{i \geq \mathbb{N}}$ is said to be *holonomic over* $\mathbb{K}$ if the terms satisfy a non-zero linear difference equation of the form

$$p_r(d)\, a_{d+r} + p_{r-1}(d)\, a_{d+r-1} + \cdots + p_1(d)\, a_{d+1} + p_0(d)\, a_d = 0, \tag{4.8}$$

where $p_0, p_1, \ldots, p_r$ are polynomials over $\mathbb{K}$. Similarly, recall that the formal power series $A(x) = \sum_{d=0}^{\infty} a_d\, x^d$ is said to be *holonomic over* $\mathbb{K}$ if it satisfies a non-zero linear differential equation of the form

$$\left[P_r(x)\, \frac{\partial^r}{\partial x^r} + P_{r-1}(x)\, \frac{\partial^{r-1}}{\partial x^{r-1}} + \cdots + P_1(x)\, \frac{\partial}{\partial x} + P_0(x) \right] A(x) = 0, \tag{4.9}$$

where $P_0, P_1, \ldots, P_r$ are polynomials over $\mathbb{K}$.

The dual use of the term "holonomic" is due to the elementary fact that the sequence $a_0, a_1, a_2, \ldots$ is holonomic over $\mathbb{K}$ if and only if the formal power series $a_0 + a_1 x + a_2 x^2 + \cdots$ is holonomic over $\mathbb{K}$. For our applications, we will use the ground field $\mathbb{K} = \mathbb{C}(\hbar)$.

Lemma 4.4.1. *A 1-point recursion exists for the numbers $n_g(d)$ in the sense of Definition 4.1.1 if and only if the formal power series*

$$F(x, \hbar) = \sum_{d=1}^{\infty} \sum_{g=0}^{\infty} n_g(d)\, \hbar^{2g-1}\, x^d$$

is holonomic over $\mathbb{C}(\hbar)$.

Proof. If $F(x, \hbar)$ is holonomic over $\mathbb{C}(\hbar)$, then there exist polynomials $P_0, P_1, \ldots, P_r$ with coefficients in $\mathbb{C}(\hbar)$ such that

$$\left[P_r(x)\, \frac{\partial^r}{\partial x^r} + P_{r-1}(x)\, \frac{\partial^{r-1}}{\partial x^{r-1}} + \cdots + P_1(x)\, \frac{\partial}{\partial x} + P_0(x) \right] F(x, \hbar) = 0.$$

One can assume that the coefficients of $P_0, P_1, \ldots, P_r$ actually lie in $\mathbb{C}[\hbar]$, by clearing denominators in the equation above. Thus, the equation has the form

$$\left[\sum_{i,j,k=0}^{\text{finite}} C_{ijk}\, \hbar^i x^j \frac{\partial^k}{\partial x^k} \right] F(x, \hbar) = 0, \tag{4.10}$$

for some complex constants C_{ijk}. Applying $C_{ijk}\, \hbar^i x^j \frac{\partial^k}{\partial x^k}$ to a term $n_g(d)\, \hbar^{2g-1}\, x^d$ in the expansion for $F(x, \hbar)$ has the effect of shifting the powers of $\hbar$ and x, while possibly introducing a factor that is polynomial in d. So after collecting terms in the resulting equation, one obtains an expression of the form of equation (4.3). Therefore, there exists a 1-point recursion for the numbers $n_g(d)$.

Conversely, suppose that there exists a 1-point recursion for the numbers $n_g(d)$, so there exists an expression of the form of equation (4.3). After multiplying both sides by $\hbar^{2g-1}\, x^d$, and summing over g and d yields

$$\sum_{d=1}^{\infty} \sum_{g=0}^{\infty} \sum_{i=0}^{i_{max}} \sum_{j=0}^{j_{max}} p_{ij}(d) n_{g-i}(d-j) \hbar^{2g-1} x^d = 0$$

Now replace $p_{ij}(d)x^d$ with $p_{ij}(x\frac{\partial}{\partial x})x^d$ and reindex the summations over d and g to obtain

$$\sum_{d=1}^{\infty} \sum_{g=0}^{\infty} \sum_{i=0}^{i_{max}} \sum_{j=0}^{j_{max}} p_{ij}(x\frac{\partial}{\partial x} x^d n_g(d) \hbar^{2i} x^j \hbar^{2g-1} x^d = 0 \Rightarrow \quad \sum_{i=0}^{i_{max}} \sum_{j=0}^{j_{max}} \left[p_{ij}(x\frac{\partial}{\partial x} x^d n_g(d) \hbar^{2i} x^j \right] = 0$$

This final equation can be expressed in the form of equation (4.10) by applying the commutation relation $[\frac{\partial}{\partial x}, x] = 1$. It then follows that $F(x, \hbar)$ is holonomic over $\mathbb{C}(\hbar)$. $\square$

The following series of elementary results provide standard tools to prove holonomicity of sequences and generating functions [67].

Theorem 4.4.2. *Let $\mathbb{K}$ be a field and let $a(z) = \sum_{d=0}^{\infty} a_d\, z^d \in \mathbb{K}[[z]]$ and $b(z) = \sum_{d=0}^{\infty} b_d\, z^d \in \mathbb{K}[[z]]$ be holonomic power series. Then*

(a) $\alpha a(z) + \beta b(z)$ is holonomic for all $\alpha, \beta \in \mathbb{K}$;
(b) the Cauchy product $a(z)\, b(z)$ and the Hadamard product $\big(a_n b_n\big)_{n=0,1,2,\ldots}$ are holonomic;
(c) the derivative $a'(z)$ and the forward shift $\big(a_{n+1}\big)_{n=0,1,2,\ldots}$ are holonomic;
(d) the integral $\int^z a(z)$ and the indefinite sum $\big(\sum_{k=0}^{n} a_k\big)_{n=0,1,2,\ldots}$ are holonomic;
(e) if $b(z)$ is algebraic with $b(0) = 0$, then $a(b(z))$ is holonomic; and
(f) $\big(a_{\lfloor un+v \rfloor}\big)_{n=0,1,2,\ldots}$ is holonomic for all non-negative rationals u and v.

Definition 4.4.3. We define the *order* and *degree* of the difference operator of equation (4.8) to be r and $\max\{\deg p_0, \deg p_1, \ldots, \deg p_r\}$, respectively. Similarly, we define the *order* and *degree* of the differential operator of equation (4.9) to be r and $\max\{\deg P_0, \deg P_1, \ldots, \deg P_r\}$, respectively.

Remark. Note that for a fixed holonomic sequence or function, there are difference or differential operators of many possible orders and degrees that annihilate it. Furthermore, it is not generally true that there exists such an operator that simultaneously minimises both the order and the degree. Thus, one does not usually refer to the order and degree of a holonomic sequence or function itself, but to the order and degree of a particular operator.

The notion of one-variable holonomic sequences functions can be generalised to multivariable sequences and functions in a couple of natural ways [104]. For our present purposes, we use the rather down-to-earth notion of D-finiteness introduced by Stanley [97]. Let $x = (x_1, \ldots, x_n)$ and let $\mathfrak{D}$ denote the ring of all linear partial differential operators in $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ with coefficients in $\mathbb{K}[x_1, \ldots, x_n]$. We say that an element f of a $\mathfrak{D}$-module $\mathfrak{M}$ is *D-finite* or *differentiably finite* if it satisfies a system of the form

$$\left[a_{in_i}(x)\Big(\frac{\partial}{\partial x_i}\Big)^{n_i} + a_{in_i-1}(x)\Big(\frac{\partial}{\partial x_i}\Big)^{n_i-1} + \cdots + a_{i0}(x)\right] f = 0, \quad \text{for } i = 1, \ldots, n, \tag{4.11}$$

where $a_{ij}(x) \in \mathbb{K}[x]$. For our purposes, we take the $\mathfrak{D}$-module $\mathfrak{M}$ to be $\mathbb{K}[[x]]$ for $\mathbb{K} = \mathbb{C}$, with the natural action of differential operators on formal power series. Equivalently, $f \in \mathbb{K}[[\mathbf{x}]]$ is D-finite if the set of derivatives $(\frac{\partial}{\partial x_1})^{i_1} \ldots (\frac{\partial}{\partial x_n})^{i_n} f$ for $(i_1, \ldots, i_n) \in \mathbb{N}^n$ generate a finite-dimensional vector space over the rational function field $\mathbb{K}(x_1, \ldots, x_n)$.

Next, we define the primitive diagonal of $f \in \mathbb{K}[[x]]$.

Definition 4.4.4. For $f = \sum a_{i_1,\ldots,i_n}\, x_1^{i_1} \ldots x_n^{i_n} \in \mathbb{K}[[x]]$ and integers $1 \le k < \ell \le n$, define the *primitive diagonal*

$$I_{k\ell}(f) := \sum a_{i_1,\ldots i_k,\ldots,i_k,\ldots i_n}\, x_1^{i_1} \cdots x_k^{i_k} \cdots \widehat{x_\ell^{i_\ell}} \cdots x_n^{i_n},$$

where the hat denotes the omission of the term involving x_ℓ.

For example, if we take $k = 1$ and $\ell = 2$, then

$$I_{12}(f) = \sum a_{i_1 i_1 i_3 \ldots i_n}\, x_1^{i_1} x_3^{i_3} \ldots x_n^{i_n}.$$

Many of the closure properties of one-variable holonomic functions from Theorem 4.4.2 carry over to the notion of D-finiteness for multivariable power series in a straightforward manner. We additionally require the following two results, which we state without proof, since they can be found in various references [97, 75].

Theorem 4.4.5. *If $f \in \mathbb{K}[[x]]$ is D-finite, and $1 \leq k < \ell \leq n$, then $I_{k\ell}(f)$ is D-finite.*

Lemma 4.4.6. *If $f = \sum a_\nu x^\nu$ and $g = \sum b_\nu x^\nu$ are D-finite functions of n variables, then the Hadamard product*

$$f \star g := \sum a_\nu b_\nu x^\nu$$

is also D-finite.

The previous result may now be used to deduce the following lemma, that will later prove useful.

Lemma 4.4.7. *If $f(x) = \sum_{m_1,\ldots,m_n} f(m_1,\ldots,m_n)\, x_1^{m_1} \ldots x_n^{m_n}$ is D-finite and $S \subset \mathbb{N}^n$ is defined by a finite set of inequalities of the form $\sum_i a_i m_i + b \geq 0$ where $a_i, b \in \mathbb{Z}$, then*

$$h(x) := \sum_{(m_1,\ldots,m_n)\in S} f(m_1,\ldots,m_n)\, x_1^{m_1} \ldots x_n^{m_n}$$

is also D-finite.

Proof. Note that it is sufficient to prove the statement when S is defined by only one inequality, since we can iterate over the set of inequalities to prove the general statement. Without loss of generality, write the inequality as

$$\sum_{i=1}^{k} \alpha_i m_i + \alpha_0 \geq \sum_{i=k+1}^{n} \beta_i m_i,$$

where $\alpha_i, \beta_i \in \mathbb{N}$. Now define the power series

$$g_1(x_1,\ldots,x_n,s,t) := s^{\alpha_0} \prod_{i=1}^{k} \frac{1}{1 - x_i s^{\alpha_i}} \prod_{i=k+1}^{n} \frac{1}{1 - x_i t^{\beta_i}},$$

$$g_2(x_1,\ldots,x_n,s,t) := \frac{1}{1-s}\frac{1}{1-st} \prod_{i=1}^{n} \frac{1}{1-x_i}.$$

Observe that $g_1(x_1,\ldots,x_n,s,t)$ and $g_2(x_1,\ldots,x_n,s,t)$ are all D-finite, due to their definition as rational functions. Now let us define the Hadamard product

$$g(x_1,\ldots,x_n,s,t) := (g_1 \star g_2)(x_1,\ldots,x_n,s,t).$$

By Lemma 4.4.6, $g(x_1,\ldots,x_n)$ is D-finite and if we substitute $s = t = 1$, we obtain that

$$\tilde{g}(x_1,\ldots,x_n) := g(x_1,\ldots,x_n,1,1) = \sum_{(m_1,\ldots,m_n)\in S} x_1^{m_1} \ldots x_n^{m_n}$$

is also D-finite. We complete the proof by noticing that $h(x_1,\ldots,x_n) = f(x_1,\ldots,x_n) \star \tilde{g}(x_1,\ldots,x_n)$ and using Lemma 4.4.6 once again. $\square$

4.5 Existence of one-point recursions

As in equation (4.4), we begin with a choice of formal power series $G(z)$ and a series $q_1, q_2, q_3, \ldots$ of complex constants. We begin by examining the simplified case in which $\mathbf{q} = (1, 0, 0, \ldots)$.

Theorem 4.5.1. *Let $G(z) \in \mathbb{C}(z)$ be a rational function and let $\mathbf{q} = (1, 0, 0, \ldots)$. Define the numbers $n_g(d) = d\, N_{g,1}(d)$ via equation (4.5). Then the generating function $\sum_{d=1}^{\infty} \sum_{g=0}^{\infty} n_g(d)\, \hbar^{2g-1}\, x^d$ is holonomic over $\mathbb{C}(\hbar)$ and it follows that the numbers $n_g(d)$ satisfy a 1-point recursion.*

Proof. Consider the following generating function for the numbers $n_g(d)$.

$$\begin{aligned} n_d(\hbar) &= \sum_{g=0}^{\infty} n_g(d)\, \hbar^{2g-1} \\ &= \sum_{k=1}^{d} (-1)^{d-k}\, s_{(k,1^{d-k})}(\tfrac{1}{\hbar}, 0, 0, \ldots) \prod_{i=1}^{d} G((k-i)\hbar) \\ &= \frac{1}{d!\, \hbar^d} \sum_{k=1}^{d} (-1)^{d-k} \binom{d-1}{k-1} \prod_{i=1}^{d} G((k-i)\hbar) \\ &= \frac{1}{d\, \hbar^d} \sum_{k=1}^{d} (-1)^{d-k} \frac{1}{(k-1)!\,(d-k)!} \prod_{i=1}^{d} G((k-i)\hbar) \end{aligned} \tag{4.12}$$

Define the sequences

$$u_k = \frac{1}{(k-1)!\, \hbar^k} \prod_{i=1}^{k} G((i-1)\hbar) \qquad \text{and} \qquad v_k = \frac{(-1)^k}{k!\, \hbar^k} \prod_{i=1}^{k} G(-i\hbar).$$

These are holonomic over $\mathbb{C}(\hbar)$ since the ratios $\frac{u_{k+1}}{u_k} = \frac{G(k\hbar)}{k\hbar}$ and $\frac{v_{k+1}}{v_k} = -\frac{G(-(k+1)\hbar)}{(k+1)\hbar}$ are rational functions of k with coefficients from $\mathbb{C}(\hbar)$. So Theorem 4.4.2 implies that the sequence

$$n_d(\hbar) = \frac{1}{d} \sum_{k=0}^{d} u_k\, v_{d-k}$$

is holonomic. Then use Lemma 4.4.1 to deduce that the numbers $n_g(d)$ satisfy a 1-point recursion. □

Next, we consider the more general case that $\mathbf{q} = (q_1, q_2, \ldots, q_r)$ for some fixed positive integer r. This requires a little more setup with holonomic functions. Let a_n, b_n, c_n, d_n be holonomic sequences and use them to define the following D-finite functions.

$$\begin{aligned} F(x_1, x_2) &:= \sum_{n=0}^{\infty} c_n (x_1 x_2)^n \\ G(x_1) &:= \sum_{n=1}^{\infty} d_n x_1^n \\ I(x_3) &:= \sum_{n=1}^{\infty} a_n x_3^n \\ J(x_4) &:= \sum_{n=0}^{\infty} b_n x_4^n \end{aligned}$$

From the closure properties of D-finite functions, we know that

$$H(x_1, x_2) := \frac{1}{1-x_2} F(x_1, x_2)\, G(x_1)$$

is D-finite. The general coefficient of the series $H(x_1, x_2)$ is given by

$$C(n,k) := [x_1^n x_2^k] H(x_1, x_2) = \sum_{\ell=0}^{k-1} c_\ell d_{n-\ell}.$$

Apply Lemma 4.4.7 with the inequality $n - k \geq 0$ to obtain the fact that

$$\tilde{H}(x_1, x_2) := \sum_{k \leq n} C_{n,k} x_1^n x - 2^k$$

is D-finite.

Lemma 4.5.2. *The series given by*

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} a_k b_{n-k} \sum_{\ell=0}^{k-1} c_\ell d_{n-\ell} \right) z^n$$

is D-finite.

Proof. Using the notation $x = (x_1, x_2, x_3, x_4)$, we have

$$L(x) := \tilde{H}(x_1, x_2) I(x_3) J(x_4) = \sum_{\substack{i,j,n,k \\ k \leq n}} a_i b_j C(n,k)\ x_1^n x_2^k x_3^i x_4^j$$

is D-finite by the closure properties of D-finite functions. Theorem 4.4.5 allows us to deduce that the primitive diagonal

$$I_{24}(L(x)) = \sum_{\substack{i,n,k \\ k \leq n}} a_i b_k C(n,k)\, x_1^n x_2^k x_3^i$$

is also D-finite. Now use Lemma 4.4.7 with the inequalities $i + k \geq n$ and $i + k \leq n$ to obtain the fact that

$$\tilde{L}(x) := \sum_{\substack{i,k,n \\ i+k=n}} a_i b_k C(n,k)\, x_1^n x_2^k x_3^i$$

is D-finite. By substituting $x_1 = z$, $x_2 = 1$ and $x_3 = 1$, we obtain the desired result. $\square$

We are now ready to prove the main result of this chapter.

Theorem 4.5.3. *Let $G(z) \in \mathbb{C}(z)$ be a rational function and let $\mathbf{q} = (q_1, q_2, \ldots, q_r, 0, 0, \ldots)$. Then the generating function $\sum_{d=1}^{\infty} \sum_{g=0}^{\infty} n_g(d)\, \hbar^{2g-1}\, x^d$ is holonomic in x with coefficients in $\mathbb{C}(\hbar, q_1, \ldots, q_r)$. It follows that the numbers $n_g(d)$ satisfy a 1-point recursion.*

Proof. Consider the following generating function for the numbers $n_g(d)$.

$$\begin{aligned}
\sum_{d=1}^{\infty}\sum_{g=0}^{\infty} n_g(d)\,\hbar^{2g-1}\,x^d &= \sum_{d=1}^{\infty} n_d(\hbar)\,x^d \\
&= \sum_{d=1}^{\infty}\sum_{k=1}^{d}(-1)^{d-k}\,s_{(k,1^{d-k})}(\tfrac{q_1}{\hbar},\tfrac{q_2}{\hbar},\ldots,\tfrac{q_r}{\hbar})\,\prod_{i=1}^{d} G((k-i)\hbar)\,x^d \qquad (\text{ Lemma 4.3.2}) \\
&= \sum_{d=1}^{\infty}\sum_{k=1}^{d}(-1)^{d-k}\prod_{i=1}^{d} G((k-i)\hbar)\sum_{i=1}^{k}(-1)^{i+1}\,h_{(k-i)}(\tfrac{q_1}{\hbar},\tfrac{q_2}{\hbar},\ldots,\tfrac{q_r}{\hbar})\,e_{(d-k+i)}(\tfrac{q_1}{\hbar},\tfrac{q_2}{\hbar},\ldots,\tfrac{q_r}{\hbar})\,x^d \\
&= \sum_{d=1}^{\infty}\sum_{k=1}^{d}\prod_{i=1}^{d} G((k-i)\hbar)\sum_{\ell=0}^{k-1}(-1)^{d-\ell+1}h_\ell(\tfrac{q_1}{\hbar},\tfrac{q_2}{\hbar},\ldots,\tfrac{q_r}{\hbar})\,e_{(d-\ell)}(\tfrac{q_1}{\hbar},\tfrac{q_2}{\hbar},\ldots,\tfrac{q_r}{\hbar})\,x^d \\
&= \sum_{d=1}^{\infty}\sum_{k=1}^{d}\prod_{i=1}^{k} G(i\hbar)\prod_{i=1}^{d-k} G(-i\hbar)\sum_{\ell=0}^{k-1}(-1)^{d-\ell+1}h_\ell(\tfrac{q_1}{\hbar},\tfrac{q_2}{\hbar},\ldots,\tfrac{q_r}{\hbar})\,e_{(d-\ell)}(\tfrac{q_1}{\hbar},\tfrac{q_2}{\hbar},\ldots,\tfrac{q_r}{\hbar})\,x^d
\end{aligned}$$

This last line will help us to identify that the generating function is D-finite from the closure properties of D-finite functions. We simply define the sequences

$$a_n := \prod_{i=1}^{n} G(i\hbar),\ b_n := \prod_{i=1}^{n} G(-i\hbar),\ c_n := h_n(\tfrac{q_1}{\hbar},\tfrac{q_2}{\hbar},\ldots,\tfrac{q_r}{\hbar}),\ d_n := (-1)^{n+1}e_n(\tfrac{q_1}{\hbar},\tfrac{q_2}{\hbar},\ldots,\tfrac{q_r}{\hbar}).$$

The first two are holonomic over $\mathbb{C}(\hbar)$ since the ratios $\frac{a_{n+1}}{a_n} = G(n\hbar)$ and $\frac{b_{n+1}}{b_n} = G(-(n+1)\hbar)$ are rational functions of n with coefficients from $\mathbb{C}(\hbar)$. The last two sequences are also holonomic over $\mathbb{C}(\hbar)$ since Lemma 4.3.3 allows us to deduce that

$$\left[\hbar\frac{\partial}{\partial x} - \sum_{k=1}^{r} q_k x^{k-1}\right]\left(\sum_{n=0}^{\infty} c_n x^n\right) = 0 \qquad \text{and} \qquad \left[\hbar\frac{\partial}{\partial x} + \sum_{k=1}^{r}(-1)^k q_k x^{k-1}\right]\left(\sum_{n=0}^{\infty} d_n x^n\right) = 0.$$

Hence, Lemma 4.5.2 implies that the sequence

$$n_d(\hbar) = \sum_{k=1}^{d} a_k b_{d-k}\sum_{\ell=0}^{k-1} c_\ell d_{d-\ell}$$

is holonomic over $\mathbb{C}(\hbar)$. It then follows from Lemma 4.4.1 that there exists a 1-point recursion for the numbers $n_g(d)$. □

4.6 Algorithms for one-point recursions

One of the features of the theory of D-finite functions is that the theoretical results can often be turned into effective algorithms [72]. In Theorem 4.5.3, we assert the existence of one-point recursions for a broad class of problems. We further claim that the existence proof can be converted into an algorithm to calculate them from the initial data of the rational function $G(z)$ and the positive integer r that records the number of non-zero weights $\mathbf{q} = (q_1, q_2, \ldots, q_r)$. For example, a naive though feasible approach would be to express the putative *1-point recursions* as

$$\sum_{i=0}^{D}\sum_{j=0}^{R} a_{ij}\,d^i\,n(d-j) = 0,$$

and treat this a linear system in the $(D+1)(R+1)$ variables $a_{ij} \in \mathbb{C}(\hbar)$. One obtains a linear constraint for each positive integer d, so a finite number of these allows for the computation of the

1-point recursions. In order to implement this approach, one requires explicit and simultaneous bounds on the degree D and the order R of such a recursion. In this section, we produce such bounds in terms of the degree of $G(z)$ and the positive integer r.

First, we begin with the operators that annihilate the generating function for the sequences a_n, b_n, c_n, d_n that appear in the proof of Theorem 4.5.3. Then, we use known bounds for the degree and order of operators that annihilates functions obtained by the holonomicity closure properties used in the proof — namely, Cauchy product, taking primitive diagonals, restricting summations, and evaluation.

Let us review the proof of the fact that the diagonal of a D-finite function is D-finite [75]. We exhibit the full proof here, as the steps involved are required to give us effective degree and order bounds. Let $f \in \mathbb{K}[x_1, \dots, x_n]$ satisfy equation (4.11) and introduce a new variable s. Let

$$F(s, x_1, x_3, \dots, x_n) := \frac{1}{s} f(s, x_1/s, \dots, x_n).$$

Note that $F(s, x_1, \dots, x_n)$ is not a formal power series in $s, x_1, \dots, x_n$ but an element of the $\mathbb{K}[s, x_1, x_3, \dots, x_n]$-module $\mathfrak{M}'$ generated by elements of the form

$$G = \sum_{\substack{j \in \mathbb{Z} \\ i_2, \dots, i_n \in \mathbb{N} \\ j + i_2 \geq -k}} a_{j i_2 \dots, i_n} s^j x_1^{i_2} x_3^{i_3} \dots x_n^{i_n}.$$

Also, note that $[s^{-1}]F(s, x_1, \dots, x_n) = I_{12}(f)$. Later, we shall need the following lemma.

Lemma 4.6.1. *If $0 \neq p \in \mathbb{K}[s, x_1, \dots, x_n]$ and $G \in \mathfrak{M}'$ satisfies $pG = 0$, then $G = 0$.*

Proof. For suitable k, $s^k G \in \mathbb{K}[[s, x_1, \dots, x_n]]$. So use the substitution $x_1 = su$, for u a new variable, to get $p(s, su, x_3, \dots, x_n) s^k G(s, u, x_3, \dots, x_n) = 0$. Now as

$$p(s, su, x_3, \dots, x_n) s^k G(s, u, x_3, \dots, x_n) \in \mathbb{K}[[s, x_1, \dots, x_n]],$$

and the substitution $x_1 = su$ forms an injective map, the conclusion follows. □

The function $F(s, x_1, x_3, \dots, x_n)$ is a D-finite function in the variables $s, x_1, x_3, \dots x_n$. This follows from the fact that f is D-finite and by applying the chain rule. Hence, there exist differential operators with polynomial coefficients

$$A\Big(s, x_1, \dots, x_n, \frac{\partial}{\partial s}\Big) = L(s, x_1, x_3, \dots, x_n)\Big(\frac{\partial}{\partial s}\Big)^m + \text{lower-order terms in } \frac{\partial}{\partial s}$$

$$B_i\Big(s, x_1, \dots, x_n, \frac{\partial}{\partial x_i}\Big) = L_i(s, x_1, x_3, \dots, x_n)\Big(\frac{\partial}{\partial x_i}\Big)^{m_i} + \text{lower-order terms in } \frac{\partial}{\partial x_i}$$

for $i \in \{1, 3, \dots, n\}$, such that

$$AF = 0, \tag{4.13}$$
$$B_i F = 0, \quad \text{for all } i \in \{1, 3, \dots, n\}. \tag{4.14}$$

Lemma 4.6.2. *There are non-zero linear partial differential operators $P_i(x_1, x_3, \dots, x_n, \frac{\partial}{\partial s}, \frac{\partial}{\partial x_i})$, for $i \in \{1, 3 \dots, n\}$ with coefficients from $\mathbb{K}[x_1, x_3, \dots, x_n]$, with P_i containing only derivatives of the form $(\frac{\partial}{\partial s})^\beta (\frac{\partial}{\partial x_i})^\gamma$, such that*

$$P_i\Big(x_1, x_3, \dots, \frac{\partial}{\partial s}, \frac{\partial}{\partial x_i}\Big) F = 0, \quad \textit{for all} \quad i \in \{1, 3, \dots, n\}.$$

Proof. Let us begin our proof by showing that there exists $P_1(x_1, x_3, \ldots, x_n, \frac{\partial}{\partial s}, \frac{\partial}{\partial x_1})$ such that

$$P_1\Big(x_1, x_3, \ldots, x_n, \frac{\partial}{\partial s}, \frac{\partial}{\partial x_1}\Big)F = 0.$$

Without loss of generality, we may assume that A and B_i in equation (4.13) have the same leading terms and denote $L_i(s, x_1, x_3, \ldots, x_n) = L(s, x_1, x_3, \ldots, x_n) = L$ for $i \in \{1, 3, \ldots, n\}$. Let all the coefficients in A and B_i have total degree bound by d. Hence, by using A, B_i we have

$$L\Big(\frac{\partial}{\partial s}\Big)^m F = \sum_{i=0}^{m-1} Q_i(s, x_1, x_3, \ldots, x_n)\Big(\frac{\partial}{\partial s}\Big)^i, \tag{4.15}$$

$$L\Big(\frac{\partial}{\partial s}\Big)^{m_1} F = \sum_{i=0}^{m_1-1} R_i(s, x_1, x_3, \ldots, x_n)(\frac{\partial}{\partial s})^i, \tag{4.16}$$

where $Q_i(s, x_1, x_3, \ldots, x_n), R_i(s, x_1, x_3, \ldots, x_n) \in \mathbb{K}[s, x_1, x_3, \ldots, x_n]$ of degree bound by d.

It is tempting to conclude that any element of the form $\big\{L(\frac{\partial}{\partial s})^k F \;|k > m\big\}$ can be written as a linear combination of $\{(\frac{\partial}{\partial s})^i F\}_{i=0}^m$ over the ring $\mathbb{K}[s, x_1, x_3, \ldots, x_n]$ but this is not true. So the important observation is that $\{L^N(\frac{\partial}{\partial s})^k|\; k > m,\; N \geq k-1\}$ can be written as a linear combination of $\{(\frac{\partial}{\partial s})^i F\}_{i=0}^m$ over the ring $\mathbb{K}[s, x_1, x_3, \ldots, x_n]$. Similarly, $\{L^N(\frac{\partial}{\partial x_1})^k|\; k > m_1,\; N \geq k-1\}$ can be written as a linear combination of $\{(\frac{\partial}{\partial x_1})^i F\}_{i=0}^{m_1}$ over the ring $\mathbb{K}[s, x_1, x_3, \ldots, x_n]$. Hence, we notice that if $k_1 + k_2 \leq N$,

$$L^N\Big(\frac{\partial}{\partial x_1}\Big)^{k_1}\Big(\frac{\partial}{\partial s}\Big)^{k_2} F = \sum_{\delta} T_\delta D_\delta, \tag{4.17}$$

where $\delta = (\delta_1, \delta_2)$ and $D_\delta = (\frac{\partial}{\partial x_1})^{\delta_1}(\frac{\partial}{\partial s})^{\delta_2}$ and the sum is over $\delta_1 < m_1, \delta_2 < m$. The degree bound d for the set of equations in equation (4.13) puts a degree bound on the coefficient T_δ — that is, the degree of T_δ is bound by Nd.

Now let

$$D := x_1^{\alpha_1} x_3^{\alpha_3} \cdots x_n^{\alpha_n}\Big(\frac{\partial}{\partial x_1}\Big)^{k_1}\Big(\frac{\partial}{\partial s}\Big)^{k_2}.$$

Note that if $\sum_i \alpha_i + k_1 + k_2 \leq N$, then

$$L^N D F = \sum_{\delta} \overline{T}_\delta D_\delta \tag{4.18}$$

where $\delta = (\delta_1, \delta_2)$ and $D_\delta = (\frac{\partial}{\partial x_1})^{\delta_1}(\frac{\partial}{\partial s})^{\delta_2}$ and the sum is over $\delta_1 < m_1, \delta_2 < m$. The total degree of $\overline{T}_\delta$ is bound by $N(d+1)$. This follows from the above discussion. The dimension of the vector space, say $\mathfrak{V}$, generated by monomials of the form $\overline{T}_\delta D_\delta$ over the field $\mathbb{K}$ that appear on the right side of equation (4.18) is $mm_1\binom{N(d+1)+n}{n}$. We obtain this by observing that the number of monomials in $s, x_1, x_3, \ldots, x_n$ of degree at most $Nd+1$ is $\binom{N(d+1)+n}{n}$ and the number of $(\frac{\partial}{\partial x_1})^{\delta_1}(\frac{\partial}{\partial s})^{\delta_2}$ is mm_1. On the other hand, the number of D appearing in equation (4.18) is $\binom{N+n+1}{n+1}$. This is also obtained by a similar count to the one made above.

We have $mm_1\binom{(Nd+1)+n}{n} \leq c_1 N^n$, where c_1 is a constant depending on d, n, m, m_1. Similarly, $\binom{N+n+1}{n+1} > N^{n+1}$, where this inequality is obtained by using the fact that $(\frac{N+n+1}{n+1})^{(n+1)} \leq \binom{n}{k}$. This implies that for N large enough, the number of terms on the left side will be more than the dimension of $\mathfrak{V}$. Hence, there is a linear dependence for large enough N. So, there are enough $a_{\alpha_1\alpha_3\ldots\alpha_n k_1 k_2}$ such that

$$L^N \sum_{\alpha_1+\alpha_3\ldots+\alpha_n+k_1+k_2\leq N} a_{\alpha_1\alpha_3\ldots\alpha_n k_1 k_2} x_1^{\alpha_1} x_3^{\alpha_3} \ldots x_n^{\alpha_n}\Big(\frac{\partial}{\partial x_1}\Big)^{k_1}\Big(\frac{\partial}{\partial s}\Big)^{k_2} F = 0.$$

Let

$$P_1 := \sum_{\alpha_1+\alpha_3 \ldots+\alpha_n+k_1+k_2 \leq N} a_{\alpha_1\alpha_3\ldots\alpha_n k_1 k_2} x_1^{\alpha_1} x_3^{\alpha_3} \ldots x_n^{\alpha_n} \Big(\frac{\partial}{\partial x_1}\Big)^{k_1} \Big(\frac{\partial}{\partial s}\Big)^{k_2}.$$

Using Lemma 4.6.1, we have $P_1 F = 0$. We can find the other $P_3, \ldots P_n$ simlarly by using $B_3, \ldots, B_n$ and this completes the proof. □

Before we prove the fact that the diagonal of a D-finite function is D-finite, we want to say a few words as to why we define $F(s, x_1, x_3, \ldots, x_n)$ with the $\frac{1}{s}$ factor. This factor makes $I_{12}(f) = [s^{-1}]F(s, x_1, x_3, \ldots, x_n)$ so, under the action of $(\frac{\partial}{\partial s})^{\alpha}$ for α a positive integer on $F(s, x_1, x_3, \ldots, x_n)$, $I_{12}(f)$ still survives as a coefficient of $s^{(-\alpha+1)}$ in $F(s, x_1, x_3, \ldots, x_n)$. If $\frac{1}{s}$ was not introduced, we could lose $I_{12}(f)$ under the action.

Theorem 4.6.3. *If $f \in \mathbb{K}[[x]]$ is D-finite, and $1 \leq i < j \leq n$, then $I_{ij}(f)$ is D-finite.*

Proof. Without the loss of generality, we take $i = 1$ and $j = 2$. Now let P_i be as in Lemma 4.6.2 and write $P_i = \sum_{j=\alpha_i}^{\beta_i} P_{ij}(x_1, x_3, \ldots, x_n, \frac{\partial}{\partial x_i})(\frac{\partial}{\partial s})^j$ with $P_{i\alpha_i} \neq 0$. Observe that the coefficient of $\frac{1}{s^{\alpha+1}}$ in $P_i F$ is $(-1)^{\alpha_i}(\alpha_i)! P_{i\alpha_i} I_{12}(f)$. Hence, $I_{12}(f)$ satisfies the equations

$$P_{i\alpha_i}\Big(x_1, x_3, \ldots, x_n, \frac{\partial}{\partial x_i}\Big) I_{12}(f) = 0 \quad \text{for all } i \in \{1, 3, \ldots, n\}.$$

Hence, $I_{12}(f)$ is D-finite. □

The next three lemmas, Lemmas 4.6.4 to 4.6.6, give bounds on the differential equation we used in the process of proving Theorem 4.6.3, all of which are crucial for the proof of our Theorem 4.6.7.

Lemma 4.6.4. *Let $f(x_1, \ldots, x_n) \in \mathbb{C}[[x_1, \ldots x_n]]$ be a D-finite function. Let $x = (x_1, \ldots, x_n)$ and suppose that we have*

$$\Big\{ a_{im_i}(x)(\frac{\partial}{\partial x_i})^{m_i} + a_{im_i-1}(x)(\frac{\partial}{\partial x_i})^{m_i-1} + \ldots + a_{i0}(x) \Big\} f(x) = 0, \quad \textit{for } i \in \{1, \ldots, n\}, \tag{4.19}$$

where $a_{ij}(x) \in \mathbb{K}[x]$. Furthermore, assume that the total degree of $a_{ij}(x)$ appears in the equations bound by a positive integer d and that $n_{max} = max\{m_1, m_2\}$. Then, there exist differential operators $P_i(x, \frac{\partial}{\partial x_i})$ such that

$$P_i\Big(x, \frac{\partial}{\partial x_i}\Big) I_{12}(f(x)) = 0, \quad \textit{for all } i \in \{1, 3, \ldots, n\},$$

satisfying

$$\operatorname{ord}(P_1) + \deg(P_1) < \frac{n_{max} m_2 (d^{n_{max}} + 1)^n - (n+1)^2}{n+1} + 1, \tag{4.20}$$

$$\operatorname{ord}(P_i) + \deg(P_i) < \frac{n_{max} m_i (d^{n_{max}} + 1)^n - (n+1)^2}{n+1} + 1, \quad \textit{for all } i \in \{3, \ldots, n\}. \tag{4.21}$$

Proof. As we see in Theorem 4.6.3, $I_{12}(f(x))$ is D-finite. In the proof, we worked with the function $F(s, x_1, x_3, \ldots, x_n)$. By Lemma 4.6.5, it satisfies the differential equations

$$\Big\{ b_{1n_{max}}(s, x)\Big(\frac{\partial}{\partial s}\Big)^{n_{max}} + b_{1n_{max}-1}(x)\Big(\frac{\partial}{\partial s}\Big)^{n_{max}-1} + \cdots + b_{10}(x) \Big\} F(s, x) = 0, \tag{4.22}$$

$$\left\{b_{2m_2}(s,x)\left(\frac{\partial}{\partial x_1}\right)^{m_2}+b_{2m_2-1}(s,x)\left(\frac{\partial}{\partial s}\right)^{m_2-1}+\cdots+b_{20}(s,x)\right\}F(s,x)=0, \tag{4.23}$$

$$\left\{b_{im_i}(s,x)\left(\frac{\partial}{\partial x_1}\right)^{m_i}+b_{im_i-1}(x)\left(\frac{\partial}{\partial s}\right)^{m_i-1}+\cdots+b_{i0}(s,x)\right\}F(s,x)=0, \quad \text{for } i=\{3,\dots,n\}. \tag{4.24}$$

Here, the total degree of $b_{ij}(s,x)$ is bound by $d^{n_{max}}$. Now we use

$$P_1 := \sum_{\alpha_1+\alpha_3\cdots+\alpha_n+k_1+k_2\le N} a_{\alpha_1\alpha_3\cdots\alpha_n k_1 k_2} x_1^{\alpha_1}x_3^{\alpha_3}\cdots x_n^{\alpha_n}\left(\frac{\partial}{\partial x_1}\right)^{k_1}\left(\frac{\partial}{\partial s}\right)^{k_2}.$$

We derive the above equation by claiming that for large enough N

$$n_{max}m_2\binom{N(d^{n_{max}}+1)+n}{n} < \binom{N+n+1}{n+1}. \tag{4.25}$$

Hence, there is a linear dependence on the left side of the equation. We want to find N in terms of m_1, m_2, d, n such that equation (4.25) holds true. Let us write $m_1m_2\binom{N(d^{n_{max}}+1)+n}{n}$ as

$$n_{max}m_2\binom{N(d^{n_{max}}+1)+n}{n} \tag{4.26}$$

$$= n_{max}m_2\frac{(N(d^{n_{max}}+1)+n)\cdots(N(d^{n_{max}}+1)+1)}{n!} \tag{4.27}$$

$$= n_{max}m_2(d^{n_{max}}+1)^n\frac{(N+n/(d^{n_{max}}+1))\cdots(N+1/(d^{n_{max}}+1))}{n!}. \tag{4.28}$$

Similarly, let us write $\binom{N+n+1}{n+1}$ as

$$\binom{N+n+1}{n+1} = \frac{(N+n+1)\cdots(N+1)}{(n+1)!} = \frac{(N+n+1)\cdots(N+1)}{(n+1)n!} \tag{4.29}$$

Now note that equation (4.26) has n terms of the form $\{(N+i/(d^{n_{max}}+1))\}_{i=1}^{n}$ and equation (4.29) have $n+1$ terms of the form $\{(N+i)\}_{i=1}^{n+1}$. As

$$N+i > N+\frac{i}{(d^{n_{max}}+1)},$$

we can make equation (4.25) true by finding N such that

$$N+n+1 > \frac{n_{max}m_2(d^{n_{max}}+1)^n}{n+1}.$$

Hence, we need

$$N > \frac{n_{max}m_2(d^{n_{max}}+1)^n}{n+1}-(n+1) = \frac{n_{max}m_2(d^{n_{max}}+1)^n-(n+1)^2}{n+1}$$

So let us take $N = \frac{n_{max}m_2(d^{n_{max}}+1)^n-(n+1)^2}{n+1}+1$. Similarly we could take P_i for $i\in\{3,\dots,n\}$ as in Theorem 4.6.3 and with the same arguments as above, we would obtain a bound on the order and degree of the differential equations $P(x_i,\frac{\partial}{\partial x_i})$ for $i\in\{3,\dots,n\}$. □

The above exhibits how to find a bound on the order and the degree of differential equations that annihilate $I_{12}(f(x))$. However, we have assumed that we are given a set of differential equations that annihilates $F(s,x_1,x_3,\dots,x_n)$ when in practice we are only given a set of differential equations that annihilates $f(x_1, x_2, \ldots, x_n)$. So suppose that $f(x_1, x_2, \ldots, x_n)$ is a D-finite function and that

$$\left[a_{in_i}(x)\Big(\frac{\partial}{\partial x_i}\Big)^{n_i} + a_{in_i-1}(x)\Big(\frac{\partial}{\partial x_i}\Big)^{n_i-1} + \cdots + a_{i0}(x)\right] f = 0, \quad \text{for } i \in \{1, \ldots, n\}, \tag{4.30}$$

where $a_{ij}(x) \in \mathbb{K}[x]$. Let d be a bound on the total degree of $a_{ij}(x)$. These equations imply that any derivatives $\left\{\frac{\partial^k f(x)}{\partial x_i} | \ k \geq n_i\right\}$ can be written in terms of $\left\langle \frac{\partial^j f(x)}{\partial x_i} | \ 0 < j < n_i \right\rangle$ over the field $K(x)$.

Lemma 4.6.5. *Let $f(x_1, \ldots, x_n) \in \mathbb{C}[[x_1, \ldots x_n]]$ be a D-finite function, with*

$$\left\{a_{im_i}(x)\Big(\frac{\partial}{\partial x_i}\Big)^{m_i} + a_{im_i-1}(x)\Big(\frac{\partial}{\partial x_i}\Big)^{m_i-1} + \ldots + a_{i0}(x)\right\} f(x) = 0, \quad \textit{for } i \in \{1, \ldots, n\}, \tag{4.31}$$

where $a_{ij}(x) \in \mathbb{K}[x]$. Also let d denote the bound on the total degree of $a_{ij}(x)$.Define

$$F(s, x_1, x_3, \ldots, x_n) := \frac{1}{s} f(s, x_1/s, x_3, \ldots, x_n),$$

and let $n_{max} = \max(m_1, \ldots, m_n)$. Then there exist differential equations

$$\left\{b_{1n_{max}}(s, x)\Big(\frac{\partial}{\partial s}\Big)^{n_{max}} + b_{1n_{max}-1}(x)\Big(\frac{\partial}{\partial s}\Big)^{n_{max}-1} + \cdots + b_{10}(x)\right\} F(s, x) = 0, \tag{4.32}$$

$$\left\{b_{2m_2}(s, x)\Big(\frac{\partial}{\partial x_1}\Big)^{m_2} + b_{2m_2-1}(s, x)\Big(\frac{\partial}{\partial s}\Big)^{m_2-1} + \cdots + b_{20}(s, x)\right\} F(s, x) = 0, \tag{4.33}$$

$$\left\{b_{im_i}(s, x)\Big(\frac{\partial}{\partial x_1}\Big)^{m_i} + b_{im_i-1}(x)\Big(\frac{\partial}{\partial s}\Big)^{m_i-1} + \cdots + b_{i0}(s, x)\right\} F(s, x) = 0, \quad \textit{for } i \in \{3, \ldots, n\}, \tag{4.34}$$

where the total degree of $b_{ij}(s, x)$ is bound by $d^{n_{max}}$.

Proof. Our aim is to find a bound on the differential operator that annihilates $F(s, x_1, x_3, \ldots, x_n)$ in terms of n_i for all $i \in \{1, 2, \ldots, n\}$ and d. By the chain rule, we have the following.

$$s\frac{\partial F(s, x_1, x_3, \ldots, x_n)}{\partial s} = -1/s f(s, x_1/s, x_3, \ldots, x_n) + \frac{\partial f(s, x_1/s, x_3, \ldots, x_n)}{\partial x_1} \tag{4.35}$$

$$- x_1/s^2 \frac{\partial f(s, x_1/s, x_3, \ldots, x_n)}{\partial x_2} \tag{4.36}$$

$$s\frac{\partial F(s, x_1, x_3, \ldots, x_n)}{\partial x_1} = \frac{1}{s}\frac{\partial f(s, x_1/s, x_3, \ldots, x_n)}{\partial x_2} \tag{4.37}$$

$$s\frac{\partial F(s, x_1, x_3, \ldots, x_n)}{\partial x_j} = \frac{1}{s}\frac{\partial f(s, x_1/s, x_3, \ldots, x_n)}{\partial x_j} \quad \text{for } j \in \{3, \ldots, n\}. \tag{4.38}$$

This above equation implies that there would be a bound on the order of $\frac{\partial^k F(s, x_1, x_3, \ldots, x_n)}{\partial s}$ by $n_{max} = \max(m_1, \ldots, m_n)$ and a bound on the order of $\frac{\partial^k F(s, x_1, x_3, \ldots, x_n)}{\partial x_1}$ by n_2 and a bound on the order of $\frac{\partial^k F(s, x_1, x_3, \ldots, x_n)}{\partial x_j}$ by n_j. Now to find the bound on the degree of the differential equations annihilating $F(s, x_1, x_3, \ldots, x_n)$, we look for an equation of the type

$$\Big(t_0 + t_1\frac{\partial}{\partial s} + \cdots + t_n\frac{\partial^{n_{max}}}{\partial s}\Big) F(s, x_1, x_3, \ldots, x_n) = 0.$$

We replace the derivative of $\frac{\partial^k F(s,x_1,x_3,\ldots,x_n)}{\partial s}$ by equation (4.35). Using Cramer's rule, we can solve for t_i and see that if the bound on the degree of equation (4.11) is given by d, then the bound on the degree of the equations annihilating $F(s,x_1,x_3,\ldots,x_n)$ is given by $d^{n_{max}}$. □

Lemma 4.6.6. *Let $G(z)$ denote a rational function of degree a. Let*

$$U(x) = 1 + \sum_{k=1} \frac{1}{k!} \prod_{j=1}^{k} G(i\hbar) x^k \in \mathbb{C}(\hbar)[[x]].$$

Then there exists a linear differential operator D of order a and degree a such that $DU(x) = 0$.

Proof. To construct a differential equation that annihilates $U(x)$ we consider $[x^k]U(x)$ and denote it as p_k.

$$\frac{p_{k+1}}{p_k} = \frac{G((k+1)\hbar)}{(k+1)} \tag{4.39}$$

$$(k+1)p_{k+1} = G((k+1)\hbar)p_k \tag{4.40}$$

Multiply both sides of the above equation by x^{k+1} and sum over all values of k. Then use the fact that $(x\frac{\partial}{\partial x})^m x^k = k^m x^k$ to create a differential operator that annihilates $U(x)$, with the order of the differential operator comeing from the degree of $G(z)$. □

Theorem 4.6.7. *Let $G(z) \in \mathbb{C}(z)$ be a rational function of degree a and let $\mathbf{q} = (q_1, q_2, \ldots, q_r)$. Then the generating function $\sum_{i=1}^{\infty} \sum_{g=0}^{\infty} n_g(d)\, \hbar^{2g-1}\, x^i$ is holonomic in x with coefficients in $\mathbb{C}(\hbar, q_1, \ldots, q_r)$. Furthermore, there exists a differential operator P that annihilates the expression $\sum_{i=1}^{\infty} \sum_{g=0}^{\infty} n_g(d)\, \hbar^{2g-1}\, x^i$ such that*

$$\operatorname{ord}(P) + \deg(P) < \frac{(d+1)^4 - 25}{5} + 1, \tag{4.41}$$

where $d = \max(a, (9r)^4, (2r^2)^4)$.

Proof. As the degree of $G(z)$ is a, Lemma 4.6.6 guarantees that there exists a differential equation of order a and degree a that annihilates $1 + \sum_{k=1} \frac{1}{k!} \prod_{j=1}^{k} G(ih)x^k$. With similar reasoning, we have a differential equation or degree a and order a that annihilates $\sum_{k=1} \frac{1}{k!} \prod_{j=1}^{k} G(ih)x^k$.

With our assumption that $q_{r+1} = q_{r+2} = \cdots = 0$, we have

$$\sum_{i=0}^{\infty} s_{(i)}(\mathbf{q})\, x^i = \exp\left(\sum_{m=1}^{r} q_m \frac{x^m}{m}\right).$$

It is clear there exists a differential equation of order 1 and degree r that annihilates this expression. The same is true for $\sum_{i=0}^{\infty} e_{(i)}(\mathbf{q})\, x^i$, since

$$\sum_{i=0}^{\infty} e_{(i)}(\mathbf{q})\, x^i = \exp\left(\sum_{m=1}^{\infty} q_m \frac{(-x)^m}{m}\right).$$

Now we want to find a bound for the differential equation in $\frac{\partial}{\partial x_3}$ and $\frac{\partial}{\partial x_4}$ that annihilates

$$H(x_1, x_2) = \frac{1}{(1-x_2)} F(x_1, x_2)\, G(x_1),$$

where we refer back to Section 4.5 for the definitions. There exists a differential equation of order 1 and degree 1 in x_2 that annihilates $\frac{1}{1-x_2}$. As the definition for $F(x_1, x_2)$ is obtained by replacing x_1 by x_1x_2, it is annihilated by a differential operator of order 1 in $\frac{\partial}{\partial x_1}$ and total degree bound by $2r$. Similarly, it is annihilated by a differential operator of order 1 in $\frac{\partial}{\partial x_2}$ and total degree bound by $2r$. Finally, we note that $G(x_1)$ is annihilated by a differential operator of order 1 in $\frac{\partial}{\partial x_1}$ and degree bound by r.

Hence, using the above three differential equations, we can calculate the bound on the order and the degree of the differential equation in $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ that annihilates $H(x_1, x_2)$. By the closure property, there exists a differential equation of order 3 in $\frac{\partial}{\partial x_1}$ that is of the form

$$u + u_0 H(x_1, x_2) + u_1 \frac{\partial H(x_1, x_2)}{\partial x_1} + u_2 \frac{\partial H(x_1, x_2)}{\partial x_1}^2 + u_3 \frac{\partial H(x_1, x_2)}{\partial x_1}^3 = 0.$$

Solving for u, u_i gives us the bound $(9r)^4$ on the degree. Similarly, by the closure property, there exists a differential equation of order 3 in $\frac{\partial}{\partial x_4}$ that is of the form

$$v + v_0 H(x_1, x_2) + v_1 \frac{\partial H(x_1, x_2)}{\partial x_2} + v_2 \frac{\partial^2 H(x_1, x_2)}{\partial x_2} + v_3 \frac{\partial^3 H(x_1, x_2)}{\partial x_2} = 0.$$

Solving for v, v_i gives us the bound on the degree $(2r^2)^4$.

Finally, $I_{24}(L(x_1, x_2, x_3, x_4))$ is holonomic by Theorem 4.6.3, so there exist $P_1(x_1, \frac{\partial}{\partial x_1})$, $P_2(x_2, \frac{\partial}{\partial x_2})$ and $P_3(x_3, \frac{\partial}{\partial x_3})$ that annihilate $I_{24}(L(x_1, x_2, x_3, x_4))$. So for $d = \max(a, (9r)^4, (2r^2)^4)$,

$$\mathrm{ord}(P_i) + \deg(P_i) < \frac{(d+1)^4 - 25}{5} + 1.$$

Now evaluation at $x_2 = x_3 = 1$ cannot increase these degree and order bounds, so we have obtained the desired result. $\square$

The bounds proven above could theoretically be turned into an algorithm to find one-point recursions, as mentioned at the start of the section. However, the existing software for calculating with holonomic functions provides a more effective approach. Let us give an example how the gfun package for computing with holonomic functions in *Maple* can be used to calculate a one-point recursion for monotone Hurwitz numbers [93].

Example 4.6.8. The proof of Theorem 4.5.3 implies that monotone Hurwitz numbers satisfy the relation

$$m(d) = \sum_{g=0}^{\infty} m_g(d) \hbar^{2g-1} = \frac{1}{d} \sum_{k=1}^{d} u_k v_{d-k}$$

where $\frac{u_{k+1}}{u_k} = \frac{G(k\hbar)}{k\hbar}$ and $\frac{v_{k+1}}{v_k} = \frac{G(-(k+1)\hbar)}{(k+1)\hbar}$. So the sequence $m(d)$ can be obtained by taking the Cauchy product of u_k and v_k, and then taking the Hadamard product of the result and the sequence $\frac{1}{k}$.

Based on this observation, the following shows several lines of hopefully self-explanatory *Maple* code that produce a 1-point recursion for monotone Hurwitz numbers.

```
> with(gfun):
> G(z) := 1/(1-z):
> rec1:={d*m(d+1)-G(d*hh)*m(d)=0, m(0)=0, m(1)=1}:
> rec2:={(d+1)*m(d+1)+G(-(d+1)*hh)*m(d)=0, m(1)=-G(-hh)}:
> rec3:={(d+1)*m(d+1)-d*m(d)=0, m(1)=1}:
> recprod:={cauchyproduct(rec1, rec2, m(d)) = 0}:
> finalrec:=`rec*rec`(recprod, rec3, m(d));
```

$\{(-2*hh+4*d*hh)*m(d)+(-d-1+hh^2*d^3+hh^2*d^2)*m(d+1), m(0)=0, m(1)=\_C[0]\}$

The output asserts that

$$(-2\hbar+4d\hbar)\,m(d)+(-d-1+\hbar^2d^3+\hbar^2d^2)\,m(d+1)=0.$$

By collecting the coefficient of $\hbar^{2g-1}$ and shifting the index, we obtain the 1-point recursion

$$dm_g(d)=2(2d-3)m_g(d-1)+d(d-1)^2m_{g-1}(d). \tag{4.42}$$

4.7 Examples and applications

In this section, we return our attention to the enumerative problems introduced in Section 4.2. In particular, we apply the methodology developed in Section 4.4 to deduce 1-point recursions for the enumeration of fatgraphs and dessins d'enfant, as well as for Bousquet-Mélou–Schaeffer numbers and monotone Hurwitz numbers. For the case of simple Hurwitz numbers, the weight generating function $G(z)$ is not a rational function, so Theorem 4.1.2 ceases to apply. As a partial converse to this theorem, we show that simple Hurwitz numbers do not satisfy a 1-point recursion. Throughout the section, we demonstrate how our calculations may yield explicit formulas and polynomial structure results for 1-point invariants.

Hypermaps and Bousquet-Mélou–Schaeffer numbers

The methodology of Section 4.4 allows one to recover the Harer–Zagier and Do–Norbury 1-point recursions for the enumeration of fatgraphs and dessins d'enfant, stated as equation (4.1) and equation (4.2), respectively. Recall that these two examples inspired the current work. It is possible to use the methodology of Section 4.4 to deduce other 1-point recursions, although the results are often rather lengthy to state. We provide the following two examples to demonstrate.

Proposition 4.7.1. *The enumeration of 3-hypermaps defined in Section 1.4 satisfies the following 1-point recursion.*

$$\begin{aligned}2d(2d+1)\,a_g^3(d) =& 3(3d-1)(3d-2)\,a_g^3(d-1)-(3d-1)(3d-2)(9d^2-8d+2)\,a_{g-1}^3(d-1)\\&+(d-1)(3d-1)(3d-2)(3d-4)(3d-5)(6d-7)\,a_{g-2}^3(d-2)\\&-(d-1)(d-2)(3d-1)(3d-2)(3d-4)(3d-5)(3d-7)(3d-8)\,a_{g-3}^3(d-3)\end{aligned}$$

The Bousquet-Mélou–Schaeffer numbers with $m=3$ defined in Section 1.1 satisfy the following 1-point recursion.

$$\begin{aligned}&2d(2d+1)(3d-1)b_g^3(d)\\&=3(3d-1)(3d-2)(3d-4)b_g^3(d-1)+(d-1)(3d+1)(9d^3-22d^2+14d-2)b_{g-1}^3(d-1)\\&\quad-(d-1)^2(d-2)(18d^4-93d^3+172d^2-127d+26)b_{g-2}^3(d-2)\\&\quad+(d-1)^2(d-2)^5(d-3)(3d-1)b_{g-3}^3(d-3)\end{aligned}$$

Hurwitz numbers

Observe that Theorem 4.1.2 does not apply in the case of Hurwitz numbers, since the weight generating function $G(z)=\exp(z)$ is not rational. As a partial converse to our main theorem, we now demonstrate that the simple Hurwitz numbers do not satisfy a 1-point recursion.

Proposition 4.7.2. *The simple Hurwitz numbers do not satisfy a 1-point recursion.*

Proof. By Lemma 4.4.1, we know that the simple Hurwitz numbers satisfy a 1-point recursion if and only if the sequence

$$\begin{aligned} h_d(\hbar) &= \frac{1}{d!\,\hbar^d} \sum_{k=1}^{d} (-1)^{d-k} \binom{d-1}{k-1} \exp(d(2k-d-1)\hbar/2) \\ &= \frac{1}{d!\,\hbar^d} \exp(-d(d+1)\hbar/2)\,(\exp(d\hbar)-1)^{d-1} \end{aligned}$$

is holonomic over $\mathbb{C}(\hbar)$. However, if this were the case, then we would have that the sequence

$$h_d(1) = \frac{1}{d!} \exp(-d(d+1)/2)\,(\exp(d)-1)^{d-1}$$

is holonomic over $\mathbb{C}$.

It is known that holonomic sequences $a_1, a_2, a_3, \ldots$ over $\mathbb{C}$ must satisfy the asymptotic growth condition $h_d = O(d!^{\alpha})$ for some constant α. On the other hand, we have

$$\begin{aligned} h_d(1) &= \frac{1}{d!} \exp(-d(d+1)/2)\,(\exp(d)-1)^{d-1} \\ &\sim \frac{1}{d!} \exp(-d(d+1)/2)\, \exp(d(d-1)) = \frac{1}{d!} \exp(d(d-3)/2). \end{aligned}$$

Applying Stirling's formula, we see that this grows too fast for $h_d(1)$ to be holonomic. So it follows that the simple Hurwitz numbers do not satisfy a 1-point recursion. □

The formula of equation (4.12) still applies to this case though, so the 1-part Hurwitz numbers satisfy

$$\begin{aligned} \sum_{g=0}^{\infty} h_g(d)\,\hbar^{2g-1} &= \frac{1}{d!\,h^d} \sum_{k=1}^{d} (-1)^{d-k} \binom{d-1}{k-1} \prod_{j=1}^{d} \exp((k-j)\hbar) \\ &= \frac{1}{d!\,\hbar^d} \sum_{k=1}^{d} (-1)^{d-k} \binom{d-1}{k-1} \exp\left(\tfrac{1}{2} d\hbar(2k-d-1)\right). \end{aligned}$$

By extracting coefficients of $\hbar$ on both sides, we recover the following formula for 1-part simple Hurwitz numbers.

Proposition 4.7.3. *The 1-part simple Hurwitz numbers satisfy the equation*

$$h_g(d) = \frac{(d/2)^{d+2g-1}}{(d+2g-1)!} \sum_{k=0}^{d-1} (-1)^k \binom{d-1}{k} (d-1-2k)^{d+2g-1}.$$

In particular, we have the structure theorem $h_g(d) = \frac{d^d}{d!} p_g(d)$, where p_g is a polynomial of degree $3g-1$. One can make sense of this statement in the case $g=0$ by taking $p_0(d) = \frac{1}{d}$.

Remark 4.7.4. We remark that the polynomial structure derived here is a direct corollary of the more general polynomial structure for simple Hurwitz numbers with any number of parts. This in turn follows from the ELSV formula, which relates simple Hurwitz numbers to intersection theory on moduli spaces of curves [44]. The formula of Proposition 4.7.3 is not new either, but appeared in the work of Shapiro, Shapiro and Vainshtein [95]. The results and proof here may generalise to other settings, as we will observe in the context of monotone Hurwitz numbers.

Monotone Hurwitz numbers

In Section 4.6, we observed that the following 1-point recursion for monotone Hurwitz numbers could be deduced from several lines of *Maple* code. As with the Harer–Zagier recursion, it would be of interest to have an independent and purely combinatorial proof of this statement.

Proposition 4.7.5. *The 1-part monotone Hurwitz numbers satisfy the 1-point recursion*

$$d\, m_g(d) = 2(2d-3)\, m_g(d-1) + d(d-1)^2\, m_{g-1}(d).$$

The 1-point function $F(x,\hbar) = \sum_{d=1}^{\infty} \sum_{g=0}^{\infty} m_g(d) \hbar^{2g-1} x^d$ *satisfies*

$$\big[2 + (1-4x)\partial - \hbar^2 x \partial^2 - \hbar^2 x^2 \partial^3\big] F(x,\hbar) = 0.$$

In the context of monotone Hurwitz numbers, equation (4.12) implies that

$$\begin{aligned}\sum_{g=0}^{\infty} m_g(d)\, \hbar^{2g-1} &= \frac{1}{d!\, \hbar^d} \sum_{k=1}^{d} (-1)^{d-k} \binom{d-1}{k-1} \prod_{j=1}^{d} \frac{1}{1-(k-j)\hbar} \\ &= \frac{(2d-2)!}{d!\,(d-1)!} \prod_{k=-d+1}^{d-1} \frac{1}{1-k\hbar}.\end{aligned}$$

The identity that leads to the second equality can be established by considering the residue at $\hbar = \frac{1}{k}$ for $-d+1 \leq k \leq d-1$.

By extracting coefficients of $\hbar$ on both sides, we recover the following formula for 1-part monotone Hurwitz numbers.

Corollary 4.7.6. *The 1-part monotone Hurwitz numbers satisfy the equation*

$$\begin{aligned} m_g(d) &= \frac{(2d-2)!}{d!(d-1)!} \sum_{k_1+\cdots+k_{d-1}=g} \prod_{i=1}^{d-1} i^{2k_i} \\ &= \frac{(2d-2)!}{d!(d-1)!} \sum_{1 \leq m_1 \leq m_2 \leq \cdots \leq m_g \leq d-1} (m_1 m_2 \cdots m_g)^2. \end{aligned}$$

The latter summation is a polynomial in d *of degree* $3g$ *that is divisible by* $2d-1$*, so we have the structure theorem* $m_g(d) = \binom{2d}{d} \tilde{p}_g(d)$*, where* $\tilde{p}_g$ *is a polynomial of degree* $3g-1$*. One can make sense of this statement in the case* $g=0$ *by taking* $\tilde{p}_0(d) = \frac{1}{d}$*.*

This is a particular case of the more general result of Goulden et al., who prove a structure theorem for monotone Hurwitz numbers [55].

$$M_{g,n}(d_1, d_2, \ldots, d_n) = \prod_{i=1}^{n} \binom{2d_i}{d_i} \times P_{g,n}(d_1, d_2, \ldots, d_n),$$

where $P_{g,n}$ is a polynomial of degree $3g-3+n$. One wonders whether the techniques of this paper can be used to prove the more general structure theorem of Goulden et al.

4.8 Relations to topological recursion and quantum curves

In this section, we aim to address the question: how universal is the notion of a 1-point recursion? Thus, one seeks a natural class of enumerative problems for which 1-point recursions exist. Such a class should include not only the fatgraph and dessin d'enfant enumerations, but also those families of problems encompassed by Theorem 4.5.3 — namely, those arising from the double Schur function expansion of equation (4.4) with $\mathbf{q} = (q_1, q_2, \ldots, q_r, 0, 0, \ldots)$ and a rational weight generating function $G(z)$. We claim that a natural candidate is the class of problems governed by topological recursion.

Topological recursion can be thought of as a vast generalisation of Tutte's recursion for the enumeration of fatgraphs. It calculates n-point functions in a recursive manner, starting from the input data of a spectral curve, as described in Chapter 3. The following result asserts that the weighted Hurwitz numbers — essentially, the $N_{g,n}(d_1, d_2, \ldots, d_n)$ defined by equations (4.4) and (4.5) — are governed by the topological recursion.

Theorem 4.8.1 (Alexandrov, Chapuy, Eynard and Harnad [2])**.** *The spectral curve given by*

$$\left(\mathbb{CP}^1,\ x(z) = \frac{z}{G(Q(z))},\ y(z) = \frac{Q(z)}{z} G(Q(z)),\ \omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1\, \mathrm{d}z_2}{(z_1 - z_2)^2}\right)$$

with $Q(z) = q_1 z + q_2 z^2 + \ldots + q_r z^r$, produces correlation differentials that satisfy

$$\omega_{g,n} = \sum_{d_1, d_2, \ldots, d_n = 1}^{\infty} N_{g,n}(d_1, d_2, \ldots, d_n) \prod_{i=1}^{n} d_i x_i^{d_i - 1}\, \mathrm{d}x_i.$$

This lends credence to the following conjecture, which states that 1-point recursions exist for rational spectral curves in general.

Conjecture 4.8.2. *Let $(\mathbb{CP}^1, x, y, \omega_{0,2})$ be a spectral curve with x and y rational functions. Let $\omega_{g,n}$ denote the correlation differentials produced by the topological recursion applied to this spectral curve.*

$$\omega_{g,n} = \sum_{d_1, d_2, \ldots, d_n = 1}^{\infty} N_{g,n}(d_1, d_2, \ldots, d_n) \prod_{i=1}^{n} d_i x_i^{d_i - 1}\, \mathrm{d}x_i$$

The the numbers $n_g(d) = d\, N_{g,1}(d)$ satisfy a 1-point recursion.

We conclude this section with an example of a problem that is governed by topological recursion and satisfies a 1-point recursion, but does not satisfy the conditions of Theorem 4.5.3. Thus, one can consider this as further evidence towards the conjecture above.

Example 4.8.3. Chekhov and Norbury [24] consider topological recursion applied to the spectral curve $x^2 y^2 - 4y^2 - 1 = 0$, given by the rational parametrisation

$$x(z) = z + \frac{1}{z} \qquad \text{and} \qquad y(z) = \frac{z}{z^2 - 1}$$

The resulting correlation differentials can be expressed as

$$\omega_{g,n} = \sum_{d_1, \ldots, d_n = 1}^{\infty} J_{g,n}(d_1, d_2, \ldots, d_n) \prod_{i=1}^{n} d_i z_i^{d_i - 1}\, \mathrm{d}z_i.$$

These are derivatives of the correlation functions for the Legendre ensemble, which arise from a particular Hermitian matrix model, as well as related models from conformal field theory. In the latter context, Gaberdiel, Klemm and Runkel use null vectors for Virasoro highest weight representations to deduce an equation [52, equation (4.18)] that is equivalent to a 1-point recursion for the numbers $j_d = dJ_{g,1}(d)$. In summary, the 1-point invariants produced by the topological recursion on the rational spectral curve above satisfy a 1-point recursion.

Recall that the notion of quantum curves is closely related to that of topological recursion, as discussed in Section 3.4. In short, they are non-commutative deformations of the spectral curves that are used as the input to the topological recursion. Under a certain polarisation, they become differential operators that should annihilate the wave function, defined to be the principal specialisation of the partition function. Although it is not currently clear when they exist, the quantum curve phenomenon has been proved or observed in many instances of the topological recursion.

In the context of the double Schur function expansions considered in this paper, the principal specialisation of the partition function is given by

$$\psi(x,\hbar) = \sum_{\lambda \in \mathcal{P}} s_\lambda(x, x^2, x^3, \ldots) s_\lambda\Big(\frac{q_1}{\hbar}, \frac{q_2}{\hbar}, \ldots\Big) \prod_{\Box \in \lambda} G(c(\Box)\hbar).$$

As in Section 4.3, the hook-content formula stated in Lemma 4.3.1 may be invoked to simplify the expression to obtain

$$\psi(x,\hbar) = \sum_{\lambda \in \mathcal{P}} x^d s_\lambda\Big(\frac{q_1}{\hbar}, \frac{q_2}{\hbar}, \ldots\Big) \prod_{k=1}^{d-1} G(k\hbar) = \sum_{d=0}^{\infty} x^d \prod_{k=1}^{d-1} G(k\hbar) [y^d] \exp\Bigg(\sum_{k=1}^{r} \frac{q_k}{k\hbar} y^k\Bigg),$$

where $[y^d]$ denotes extraction of the coefficient of y^d. The quantum curve may then be calculated from this expression, and the result appears in the work of Alexandrov, Chapuy, Eynard and Harnad [2].

Our calculation of 1-point invariants from the partition function in Section 4.3 bears a strong resemblance to the calculation of the quantum curve form the partition function [2, 3, 80]. In the former case, the partition function reduces to a sum over hook partitions, while in the latter case it reduces to a sum over 1-part partitions. One may wonder whether there may be a deeper connection here.

Chapter 5

Topological recursion for lattice points in $\overline{\mathcal{M}}_{g,n}$

Do and Norbury initiated the enumeration of lattice points in the Deligne–Mumford compactifications of moduli spaces of curves. They showed that the enumeration may be expressed in terms of polynomials, whose top and bottom degree coefficients store psi-class intersection numbers and orbifold Euler characteristics of $\overline{\mathcal{M}}_{g,n}$, respectively. Furthermore, they ask whether the enumeration is governed by the topological recursion and whether the intermediate coefficients also store algebro-geometric information. In this chapter, we prove that the enumeration does indeed satisfy the topological recursion, although with a modification to the initial spectral curve data. Thus, one can consider this to be one of few known instances of a natural enumerative problem governed by the so-called *local topological recursion.* Combining the present work with the known relation between local topological recursion and cohomological field theory should uncover the geometric meaning of the intermediate coefficients of the aforementioned polynomials.

5.1 Motivation

Norbury proved that a certain count of lattice points in the moduli space of curves $\mathcal{M}_{g,n}$ stores information about its intersection theory and orbifold Euler characteristic [81]. He furthermore showed that the enumeration is governed by the topological recursion of Chekhov, Eynard and Orantin [23, 47, 82]. More recently, Andersen, Chekhov, Norbury and Penner use the general theory that identifies topological recursion with the Givental formalism to relate this enumeration to cohomological field theory [5, 6, 43].

Do and Norbury introduced the related count of lattice points in $\overline{\mathcal{M}}_{g,n}$, the Deligne–Mumford compactification of the moduli space of curves [29]. For positive integers $b_1, b_2, \ldots, b_n$, they define the subset

$$\overline{\mathcal{Z}}_{g,n}(b_1, b_2, \ldots, b_n) \subset \overline{\mathcal{M}}_{g,n},$$

as discussed in Section 2.2. The set $\overline{\mathcal{Z}}_{g,n}(b_1, b_2, \ldots, b_n)$ is typically the union of a finite set of discrete points in $\overline{\mathcal{M}}_{g,n}$ with higher-dimensional components that are naturally products of moduli spaces of curves. The latter arise from maps $f : \Sigma \to \mathbb{CP}^1$ that have *ghost components* — that is, irreducible components of Σ that map entirely to $0 \in \mathbb{CP}^1$. To properly "count" points in $\overline{\mathcal{Z}}_{g,n}(b_1, b_2, \ldots, b_n)$, one needs to account for both the orbifold nature of $\overline{\mathcal{M}}_{g,n}$ and the existence of these ghost components. This can be conveniently expressed via the orbifold Euler characteristic as follows.

Definition 5.1.1. For positive integers $b_1, b_2, \ldots, b_n$, define

$$\overline{N}_{g,n}(b_1, b_2, \ldots, b_n) = \chi\left(\overline{\mathcal{Z}}_{g,n}(b_1, b_2, \ldots, b_n)\right).$$

The enumeration $\overline{N}_{g,n}$ enjoys the following properties, which can be found in the existing literature [29] and are explained in greater detail in Section 2.2.

- *Quasi-polynomiality.* For $(g,n) \neq (0,1)$ or $(0,2)$, $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ is a symmetric quasi-polynomial in $b_1^2, b_2^2, \ldots, b_n^2$ of degree $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$. We use the term *quasi-polynomial* to refer to a function on $\mathbb{Z}_+^n$ that is polynomial on each fixed parity class. Observe that this allows us to extend $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ to evaluation at $b_i = 0$.
- *Combinatorial recursion.* The enumeration $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ can be interpreted as a weighted count of combinatorial objects known as *stable fatgraphs*. From this interpretation, one can deduce an effective recursion to calculate $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$.
- *Psi-class intersection numbers.* The top degree coefficients of the quasi-polynomial $\overline{N}_{g,n}$ store psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$.
- *Orbifold Euler characteristics.* The quasi-polynomial $\overline{N}_{g,n}$ satisfies $\overline{N}_{g,n}(0, 0, \ldots, 0) = \chi(\overline{\mathcal{M}}_{g,n})$.

We previously mentioned that the enumeration of lattice points in the uncompactified moduli space of curves $\mathcal{M}_{g,n}$ is governed by the topological recursion and consequently, related to cohomological field theory. It is certainly natural to seek analogous results in the context of the compactified enumeration $\overline{N}_{g,n}$. In this regard, Do and Norbury originally state the following.

(a) *"It would be interesting to know whether the compactified lattice point polynomials can be used to define multidifferentials which also satisfy a topological recursion."* [29, p. 2343]

(b) *"We remark that it is currently unknown whether or not the intermediate coefficients of $\overline{N}_{g,n}(\mathbf{b})$ store topological information about $\overline{\mathcal{M}}_{g,n}$."* [29, p. 2323]

In Section 5.2, we settle problem (a) above by proving that the enumeration $\overline{N}_{g,n}$ is indeed governed by the topological recursion, although with a modification to the initial spectral curve data that is explained below. Although problem (b) above remains unresolved, our main theorem should allow one to invoke the general theory that identifies topological recursion with the Givental formalism to yield a connection to cohomological field theory [43]. This would then provide a relation between the intermediate coefficients of $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ and the intersection theory of $\overline{\mathcal{M}}_{g,n}$. We aim to report on this work in the future.

The main result of the present chapter is the following.

Theorem 5.1.2. *Topological recursion applied to the* **local spectral curve** $\mathbb{C}^*$ *equipped with the data*

$$x(z) = z + \frac{1}{z}, \quad y(z) = z \quad \text{and} \quad \omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{(z_1 - z_2)^2} + \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{z_1 z_2} \tag{5.1}$$

produces multidifferentials whose expansions at $z_i = 0$ satisfy

$$\omega_{g,n}(z_1, z_2, \ldots, z_n) = \sum_{b_1, b_2, \ldots, b_n = 0}^{\infty} \overline{N}_{g,n}(b_1, b_2, \ldots, b_n) \prod_{i=1}^{n} [b_i] z_i^{b_i - 1} \, \mathrm{d}z_i, \text{for } (g,n) \neq (0,1) \text{ or } (0,2).$$

Here, we use the notation $[b] = b$ for b positive and $[0] = 1$.

The most notable aspect of the theorem is the nature of the spectral curve involved, which can be considered local rather than global, in the sense discussed in Section 3.2. The following is a short summary of the difference between the global and local topological recursion.

- *Global topological recursion.*[1] In the foundational literature on topological recursion, a spectral curve is defined to be the data $(\mathcal{C}, x, y, T)$, where $\mathcal{C}$ is a compact Riemann surface, x and y are meromorphic functions on $\mathcal{C}$, and T is a Torelli marking on $\mathcal{C}$ — that is, a choice of symplectic basis for $H_1(\mathcal{C}; \mathbb{Z})$ [47, 48]. One usually also imposes some mild regularity conditions on this data, although they play no role in the present discussion. The global topological recursion then recursively produces so-called correlation differentials $\omega_{g,n}$ for integers $g \geq 0$ and $n \geq 1$. In particular, $\omega_{0,2}(z_1, z_2)$ is defined implicitly by the fact that it has double poles without residue along the diagonal $z_1 = z_2$, is holomorphic away from the diagonal, and is normalised on the $\mathcal{A}$-cycles of the Torelli marking via the equation

$$\oint_{\mathcal{A}_i} \omega_{0,2}(z_1, z_2) = 0, \qquad \text{for } i = 1, 2, \ldots, \text{genus}(\mathcal{C}).$$

  The compact nature of $\mathcal{C}$ ensures that $\omega_{0,2}$ is uniquely defined from the spectral curve data. A consequence of the global topological recursion is that for $(g, n) \neq (0, 1)$ or $(0, 2)$, the correlation differentials $\omega_{g,n}$ have poles only at the ramification points of the spectral curve, where $\mathrm{d}x$ vanishes.

- *Local topological recursion.* One can observe that the global topological recursion actually only requires the local information of the meromorphic functions x, y and the bidifferential $\omega_{0,2}$ at the ramification points of the spectral curve, in order to produce the correlation differentials. Thus, one can more generally define topological recursion on spectral curves comprising isolated local germs of x, y and $\omega_{0,2}$, without requiring the existence of a global compact Riemann surface on which this data can be defined. In particular, the local topological recursion requires $\omega_{0,2}$ to become part of the spectral curve data. This viewpoint was promoted by Dunin-Barkowksi, Orantin, Shadrin and Spitz in their work relating topological recursion to Givental's approach to cohomological field theory [43].

The spectral curve of Theorem 5.1.2 is local in the sense that the data cannot be extended to $\mathbb{CP}^1$ such that $\omega_{0,2}$ satisfies the conditions of the global topological recursion. The simple poles of $\omega_{0,2}$ at $z_1 = 0$ and $z_2 = 0$ lead to the correlation differentials $\omega_{g,n}(z_1, z_2, \ldots, z_n)$ having simple poles at $z_i = 0$ more generally. This departs from the usual behaviour exhibited by the global topological recursion, in which the poles appear only at the ramification points of the spectral curve, which correspond to $z_i = \pm 1$ in our case. Although we do not take this approach here, the spectral curve could alternatively have been presented more abstractly as the disjoint union of two small disks, corresponding to the two ramification points.

It was previously unclear whether there were benefits to using the local version of the topological recursion beyond the more general viewpoint it afforded. Indeed, Dunin-Barkowski [39] states that *"local topological recursion (to the moment) lacks interesting applications or profound meaning separate from what originates from ordinary (global) spectral curve topological recursion"*. Theorem 5.1.2 above provides an instance of the local topological recursion applied to a natural enumerative problem. It is one of a number of known examples of local topological recursion in which $\omega_{0,2}$ is deformed away from its usual global form. These include: the enumeration of maps with self-avoiding loops in the critical regime [10]; the Chern–Simons invariants of S^3/Γ for certain non-abelian Γ [12]; random tensor models governed by the blobbed topological recursion [9, 13]; and recent work on Masur–Veech volumes [4].

The proof of Theorem 5.1.2 adopts a general strategy that has been previously employed to show that topological recursion governs enumerative problems, such as counting lattice points in $\mathcal{M}_{g,n}$ [82] and several variants of Hurwitz numbers [15, 31, 34, 50]. Minor technical difficulties

[1] We use the expression *global topological recursion* to contrast it with its local counterpart. However, this is not to be confused with the global version of topological recursion introduced by Bouchard and Eynard [16].

arise from the modification to $\omega_{0,2}$, which introduces logarithmic terms into the topological recursion kernel.

The general theory of topological recursion allows one to calculate so-called symplectic invariants $F_g \in \mathbb{C}$ for $g = 0, 1, 2, \ldots$ and to deduce relations known as string and dilaton equations. Thus, we have the following immediate consequence of our main result, which previously appeared in the literature with an alternative proof [29].

Corollary 5.1.3. *The string and dilaton equations for the topological recursion imply the following known relations, respectively, for $(g, n) \neq (0, 1)$ or $(0, 2)$ and $b_1, b_2, \ldots, b_n \geq 0$. The hat over b_k in the first equation denotes the fact that we omit it as an argument.*

$$\overline{N}_{g,n+1}(1, b_1, b_2, \ldots, b_n) = \sum_{k=1}^{n} \sum_{a=0}^{b_k - 1} [a] \, \overline{N}_{g,n}(a, b_1, \ldots, \widehat{b}_k, \ldots, b_n)$$

$$\overline{N}_{g,n+1}(2, b_1, b_2, \ldots, b_n) - \overline{N}_{g,n+1}(0, b_1, b_2, \ldots, b_n) = (2g - 2 + n) \, \overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$$

One potential application of the present work is to give an explicit relation between the enumeration $\overline{N}_{g,n}$ and the algebraic geometry of $\overline{\mathcal{M}}_{g,n}$. A priori, one might expect such a relation due to the definition of $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ as a virtual count of the set $\overline{\mathcal{Z}}_{g,n}(b_1, b_2, \ldots, b_n) \subset \overline{\mathcal{M}}_{g,n}$. Furthermore, we note that $\overline{\mathcal{Z}}_{g,n}(b_1, b_2, \ldots, b_n)$ may alternatively be interpreted as a subset of $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1; \sum b_i)$, the moduli space of stable maps, making a connection with the Gromov–Witten theory of the sphere. Theorem 5.1.2 now provides a promising pathway towards the ultimate proof of a relation between the enumeration $\overline{N}_{g,n}$ and the intersection theory of $\overline{\mathcal{M}}_{g,n}$ via the identification of topological recursion with Givental's formula [43].

Example 5.1.4. Recall that the local spectral curve of Theorem 5.1.2 is $\mathbb{C}^*$ equipped with the data

$$x(z) = z + \frac{1}{z}, \qquad y(z) = z \qquad \text{and} \qquad \omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{(z_1 - z_2)^2} + \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{z_1 z_2}.$$

The ramification points are the zeroes of $\mathrm{d}x$ — namely, $z = 1$ and $z = -1$. At both of these ramification points, the local involution $s(z)$ is given by $s(z) = \frac{1}{z}$. Thus, the recursion kernel can be taken to be

$$\begin{aligned} K(z_1, z) &= -\frac{\int_o^z \omega_{0,2}(z_1, \cdot\,)}{[y(z) - y(s(z))] \, \mathrm{d}x(z)} \\ &= -\frac{\int_\infty^z \frac{\mathrm{d}z_1 \, \mathrm{d}t}{(z_1 - t)^2} + \int_1^z \frac{\mathrm{d}z_1 \, \mathrm{d}t}{z_1 \, t}}{[y(z) - y(s(z))] \, \mathrm{d}x(z)} = -\left[\frac{1}{z_1 - z} + \frac{\log(z)}{z_1} \right] \frac{z^3}{(1 - z^2)^2} \frac{\mathrm{d}z_1}{\mathrm{d}z}. \end{aligned}$$

In the equation above, we split the integral into two and integrate the contributions with two different base points. Note that this is justified as the extra term will not contribute to the residue.

The recursion produces the following formulas in the cases $(g, n) = (0, 3)$ and $(1, 1)$.

$$\begin{aligned}
&\frac{\omega_{0,3}(z_1, z_2, z_3)}{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3} \\
&= \sum_{\alpha=\pm1} \operatorname*{Res}_{z=\alpha} \frac{K(z_1, z)}{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3} \left[\omega_{0,2}(z, z_2)\,\omega_{0,2}(\tfrac{1}{z}, z_3) + \omega_{0,2}(z, z_3)\,\omega_{0,2}(\tfrac{1}{z}, z_2)\right] \\
&= \sum_{\alpha=\pm1} \operatorname*{Res}_{z=\alpha} \left[\frac{1}{z_1 - z} + \frac{\log(z)}{z_1}\right] \frac{z^3}{(1-z^2)^2} \left[\frac{\mathrm{d}z}{(z-z_2)^2\,(1-zz_3)^2} + \frac{\mathrm{d}z}{(z-z_3)^2\,(1-zz_2)^2}\right] \\
&= \frac{1}{2z_1z_2z_3} \left[\prod_{i=1}^{3} \frac{z_i^2 - z_i + 1}{(z_i - 1)^2} + \prod_{i=1}^{3} \frac{z_i^2 + z_i + 1}{(z_i + 1)^2}\right] \\
&\frac{\omega_{1,1}(z_1)}{\mathrm{d}z_1} = \sum_{\alpha=\pm1} \operatorname*{Res}_{z=\alpha} \frac{K(z_1, z)}{\mathrm{d}z_1}\,\omega_{0,2}(z, \tfrac{1}{z}) \\
&= \sum_{\alpha=\pm1} \operatorname*{Res}_{z=\alpha} \left[\frac{1}{z_1 - z} + \frac{\log(z)}{z_1}\right] \frac{z^3}{(1-z^2)^2} \left(\frac{1}{(z^2-1)^2} + 1\right) \mathrm{d}z \\
&= \frac{5z_1^8 - 8z_1^6 + 18z_1^4 - 8z_1^2 + 5}{12z_1(z_1^2 - 1)^4}
\end{aligned}$$

5.2 Proof of the main theorem

For the proof of Theorem 5.1.2, we adopt a general strategy that has been previously used to prove the topological recursion for enumerative problems, such as counting lattice points in uncompactified moduli spaces of curves [82] and various kinds of Hurwitz numbers [50, 15, 34, 31]. The modification to $\omega_{0,2}$ in our result adds minor technical difficulties, since logarithmic terms are introduced into the topological recursion kernel. We break down the proof into the following parts.

1. Define natural multidifferentials $\Omega_{g,n}(z_1, z_2, \ldots, z_n)$ for the enumerative problem and use the quasi-polynomiality of Theorem 2.2.12 to deduce analytic and symmetry properties for $\Omega_{g,n}(z_1, z_2, \ldots, z_n)$ (Proposition 5.2.4).

2. Express the combinatorial recursion of Theorem 2.2.12 in terms of the aforementioned multidifferentials $\Omega_{g,n}(z_1, z_2, \ldots, z_n)$ (Proposition 5.2.6).

3. Break the natural symmetry of the recursion for $\Omega_{g,n}(z_1, z_2, \ldots, z_n)$ by taking the symmetric part with respect to z_1, using the symmetry properties of Proposition 5.2.4 (Proposition 5.2.7).

4. Use the fact that a rational differential form is equal to the sum of its principal parts, where the principal part of $\Omega(z_1)$ at $z_1 = \alpha$ may be defined by

$$\operatorname*{Res}_{z=\alpha} \frac{\mathrm{d}z_1}{z_1 - z}\Omega(z). \tag{5.2}$$

   Finally, match the resulting recursion for the multidifferentials $\Omega_{g,n}(z_1, z_2, \ldots, z_n)$ with the topological recursion for the correlation differentials $\omega_{g,n}(z_1, z_2, \ldots, z_n)$.

These four steps are carried out in the following four subsections, respectively.

Structure of the enumeration

From the enumeration $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ of Definition 5.1.1, we define the following formal multidifferentials.

$$\Omega_{g,n}(z_1, z_2, \ldots, z_n) = \sum_{b_1,b_2,\ldots,b_n=0}^{\infty} \overline{N}_{g,n}(b_1, b_2, \ldots, b_n) \prod_{i=1}^{n} [b_i] z_i^{b_i-1} \, \mathrm{d}z_i \tag{5.3}$$

Theorem 5.1.2 is essentially the statement that the correlation differentials produced by the topological recursion applied to the spectral curve of equation (5.1) satisfy

$$\Omega_{g,n}(z_1, z_2, \ldots, z_n) = \omega_{g,n}(z_1, z_2, \ldots, z_n), \qquad \text{for } (g, n) \neq (0, 1) \text{ or } (0, 2).$$

The primary aim is to understand the structure of $\Omega_{g,n}(z_1, z_2, \ldots, z_n)$, which will play a crucial role in the proof of Theorem 5.1.2. The quasi-polynomiality of $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ stated in Theorem 2.2.12 is equivalent to the fact that for $(g, n) \neq (0, 1)$ or $(0, 2)$,

$$\Omega_{g,n}(z_1, z_2, \ldots, z_n) \in V(z_1) \otimes V(z_2) \otimes \cdots \otimes V(z_n), \tag{5.4}$$

where we define the vector space $V(z)$ as follows.

Definition 5.2.1. Define the complex vector space of differential forms

$$V(z) = \left\{ \sum_{b=0}^{\infty} [b] Q(b) z^{b-1} \, \mathrm{d}z \;\middle|\; Q(b) \text{ is a quasi-polynomial in } b^2 \right\}.$$

Lemma 5.2.2. *The vector space $V(z)$ has the basis $\{\xi_k^{\text{even}}(z), \xi_k^{\text{odd}}(z)\}$, where $k \geq 0$ and*

$$\xi_k^{\text{even}}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} \frac{z^2}{1-z^2} \, \mathrm{d}z + \frac{\delta_{k,0}}{z} \, \mathrm{d}z \qquad \textit{and} \qquad \xi_k^{\text{odd}}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} \frac{z}{1-z^2} \, \mathrm{d}z.$$

Proof. Begin by observing that a quasi-polynomial is a unique linear combination of monomials, acting on either even or odd arguments. So we have the following basis for $V(z)$, as k varies over the non-negative integers.

$$\begin{aligned} \xi_k^{\text{even}}(z) &= \sum_{\substack{b \geq 0 \\ b \text{ even}}} [b] \cdot b^{2k} z^{b-1} \, \mathrm{d}z \\ &= \frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} \sum_{\substack{b > 0 \\ b \text{ even}}} z^b \, \mathrm{d}z + \frac{\delta_{k,0}}{z} \, \mathrm{d}z \\ &= \frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} \frac{z^2}{1-z^2} \, \mathrm{d}z + \frac{\delta_{k,0}}{z} \, \mathrm{d}z \end{aligned} \qquad \begin{aligned} \xi_k^{\text{odd}}(z) &= \sum_{\substack{b \geq 0 \\ b \text{ odd}}} [b] \cdot b^{2k} z^{b-1} \, \mathrm{d}z \\ &= \frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} \sum_{\substack{b > 0 \\ b \text{ odd}}} z^b \, \mathrm{d}z \\ &= \frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} \frac{z}{1-z^2} \, \mathrm{d}z \end{aligned}$$

□

Example 5.2.3. It was previously shown that [29]

$$\overline{N}_{0,3}(b_1, b_2, b_3) = \begin{cases} 1, & b_1 + b_2 + b_3 \text{ even}, \\ 0, & b_1 + b_2 + b_3 \text{ odd}, \end{cases} \qquad \text{and} \qquad \overline{N}_{1,1}(b_1) = \begin{cases} \frac{1}{48}(b_1^2 + 20), & b_1 \text{ even}, \\ 0, & b_1 \text{ odd}. \end{cases}$$

Hence, we can express the corresponding generating differentials in terms of the basis elements of Lemma 5.2.2.

$$\begin{aligned}
\Omega_{0,3}(z_1,z_2,z_3) &= \zeta_0^{\text{even}}(z_1)\,\zeta_0^{\text{even}}(z_2)\,\zeta_0^{\text{even}}(z_3) + \zeta_0^{\text{even}}(z_1)\,\zeta_0^{\text{odd}}(z_2)\,\zeta_0^{\text{odd}}(z_3) \\
&\quad + \zeta_0^{\text{odd}}(z_1)\,\zeta_0^{\text{even}}(z_2)\,\zeta_0^{\text{odd}}(z_3) + \zeta_0^{\text{odd}}(z_1)\,\zeta_0^{\text{odd}}(z_2)\,\zeta_0^{\text{even}}(z_3) \\
&= \frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3}{2z_1z_2z_3}\left[\prod_{i=1}^{3}\frac{z_i^2-z_i+1}{(z_i-1)^2} + \prod_{i=1}^{3}\frac{z_i^2+z_i+1}{(z_i+1)^2}\right] \\
\Omega_{1,1}(z_1) &= \frac{1}{48}(\zeta_1^{\text{even}}(z_1) + 20\,\zeta_0^{\text{even}}(z_1)) \\
&= \frac{\mathrm{d}z_1}{12z_1}\,\frac{5z_1^8-8z_1^6+18z_1^4-8z_1^2+5}{(z_1^2-1)^4}
\end{aligned}$$

A consequence of Lemma 5.2.2 is that elements of $V(z)$ are rational differential forms. The next result reveals that they possess interesting pole structure and symmetry.

Proposition 5.2.4. *For all $\Omega(z) \in V(z)$,*

- *$\Omega(z)$ has poles only at $z = 1$, $z = -1$ and $z = 0$, with only simple poles occurring at $z = 0$; and*
- *$\Omega(z) + \Omega(\frac{1}{z}) = 0$.*

Proof. The first statement is immediate from Lemma 5.2.2, since the operator $\frac{\mathrm{d}}{\mathrm{d}z}(z\,\cdot)$ cannot introduce new poles on $\mathbb{CP}^1$. The second statement can be verified on the basis elements $\xi_k^{\text{even}}(z)$ and $\xi_k^{\text{odd}}(z)$, then deduced for all $\Omega(z) \in V(z)$ by linearity. The verification on basis elements is as follows, using the observation that $\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}(1/z)} = -z\frac{\mathrm{d}}{\mathrm{d}z}$.

$$\begin{aligned}
\xi_k^{\text{even}}(z) + \xi_k^{\text{even}}(\tfrac{1}{z}) &= \mathrm{d}\left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k}\frac{z^2}{1-z^2} + \delta_{k,0}\log(z)\right] + \mathrm{d}\left[\left(-z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k}\frac{(\frac{1}{z})^2}{1-(\frac{1}{z})^2} + \delta_{k,0}\log(\tfrac{1}{z})\right] \\
&= \mathrm{d}\left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k}\left(\frac{z^2}{1-z^2}+\frac{1}{z^2-1}\right) + \delta_{k,0}\big(\log(z)+\log(\tfrac{1}{z})\big)\right] = 0
\end{aligned}$$

$$\begin{aligned}
\xi_k^{\text{odd}}(z) + \xi_k^{\text{odd}}(\tfrac{1}{z}) &= \mathrm{d}\left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k}\frac{z}{1-z^2}\right] + \mathrm{d}\left[\left(-z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k}\frac{\frac{1}{z}}{1-(\frac{1}{z})^2}\right] \\
&= \mathrm{d}\left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k}\left(\frac{z}{1-z^2}+\frac{z}{z^2-1}\right)\right] = 0
\end{aligned}$$

□

We next state a lemma concerning $V(z)$ that will be necessary for the subsequent proof of Theorem 5.1.2.

Lemma 5.2.5. *For all $\Omega(z) \in V(z)$,*

$$\sum_{\alpha=\pm1}\operatorname*{Res}_{z=\alpha}\Omega(z)\log(z) = \operatorname*{Res}_{z=0}\Omega(z). \tag{5.5}$$

Proof. We simply verify the equation for the basis elements $\xi_k^{\text{even}}(z)$ and $\xi_k^{\text{odd}}(z)$, then deduce it for all $\Omega(z) \in V(z)$ by linearity. Note that the residue on the right side is 0 for each basis element, apart from $\xi_0^{\text{even}}(z)$. So let us first suppose that $k \geq 1$ and verify the equation for $\xi_k^{\text{even}}(z)$.

$$\begin{aligned}\sum_{\alpha=\pm1} \operatorname*{Res}_{z=\alpha} \xi_k^{\text{even}}(z) \log(z) &= -\sum_{\alpha=\pm1} \operatorname*{Res}_{z=\alpha} \left[\int \xi_k^{\text{even}}(z)\right] \mathrm{d}\log(z) = -\sum_{\alpha=\pm1} \operatorname*{Res}_{z=\alpha} \left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} \frac{z^2}{1-z^2}\right] \frac{\mathrm{d}z}{z} \\ &= -\sum_{\alpha=\pm1} \operatorname*{Res}_{z=\alpha} \left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} \frac{1}{1-z^2}\right] \frac{\mathrm{d}z}{z} \\ &= \operatorname*{Res}_{z=0} \left[\frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k-1} \frac{1}{1-z^2}\right] \mathrm{d}z\end{aligned}$$

The first line uses the fact that a function $F(z)$ that is meromorphic at $z = \alpha$ satisfies $\operatorname*{Res}_{z=\alpha} \mathrm{d}F = 0$. It follows that $\operatorname*{Res}_{z=\alpha} f \,\mathrm{d}g = -\operatorname*{Res}_{z=\alpha} g \,\mathrm{d}f$ for any two functions $f(z)$ and $g(z)$ that are meromorphic at $z = \alpha$. The second line uses the fact that k is positive. The third line uses the fact that the sum of the residues of a rational differential form is equal to 0. It is clear that the final expression obtained is equal to 0, since the argument is holomorphic at $z = 0$. This completes the proof in this case.

The analogous calculation for $\xi_k^{\text{odd}}(z)$ and $k \geq 0$ is almost identical to the previous and is omitted for brevity. It remains to treat the case $\xi_0^{\text{even}}(z)$, in which case the residue on the right side of the equation is evidently equal to 1. We calculate the left side as follows.

$$\begin{aligned}&\sum_{\alpha=\pm1} \operatorname*{Res}_{z=\alpha} \xi_0^{\text{even}}(z) \log(z) \\ &= \sum_{\alpha=\pm1} \operatorname*{Res}_{z=\alpha} \left[\frac{\mathrm{d}}{\mathrm{d}z}\frac{z^2}{1-z^2}\,\mathrm{d}z + \frac{\mathrm{d}z}{z}\right] \log(z) = \sum_{\alpha=\pm1} \operatorname*{Res}_{z=\alpha} \left[\frac{\mathrm{d}}{\mathrm{d}z}\frac{z^2}{1-z^2}\,\mathrm{d}z\right] \log(z) \\ &= -\sum_{\alpha=\pm1} \operatorname*{Res}_{z=\alpha} \left[\frac{z^2}{1-z^2}\right] \frac{\mathrm{d}z}{z} = -\sum_{\alpha=\pm1} \operatorname*{Res}_{z=\alpha} \frac{z}{1-z^2}\,\mathrm{d}z\end{aligned}$$

The first line uses the definition of $\xi_0^{\text{even}}(z)$ and removes a summand that is clearly holomorphic at $z = \pm 1$. The second line uses the fact that $\operatorname*{Res}_{z=\alpha} f \,\mathrm{d}g = -\operatorname*{Res}_{z=\alpha} g \,\mathrm{d}f$ for any two functions $f(z)$ and $g(z)$ that are meromorphic at $z = \alpha$. It is then straightforward to calculate that the final expression obtained is equal to 1. This completes the proof in this case. □

Combinatorial recursion

We now express the combinatorial recursion of Theorem 2.2.12 in terms of natural generating functions. Rather than using the multidifferentials $\Omega_{g,n}(z_1, z_2, \ldots, z_n)$ defined earlier, it will be convenient to work with the closely related generating functions

$$W_{g,n}(z_1, z_2, \ldots, z_n) = \frac{\Omega_{g,n}(z_1, z_2, \ldots, z_n)}{\mathrm{d}z_1\,\mathrm{d}z_2 \,\cdots\, \mathrm{d}z_n} = \sum_{b_1, b_2, \ldots, b_n = 0}^{\infty} \overline{N}_{g,n}(b_1, b_2, \ldots, b_n) \prod_{i=1}^{n} [b_i] z_i^{b_i - 1}.$$

Proposition 5.2.6. *For $2g - 2 + n \geq 2$, we have the following equation, where $S = \{1, 2, \ldots, n\}$*

and $\mathbf{z}_I = (z_{i_1}, z_{i_2}, \ldots, z_{i_k})$ *for* $I = \{i_1, i_2, \ldots, i_k\}$.

$$\begin{aligned}\sum_{i=1}^{n} \frac{\partial}{\partial z_i} z_i W_{g,n}(\mathbf{z}_S) &= \sum_{i<j} \left(\frac{\partial}{\partial z_i}\left[\frac{2}{z_j}\frac{z_i^3}{(1-z_i^2)^2} W_{g,n-1}(\mathbf{z}_{S\setminus\{j\}})\right] + \frac{\partial}{\partial z_j}\left[\frac{2}{z_i}\frac{z_j^3}{(1-z_j^2)^2} W_{g,n-1}(\mathbf{z}_{S\setminus\{i\}})\right]\right.\\ &\quad \left. + 2\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}\left[\frac{z_j}{z_i-z_j}\frac{z_i^3}{(1-z_i^2)^2} W_{g,n-1}(\mathbf{z}_{S\setminus\{j\}}) - \frac{z_i}{z_i-z_j}\frac{z_j^3}{(1-z_j^2)^2} W_{g,n-1}(\mathbf{z}_{S\setminus\{i\}})\right]\right)\\ &\quad + \sum_{i=1}^{n} \frac{\partial}{\partial z_i}\frac{z_i^4}{(1-z_i^2)^2}\left[W_{g-1,n+1}(z_i, z_i, \mathbf{z}_{S\setminus\{i\}}) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=S\setminus\{i\}}}^{\text{stable}} W_{g_1,|I|+1}(z_i, \mathbf{z}_I)\, W_{g_2,|J|+1}(z_i, \mathbf{z}_J)\right]\end{aligned}$$

Proof. The combinatorial recursion of Theorem 2.2.12 states that for $2g-2+n \geq 2$ and $b_1, b_2, \ldots, b_n \geq 0$, we have the following equation.

$$\begin{aligned}\left(\sum_{i=1}^{n} b_i\right)\overline{N}_{g,n}(\mathbf{b}_S) &= \sum_{i<j}\sum_{\substack{p+q=b_i+b_j \\ q \text{ even}}} [p]q\, \overline{N}_{g,n-1}(p, \mathbf{b}_{S\setminus\{i,j\}})\\ &\quad + \frac{1}{2}\sum_{i}\sum_{\substack{p+q+r=b_i \\ r \text{ even}}} [p][q]r\left[\overline{N}_{g-1,n+1}(p, q, \mathbf{b}_{S\setminus\{i\}}) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=S\setminus\{i\}}}^{\text{stable}} \overline{N}_{g_1,|I|+1}(p, \mathbf{b}_I)\, \overline{N}_{g_2,|J|+1}(q, \mathbf{b}_J)\right]\end{aligned}$$

Let us define the operators

$$\mathcal{O} = \sum_{b_1,b_2,\ldots,b_n=0}^{\infty} [\ \cdot\ ]\prod_{i=1}^{n}[b_i]z_i^{b_i-1} \qquad \text{and} \qquad \mathcal{O}_J = \sum_{b_i=0:i\notin J}^{\infty} [\ \cdot\ ]\prod_{i\notin J}[b_i]z_i^{b_i-1}.$$

The result arises from applying the operator $\mathcal{O}$ to both sides of the combinatorial recursion. The left side becomes

$$\begin{aligned}\sum_{b_1,b_2,\ldots,b_n=0}^{\infty}\left(\sum_{i=1}^{n} b_i\right)\overline{N}_{g,n}(\mathbf{b}_S)\prod_{i=1}^{n}[b_i]z_i^{b_i-1} &= \sum_{i=1}^{n}\sum_{b_1,b_2,\ldots,b_n=0}^{\infty}\frac{\partial}{\partial z_i}z_i\left(\overline{N}_{g,n}(\mathbf{b}_S)\prod_{i=1}^{n}[b_i]z_i^{b_i-1}\right)\\ &= \sum_{i=1}^{n}\frac{\partial}{\partial z_i}z_i W_{g,n}(\mathbf{z}_S). \qquad (*)\end{aligned}$$

Applying the operator $\mathcal{O}$ to the (i,j)th summand in the first term on the right side of the combinatorial recursion yields

$$\begin{aligned}
&\sum_{b_1,b_2,\ldots,b_n=0}^{\infty} \sum_{\substack{p+q=b_i+b_j \\ q \text{ even}}} [p]q\,\overline{N}_{g,n-1}(p,\mathbf{b}_{S\setminus\{i,j\}}) \prod_{i=1}^{n} [b_i] z_i^{b_i-1} \\
&= \mathcal{O}_{i,j} \sum_{b_i,b_j=0}^{\infty} \sum_{\substack{p+q=b_i+b_j \\ q \text{ even}}} [p]q\,\overline{N}_{g,n-1}(p,\mathbf{b}_{S\setminus\{i,j\}})\,[b_i]\,[b_j] z_i^{b_i-1} z_j^{b_j-1} \\
&= \mathcal{O}_{i,j} \sum_{\substack{p,q=0 \\ q \text{ even}}}^{\infty} [p]q\,\overline{N}_{g,n-1}(p,\mathbf{b}_{S\setminus\{i,j\}}) \sum_{k=0}^{p+q} [k]\,[p+q-k] z_i^{k-1} z_j^{p+q-k-1} \\
&= \mathcal{O}_{i,j} \sum_{\substack{p,q=0 \\ q \text{ even}}}^{\infty} [p]q\,\overline{N}_{g,n-1}(p,\mathbf{b}_{S\setminus\{i,j\}}) \left[\frac{\partial}{\partial z_i} z_i^{p+q} z_j^{-1} + \frac{\partial}{\partial z_j} z_i^{-1} z_j^{p+q} \right] \\
&\quad + \mathcal{O}_{i,j} \sum_{\substack{p,q=0 \\ q \text{ even}}}^{\infty} [p]q\,\overline{N}_{g,n-1}(p,\mathbf{b}_{S\setminus\{i,j\}}) \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j} \big(z_i^{p+q-1} z_j^1 + z_i^{p+q-2} z_j^2 + \cdots + z_i^1 z_j^{p+q-1} \big).
\end{aligned}$$

Consider the first of the two terms in this last expression and use $\sum\limits_{q \text{ even}} q z^q = \frac{2z^2}{(1-z^2)^2}$ to obtain the following.

$$\begin{aligned}
&\mathcal{O}_{i,j} \sum_{\substack{p,q=0 \\ q \text{ even}}}^{\infty} [p]q\,\overline{N}_{g,n-1}(p,\mathbf{b}_{S\setminus\{i,j\}}) \left[\frac{\partial}{\partial z_i} z_i^{p+q} z_j^{-1} + \frac{\partial}{\partial z_j} z_i^{-1} z_j^{p+q} \right] \\
&\quad = \mathcal{O}_{i,j} \frac{\partial}{\partial z_i} z_j^{-1} z_i \sum_{\substack{q=0 \\ q \text{ even}}}^{\infty} q z_i^q \sum_{p=0}^{\infty} [p]\,\overline{N}_{g,n-1}(p,\mathbf{b}_{S\setminus\{i,j\}})\, z_i^{p-1} \\
&\quad\quad + \mathcal{O}_{i,j} \frac{\partial}{\partial z_j} z_i^{-1} z_j \sum_{\substack{q=0 \\ q \text{ even}}}^{\infty} q z_j^q \sum_{p=0}^{\infty} [p]\,\overline{N}_{g,n-1}(p,\mathbf{b}_{S\setminus\{i,j\}})\, z_j^{p-1} \\
&\quad = \frac{\partial}{\partial z_i} \left[\frac{2}{z_j} \frac{z_i^3}{(1-z_i^2)^2} W_{g,n-1}(\mathbf{z}_{S\setminus\{j\}}) \right] + \frac{\partial}{\partial z_j} \left[\frac{2}{z_i} \frac{z_j^3}{(1-z_j^2)^2} W_{g,n-1}(\mathbf{z}_{S\setminus\{i\}}) \right] \qquad (*)
\end{aligned}$$

Now consider the second term in a similar fashion to obtain the following.

$$\begin{aligned}
&\mathcal{O}_{i,j} \sum_{\substack{p,q=0 \\ q \text{ even}}}^{\infty} [p]q\,\overline{N}_{g,n-1}(p,\mathbf{b}_{S\setminus\{i,j\}}) \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j} \big(z_i^{p+q-1} z_j^1 + z_i^{p+q-2} z_j^2 + \cdots + z_i^1 z_j^{p+q-1} \big) \\
&\quad = \mathcal{O}_{i,j} \sum_{\substack{p,q=0 \\ q \text{ even}}}^{\infty} [p]q\,\overline{N}_{g,n-1}(p,\mathbf{b}_{S\setminus\{i,j\}}) \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j} \frac{z_i^{p+q} z_j - z_i z_j^{p+q}}{z_i - z_j} \\
&\quad = \mathcal{O}_{i,j} \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j} \left[\frac{1}{z_i - z_j} \sum_{p,q=0}^{\infty} [p]q\,\overline{N}_{g,n-1}(p,\mathbf{b}_{S\setminus\{i,j\}}) \big(z_i^{p+q} z_j - z_i z_j^{p+q} \big) \right] \\
&\quad = 2 \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j} \left[\frac{z_j}{z_i - z_j} \frac{z_i^3}{(1-z_i^2)^2} W_{g,n-1}(\mathbf{z}_{S\setminus\{j\}}) - \frac{z_i}{z_i - z_j} \frac{z_j^3}{(1-z_j^2)^2} W_{g,n-1}(\mathbf{z}_{S\setminus\{i\}}) \right] \qquad (*)
\end{aligned}$$

Applying the operator $\mathcal{O}$ to twice the ith summand in the second term on the right side of the combinatorial recursion yields

$$\sum_{\substack{b_1,b_2,\ldots,b_n=0 \\ p+q+r=b_i \\ r \text{ even}}}^{\infty} [p][q]r \left[\overline{N}_{g-1,n+1}(p,q,\mathbf{b}_{S\setminus\{i\}}) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S\setminus\{i\}}}^{\text{stable}} \overline{N}_{g_1,|I|+1}(p,\mathbf{b}_I)\,\overline{N}_{g_2,|J|+1}(q,\mathbf{b}_J) \right] \prod_{i=1}^{n} [b_i] z_i^{b_i-1}$$

$$= \mathcal{O}_i \sum_{\substack{b_i=0 \\ p+q+r=b_i \\ r \text{ even}}} [p][q]r \left[\overline{N}_{g-1,n+1}(p,q,\mathbf{b}_{S\setminus\{i\}}) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S\setminus\{i\}}}^{\text{stable}} \overline{N}_{g_1,|I|+1}(p,\mathbf{b}_I)\,\overline{N}_{g_2,|J|+1}(q,\mathbf{b}_J) \right] [b_i] z_i^{b_i-1}$$

$$= \mathcal{O}_i \frac{\partial}{\partial z_i} z_i \sum_{\substack{p,q,r=0 \\ r \text{ even}}}^{\infty} [p][q]r \left[\overline{N}_{g-1,n+1}(p,q,\mathbf{b}_{S\setminus\{i\}}) \right.$$

$$\left. + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S\setminus\{i\}}}^{\text{stable}} \overline{N}_{g_1,|I|+1}(p,\mathbf{b}_I)\,\overline{N}_{g_2,|J|+1}(q,\mathbf{b}_J) \right] z_i^{p+q+r-1}$$

$$= \frac{\partial}{\partial z_i} \frac{2z_i^4}{(1-z_i^2)^2} \left[W_{g-1,n+1}(z_i,z_i,\mathbf{z}_{S\setminus\{i\}}) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S\setminus\{i\}}}^{\text{stable}} W_{g_1,|I|+1}(z_i,\mathbf{z}_I)\,W_{g_2,|J|+1}(z_i,\mathbf{z}_J) \right] \qquad (*)$$

Finally, combine all of the contributions from the expressions marked by $(*)$ to obtain the desired result. □

Breaking the symmetry

A feature of the topological recursion is that it produces symmetric meromorphic multidifferentials from a recursion that is manifestly asymmetric, with a special role played by the variable z_1. We break the symmetry in the recursion of Proposition 5.2.6 by applying the operator

$$F(z_1) \mapsto F(z_1) - \frac{1}{z_1^2} F(\tfrac{1}{z_1})$$

to every term appearing. In a precise sense, this amounts to taking the symmetric part with respect to the involution $s(z) = \frac{1}{z}$ appearing in the topological recursion, stated at the level of functions rather than differentials.

Recall that equation (5.4) combined with Proposition 5.2.4 assert that

$$\Omega_{g,n}(z_1,z_2,\ldots,z_n) + \Omega_{g,n}(\tfrac{1}{z_1},z_2,\ldots,z_n) = 0.$$

At the level of generating functions, this translates to the property

$$W_{g,n}(z_1,z_2,\ldots,z_n) - \frac{1}{z_1^2} W_{g,n}(\tfrac{1}{z_1},z_2,\ldots,z_n) = 0, \qquad (5.6)$$

which will be useful in subsequent calculations.

Proposition 5.2.7. *For $2g-2+n \geq 2$, we have the following equation, where $S = \{2,3,\ldots,n\}$*

and $\mathbf{z}_I = (z_{i_1}, z_{i_2}, \ldots, z_{i_k})$ *for* $I = \{i_1, i_2, \ldots, i_k\}$.

$$\begin{aligned}
&W_{g,n}(z_1, \mathbf{z}_S) - \operatorname*{Res}_{p=0} \frac{W_{g,n}(p, \mathbf{z}_S)\,\mathrm{d}p}{z_1} \\
&= \sum_{j=2}^{n} \left(\frac{2}{z_1 z_j} + \frac{1}{(z_1 - z_j)^2} + \frac{1}{(1 - z_1 z_j)^2} \right) \frac{z_1^3}{(1 - z_1^2)^2} W_{g,n-1}(z_1, \mathbf{z}_{S \setminus \{j\}}) \\
&\qquad - \sum_{j=2}^{n} \frac{\partial}{\partial z_j} \left[\left(\frac{1}{z_1 - z_j} + \frac{z_j}{1 - z_1 z_j} \right) \frac{z_j^3}{(1 - z_j^2)^2} W_{g,n-1}(\mathbf{z}_S) \right] \\
&\qquad + \frac{z_1^3}{(1 - z_1^2)^2} \left[W_{g-1,n+1}(z_1, z_1, \mathbf{z}_S) + \sum_{\substack{g_1 + g_2 = g \\ I \sqcup J = S}}^{\text{stable}} W_{g_1,|I|+1}(z_1, \mathbf{z}_I)\, W_{g_2,|J|+1}(z_1, \mathbf{z}_J) \right]
\end{aligned}$$

Proof. As mentioned above, we apply the operator $F(z_1) \mapsto F(z_1) - \frac{1}{z_1^2} F(\frac{1}{z_1})$ to all terms appearing in the recursion of Proposition 5.2.6. The left side becomes

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{\partial}{\partial z_i} z_i W_{g,n}(z_1, \mathbf{z}_S) - \frac{1}{z_1^2} \left[\sum_{i=1}^{n} \frac{\partial}{\partial z_i} z_i W_{g,n}(z_1, \mathbf{z}_S) \right]_{z_1 \mapsto \frac{1}{z_1}} \\
&= \frac{\partial}{\partial z_1} z_1 W_{g,n}(z_1, \mathbf{z}_S) - \frac{1}{z_1^2} \frac{\partial}{\partial (\frac{1}{z_1})} \frac{1}{z_1} W_{g,n}(\tfrac{1}{z_1}, \mathbf{z}_S) + \sum_{i=2}^{n} \frac{\partial}{\partial z_i} z_i \left[W_{g,n}(z_1, \mathbf{z}_S) - \frac{1}{z_1^2} W_{g,n}(\tfrac{1}{z_1}, \mathbf{z}_S) \right] \\
&= 2 \frac{\partial}{\partial z_1} z_1 W_{g,n}(z_1, \mathbf{z}_S). \qquad (**)
\end{aligned}$$

Here, we have used the symmetry property of equation (5.6) to deduce that the summands with $2 \leq i \leq n$ are equal to 0 and to express $W_{g,n}(\frac{1}{z_1}, \mathbf{z}_S)$ in terms of $W_{g,n}(z_1, \mathbf{z}_S)$.

In the summation over $i < j$ on the right side, the symmetry property of equation (5.6) ensures that a non-zero contribution arises only for the summands with $i = 1$ and $j = 2, 3, \ldots, n$. For such a summand, the first line on the right side contributes

$$\begin{aligned}
&\frac{\partial}{\partial z_1} \left[\frac{2}{z_j} \frac{z_1^3}{(1 - z_1^2)^2} W_{g,n-1}(\mathbf{z}_{S \setminus \{j\}}) \right] - \frac{1}{z_1^2} \frac{\partial}{\partial (\frac{1}{z_1})} \left[\frac{2}{z_j} \frac{\frac{1}{z_1^3}}{(1 - \frac{1}{z_1^2})^2} W_{g,n-1}(\tfrac{1}{z_1}, \mathbf{z}_{S \setminus \{1,j\}}) \right] \\
&+ \frac{\partial}{\partial z_j} \left[\frac{1}{z_1} \frac{z_j^3}{(1 - z_j^2)^2} W_{g,n-1}(\mathbf{z}_S) \right] - \frac{1}{z_1^2} \frac{\partial}{\partial z_j} \left[z_1 \frac{z_j^3}{(1 - z_j^2)^2} W_{g,n-1}(\mathbf{z}_S) \right] \\
&\qquad = 2 \frac{\partial}{\partial z_1} \left[\frac{2}{z_j} \frac{z_1^3}{(1 - z_1^2)^2} W_{g,n-1}(\mathbf{z}_{S \setminus \{j\}}) \right]. \qquad (**)
\end{aligned}$$

The second line on the right side contributes

$$\begin{aligned}
&2 \frac{\partial}{\partial z_1} \frac{\partial}{\partial z_j} \left[\frac{1}{z_1 - z_j} \frac{z_1^3 z_j}{(1 - z_1^2)^2} W_{g,n-1}(z_1, \mathbf{z}_{S \setminus \{j\}}) \right] \\
&- 2 \frac{1}{z_1^2} \frac{\partial}{\partial (\frac{1}{z_1})} \frac{\partial}{\partial z_j} \left[\frac{1}{\frac{1}{z_1} - z_j} \frac{\frac{1}{z_1^3} z_j}{(1 - \frac{1}{z_1^2})^2} W_{g,n-1}(\tfrac{1}{z_1}, \mathbf{z}_{S \setminus \{j\}}) \right] \\
&- 2 \frac{\partial}{\partial z_1} \frac{\partial}{\partial z_j} \left[\frac{1}{z_1 - z_j} \frac{z_1 z_j^3}{(1 - z_j^2)^2} W_{g,n-1}(\mathbf{z}_S) \right] + 2 \frac{1}{z_1^2} \frac{\partial}{\partial (\frac{1}{z_1})} \frac{\partial}{\partial z_j} \left[\frac{1}{\frac{1}{z_1} - z_j} \frac{\frac{1}{z_1} z_j^3}{(1 - z_j^2)^2} W_{g,n-1}(\mathbf{z}_S) \right] \\
&\quad = 2 \frac{\partial}{\partial z_1} \frac{\partial}{\partial z_j} \left[\left(\frac{z_j}{z_1 - z_j} + \frac{z_1 z_j}{1 - z_1 z_j} \right) \frac{z_1^3}{(1 - z_1^2)^2} W_{g,n-1}(z_1, \mathbf{z}_{S \setminus \{j\}}) \right] \\
&\qquad - 2 \frac{\partial}{\partial z_1} \frac{\partial}{\partial z_j} \left[\left(\frac{z_1}{z_1 - z_j} + \frac{1}{1 - z_1 z_j} \right) \frac{z_j^3}{(1 - z_j^2)^2} W_{g,n-1}(\mathbf{z}_S) \right]. \qquad (**)
\end{aligned}$$

In the summation over i on the right side, the symmetry property of equation (5.6) ensures that a non-zero contribution arises only for the summands with $i = 1$. So the third line on the right side contributes

$$\frac{\partial}{\partial z_1} \frac{z_1^4}{(1-z_1^2)^2} \left[W_{g-1,n+1}(z_1, z_1, \mathbf{z}_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}}^{\text{stable}} W_{g_1,|I|+1}(z_1, \mathbf{z}_I)\, W_{g_2,|J|+1}(z_1, \mathbf{z}_J) \right]$$
$$- \frac{1}{z_1^2} \frac{\partial}{\partial(\frac{1}{z_1})} \frac{\frac{1}{z_1^4}}{(1-\frac{1}{z_1^2})^2} \left[W_{g-1,n+1}(\tfrac{1}{z_1}, \tfrac{1}{z_1}, \mathbf{z}_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}}^{\text{stable}} W_{g_1,|I|+1}(\tfrac{1}{z_1}, \mathbf{z}_I)\, W_{g_2,|J|+1}(\tfrac{1}{z_1}, \mathbf{z}_J) \right]$$
$$= 2 \frac{\partial}{\partial z_1} \frac{z_1^4}{(1-z_1^2)^2} \left[W_{g-1,n+1}(z_1, z_1, \mathbf{z}_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}}^{\text{stable}} W_{g_1,|I|+1}(z_1, \mathbf{z}_I)\, W_{g_2,|J|+1}(z_1, \mathbf{z}_J) \right]. \quad (**)$$

Gather together all of the terms marked by $(**)$ and perform some mild algebraic simplification to obtain the following.

$$\frac{\partial}{\partial z_1} z_1 W_{g,n}(z_1, \mathbf{z}_S) = \sum_{j=2}^{n} \frac{\partial}{\partial z_1} \left[z_1 \left(\frac{2}{z_1 z_j} + \frac{1}{(z_1 - z_j)^2} + \frac{1}{(1 - z_1 z_j)^2} \right) \frac{z_1^3}{(1-z_1^2)^2} W_{g,n-1}(z_1, \mathbf{z}_{S \setminus \{j\}}) \right]$$
$$- \sum_{j=2}^{n} \frac{\partial}{\partial z_1} \frac{\partial}{\partial z_j} \left[z_1 \left(\frac{1}{z_1 - z_j} + \frac{z_j}{1 - z_1 z_j} \right) \frac{z_j^3}{(1-z_j^2)^2} W_{g,n-1}(\mathbf{z}_S) \right]$$
$$+ \frac{\partial}{\partial z_1} \frac{z_1^4}{(1-z_1^2)^2} \left[W_{g-1,n+1}(z_1, z_1, \mathbf{z}_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}}^{\text{stable}} W_{g_1,|I|+1}(z_1, \mathbf{z}_I)\, W_{g_2,|J|+1}(z_1, \mathbf{z}_J) \right]$$

One can remove the operator $\frac{\partial}{\partial z_1} z_1$ from every term to recover an equality of the following form, where [correction] is independent of z_1.

$$W_{g,n}(z_1, \mathbf{z}_S) + \frac{[\text{correction}]}{z_1}$$
$$= \sum_{j=2}^{n} \left[\left(\frac{2}{z_1 z_j} + \frac{1}{(z_1 - z_j)^2} + \frac{1}{(1 - z_1 z_j)^2} \right) \frac{z_1^3}{(1-z_1^2)^2} W_{g,n-1}(z_1, \mathbf{z}_{S \setminus \{j\}}) \right]$$
$$- \sum_{j=2}^{n} \frac{\partial}{\partial z_j} \left[\left(\frac{1}{z_1 - z_j} + \frac{z_j}{1 - z_1 z_j} \right) \frac{z_j^3}{(1-z_j^2)^2} W_{g,n-1}(\mathbf{z}_S) \right]$$
$$+ \frac{z_1^3}{(1-z_1^2)^2} \left[W_{g-1,n+1}(z_1, z_1, \mathbf{z}_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}}^{\text{stable}} W_{g_1,|I|+1}(z_1, \mathbf{z}_I)\, W_{g_2,|J|+1}(z_1, \mathbf{z}_J) \right]$$

Finally, recall that $W_{g,n}(z_1, z_2, \ldots, z_n)$ has at worst a simple pole at $z_1 = 0$. It follows that the right side of this equation has no pole at $z_1 = 0$, so the correction term is given by

$$[\text{correction}] = - \operatorname*{Res}_{p=0} W_{g,n}(p, \mathbf{z}_S)\, \mathrm{d}p,$$

and this completes the proof. □

Proof of the main theorem

We now have all of the pieces in place to prove our main result.

Proof of Theorem 5.1.2. Recall that we wish to prove that $\Omega_{g,n} = \omega_{g,n}$ for all $(g,n) \neq (0,1)$ or $(0,2)$, where the former is defined via the enumeration $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ and equation (5.3), while the latter is defined via the topological recursion applied to the local spectral curve of equation (5.1). We use an inductive approach with base cases $(g,n) = (0,3)$ and $(1,1)$. One may verify directly that $\Omega_{0,3} = \omega_{0,3}$ and $\Omega_{1,1} = \omega_{1,1}$ by comparing the calculations of Example 5.1.4 and Example 5.2.3. So the theorem is true whenever $2g - 2 + n = 1$. Now consider (g,n) satisfying $2g-2+n \geq 2$ and assume the inductive hypothesis that $\Omega_{g',n'} = \omega_{g',n'}$ whenever $1 \leq 2g' - 2 + n' < 2g - 2 + n$ and $(g',n') \neq (0,1)$ or $(0,2)$.

We begin by rewriting Proposition 5.2.7 in terms of multidifferentials by multiplying by $\mathrm{d}z_1 \cdots \mathrm{d}z_n$.

$$\Omega_{g,n}(z_1, \mathbf{z}_S) - \frac{\mathrm{d}z_1}{z_1} \operatorname*{Res}_{p=0} \Omega_{g,n}(p, \mathbf{z}_S) \tag{5.7}$$

$$= \sum_{j=2}^{n} \left(\frac{2\,\mathrm{d}z_j}{z_1 z_j} + \frac{\mathrm{d}z_j}{(z_1 - z_j)^2} + \frac{\mathrm{d}z_j}{(1 - z_1 z_j)^2} \right) \frac{z_1^3}{(1 - z_1^2)^2}\, \Omega_{g,n-1}(z_1, \mathbf{z}_{S \setminus \{j\}})$$

$$- \sum_{j=2}^{n} \frac{\partial}{\partial z_j} \left[\left(\frac{1}{z_1 - z_j} + \frac{z_j}{1 - z_1 z_j} \right) \frac{z_j^3}{(1 - z_j^2)^2} W_{g,n-1}(\mathbf{z}_S) \right] \mathrm{d}z_1\, \mathrm{d}z_2 \cdots \mathrm{d}z_n$$

$$+ \frac{z_1^3}{(1 - z_1^2)^2} \frac{1}{\mathrm{d}z_1} \left[\Omega_{g-1,n+1}(z_1, z_1, \mathbf{z}_S) + \sum_{\substack{g_1 + g_2 = g \\ I \sqcup J = S}}^{\text{stable}} \Omega_{g_1,|I|+1}(z_1, \mathbf{z}_I)\, \Omega_{g_2,|J|+1}(z_1, \mathbf{z}_J) \right] \tag{5.8}$$

By Proposition 5.2.4, $\Omega_{g,n}(z_1, \mathbf{z}_S)$ has at worst a simple pole at $z_1 = 0$ and poles at $z_1 = 1$ and $z_1 = -1$. Hence, the left side of the previous equation only has poles at $z_1 = 1$ and $z_1 = -1$. Now use the fact that a rational differential is equal to the sum of its principal parts, each of which may be expressed by equation (5.2), to obtain the following.$\mathbf{z}$

$$\Omega_{g,n}(z_1, \mathbf{z}_S) - \frac{\mathrm{d}z_1}{z_1} \operatorname*{Res}_{p=0} \Omega_{g,n}(p, \mathbf{z}_S) = \sum_{\alpha = \pm 1} \operatorname*{Res}_{z=\alpha} \frac{\mathrm{d}z_1}{z_1 - z} \left[\Omega_{g,n}(z, \mathbf{z}_S) - \frac{\mathrm{d}z}{z} \operatorname*{Res}_{p=0} \Omega_{g,n}(p, \mathbf{z}_S) \right]$$

Substituting equation (5.7) into the right side of the previous equation yields the following.

$$\Omega_{g,n}(z_1, \mathbf{z}_S) - \frac{\mathrm{d}z_1}{z_1} \operatorname*{Res}_{p=0} \Omega_{g,n}(p, \mathbf{z}_S)$$

$$= \sum_{\alpha = \pm 1} \operatorname*{Res}_{z=\alpha} \frac{1}{z_1 - z} \frac{z^3}{(1 - z^2)^2} \frac{\mathrm{d}z_1}{\mathrm{d}z} \left[\sum_{j=2}^{n} \left(\frac{2\,\mathrm{d}z\,\mathrm{d}z_j}{z z_j} + \frac{\mathrm{d}z\,\mathrm{d}z_j}{(z - z_j)^2} + \frac{\mathrm{d}z\,\mathrm{d}z_j}{(1 - z z_j)^2} \right) \omega_{g,n-1}(z, \mathbf{z}_{S \setminus \{j\}}) \right.$$

$$\left. + \,\omega_{g-1,n+1}(z, z, \mathbf{z}_S) + \sum_{\substack{g_1 + g_2 = g \\ I \sqcup J = S}}^{\text{stable}} \omega_{g_1,|I|+1}(z, \mathbf{z}_I)\, \omega_{g_2,|J|+1}(z, \mathbf{z}_J) \right] \tag{5.9}$$

Since the entire second line on the right side of equation (5.7) is evidently analytic at $z_1 = \alpha$ for all $\alpha \in \mathbb{C}$, we may omit it from this equation. Furthermore, we have invoked the induction hypothesis to replace each $\Omega_{g',n'}$ on the right side with $\omega_{g',n'}$.

Recalling the definition of $\omega_{0,2}$, we have

$$\omega_{0,2}(z, z_2) = \frac{\mathrm{d}z\,\mathrm{d}z_2}{(z - z_2)^2} + \frac{\mathrm{d}z\,\mathrm{d}z_2}{z\, z_2} \quad \Rightarrow \quad \omega_{0,2}(z, z_2) - \omega_{0,2}(\tfrac{1}{z}, z_2) = \frac{2\,\mathrm{d}z\,\mathrm{d}z_2}{z\, z_2} + \frac{\mathrm{d}z\,\mathrm{d}z_2}{(z - z_2)^2} + \frac{\mathrm{d}z\,\mathrm{d}z_2}{(1 - z z_2)^2}.$$

Therefore, equation (5.9) above can be written equivalently as follows, where we have also used the induction hypothesis and Proposition 5.2.4 to deduce that $\omega_{g',n'}(z, \mathbf{z}) = -\omega_{g',n'}(\frac{1}{z}, \mathbf{z})$ for various terms on the right side.

$$\begin{aligned}
&\Omega_{g,n}(z_1, \mathbf{z}_S) \\
&= \frac{\mathrm{d}z_1}{z_1} \operatorname*{Res}_{p=0} \Omega_{g,n}(p, \mathbf{z}_S) + \sum_{\alpha=\pm 1} \operatorname*{Res}_{z=\alpha} \frac{-1}{z_1 - z} \frac{z^3}{(1-z^2)^2} \frac{\mathrm{d}z_1}{\mathrm{d}z} \Bigg[\sum_{j=2}^{n} \omega_{0,2}(z, z_2)\, \omega_{g,n-1}(\tfrac{1}{z}, \mathbf{z}_{S\setminus\{j\}}) \\
&+ \sum_{j=2}^{n} \omega_{0,2}(\tfrac{1}{z}, z_2)\, \omega_{g,n-1}(z, \mathbf{z}_{S\setminus\{j\}}) + \omega_{g-1,n+1}(z, \tfrac{1}{z}, \mathbf{z}_S) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=S}}^{\text{stable}} \omega_{g_1,|I|+1}(z, \mathbf{z}_I)\, \omega_{g_2,|J|+1}(\tfrac{1}{z}, \mathbf{z}_J) \Bigg],
\end{aligned}$$

Now absorb the terms in the two summations over j into the summation over $g_1 + g_2$ and $I \sqcup J = S$. Recall that the symbol $\circ$ over the inner summation denotes that we exclude all terms with $\omega_{0,1}$.

$$\begin{aligned}
\Omega_{g,n}(z_1, \mathbf{z}_S) = \frac{\mathrm{d}z_1}{z_1} \operatorname*{Res}_{p=0} \Omega_{g,n}(p, \mathbf{z}_S) + \sum_{\alpha=\pm 1} \operatorname*{Res}_{z=\alpha} \frac{-1}{z_1 - z} \frac{z^3}{(1-z^2)^2} \frac{\mathrm{d}z_1}{\mathrm{d}z} \Bigg[\omega_{g-1,n+1}(z, \tfrac{1}{z}, \mathbf{z}_S) \\
+ \sum_{\substack{g_1+g_2=g \\ I\sqcup J=S}}^{\circ} \omega_{g_1,|I|+1}(z, \mathbf{z}_I)\, \omega_{g_2,|J|+1}(\tfrac{1}{z}, \mathbf{z}_J) \Bigg]. \qquad (5.10)
\end{aligned}$$

By construction we have $\Omega_{g,n}(p, \mathbf{z}_S) \in V(p) \otimes V(z_2) \otimes \cdots \otimes V(z_n)$, so Lemma 5.2.5 asserts that

$$\operatorname*{Res}_{p=0} \Omega_{g,n}(p, \mathbf{z}_S) = \sum_{\alpha=\pm 1} \operatorname*{Res}_{z=\alpha} \Omega_{g,n}(z, \mathbf{z}_S) \log(z).$$

Multiply both sides of this equation by $\frac{\mathrm{d}z_1}{z_1}$ and use equation (5.7) to substitute for $\Omega_{g,n}(z, \mathbf{z}_S)$ on the right side. Observing that the terms

$$\frac{\mathrm{d}z_1}{z_1} \operatorname*{Res}_{p=0} \Omega_{g,n}(p, \mathbf{z}_S) \qquad \text{and} \qquad \sum_{j=2}^{n} \frac{\partial}{\partial z_j} \left[\left(\frac{1}{z_1 - z_j} + \frac{z_j}{1 - z_1 z_j} \right) \frac{z_j^3}{(1-z_j^2)^2} W_{g,n-1}(\mathbf{z}_S) \right]$$

are analytic at $z_1 = 1$ and $z_1 = -1$, we obtain the following.

$$\begin{aligned}
&\frac{\mathrm{d}z_1}{z_1} \operatorname*{Res}_{p=0} \Omega_{g,n}(p, \mathbf{z}_S) \\
&= \sum_{\alpha=\pm 1} \operatorname*{Res}_{z=\alpha} \Omega_{g,n}(z, \mathbf{z}_S) \frac{\log(z)}{z_1} \mathrm{d}z_1 \\
&= \sum_{\alpha=\pm 1} \operatorname*{Res}_{z=\alpha} \frac{\log(z)}{z_1} \frac{z^3}{(1-z^2)^2} \frac{\mathrm{d}z_1}{\mathrm{d}z} \Bigg[\sum_{j=2}^{n} \left(\frac{2\,\mathrm{d}z\,\mathrm{d}z_j}{z z_j} + \frac{\mathrm{d}z\,\mathrm{d}z_j}{(z - z_j)^2} + \frac{\mathrm{d}z\,\mathrm{d}z_j}{(1 - z z_j)^2} \right) \Omega_{g,n-1}(z, \mathbf{z}_{S\setminus\{j\}}) \\
&+ \Omega_{g-1,n+1}(z_1, z_1, \mathbf{z}_S) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=S}}^{\text{stable}} \Omega_{g_1,|I|+1}(z_1, \mathbf{z}_I)\, \Omega_{g_2,|J|+1}(z_1, \mathbf{z}_J) \Bigg] \\
&= \sum_{\alpha=\pm 1} \operatorname*{Res}_{z=\alpha} \frac{-\log(z)}{\mathrm{d}z} \frac{z^3}{(1-z^2)^2} \frac{\mathrm{d}z_1}{\mathrm{d}z} \Bigg[\omega_{g-1,n+1}(z, \tfrac{1}{z}, \mathbf{z}_S) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=S}}^{\circ} \omega_{g_1,|I|+1}(z, \mathbf{z}_I)\, \omega_{g_2,|J|+1}(\tfrac{1}{z}, \mathbf{z}_J) \Bigg]
\end{aligned}$$

Here, we have used the induction hypothesis and the same algebraic trickery that was used previously to deduce equation (5.10) from equation (5.9).

Substituting the previous equation into equation (5.10) results in

$$\Omega_{g,n}(z_1, \mathbf{z}_S) = \sum_{\alpha=\pm1} \operatorname*{Res}_{z=\alpha} K(z_1, z) \left[\omega_{g-1,n+1}(z, \tfrac{1}{z}, \mathbf{z}_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}}^{\circ} \omega_{g_1,|I|+1}(z, \mathbf{z}_I)\, \omega_{g_2,|J|+1}(\tfrac{1}{z}, \mathbf{z}_J) \right],$$

where we have recognised the recursion kernel $K(z_1, z)$ calculated in Example 5.1.4. The right side of this equation coincides precisely with the topological recursion as defined in Section 3.2, so we have finally deduced that $\Omega_{g,n} = \omega_{g,n}$. By induction, we conclude that $\Omega_{g,n} = \omega_{g,n}$ for all $(g, n) \neq (0, 1)$ or $(0, 2)$. □

5.3 Asymmetric combinatorial recursion for stable fatgraphs

In the previous section, we proved Theorem 5.1.2 by using the symmetric combinatorial recursion of Theorem 2.2.12 to derive a recursion at the level of generating functions. We then proceeded to break the symmetry to obtain an equation in Proposition 5.2.7 that is asymmetric in the sense that the variable z_1 does not appear in the same way as the remaining variables $z_2, \ldots, z_n$. In this section, we observe that one could instead derive an asymmetric combinatorial recursion in order to lead directly to Proposition 5.2.7. This asymmetric combinatorial recursion is stated and proved below.

Proposition 5.3.1. *For $2g - 2 + n \geq 2$ and $b_1, b_2, \ldots, b_n \geq 0$, we have the following equation, where $S = \{2, 3, \ldots, n\}$ and $\mathbf{b}_I = (b_{i_1}, b_{i_2}, \ldots, b_{i_k})$ for $I = \{i_1, i_2, \ldots, i_k\}$.*

$$\begin{aligned}
&2b_1 \overline{N}_{g,n}(b_1, b_S) \\
&= \sum_{j=2}^{n} \left[\sum_{\substack{p+q=b_1+b_j \\ q \text{ even}}} [p]q\, \overline{N}_{g,n-1}(p, \mathbf{b}_{S\setminus\{j\}}) + \operatorname{sgn}(b_1 - b_j) \sum_{\substack{p+q=|b_1-b_j| \\ q \text{ even}}} [p]q\, \overline{N}_{g,n-1}(p, \mathbf{b}_{S\setminus\{j\}}) \right] \\
&\quad + \sum_{\substack{p+q+r=b_1 \\ r \text{ even}}} [p][q]r \left[\overline{N}_{g-1,n+1}(p, q, \mathbf{b}_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}}^{\text{stable}} \overline{N}_{g_1,|I|+1}(p, \mathbf{b}_I)\, \overline{N}_{g_2,|J|+1}(q, \mathbf{b}_J) \right]
\end{aligned} \tag{5.11}$$

Proof. We use the notation and ideas of the previous work of Norbury [81]. First, we notice that

$$\overline{S}_m(k) := \sum_{\substack{p+q=k \\ q \text{ even}}} [p]^{2m+1} q \tag{5.12}$$

is an odd polynomial in k of degree $2m + 3$, which follows directly from Lemma 1 of [81].

The recursion of Theorem 2.2.12 can be used to produce the numbers $\overline{N}_{g,n}(b_1, \ldots, b_n)$ from the base cases $\overline{N}_{0,3}(b_1, b_2, b_3)$ and $\overline{N}_{1,1}(b_1)$. Suppose that the recursion of Proposition 5.3.1 produces the numbers $\overline{N}'_{g,n}(b_1, \ldots, b_n)$ from the base cases. If we then show that $\overline{N}'_{g,n}(b_1, \ldots, b_n)$ also satisfy the recursion of Theorem 2.2.12, then we are done.

Using equation (5.11), we calculate $b_i \overline{N}'_{g,n}(b_1, \ldots, b_n)$ for $i = 1, 2, \ldots, n$ and then add them together to obtain the following equation.

$$\Big(\sum_{i=1}^{n} b_i\Big)\overline{N}'_{g,n}(b_1,b_2,\ldots,b_n) = \frac{1}{2}\sum_{i\neq j}\Bigg[\sum_{\substack{p+q=b_1+b_j\\ q \text{ even}}}[p]q\overline{N}'_{g,n-1}(p,b_1,\ldots\hat{b}_i,\ldots,\hat{b}_j,\ldots,b_n)$$
$$+\operatorname{sgn}(b_i-b_j)\sum_{\substack{p+q=|b_1-b_j|\\ q \text{ even}}}[p]q\overline{N}'_{g,n-1}(p,b_1,\ldots,b_i,\ldots,\hat{b}_j,\ldots,b_n)\Bigg]$$
$$+\frac{1}{2}\sum_{\substack{p+q+r=b_1\\ r \text{ even}}}[p][q]r\Bigg[N'_{g-1,n+1}(p,q,b_2,\ldots,b_n)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S}}^{\text{stable}}\overline{N}'_{g_1,|I|+1}(p,b_I)\,\overline{N}_{g_2,|J|+1}(q,b_J)\Bigg]$$

Now the contribution from the second line of this equation satisfies

$$\frac{1}{2}\sum_{i<j}\Bigg(\sum_{\substack{p+q=b_1+b_j\\ q \text{ even}}} + sgn(b_i-b_j)\sum_{\substack{p+q=|b_1-b_j|\\ q \text{ even}}}\Bigg)[p]q\overline{N}'_{g,n-1}(p,b_1,\ldots,\hat{b}_i,\ldots,b_j\ldots,b_n) = 0.$$

This is since $\overline{S}_m(k)$ is an odd polynomial, which implies that $S_m(b_i - b_j) = -S_m(b_j - b_i)$. Therefore, the numbers $\overline{N}'_{g,n}(b_1,\ldots,b_n)$ do indeed satisfy the recursion of Theorem 2.2.12, so we have

$$\overline{N}'_{g,n}(b_1,\ldots,b_n) = \overline{N}_{g,n}(b_1,\ldots,b_n),$$

and this completes the proof. □

We conclude the section by remarking that the generating function form of the recursion of Proposition 5.3.1 is essentially equivalent to the statement of Proposition 5.2.7.

5.4 Applications and remarks

String and dilaton equations

The correlation differentials produced by the topological recursion satisfy *string and dilaton equations* [47].

$$\sum_{\alpha}\operatorname*{Res}_{z=\alpha} y(z)\,\omega_{g,n+1}(z,\mathbf{z}_S) = -\sum_{k=1}^{n}\mathrm{d}z_k\frac{\partial}{\partial z_k}\left(\frac{\omega_{g,n}(\mathbf{z}_S)}{\mathrm{d}x(z_k)}\right) \tag{5.13}$$
$$\sum_{\alpha}\operatorname*{Res}_{z=\alpha} \Phi(z)\,\omega_{g,n+1}(z,\mathbf{z}_S) = (2g-2+n)\,\omega_{g,n}(\mathbf{z}_S) \tag{5.14}$$

Each left side is a summation over the zeroes α of $\mathrm{d}x$, S denotes the set $\{1,2,\ldots,n\}$, and $\Phi(z)$ is any function satisfying $\mathrm{d}\Phi(z) = y(z)\,\mathrm{d}x(z)$. Although these were originally proven in the context of global topological recursion, we show below that they also hold for the spectral curve of Theorem 5.1.2. In that case, we immediately obtain the relations of Corollary 5.1.3, which are known due to the previous work of Do and Norbury [29].

Proof of Corollary 5.1.3. First, we deal with the string equation. Consider the left side of equation (5.13) and use the fact that the sum of the residues at the poles of $y(z)\,\omega_{g,n+1}(z,\mathbf{z}_S)$ is 0. Multiplying $\omega_{g,n+1}(z,\mathbf{z}_S)$ by $y(z) = z$ removes the simple pole and introduces a pole at $z = \infty$.

So using Proposition 5.2.4, we have

$$\begin{aligned}
&\sum_{\alpha=\pm 1} \operatorname*{Res}_{z=\alpha} y(z)\, \omega_{g,n+1}(z, \mathbf{z}_S) \\
&= -\operatorname*{Res}_{z=\infty} z\, \omega_{g,n+1}(z, \mathbf{z}_S) = -\operatorname*{Res}_{z=0} \frac{1}{z} \omega_{g,n+1}(\tfrac{1}{z}, \mathbf{z}_S) = \operatorname*{Res}_{z=0} \frac{1}{z} \omega_{g,n+1}(z, \mathbf{z}_S) \\
&= \sum_{b_1, b_2, \ldots, b_n = 0}^{\infty} \overline{N}_{g,n+1}(1, \mathbf{z}\mathbf{b}_S) \prod_{i=1}^{n} [b_i] z_i^{b_i - 1}\, \mathrm{d}z_i.
\end{aligned}$$

Next, consider the kth summand of the right side of equation (5.13).

$$\begin{aligned}
-\,\mathrm{d}z_k \frac{\partial}{\partial z_k} \left(\frac{\omega_{g,n}(\mathbf{z}_S)}{\mathrm{d}x(z_k)} \right) &= \mathrm{d}z_k \frac{\partial}{\partial z_k} \left(\frac{1}{\mathrm{d}z_k} \frac{z_k^2}{1 - z_k^2} \sum_{b_1, b_2, \ldots, b_n = 0}^{\infty} \overline{N}_{g,n}(\mathbf{z}\mathbf{b}_S) \prod_{i=1}^{n} [b_i] z_i^{b_i - 1}\, \mathrm{d}z_i \right) \\
&= \mathrm{d}z_k \sum_{a=0}^{\infty} \sum_{m=1}^{\infty} \sum_{b_1, \ldots, \widehat{b}_k, \ldots, b_n = 0}^{\infty} \overline{N}_{g,n}(a, \mathbf{z}\mathbf{b}_{S \setminus \{k\}})\, [a] (a + 2m - 1) z_i^{a+2m-2} \prod_{i \in S \setminus \{k\}} [b_i] z_i^{b_i - 1}\, \mathrm{d}z_i
\end{aligned}$$

Hence, extracting the coefficient of $\prod_{i=1}^{n} [b_i]\, z_i^{b_i - 1}\, \mathrm{d}z_i$ from the two sides of equation (5.13) leads to the first relation of Corollary 5.1.3.

$$\overline{N}_{g,n+1}(1, b_1, b_2, \ldots, b_n) = \sum_{k=1}^{n} \sum_{a=0}^{b_k} [a]\, \overline{N}_{g,n}(a, b_1, \ldots, \widehat{b}_k, \ldots, b_n)$$

Next, we deal with the dilaton equation, in which case we take $\Phi(z) = \frac{1}{2} z^2 - \log(z)$. Consider the left side of equation (5.14) and use Lemma 5.2.5 to deal with the logarithmic term that arises.

$$\begin{aligned}
\sum_{\alpha=\pm 1} \operatorname*{Res}_{z=\alpha} \Phi(z)\, \omega_{g,n+1}(z, \mathbf{z}_S) &= \sum_{\alpha=\pm 1} \operatorname*{Res}_{z=\alpha} \frac{1}{2} z^2 \omega_{g,n+1}(z, \mathbf{z}_S) - \sum_{\alpha=\pm 1} \operatorname*{Res}_{z=\alpha} \log(z)\, \omega_{g,n+1}(z, \mathbf{z}_S) \\
&= -\operatorname*{Res}_{z=\infty} \frac{1}{2} z^2 \omega_{g,n+1}(z, \mathbf{z}_S) - \operatorname*{Res}_{z=0} \omega_{g,n+1}(z, \mathbf{z}_S) \\
&= \operatorname*{Res}_{z=0} \frac{1}{2z^2} \omega_{g,n+1}(z, \mathbf{z}_S) - \operatorname*{Res}_{z=0} \omega_{g,n+1}(z, \mathbf{z}_S) \\
&= \sum_{b_1, b_2, \ldots, b_n = 0}^{\infty} \left[\overline{N}_{g,n+1}(2, \mathbf{z}\mathbf{b}_S) - \overline{N}_{g,n+1}(0, \mathbf{z}\mathbf{b}_S) \right] \prod_{i=1}^{n} [b_i] z_i^{b_i - 1}\, \mathrm{d}z_i
\end{aligned}$$

So extracting the coefficient of of $\prod_{i=1}^{n} [b_i] z_i^{b_i - 1}\, \mathrm{d}z_i$ from the two sides of equation (5.14) leads to the second relation of Corollary 5.1.3.

$$\overline{N}_{g,n+1}(2, b_1, b_2, \ldots, b_n) - \overline{N}_{g,n+1}(0, b_1, b_2, \ldots, b_n) = (2g - 2 + n)\, \overline{N}_{g,n}(b_1, b_2, \ldots, b_n) \qquad \square$$

Quantum curves

The notion of topological recursion is closely related to the notion of quantum curve [83]. Briefly speaking, one integrates the correlation differentials and stores them in the following so-called *wave function.*

$$\psi(x, \hbar) = \exp \left[\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g-2+n}}{n!} \int_a^x \int_a^x \cdots \int_a^x \omega_{g,n}(z_1, z_2, \ldots, z_n) \right]$$

Different choices of the base point a for integration may result in different quantum curves, though the base point should in general be a pole of $x(z)$.

The wave function satisfies differential equations of the form

$$\hat{P}(\hat{x}, \hat{y})\, \psi(x, \hbar) = 0,$$

where $\hat{x} = x$, $\hat{y} = -\hbar \frac{\partial}{\partial x}$ and $\hat{P}$ is a non-commutative polynomial. It has been empirically observed and proved in a variety of contexts that there is natural choice of $\hat{P}(\hat{x}, \hat{y})$ whose semi-classical limit $P(x, y) = 0$ recovers the underlying spectral curve for the topological recursion. Of course, this phenomenon most naturally applies to the case of global spectral curves. As an example, it is known that the enumeration of lattice points in $\mathcal{M}_{g,n}$ is governed by the global rational spectral curve $x(z) = z + \frac{1}{z}$ and $y(z) = z$ and that the corresponding quantum curve is given by the following operator $\hat{P}(\hat{x}, \hat{y}) = \hat{y}^2 - \hat{x}\hat{y} + 1$ [35].

It would be interesting to construct a natural wave function for the topological recursion of Theorem 5.1.2 and to find a quantum curve operator that annihilates it. Although the spectral curve is not global in the usual sense, it has the same underlying algebraic curve as for the enumeration of lattice points in $\mathcal{M}_{g,n}$. Thus, one might expect a different quantum curve operator to the one above, which still recovers $y^2 - xy + 1 = 0$ in the semi-classical limit. Examples of this nature may help to shed further light on the still mysterious phenomenon of quantum curves.

Where did the spectral curve come from?

It is natural to ask where the spectral curve of Theorem 5.1.2 came from. In particular, it would be useful to be able to identify other problems that are governed by local topological recursion, perhaps with a modified $\omega_{0,2}$ as in the case here. Typically, one can speculate the form of a global spectral curve attached to an enumerative problem from the case $(g, n) = (0, 1)$, given that $\omega_{0,1}(z_1) = -y(z_1)\, \mathrm{d}x(z_1)$. The enumeration of lattice points in $\overline{\mathcal{M}}_{g,n}$ for $(g, n) = (0, 1)$ and $(0, 2)$ matches the enumeration of lattice points in $\mathcal{M}_{g,n}$, which indicates using the same $x(z)$ and $y(z)$ in the spectral curve data.[2]

The spectral curve of Theorem 5.1.2 arises from a modification to $\omega_{0,2}$ for the enumeration of lattice points in $\mathcal{M}_{g,n}$. One moves to the compactified version of the count by allowing nodes and there is a sense in which nodes correspond to $(0, 2)$ information. For example, the stabilisation of a nodal curve contracts $(0, 2)$ components — that is, components with genus zero and two nodal points — to nodes. Alternatively, consider the graphical interpretation of topological recursion, which expresses each correlation differential $\omega_{g,n}$ as a weighted sum over decorated graphs [43, 47]. For each such graph, the vertices are weighted by intersection numbers on $\overline{\mathcal{M}}_{g,n}$ and the edges by so-called *jumps*, which are essentially the coefficients of $\omega_{0,2}$. These decorated graphs bear a close relation to the graphs arising from the stratification of $\overline{\mathcal{M}}_{g,n}$, so that edges correspond to nodes. So again, we see that $\omega_{0,2}$ controls nodal behaviour and it should come as less of a surprise that the enumeration of lattice points in $\overline{\mathcal{M}}_{g,n}$ requires a modification to $\omega_{0,2}$. That the extra contribution to $\omega_{0,2}$ is of the form $\frac{\mathrm{d}z_1\, \mathrm{d}z_2}{z_1 z_2}$ corresponds to the fact that we should take $\overline{N}_{0,2}(0, 0) = 1$.

It would be interesting to take standard enumerative problems governed by global topological recursion — such as the psi-class intersection numbers on $\mathcal{M}_{g,n}$, simple Hurwitz numbers and

[2] This statement is somewhat subtle, since the natural definitions would lead to $\overline{N}_{0,1}(b) = 0$ for $b > 0$. Instead, consider the enumeration of lattice points in $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ as the enumeration of ordinary and stable fatgraphs, in which all vertices have degree at least two. One can pass to the analogous problems in which this degree condition is relaxed using the pruning correspondence [30]. The resulting problems are stored in the same correlation differentials, but as coefficients in the expansion at $x = \infty$, rather than at $z = 0$. It is the alignment of these problems for $(g, n) = (0, 1)$ and $(0, 2)$ that suggests using the same $x(z)$ and $y(z)$ in the spectral curve data.

the Gromov–Witten theory of $\mathbb{CP}^1$ — and consider the effect of a modification to $\omega_{0,2}$ on the associated correlation differentials.

Data and positivity conjecture

Theorem 2.2.12 asserts that $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ is a symmetric quasi-polynomial that is non-zero only when $b_1 + b_2 + \cdots + b_n$ is even. Hence, $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ can be described by the underlying polynomials $\overline{N}^{(k)}_{g,n}(b_1, b_2, \ldots, b_n)$ that determine it in the case $b_1, b_2, \ldots, b_k$ are odd and $b_{k+1}, b_{k+2}, \ldots, b_n$ are even, where we may restrict to k even. The following table is replicated from the literature [29] and stores this information for some small values of g and n.

The data provides strong evidence towards the following conjecture, which also supports the speculation that the coefficients of $\overline{N}_{g,n}$ store algebro-geometric content.

Conjecture 5.4.1. *The polynomials underlying the quasi-polynomial $\overline{N}_{g,n}$ have positive coefficients.*

g	n	k	$\overline{N}^{(k)}_{g,n}(b_1, b_2, \ldots, b_n)$
0	3	0	1
0	3	2	1
1	1	0	$\frac{1}{48}(b_1^2 + 20)$
0	4	0	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 + 8)$
0	4	2	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 + 2)$
0	4	4	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 + 8)$
1	2	0	$\frac{1}{384}(b_1^4 + b_2^4 + 2b_1^2b_2^2 + 36b_1^2 + 36b_2^2 + 192)$
1	2	2	$\frac{1}{384}(b_1^4 + b_2^4 + 2b_1^2b_2^2 + 36b_1^2 + 36b_2^2 + 84)$
0	5	0	$\frac{1}{32}\sum b_i^4 + \frac{1}{8}\sum b_i^2b_j^2 + \frac{7}{8}\sum b_i^2 + 7$
0	5	2	$\frac{1}{32}\sum b_i^4 + \frac{1}{8}\sum b_i^2b_j^2 + \frac{5}{16}(b_1^2 + b_2^2) + \frac{1}{8}(b_3^2 + b_4^2 + b_5^2) + \frac{19}{16}$
0	5	4	$\frac{1}{32}\sum b_i^4 + \frac{1}{8}\sum b_i^2b_j^2 + \frac{5}{16}(b_1^2 + b_2^2 + b_3^2 + b_4^2) + \frac{7}{8}b_5^2 + \frac{7}{8}$
1	3	0	$\frac{1}{4608}\sum b_i^6 + \frac{1}{768}\sum b_i^4b_j^2 + \frac{1}{384}b_1^2b_2^2b_3^2 + \frac{13}{1152}\sum b_i^4 + \frac{1}{24}\sum b_i^2b_j^2 + \frac{29}{144}\sum b_i^2 + \frac{17}{12}$
1	3	2	$\frac{1}{4608}\sum b_i^6 + \frac{1}{768}\sum b_i^4b_j^2 + \frac{1}{384}b_1^2b_2^2b_3^2 + \frac{43}{4608}\sum b_i^4 + \frac{1}{24}\sum b_i^2b_j^2 + \frac{277}{4608}\sum b_i^2 + \frac{1}{512}b_3^4 + \frac{1}{1536}b_3^2 + \frac{81}{256}$
2	1	0	$\frac{1}{1769472}b_1^8 + \frac{3}{40960}b_1^6 + \frac{133}{61440}b_1^4 + \frac{1087}{34560}b_1^2 + \frac{247}{1440}$
0	6	0	$\frac{1}{384}\sum b_i^6 + \frac{3}{28}\sum b_i^4b_j^2 + \frac{3}{32}\sum b_i^2b_j^2b_k^2 + \frac{1}{6}\sum b_i^4 + \frac{9}{6}\sum b_i^2b_j^2 + \frac{109}{24}b_i^2 + 34$

Chapter 6

Towards the topological recursion for Gromov–Witten invariants of $\mathbb{CP}^1$

Gromov–Witten theory deals with the enumeration of maps from complex algebraic curves into a complex variety. This theory was motivated by theoretical physics, acting as a mathematical interpretation for certain models of string theory. In the case that the target variety is a non-singular curve, Okounkov and Pandharipande relate Gromov–Witten invariants to classical Hurwitz numbers, giving an explicit way to compute them. In this chapter, we introduce a conjecture that states that certain relative Gromov–Witten invariants of $\mathbb{CP}^1$ are governed by the topological recursion. This conjecture can be seen as a vast generalisation of the Bouchard–Mariño conjecture relating simple Hurwitz numbers with topological recursion. After stating the conjecture and its motivation, we discuss the Gromov–Witten/Hurwitz correspondence, which is then used to deduce a quantum curve for the enumerative problem under consideration. This can be considered strong evidence towards the conjecture.

6.1 Conjecture

The main goal of this chapter is to state and give evidence towards a new conjecture — Conjecture 6.1.2 — which asserts that certain relative Gromov–Witten invariants of $\mathbb{CP}^1$ are governed by topological recursion. The main inspiration behind this conjecture is the ongoing series of results concerning simple Hurwitz numbers and their generalisations. The study of Hurwitz numbers dates back to the nineteenth century, yet their remarkably rich structure only became apparent towards the end of the twentieth century. Goulden, Jackson and Vainshtein observed certain polynomiality structure underlying simple Hurwitz numbers and brought attention back to their study [57]. Loosely speaking, simple Hurwitz numbers enumerated branched covers of $\mathbb{CP}^1$ with prescribed ramification over $\infty \in \mathbb{CP}^1$ and simple branching elsewhere. This polynomiality of simple Hurwitz numbers was later proved by Ekedahl, Lando, Shapiro and Vainshtein, who showed that simple Hurwitz numbers are equal to Hodge integrals over the Deligne–Mumford compactification of the moduli space of curves [44]. Their so-called ELSV formula not only makes the polynomial structure of Hurwitz numbers apparent, but also connects them to the realms of enumerative geometry and mathematical physics. More recently, work motivated by topological string theory led Bouchard and Mariño to conjecture that simple Hurwitz numbers are governed by topological recursion [19]. This was subsequently proved by Eynard, Mulase and Safnuk [50] and it has furthermore been demonstrated that the ELSV formula and the Bouchard Mariño conjecture are in some sense equivalent [40].

Simple Hurwitz numbers have been generalised in a variety of ways, and we consider two of them here. One generalisation is to consider branched covers whose ramification over 0 has profile $(a, a, \ldots, a)$ and the resulting counts are known as *orbifold Hurwitz numbers*. These are essentially Gromov–Witten invariants relative to the partition $(a, a, \ldots, a)$ over 0. Orbifold Hurwitz numbers are known to satisfy an ELSV-type formula [65] and topological recursion [18, 34]. Another generalisation is to consider branched covers, where the simple branching is changed to order r branching using completed $(r+1)$-cycles and the resulting counts are known as *spin Hurwitz numbers*. These are essentially Gromov–Witten invariants with τ_r insertions rather than τ_1 insertions. Spin Hurwitz numbers are also known to satisfy an ELSV-type formula and topological recursion, both of which were recently proved as the culmination of a series of papers [41].

Double Hurwitz numbers enumerate branched covers of the Riemann sphere with specified genus, prescribed ramification over both zero and infinity, and simple branching elsewhere. They possess a piecewise polynomial structure and are conjectured to relate to intersection theory on moduli spaces [58]. Do and Karev package double Hurwitz numbers in a particular way, recording the branching over 0 via certain monomial weights [33]. They conjectured that they too satisfied topological recursion, a conjecture that was recently proved by Borot, Do, Karev, Lewański and Moskovsky [14]. Our Conjecture 6.1.2 takes this idea and applies it also to the insertions, allowing them to be of arbitrary order and storing them in a separate system of weights. So we are interested in Gromov–Witten invariants of $\mathbb{CP}^1$ of the form

$$\langle\, \nu \mid \tau_{\lambda_1}(\omega)\tau_{\lambda_2}(\omega)\cdots\tau_{\lambda_m}(\omega) \mid \mu \,\rangle_{g,m},$$

where μ and ν are arbitrary partitions of the same size and $\lambda_1, \lambda_2, \ldots, \lambda_m$ are arbitrary positive integers such that the degree condition $\sum \lambda_i = 2g-2+\ell(\mu)+\ell(\nu)$ is satisfied. We package these Gromov–Witten invariants in the following way.

Definition 6.1.1. For $q_1, q_2, \ldots,$ $w_1, w_2, \ldots$ complex parameters and $\mu_1, \mu_2, \ldots, \mu_n$ positive integers with sum d, we define

$$GW_{g,n}(\mu_1,\ldots,\mu_n) = |\mathrm{Aut}(\mu)| \sum_{\substack{\nu \vdash d \\ \lambda \vdash 2g-2+n+\ell(\nu)}} \langle\, \nu \mid \tau_{\lambda_1}(\omega)\tau_{\lambda_2}(\omega)\cdots\tau_{\lambda_m}(\omega) \mid \mu \,\rangle_{g,m} \frac{q_\nu\, w_\lambda}{|\mathrm{Aut}(\lambda)|},$$

$$GW^{\bullet}_{g,n}(\mu_1,\ldots,\mu_n) = |\mathrm{Aut}(\mu)| \sum_{\substack{\nu \vdash d \\ \lambda \vdash 2g-2+n+\ell(\nu)}} \langle\, \nu \mid \tau_{\lambda_1}(\omega)\tau_{\lambda_2}(\omega)\cdots\tau_{\lambda_m}(\omega) \mid \mu \,\rangle^{\bullet}_{g,m} \frac{q_\nu\, w_\lambda}{|\mathrm{Aut}(\lambda)|}.$$

Here, $q_\nu = q_{\nu_1}q_{\nu_2}\cdots q_{\nu_{\ell(\nu)}}$ and $w_\lambda = w_{\lambda_1}w_{\lambda_2}\cdots w_{\lambda_{\ell(\lambda)}}$.

Observe that $GW_{g,n}(\mu_1,\ldots,\mu_n)$ and $GW^{\bullet}_{g,n}(\mu_1,\ldots,\mu_n)$ are both polynomials in $q_1, q_2, \ldots$ and $w_1, w_2, \ldots$. In the following, we assume that $q_{d+1} = q_{d+2} = \cdots = 0$ and $w_{k+1} = w_{k+2} = \cdots = 0$ in order to guarantee that they both in fact belong to $\mathbb{Q}[q_1, q_2, \ldots q_d; w_1, w_2, \ldots, w_k]$.

Conjecture 6.1.2. *Let $Q(z) = q_1 z + q_2 z^2 + \cdots + q_d z^d$ and $W(z) = w_1 z + w_2 z^2 + \cdots + w_k z^k$ be complex polynomials. The correlation differentials obtained by topological recursion applied to the spectral curve*

$$\left(\mathbb{CP}^1,\ x(z) = -W(Q(z)) + \log(z),\ y(z) = Q(z),\ \omega_{0,2} = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1-z_2)^2}\right)$$

satisfy, for $(g,n) \neq (0,2)$,

$$\omega_{g,n}(z_1,\ldots,z_n) = \mathrm{d}_1\cdots\mathrm{d}_n \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} GW_{g,n}(\mu_1,\ldots,\mu_n)\, e^{\mu_1 x(z_1)}\cdots e^{\mu_n x(z_n)}.$$

To obtain an unparametrised description of the spectral curve, observe that

$$e^x = ze^{-W(y)} \quad \Rightarrow \quad z = e^{x+W(y)} \quad \Rightarrow \quad y = Q(e^{x+W(y)}). \tag{6.1}$$

For basic evidence that supports Conjecture 6.1.2, we show the existence of a quantum curve in Theorem 6.3.2. One can also calculate rather explicitly with the topological recursion in low genus and verify that the series expansions of the correlation differentials store relative Gromov–Witten invariants of $\mathbb{CP}^1$. For example, using

$$\omega_{0,1}(z) = y(z)\,\mathrm{d}x(z) = \mathrm{d}\sum_{\mu=1}^{\infty} GW_{0,1}(\mu)\,e^{\mu x(z)} = \sum_{\mu=1}^{\infty} \mu\, GW_{0,1}(\mu)\,e^{\mu x(z)}\,\mathrm{d}x(z)$$

one can then extract coefficients by taking appropriate residues, which leads to the following data. The coefficients appearing here are all relative Gromov–Witten invariants of $\mathbb{CP}^1$ and can be calculated using the Gromov–Witten/Hurwitz correspondence. The numbers match, which provides evidence towards Conjecture 6.1.2.

$$\begin{aligned}
GW_{0,1}(1) &= q_1\\
GW_{0,1}(2) &= \tfrac{1}{2}q_2 + \tfrac{1}{2}w_1q_1^2\\
GW_{0,1}(3) &= \tfrac{1}{3}q_3 + w_1q_2q_1 + (\tfrac{1}{2}w_1^2 + \tfrac{1}{3}w_2)q_1^3\\
GW_{0,1}(4) &= \tfrac{1}{4}q_4 + w_1q_3q_1 + \tfrac{1}{2}w_1q_2^2 + (w_2 + 2w_1^2)q_2q_1^2 + (\tfrac{1}{4}w_3 + w_2w_1 + \tfrac{2}{3}w_1^3){q_1}^4\\
GW_{0,1}(5) &= \tfrac{1}{5}q_5 + w_1q_4q_1 + w_1q_3q_2 + (w_2 + \tfrac{5}{2}w_1^2)q_3q_1^2 + (w_2 + \tfrac{5}{2}w_1^2)q_2^2q_1\\
&\quad + (w_3 + 5w_2w_1 + \tfrac{25}{6}w_1^3)q_2q_1^3 + (\tfrac{1}{5}w_4 + w_3w_1 + \tfrac{1}{2}w_2^2 + \tfrac{5}{2}w_2w_1^2 + \tfrac{5}{4}w_1^4)q_1^5\\
GW_{0,1}(6) &= \tfrac{1}{6}q_6 + w_1q_5q_1 + w_1q_4q_2 + (w_2 + 3w_1^2)q_4q_1^2 + \tfrac{1}{2}w_1q_3^2 + (2w_2 + 6w_1^2)q_3q_2q_1\\
&\quad + (w_3 + 6w_2w_1 + 6w_1^3)q_3q_1^3 + (\tfrac{1}{3}w_2 + w_1^2)q_2^3 + (\tfrac{3}{2}w_3 + 9w_2w_1 + 9w_1^3)q_2^2q_1^2\\
&\quad + (w_4 + 6w_3w_1 + 3w_2^2 + 18w_2w_1^2 + 9w_1^4)q_2q_1^4\\
&\quad + (\tfrac{1}{6}w_5 + w_4w_1 + w_3w_2 + 3w_3w_1^2 + 3w_2^2w_1 + 6w_2w_1^3 + \tfrac{9}{5}w_1^5)q_1^6
\end{aligned}$$

The following table gives a summary of the various types of Hurwitz numbers along with the corresponding spectral curves. The shaded entries refer to known results in which the spectral curve does indeed govern the enumeration, while the unshaded entries are conjectural. Each entry in the table is a generalisation of those entries above or to the left of it. The original Bouchard–Mariño conjecture lies in the top-left entry while Conjecture 6.1.2 lies in the bottom-right entry.

It is worth mentioning the conjecture of Norbury and Scott, which states that stationary Gromov–Witten invariants of $\mathbb{CP}^1$ are governed by topological recursion [86]. This was later proved by Dunin-Barkowski, Orantin, Shadrin and Spitz in their work relating topological recursion to cohomological field theories [43]. The result states that the correlation differentials obtained by topological recursion applied to the spectral curve

$$\left(\mathbb{CP}^1,\ x(z) = z + \frac{1}{z},\ y(z) = \log(z),\ \omega_{0,2} = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1 - z_2)^2}\right)$$

satisfy, for $(g, n) \neq (0, 2)$,

$$\omega_{g,n}(z_1, \ldots, z_n) = \mathrm{d}_1 \cdots \mathrm{d}_n \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \langle \tau_{\mu_1-1}(\omega) \cdots \tau_{\mu_n-1}(\omega)\rangle_{g,n} \frac{(\mu_1 - 1)!}{x(z_1)^{\mu_1}} \cdots \frac{(\mu_n - 1)!}{x(z_n)^{\mu_n}}.$$

The Gromov–Witten invariants appearing here are simply the stationary Gromov–Witten invariants of $\mathbb{CP}^1$, all of which arise in Conjecture 6.1.2. However, our conjecture does not appear to be a direct generalisation of the Norbury–Scott conjecture.

	unramified over 0	$(a, a, \ldots, a)$ over 0	any ramification over 0
simple	simple $x(z) = z\exp(-z)$ $y(z) = z$	orbifold $x(z) = z\exp(-z^a)$ $y(z) = z^a$	double $x(z) = z\exp(-Q(z))$ $y(z) = Q(z)$
r-spin	spin $x(z) = z\exp(-z^r)$ $y(z) = z$	spin orbifold $x(z) = z\exp(-z^{ar})$ $y(z) = z^a$	spin double $x(z) = z\exp(-Q(z)^r)$ $y(z) = Q(z)$
insertions	every spin $x(z) = z\exp(-W(z))$ $y(z) = z$	every spin orbifold $x(z) = z\exp(-W(z^a))$ $y(z) = z^a$	every spin double $x(z) = z\exp(-W(Q(z)))$ $y(z) = Q(z)$

6.2 Gromov–Witten/Hurwitz correspondence

Hurwitz theory

The work in this chapter relies crucially on the Gromov–Witten/Hurwitz correspondence of Okounkov and Pandharipande [87]. We begin by discussing general Hurwitz theory, which counts branched covers over a curve X with specified ramification. Let d be a positive integer and let $\eta_1, \ldots, \eta_m$ be partitions of d, which we assign to m fixed points $q_1, \ldots, q_m \in X$. A genus g *Hurwitz cover* of X of degree d, with monodromy η_i at q_i, is a morphism $f : C \to X$ satisfying

- C is a smooth genus g curve;
- f has ramification profile η_i over q_i; and
- f is unramified over $X \setminus \{q_1, \ldots, q_m\}$.

Hurwitz covers exist for connected or disconnected domains, as discussed in Section 1.1. Two covers $f : C \to X$ and $f' : C' \to X$ are equivalent if there exists an isomorphism of curves $\phi : C \to C'$ satisfying $f' \circ \phi = f$. Up to equivalence, there are only finitely many genus g Hurwitz covers of X of degree d, with monodromy η_i at q_i. Each such cover f has a finite group of automorphism, which we denote by $\mathrm{Aut}(f)$. The Hurwitz number

$$H_d^X(\eta_1, \ldots, \eta_m)$$

is defined to be the weighted count of possibly disconnected Hurwitz covers of degree d, with monodromy η_i at q_i. The weight of such a cover is defined to by $\frac{1}{|\mathrm{Aut}(f)|}$. Observe that the genus of such a map is recovered by the Riemann–Hurwitz formula from the other data.

The above definition of Hurwitz number $H_d^X(\eta_1, \ldots, \eta_m)$ can be seen as a function on tuples of partitions of d. We extend the definition of $H_d^X(\eta_1, \ldots, \eta_m)$ to arbitrary tuples of partitions with the following rules.

- We set $H_d^X(\emptyset, \ldots, \emptyset) = 1$, where $\emptyset$ denotes the empty partition.
- If $|\eta_i| > d$ for some i, then the Hurwitz number vanishes.
- If $|\eta_i| \leq d$ for all i, then we set

$$H_d^X(\eta_1, \ldots, \eta_m) = \prod_{i=1}^{m} \binom{m_1(\boldsymbol{\eta}_i)}{m_1(\eta_i)} H_d^X(\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_m), \tag{6.2}$$

where $m_i(\eta)$ denotes the multiplicity of the part i and $\boldsymbol{\eta}$ is the partition of size d obtained from η by adding $d - |\eta|$ parts of size 1.

In Section 1.2, we discussed the simple Hurwitz numbers over $\mathbb{CP}^1$ and its relation to monodromy representations and the character theory of symmetric groups. In particular, we defined the function $f_2(\lambda)$ in Proposition 1.2.3 and we can extend this idea to handle the more general Hurwitz numbers defined above. First, we state the monodromy representation interpretation of Hurwitz numbers over $\mathbb{CP}^1$.

Proposition 6.2.1. *The number $H_d^{\mathbb{CP}^1}(\eta_1, \eta_2 \ldots, \eta_m)$ is $\frac{1}{d!}$ multiplied by the number of tuples $(\rho_1, \rho_2, \ldots, \rho_m)$ of permutations in S_d such that*

- *ρ_i has cycle type $\boldsymbol{\eta}_i$; and*
- *the product $\rho_1 \circ \rho_2 \circ \cdots \circ \rho_m$ is the identity.*

For λ a partition of d and η an arbitrary partition, let us define

$$\mathbf{f}_\eta(\lambda) = \binom{|\lambda|}{|\eta|} |C_\eta| \frac{\chi^\lambda_{\boldsymbol{\eta}}}{\dim(\lambda)}. \tag{6.3}$$

Note that for $|\eta| > |\lambda|$, the binomial coefficient in equation (6.3) vanishes. In the case $\eta = \emptyset$, we interpret the formula as $\mathbf{f}_\emptyset(\lambda) = 1$.

As in Section 1.2, we have a character formula as follows.

$$H_d^{\mathbb{CP}^1}(\eta_1, \ldots, \eta_m) = \sum_{|\lambda|=d} \frac{\dim(\lambda)^2}{d!} \prod_{i=1}^m \mathbf{f}_{\eta_i}(\lambda) \tag{6.4}$$

In fact, this can be extended to a target curve X of any genus, although we won't be concerned about such targets. The interested reader can see the discussion in the original paper of Okounkov and Pandharipande [87].

Shifted symmetric functions and completed cycles

The Gromov–Witten/Hurwitz correspondence passes through the algebra of shifted symmetric functions in order to define the notion of completed cycles. We begin by recalling the definition of Λ, the algebra of symmetric functions. Let $\Lambda(n)$ denote the algebra of symmetric polynomials in $x_1, x_2, \ldots, x_n$. The specialisation $x_{n+1} = 0$ is a morphism of graded algebras

$$\Lambda(n+1) \to \Lambda(n), \tag{6.5}$$

and we define Λ to be the projective limit as follows.

Definition 6.2.2. Let

$$\Lambda := \varprojlim_{n \to \infty} \Lambda(n),$$

taken in the category of graded algebras with respect to the morphisms of equation (6.5). An element $f \in \Lambda$ is by definition a sequence $f_1, f_2, f_3, \ldots$ such that

- $f_n \in \Lambda(n)$ for $n = 1, 2, 3, \ldots$,
- $f_{n+1}(x_1, \ldots, x_n, 0) = f_n(x_1, \ldots, x_n)$, and
- $\sup_n \deg f_n < \infty$.

Now let us denote by $\Lambda^*(n)$ the algebra of polynomials in $x_1, x_2, \ldots, x_n$ that become symmetric in the new variables

$$x_i' = x_i - i + c, \qquad \text{for } i = 1, 2, \ldots, n.$$

Here, c is an arbitrary fixed number and the definition does not depend on its choice. We call such polynomials *shifted symmetric*. The algebra $\Lambda^*(n)$ is filtered by polynomial degree and the specialisation $x_{n+1} = 0$ is a morphism of filtered algebras

$$\Lambda^*(n+1) \to \Lambda^*(n). \tag{6.6}$$

Definition 6.2.3. Let

$$\Lambda^* := \varprojlim_{n\to\infty} \Lambda^*(n),$$

taken in the category of filtered algebras with respect to the morphisms of equation (6.3). We call Λ^* the *algebra of shifted symmetric functions*.

The algebra Λ^* is filtered by degree and the associated graded algebra $\operatorname{gr}\Lambda^*$ is canonically isomorphic to the Λ, the usual algebra of symmetric functions.

Define

$$\mathbf{p}_k(\lambda) = \sum_{i=1}^{\infty} \left[\left(\lambda_i - i + \tfrac{1}{2}\right)^k - \left(-i + \tfrac{1}{2}\right)^k \right] + (1 - 2^{-k})\zeta(-k), \tag{6.7}$$

where $(1 - 2^{-k})\zeta(-k)$ is a regularisation term. Then $\mathbf{p}_k$ is shifted symmetric and we refer to it as the *power-sum shifted symmetric function*. The canonical isomorphism between Λ^* and Λ sends $\mathbf{p}_k$ to the power-sum symmetric function p_k. As the power-sum symmetric functions are free commutative generators of Λ, we conclude that

$$\Lambda^* = \mathbb{Q}[\mathbf{p_1}, \mathbf{p_2}, \ldots].$$

Due to a deep result of Kerov and Olshanki [68], the functions $\mathbf{f}_\mu$ define earlier are shifted symmetric and form a vector space basis for the algebra of shifted symmetric functions. We define a map from

$$\phi : \bigoplus_{d=0}^{\infty} Z(\mathbb{C}[S_d]) \to \Lambda^*$$

By a theorem of Kerov and Vershik [101], the highest degree term of $\mathbf{f}_\mu$ can be identified with $\frac{\mathbf{p}_\mu}{\prod \mu_i}$, where $\mathbf{p}_\mu := \prod \mathbf{p}_{\mu_i}$. Following [87], we can define the completed conjugacy classes by

$$\overline{C}_\mu = \frac{1}{\prod \mu_i} \phi^{-1}(\mathbf{p}_\mu) \in \bigoplus_{d=0}^{|\mu|} Z(\mathbb{C}[S_d]).$$

As the basis $\mathbf{p}_\mu$ is multiplicative, a special role is played by the classes

$$\overline{(k)} := \overline{C}_{(k)}, \qquad k = 1, 2, \ldots,$$

which are called *completed cycles* in [87].

Consider the series expansions

$$\mathcal{S}(z) = \frac{\sinh(z/2)}{z/2} = \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}(2k+1)!} \qquad \text{and} \qquad \frac{1}{\mathcal{S}(z)} = \sum_{k=0}^{\infty} c_k z^k,$$

the latter of which defines the coefficients $c_0, c_1, c_2, \ldots$. The generating function for $\mathbf{p}_k$ is given by

$$\exp(\lambda, z) = \sum_{i=0}^{\infty} e^{z(\lambda_i - i + \frac{1}{2})} = \sum_{k=0}^{\infty} \mathbf{p}_k(\lambda)\, z^k.$$

that $\mathbf{p}_k(\lambda)$ appear as the coefficient of z^k in the series expansion of the meromorphic function $\exp(\lambda, z)$ at $z = 0$.

The completed cycle can be expressed in terms of conjugacy classes via

$$\overline{(k)} = \sum_{\mu} \rho_{k,\mu} C_\mu,$$

where $\rho_{k,\emptyset} = (k-1)! c_{k+1}$ and for $\mu \neq \emptyset$,

$$\rho_{k,\mu} = (k-1)! \frac{\prod \mu_i}{|\mu|!} [z^{k+1-|\mu|-\ell(\mu)}] \mathcal{S}(z)^{|\mu|-1} \prod \mathcal{S}(\mu_i z).$$

Gromov–Witten/Hurwitz correspondence

In the work of Eskin and Okounkov on asymptotics of the enumeration of certain branched covers of the torus [45], it was understood that shifted symmetric functions play a role in accounting for degenerations of Hurwitz covers. In particular, this connection uses the interplay between two natural bases for Λ^*, the space of shifted symmetric functions Λ^*, given by $\{\mathbf{f}_\mu\}$ and $\{\mathbf{p}_\mu\}$. equation (6.4) gives a formula for classical Hurwitz numbers over $\mathbb{CP}^1$ in terms of the shifted symmetric functions $\mathbf{f}_\mu$. Replacing them with $\mathbf{p}_\mu$ then produces a formula for Gromov–Witten invariants of $\mathbb{CP}^1$ that we state explicitly below. The Gromov–Witten/Hurwitz correspondence explicitly describes this relationship and furthermore, explains the geometric meaning of the completed cycles as as contributions from the boundary of the moduli space of stable maps.

Proposition 1.1 of the paper of Okounkov and Pandharipande [87] explains that

$$\left(\prod_{i=1}^{n} k_i!\, \psi_i^{k_i} \mathrm{ev}_i^*(\omega) \right) \cap [\mathcal{M}_{g,n}(X,d)]$$

is represented by the locus of covers enumerated by $H_d^X((k_1+1), \ldots, (k_n+1))$. This result shows a connection between descendent classes on open moduli space of stable maps and the enumeration of classical Hurwitz covers. One then expects a geometric formuala

$$\langle \tau_{k_1}(\omega), \cdots \tau_{k_n}\omega) \rangle_d^X = \frac{H_d^X((k_1+1), \ldots, (k_n+1))}{\prod k_i!} + \Delta,$$

where Δ denotes the correction terms coming from the boundary

$$\overline{\mathcal{M}}_{g,n}(X,d) \setminus \mathcal{M}_{g,n}(X,d).$$

To understand the Gromov–Witten invariants arising in this manner, it is necessary to consider the richer context of relative Gromov–Witten theory. We thus consider the moduli space of stable maps, relative to $\eta_1, \ldots, \eta_m$, which are partitions of some positive integer d. It is denoted by

$$\overline{\mathcal{M}}_{g,n}(X, \eta_1, \ldots, \eta_m)$$

and parametrises stable maps $f : C \to X$ from a genus g curve with n marked points, with ramification profile η_i over the fixed point $q_i \in X$ for $i = 1, 2, \ldots, m$. Note that we drop the class

d from the notation, since we can recover it from the partitions $\eta_1, \ldots, \eta_m$. The Gromov–Witten invariants that we are interested in are the integrals of descendents of $\omega \in H^2(X;\mathbb{Q})$, the Poincaré dual of the point class, relative to points $q_1, \ldots, q_m \in X$.

$$\Big\langle \prod_{i=1}^{n} \tau_{b_i}(\omega) \mid \eta_1, \ldots, \eta_m \Big\rangle_{g,n}^{X} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\eta_1,\ldots,\eta_m)]^{\mathrm{vir}}} \prod_{i=1}^{n} \psi_i^{k_i} \mathrm{ev}_i^*(\omega), \tag{6.8}$$

The Gromov–Witten/Hurwitz correspondence may then be stated as follows.

Theorem 6.2.4 (Gromov–Witten/Hurwitz correspondence [87]). *The disconnected stationary Gromov–Witten invariants of a non-singular target curve X relative to m points are given by the following formula.*

$$\Big\langle \prod_{i=1}^{n} \tau_{b_i}(\omega) \mid \eta_1, \ldots, \eta_m \Big\rangle_{g,n}^{\bullet X} = \frac{1}{\prod k_i!} H_d^X\left(\overline{(k_1+1)}, \ldots, \overline{(k_n+1)}, \eta_1, \ldots, \eta_m\right)$$

One is required to interpret the right side by linearity, after expressing the completed cycles as linear combinations of conjugacy classes.

We will be particularly interested in the Gromov–Witten theory of $\mathbb{CP}^1$ relative to two points. Let μ, ν be partitions of d, prescribing the ramification profiles over $0, \infty \in \mathbb{CP}^1$. In the following, we will use the notations

$$\langle \mu \mid \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega) \mid \nu \rangle_{g,n} \qquad \text{and} \qquad \langle \mu \mid \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega) \mid \nu \rangle_{g,n}^{\bullet} \tag{6.9}$$

for the connected and disconnected relative Gromov–Witten invariants of $\mathbb{CP}^1$, respectively.

6.3 A quantum curve for the relative Gromov–Witten invariants of $\mathbb{CP}^1$

Infinite wedge space and character formula for the partition function

Before deriving the quantum curve for the enumerative problem $GW_{g,n}(\mu_1, \ldots, \mu_n)$, we write the associated partition function in a form that is easier to work with, based on the Gromov–Witten/Hurwitz correspondence and the character theory of symmetric groups. The primary tool for this is the infinite wedge space.

As usual, we define the partition function for the enumerative problem $GW_{g,n}(\mu_1, \ldots, \mu_n)$ defined earlier in the following way.

$$\begin{aligned} Z(p_1, p_2, \ldots; \hbar) &= \exp\left[\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} GW_{g,n}(\mu_1, \ldots, \mu_n) \frac{\hbar^{2g-2+n}}{n!} p_{\mu_1} \cdots p_{\mu_n}\right] \\ &= 1 + \sum_{g=-\infty}^{\infty} \sum_{n=1}^{\infty} \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} GW_{g,n}^{\bullet}(\mu_1, \ldots, \mu_n) \frac{\hbar^{2g-2+n}}{n!} p_{\mu_1} \cdots p_{\mu_n} \end{aligned}$$

The partition function can be expressed naturally in terms of a vacuum expectation on the infinite wedge space. Some of the notation for the infinite wedge space was introduced in Section 1.5 and we refer the reader to the literature for further details [87, 96]. The following result requires the operator on the infinite wedge space defined by

$$\mathcal{F}_{r+1} = \sum_{k \in \mathbb{Z}+\frac{1}{2}} \frac{k^{r+1}}{(r+1)!} : \psi_k \psi_k^* : .$$

Proposition 6.3.1.

$$Z(p_1,p_2,\ldots;\hbar)=\left\langle \exp\left(\sum_{m=1}^{\infty}\frac{q_m\alpha_m}{m\hbar}\right)\exp\left(\sum_{r=1}^{\infty}w_r\hbar^r\mathcal{F}_{r+1}\right)\exp\left(\sum_{m=1}^{\infty}\frac{p_m\alpha_{-m}}{m}\right)\right\rangle$$

Proof. For integers g and n and a partition μ, the coefficient of $\mathbf{p}_\mu\hbar^{2g-2+n}$ of the partition function is

$$[\mathbf{p}_\mu\hbar^{2g-2+n}]Z(p_1,p_2,\ldots;\hbar)=\frac{GW^{\bullet}_{g,n}(\mu_1,\mu_2,\ldots,\mu_n)}{|\mathrm{Aut}(\mu)|}.$$

We now proceed to calculate the coefficient of $\mathbf{p}_\mu\hbar^{2g-2+n}$ in the vacuum expectation above and verify that it is indeed equal to this expression. The rightmost operator in the vacuum expectation can be rewritten as follows, where μ represents the partition $(1^{k_1},2^{k_2},3^{k_3},\ldots)$ of some non-negative integer d.

$$\begin{aligned}\exp\left(\sum_{m=1}^{\infty}\frac{p_m\alpha_{-m}}{m}\right)&=\prod_{m=1}^{\infty}\exp\left(\frac{p_m\alpha_{-m}}{m}\right)=\prod_{m=1}^{\infty}\left(\sum_{k=0}^{\infty}\frac{p_m^k\alpha_{-m}^k}{m^k\,k!}\right)\\&=\sum_{k_1,k_2,\ldots}\frac{p_1^{k_1}p_2^{k_2}p_3^{k_3}\cdots\alpha_{-1}^{k_1}\alpha_{-2}^{k_2}\alpha_{-3}^{k_3}\cdots}{1^{k_1}2^{k_2}3^{k_3}\cdots k_1!k_2!k_3!\cdots}\\&=\sum_{\mu}\frac{\mathbf{p}_\mu\prod\alpha_{-\mu_i}}{|\mathrm{Aut}(\mu)|\prod\mu_i}\end{aligned}$$

Using the same process, we can rewrite the other two operators in the vacuum expectation as sums over partitions as well.

$$\begin{aligned}\exp\left(\sum_{r=1}^{\infty}w_r\hbar^r\mathcal{F}_{r+1}\right)&=\sum_{\lambda}\frac{\mathbf{w}_\lambda\hbar^{|\lambda|}\prod\mathcal{F}_{\lambda_i+1}}{|\mathrm{Aut}(\lambda)|}\\\exp\left(\sum_{m=1}^{\infty}\frac{q_m\alpha_m}{m\hbar}\right)&=\sum_{\nu}\frac{\mathbf{q}_\nu\prod\alpha_{\nu_i}}{|\mathrm{Aut}(\nu)|\,\hbar^{\ell(\nu)}\prod\nu_i}\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\left\langle \exp\left(\sum_{m=1}^{\infty}\frac{q_m\alpha_m}{m\hbar}\right)\exp\left(\sum_{r=1}^{\infty}w_r\hbar^r\mathcal{F}_{r+1}\right)\exp\left(\sum_{m=1}^{\infty}\frac{p_m\alpha_{-m}}{m}\right)\right\rangle\\&=\left\langle\sum_{\nu}\frac{\mathbf{q}_\nu\prod\alpha_{\nu_i}}{|\mathrm{Aut}(\nu)|\,\hbar^{\ell(\nu)}\prod\nu_i}\sum_{\lambda}\frac{\mathbf{w}_\lambda\hbar^{|\lambda|}\prod\mathcal{F}_{\lambda_i+1}}{|\mathrm{Aut}(\lambda)|}\sum_{\mu}\frac{\mathbf{p}_\mu\prod\alpha_{-\mu_i}}{|\mathrm{Aut}(\mu)|\prod\mu_i}\right\rangle\\&=\sum_{\nu,\lambda,\mu}\frac{\mathbf{q}_\nu\mathbf{w}_\lambda\mathbf{p}_\mu\hbar^{|\lambda|-\ell(|\nu)}}{|\mathrm{Aut}(\nu)|\,|\mathrm{Aut}(\lambda)|\,|\mathrm{Aut}(\mu)|\prod\nu_i\prod\mu_i}\left\langle\prod\alpha_{\nu_i}\prod\mathcal{F}_{\lambda_i+1}\prod\alpha_{-\mu_i}\right\rangle.\end{aligned}$$

Consider now the vacuum expectation in this last expression and apply first the rightmost operator to the vacuum vector and the leftmost operator to the covacuum vector. By the Murnaghan–Nakayama rule [92], we have

$$\left\langle\prod\alpha_{\nu_i}\prod\mathcal{F}_{\lambda_i+1}\prod\alpha_{-\mu_i}v_\emptyset\right\rangle=\sum_{|\sigma|=|\rho|=|\mu|}\chi^\sigma_\nu\chi^\rho_\mu\left\langle\sigma\,\middle|\,\prod\mathcal{F}_{\lambda_i+1}\,\middle|\,\rho\right\rangle.$$

Next, we use the fact that

$$\mathcal{F}_{r+1}v_\lambda=\frac{\mathbf{p}_{r+1}(\lambda)}{(r+1)!}v_\lambda,$$

where $\mathbf{p}_{r+1}$ denotes the power-sum shifted symmetric function. In particular, $\mathcal{F}_{r+1}$ acts diagonally and the vacuum expectation reduces to

$$\begin{aligned}&\left\langle \prod \alpha_{\nu_i} \prod \mathcal{F}_{\lambda_i+1} \prod \alpha_{-\mu_i} v_\emptyset \right\rangle \\ &= \sum_{|\rho|=|\mu|} \chi^\rho_\nu \chi^\rho_\mu \prod \frac{\mathbf{p}_{\lambda_i+1}(\rho)}{(\lambda_i+1)!} = \frac{d!^2}{|C_\mu|\,|C_\nu|} \langle \nu \mid \tau_{\lambda_1}(\omega) \cdots \tau_{\lambda_{\ell(\lambda)}}(\omega) \mid \mu \rangle^\bullet .\end{aligned}$$

The second equality here is precisely the Gromov–Witten/Hurwitz correspondence.

So extracting the coefficient of $\mathbf{p}_\mu \hbar^{2g-2+n}$ in the entire vacuum expectation gives us the following.

$$\begin{aligned}&[\mathbf{p}_\mu \hbar^{2g-2+n}] \sum_{\nu,\lambda,\mu} \frac{\mathbf{q}_\nu \mathbf{w}_\lambda \mathbf{p}_\mu \hbar^{|\lambda|-\ell(|\nu)}}{|\mathrm{Aut}(\nu)|\,|\mathrm{Aut}(\lambda)|\,|\mathrm{Aut}(\mu)| \prod \nu_i \prod \mu_i} \frac{d!^2}{|C_\mu|\,|C_\nu|} \langle \nu \mid \tau_{\lambda_1}(\omega) \cdots \tau_{\lambda_{\ell(\lambda)}}(\omega) \mid \mu \rangle^\bullet \\ &= \frac{1}{|\mathrm{Aut}(\mu)| \prod \mu_i} \frac{d!^2}{|C_\mu|\,|C_\nu|} \sum_{\substack{\nu \vdash d \\ \lambda \vdash 2g-2+n+\ell(\nu)}} \frac{\mathbf{q}_\nu \mathbf{w}_\lambda}{|\mathrm{Aut}(\nu)|\,|\mathrm{Aut}(\lambda)| \prod \nu_i} \langle \nu \mid \tau_{\lambda_1}(\omega) \cdots \tau_{\lambda_{\ell(\lambda)}}(\omega) \mid \mu \rangle^\bullet \\ &= \sum_{\substack{\nu \vdash d \\ \lambda \vdash 2g-2+n+\ell(\nu)}} \frac{\mathbf{q}_\nu \mathbf{w}_\lambda}{|\mathrm{Aut}(\lambda)|} \langle \nu \mid \tau_{\lambda_1}(\omega) \cdots \tau_{\lambda_{\ell(\lambda)}}(\omega) \mid \mu \rangle^\bullet \\ &= \frac{GW^\bullet_{g,n}(\mu_1, \ldots, \mu_n)}{|\mathrm{Aut}(\mu)|}\end{aligned}$$

This concludes the proof. □

Quantum curve

We are now in a position to derive the quantum curve underlying the Gromov–Witten invariants $GW_{g,n}(\mu_1, \ldots, \mu_n)$. As we did earlier, assume that we only have non-zero parmaters $q_1, q_2, \ldots, q_d$ and $w_1, w_2, \ldots, w_k$.

Theorem 6.3.2. *Let $\hat{x} = x$ and $\hat{y} = \hbar x \frac{\partial}{\partial x}$. The wave function $\psi(x, \hbar) = Z(p_1, p_2, \ldots ; \hbar)|_{p_i = x^i}$ satisfies the quantum curve equation*

$$\left[\hat{y} - (q_1 \hat{x} \mathcal{A}_1 + q_2 \hat{x}^2 \mathcal{A}_2 + \cdots + q_d \hat{x}^d \mathcal{A}_d) \right] \psi(x, \hbar) = 0,$$

where

$$\mathcal{A}_m := x^{\frac{1}{2}} \exp\left(m \sum_{r=1}^{k} \frac{w_r}{r+1} \sum_{i=0}^{r} \hat{x}^{-m} \hat{y}^i \hat{x}^m \hat{y}^{r-i} \right) \hat{x}^{-\frac{1}{2}}.$$

Proof. We first take the infinite wedge expression for the partition function of Proposition 6.3.1 and express it as a double Schur function expansion. This uses the fact that the outer operators are vertex operators while the middle operator is diagonal. In particular, the outer operators are examples of vertex operators and produce Schur functions on the vacuum and covacuum [88]. Thus, we obtain

$$\begin{aligned} Z(p_1, p_2, \ldots ; \hbar) &= \left\langle \exp\left(\sum_{m=1}^{d} \frac{q_m \alpha_m}{m\hbar} \right) \exp\left(\sum_{r=1}^{k} w_r \hbar^r \mathcal{F}_{r+1} \right) \exp\left(\sum_{m=1}^{\infty} \frac{p_m \alpha_{-m}}{m} \right) \right\rangle \\ &= \sum_\lambda s_\lambda(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots) f_\lambda(w_1, w_2, \ldots ; \hbar) s_\lambda(p_1, p_2, \ldots). \end{aligned}$$

Here, s_λ denotes the Schur function expressed in terms of the power-sum symmetric functions and $f_\lambda(w_1, w_2, \ldots; \hbar)$ denotes the eigenvalue of the operator

$$\exp\left(\sum_{r=1}^{k} w_r \hbar^r \mathcal{F}_{r+1}\right)$$

acting on the basis vector v_λ of the infinite wedge space.

Perform the principal specialisation $p_i \mapsto x^i$ and use the result that

$$s_\lambda(p_1, p_2, \ldots)\big|_{p_i = x^i} = \begin{cases} x^\ell, & \text{if } \lambda = (\ell), \\ 0, & \text{otherwise.} \end{cases}$$

This reduces the wave function $\psi(x, \hbar)$ to the following sum over non-negative integers, rather than a sum over partitions.

$$\psi(x, \hbar) = Z(p_1, p_2, \ldots; \hbar)\Big|_{p_i = x^i} = \sum_{\ell=0}^{\infty} s_{(\ell)}(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots, \tfrac{q_d}{\hbar}) f_{(\ell)}(w_1, w_2, \ldots, w_k; \hbar) x^\ell$$

Furthermore, we have the following expression for the eigenvalue $f_\lambda(w_1, w_2, \ldots; \hbar)$ for the case that $\lambda = (\ell)$, since the operator $\mathcal{F}_{r+1}$ acts diagonally with eigenvalue the shifted power-sum symmetric function $\frac{\mathbf{p}_{r+1}(\lambda)}{r+1}$ acting on the basis vector v_λ.

$$f_{(\ell)}(w_1, w_2, \ldots; \hbar) = \exp\left(\sum_{r=1}^{k} w_r \hbar^r \frac{(\ell - \frac{1}{2})^{r+1} - (-\frac{1}{2})^{r+1}}{r+1}\right)$$

So we obtain the following expression for the wave function.

$$\psi(x, \hbar) = \sum_{\ell=0}^{\infty} s_{(\ell)}(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots, \tfrac{q_d}{\hbar}) \exp\left(\sum_{r=1}^{k} w_r \hbar^r \frac{\left(\ell - \frac{1}{2}\right)^{r+1} - \left(-\frac{1}{2}\right)^{r+1}}{r+1}\right) x^\ell$$

Now let us consider the action of the operator $\hat{y} = \hbar x \frac{\partial}{\partial x}$ on $\psi(x, \hbar)$.

$$\begin{aligned}
\hat{y}\psi(x, \hbar) &= \sum_{\ell=0}^{\infty} \hbar \ell s_{(\ell)}(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots, \tfrac{q_d}{\hbar}) \exp\left(\sum_{r=1}^{k} w_r \hbar^r \frac{\left(\ell - \frac{1}{2}\right)^{r+1} - \left(-\frac{1}{2}\right)^{r+1}}{r+1}\right) x^\ell \\
&= \sum_{\ell=0}^{\infty} \hbar \left(\sum_{m=1}^{d} \tfrac{q_m}{\hbar} s_{(\ell-m)}(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots, \tfrac{q_d}{\hbar})\right) \exp\left(\sum_{r=1}^{k} w_r \hbar^r \frac{\left(\ell - \frac{1}{2}\right)^{r+1} - \left(-\frac{1}{2}\right)^{r+1}}{r+1}\right) x^\ell \\
&= \sum_{m=1}^{d} \sum_{\ell=0}^{\infty} q_m s_{(\ell)}(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots, \tfrac{q_d}{\hbar}) \exp\left(\sum_{r=1}^{k} w_r \hbar^r \frac{\left(\ell + m - \frac{1}{2}\right)^{r+1} - \left(-\frac{1}{2}\right)^{r+1}}{r+1}\right) x^{\ell+m}
\end{aligned}$$

The second equality here uses a standard relation between the completely homogeneous symmetric function $s_{(\ell)}$ and the power-sum symmetric functions.

Consider now the action of the operator $\mathcal{A}_m$ on a monomial x^ℓ.

$$\begin{aligned}
\mathcal{A}_m x^\ell &= x^{\frac{1}{2}} \exp\left(m \sum_{r=1}^{k} \frac{w_r}{r+1} \sum_{i=0}^{r} \hat{x}^{-m} \hat{y}^i \hat{x}^m \hat{y}^{r-i} \right) \hat{x}^{-\frac{1}{2}} x^\ell \\
&= x^{\frac{1}{2}} \exp\left(m \sum_{r=1}^{k} \frac{w_r}{r+1} \sum_{i=0}^{r} \hat{x}^{-m} \hat{y}^i \hat{x}^m \hat{y}^{r-i} \right) x^{\ell-\frac{1}{2}} \\
&= x^{\frac{1}{2}} \exp\left(m \sum_{r=1}^{k} \frac{w_r \hbar^r}{r+1} \sum_{i=0}^{r} (\ell + m - \tfrac{1}{2})^i (\ell - \tfrac{1}{2})^{r-i} \right) x^{\ell-\frac{1}{2}} \\
&= \exp\left(\sum_{r=1}^{k} w_r \hbar^r \frac{(\ell + m - \frac{1}{2})^{r+1} - (\ell - \frac{1}{2})^{r+1}}{r+1} \right) x^\ell
\end{aligned}$$

Now use this to calculate the action of the operator $\left(\sum_{m=1}^{d} q_m \hat{x}^m \mathcal{A}_m \right)$ on the wave function as follows.

$$\begin{aligned}
&\left(\sum_{m=1}^{d} q_m \hat{x}^m \mathcal{A}_m \right) \psi(x, \hbar) \\
&= \left(\sum_{m=1}^{d} q_m \hat{x}^m \mathcal{A}_m \right) \left(\sum_{\ell=0}^{\infty} s_{(\ell)}(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots, \tfrac{q_d}{\hbar}) \exp\left(\sum_{r=1}^{k} w_r \hbar^r \frac{\big(\ell - \frac{1}{2}\big)^{r+1} - \big(-\frac{1}{2}\big)^{r+1}}{r+1} \right) x^\ell \right) \\
&= \sum_{m=1}^{d} \sum_{\ell=0}^{\infty} q_m s_{(\ell)}(\tfrac{q_1}{\hbar}, \tfrac{q_2}{\hbar}, \ldots, \tfrac{q_d}{\hbar}) \exp\left(\sum_{r=1}^{k} w_r \hbar^r \frac{\big(\ell + m - \frac{1}{2}\big)^{r+1} - \big(-\frac{1}{2}\big)^{r+1}}{r+1} \right) x^{\ell+m}
\end{aligned}$$

Since this matches the expression derived earlier for $\hat{y}\psi(x, \hbar)$, this concludes the proof. $\square$

Chapter 7

Further results on monotone Hurwitz numbers

In this final chapter of the thesis, we discuss some further results on monotone Hurwitz numbers. The first section involves a derivation of the quantum Airy structure for monotone Hurwitz numbers, starting from the result of Do, Dyer and Mathews that proves they are governed by the usual topological recursion. We derive Virasoro constraints for monotone Hurwitz numbers, which highlights the interplay between topological recursion and the Virasoro algebra. The second section involves a certain identity involving weakly and strictly monotone Hurwitz numbers, which was recently proved by Cunden, Dahlqvist and O'Connell in the context of random matrices. We rederive their result using the holonomic tools introduced earlier in the thesis.

7.1 Kontsevich–Soibelman topological recursion for monotone Hurwitz numbers

It was shown by Do, Dyer and Mathews [32] that monotone Hurwitz numbers are governed by the usual topological recursion, described in Section 3.2. In this case, the spectral curve is given by

$$\left(\mathbb{CP}^1,\ x(z) = z(1-z),\ y(z) = \frac{1}{z-1},\ \omega_{0,2} = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1-z_2)^2}\right). \tag{7.1}$$

We need the following series expansion of $\omega_{0,2}$ at $z=0$.

$$\omega_{0,2}(z,z_2) = \left(\frac{1}{{z_2}^2} + 2\,\frac{z}{{z_2}^3} + 3\,\frac{z^2}{{z_2}^4} + 4\,\frac{z^3}{{z_2}^5} + 5\,\frac{z^4}{{z_2}^6} + 6\,\frac{z^5}{{z_2}^7} + \cdots\right)\mathrm{d}z\,\mathrm{d}z_2 \tag{7.2}$$

The spectral curve has one ramification point at $z=\frac{1}{2}$, so we send it to 0 for convenience, using the coordinate change $z \mapsto z - \frac{1}{2}$.Now using the notation of Section 9 from the paper [7], we calculate the tensors A, B, C, D involved in Kontsevich–Soibelman topological recursion, as carried out for the Airy and Bessel curves in Section 3.3.

$$\begin{aligned}
A^{k_1k_2k_3} &= \operatorname*{Res}_{z=0} \xi^*_{k_1}(z)\,\mathrm{d}\xi^*_{k_2}(z)\,\mathrm{d}\xi^*_{k_3}(z)\,\theta(z) \\
&= \operatorname*{Res}_{z=0} \frac{\mathrm{d}z}{z}\; z^{2k_1+2k_2+2k_3+2}\;\left(-\tfrac{1}{2}+\tfrac{1}{8}z^{-2}\right) \qquad (7.3)\\
B^{k_1k_2}_{k_3} &= \operatorname*{Res}_{z=0} \xi^*_{k_1}(z)\,\mathrm{d}\xi^*_{k_2}(z)\,\xi_{k_3}(z)\,\theta(z) \\
&= \operatorname*{Res}_{z=0} \frac{\mathrm{d}z}{z}\frac{2k_3+1}{2k_1+1} z^{2k_1+2k_2-2k_3}\;\left(-\tfrac{1}{2}+\tfrac{1}{8}z^{-2}\right) \qquad (7.4)\\
C^{k_1}_{k_2k_3} &= \operatorname*{Res}_{z=0} \xi^*_{k_1}(z)\,\xi_{k_2}(z)\,\xi_{k_3}(z)\,\theta(z) \\
&= \operatorname*{Res}_{z=0} \frac{\mathrm{d}z}{z}\frac{(2k_3+1)(2k_2+1)}{2k_1+1} z^{2k_1-2k_2-2k_3-2}\;\left(-\tfrac{1}{2}+\tfrac{1}{8}z^{-2}\right) \qquad (7.5)\\
D^{k_1} &= -\frac{\delta_{k_1,0}}{16}+\frac{\delta_{k_1,1}}{192} \qquad (7.6)
\end{aligned}$$

The tensors A, B, C, D derived above allow us to write down the following operators for $i = 0, 1, 2, \ldots$, which annihilate the partition function for monotone Hurwitz numbers.

$$L_i = \hbar\partial_i - \sum_{j\geq 0}\frac{1}{8}\delta_{ij,0}x_ix_j + \sum_{j\geq 0}\frac{\hbar}{2}\frac{2i+2j+1}{2i+1}x_i\partial_{i+j} - \sum_{j\geq 0}\frac{\hbar}{8}\frac{2i+2j+1}{2i+1}x_j\partial_{i+j-1}$$
$$+\sum_{j\geq 0}\frac{\hbar^2}{4}\frac{(2j+1)(2i-2j-1)}{2i+1}\partial_j\partial_{i-j-1} - \sum_{j\geq 0}\frac{\hbar^2}{8}\frac{(2j+1)(2i-2j-3)}{2i+1}\partial_i\partial_{i-j-2} + \frac{\hbar}{16}\delta_{i,0} - \frac{\hbar}{192}\delta_{i,1}$$

Next, we describe how these operators lead to Virasoro constraints under a suitable change of basis. By this, we mean that there exist operators $\{V_i\}$ that annihilate the partition function defined equation (3.11) and satisfy the Virasoro commutation relation

$$[V_m, V_n] = (m-n)V_{m+n}.$$

In this section, we provide an example of the relation between Virasoro constraints and topological recursion, but the techniques used apply more generally.

One can directly compute the commutator of two of the operators above to obtain

$$[L_i, L_j] = \frac{1}{4}\frac{(2i+2j-1)(j-i)}{(2j+1)(2i+1)}L_{i+j-1} - \frac{(2i+2j+1)(j-i)}{(2j+1)(2i+1)}L_{i+j}.$$

Under the obvious rescaling $\tilde{L}_i = (2i+1)L_i$, we obtain the commutation relation

$$[\tilde{L}_i, \tilde{L}_j] = (i-j)\big(\tilde{L}_{i+j} - \tfrac{1}{4}\tilde{L}_{i+j-1}\big).$$

We proceed to show that these operators span a Lie subalgebra of the Virasoro algebra. Let us start by asking for a change of basis

$$V_n = \sum_{k=0}^{n} a_{n,k}\tilde{L}_k$$

and impose the Virasoro commutation relation to determine the coefficients $a_{n,k}$. For example, consider the following calculation, where we assume $a_{0,0} = 1$ and set $\lambda = \frac{1}{4}$.

$$\begin{aligned}
[V_3, V_0] &= 3V_3 \\
[a_{3,0}\tilde{L}_0 + a_{3,1}\tilde{L}_1 + a_{3,2}\tilde{L}_2 + a_{3,3}\tilde{L}_3, \tilde{L}_0] &= 3(a_{3,0}\tilde{L}_0 + a_{3,1}\tilde{L}_1 + a_{3,2}\tilde{L}_2 + a_{3,3}\tilde{L}_3) \\
a_{3,1}(\tilde{L}_1 - \lambda\tilde{L}_0) + 2a_{3,2}(\tilde{L}_2 - \lambda\tilde{L}_1) + 3a_{3,3}(\tilde{L}_3 - \lambda\tilde{L}_2) &= 3a_{3,0}\tilde{L}_0 + 3a_{3,1}\tilde{L}_1 + 3a_{3,2}\tilde{L}_2 + 3a_{3,3}\tilde{L}_3
\end{aligned}$$

Hence, we obtain the following constraints on $a_{n,k}$ by comparing the coefficients of $\tilde{L}_i$.

$$\begin{aligned} a_{3,3} &= a_{3,3} \\ a_{3,2} &= -3\lambda a_{3,3} \\ a_{3,1} &= 3\lambda^2 a_{3,3} \\ a_{3,0} &= -\lambda^3 a_{3,3} \end{aligned}$$

More generally, the same analysis for $[V_n, V_0] = nV_n$ leads to $a_{n,k} = \binom{n}{k}(-\lambda)^{n-k}a_{n,n}$. We can set $a_{n,n} = 1$ and check in general that taking

$$a_{n,k} = \binom{n}{k}\big(-\tfrac{1}{4}\big)^{n-k}$$

produces operators that satisfy the general Virasoro commutation relation $[V_m, V_n] = (m-n)V_{m+n}$, as a result of Vandermonde's identity for binomial coefficients. Thus, we obtain the following.

Proposition 7.1.1. *For $n = 0, 1, 2, \ldots$, the operators*

$$V_n = \sum_{k=0}^{n} \binom{n}{k}\big(-\tfrac{1}{4}\big)^{n-k}(2k+1)L_k$$

annihilate the partition function for monotone Hurwitz numbers and satisfy the Virasoro commutation relation.

7.2 Cunden–Dahlqvist–O'Connell identity for monotone Hurwitz numbers

Monotone Hurwitz numbers appeared in the recent work of Cunden, Dahlqvist and O'Connell in their recent work on integer moments of complex Wishart matrices [26]. In particular, their matrix model analysis led to an interesting identity relating weakly monotone Hurwitz numbers with strictly monotone Hurwitz numbers, which appears below as equation (7.7). Cunden, Dahlqvist and O'Connell ask for a combinatorial proof for this identity. In this section, we give a different proof of the identity, though not particularly combinatorial, using the holonomic techniques of Chapter 4. In ongoing work, we have made some steps towards a purely combinatorial proof in low genus although we do not report on that work in this thesis.

First, we introduce an enumeration for monotone Hurwitz numbers that is different, but related to, the one defined in Section 1.3.

Definition 7.2.1. Let $H_g^{\leq}(n,k)$ denote the number of tuples $(\tau_1, \tau_2, \ldots, \tau_m)$ of transpositions in S_n such that

- the product $(1\ 2\ \cdots\ n) \circ \tau_1 \circ \tau_2 \circ \cdots \circ \tau_m$ has k cycles;
- $m = k - 1 + 2g$; and
- if $\tau_i = (a_i, b_i)$ with $a_i < b_i$, then $b_1 \leq b_2 \leq \cdots \leq b_m$.

We define $H_g^{<}(n,k)$ similarly, but using the strict inequalities $b_1 < b_2 \leq \cdots \leq b_m$.

We package these numbers into the following generating functions.

$$\begin{aligned} H_g^{\leq}(n;x) &= \sum_{k=1}^{n} H_g^{\leq}(n,k)\, x^{-k} \\ H_g^{<}(n;x) &= \sum_{k=1}^{n} H_g^{<}(n,k)\, x^{-k} \end{aligned}$$

Theorem 7.2.2 (Cunden–Dahlqvist–O'Connell identity [26]). *The generating functions for weakly and strictly monotone Hurwitz numbers satisfy the following, where $h_m(1^2, 2^2, \ldots, n^2)$ denotes the complete homogeneous symmetric function in $1^2, 2^2, \ldots, n^2$.*

$$\Big(\frac{x-1}{x}\Big)^{n+1} H_g^{\leq}(n+1; x-1) = n \sum_{j=0}^{g} \Big(\frac{x-1}{x}\Big)^{2j} h_{g-j}(1^2, 2^2, \ldots, n^2)\, H_j^{<}(n; x) \tag{7.7}$$

Proof. The techniques of Section 4.3 allow us to write down generating functions for these monotone Hurwitz numbers, where we include a variable $\hbar$ to keep track of genus in the following way. These equations follow from the expressions for monotone Hurwitz numbers in terms of Schur functions, along with the hook-content formula of equation (4.7).

$$\sum_{g=0}^{\infty} H_g^{\leq}(n; x)\, \hbar^{2g-1} = \sum_{k=1}^{n} \frac{(-1)^{n-k}\,(n-1)!}{n\,(k-1)!\,(n-k)!} \prod_{i=1}^{n} \big(\tfrac{1}{\hbar x} + (k-i)\big) \prod_{i=1}^{n} \frac{1}{1-(k-i)\hbar}$$
$$\sum_{g=0}^{\infty} H_g^{<}(n; x)\, \hbar^{2g-1} = \sum_{k=1}^{n} \frac{(-1)^{n-k}\,(n-1)!}{n\,(k-1)!\,(n-k)!} \prod_{i=1}^{n} \big(\tfrac{1}{\hbar x} + (k-i)\big) \prod_{i=1}^{n} (1+(k-i)\hbar)$$

Now let us multiply equation (7.7) by $\hbar^{2g-1}$ and sum over all non-negative integers g. The left side then takes the following form.

$$\sum_{g=0}^{\infty} H_g^{\leq}(n+1; x-1)\, \hbar^{2g-1}$$
$$= \Big(\frac{x-1}{x}\Big)^{n+1} \sum_{k=1}^{n+1} \frac{(-1)^{n+1-k}\, n!}{(n+1)\,(k-1)!\,(n+1-k)!} \prod_{i=1}^{n+1} \Big(\frac{1}{\hbar(x-1)} + (k-i)\Big) \prod_{i=1}^{n+1} \frac{1}{(1-(k-i)\hbar)}$$

The right side then takes the following form, where we use the generating function for the complete homogeneous symmetric functions.

$$\sum_{g=0}^{\infty} n \sum_{j=0}^{g} \Big(\frac{x-1}{x}\Big)^{2j} h_{g-j}(1^2, 2^2, \ldots, n^2)\, H_j^{<}(n; x)\, \hbar^{2g-1}$$
$$= n\Big(\frac{x-1}{x}\Big) \left[\sum_{j=0}^{\infty} H_j^{<}(n; x) \Big(\frac{\hbar(x-1)}{x}\Big)^{2j-1}\right] \times \left[\sum_{\ell=0}^{\infty} h_\ell(1^2, 2^2, \ldots, n^2)\, \hbar^{2\ell}\right]$$
$$= n\Big(\frac{x-1}{x}\Big) \left[\sum_{k=1}^{n} \frac{(-1)^{n-k}\,(n-1)!}{n\,(k-1)!\,(n-k)!} \prod_{i=1}^{n} \Big(\frac{1}{\hbar(x-1)} + (k-i)\Big) \prod_{i=1}^{n} \Big(1 + (k-i)\hbar\frac{x-1}{x}\Big)\right]$$
$$\times \left[\prod_{i=-n}^{n} \frac{1}{(1+i\hbar)}\right].$$

Equating these two expressions, we see that the desired result is equivalent to the following.

$$\frac{n\,(x-1)^n}{n+1} \sum_{k=1}^{n+1} \frac{(-1)^{n+1-k}}{(k-1)!\,(n+1-k)!} \prod_{i=1}^{n+1} \Big(\frac{1}{\hbar(x-1)} + (k-i)\Big) \prod_{j=k}^{n} (1-j\hbar) \prod_{\ell=n-k+2}^{n} (1+\ell\hbar)$$
$$= x^n \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(k-1)!\,(n-k)!} \prod_{i=1}^{n} \Big(\frac{1}{\hbar(x-1)} + (k-i)\Big) \prod_{i=1}^{n} \Big(1 + (k-i)\hbar\frac{x-1}{x}\Big)$$

Write $a = \frac{1}{\hbar(x-1)}$ and $b = -\frac{1}{\hbar}$, and retain the usual notation $\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$ for real x. Then some straightforward algebraic manipulations allow us to express the above equation as

$$F_n(a, a-b) = (-1)^{n+1} \binom{2n+1}{n} \binom{b+n}{2n+1} G_{n+1}(a, b), \tag{7.8}$$

where we define the functions

$$F_n(a,b) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \binom{a+k}{n} \binom{b+k}{n},$$
$$G_n(a,b) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \binom{a+k}{n} \Big/ \binom{b+k}{n}.$$

The result then follows once we prove equation (7.8). We use Zeilberger's algorithm, which finds a polynomial recursion for hypergeometric-type functions such as F_n and G_n defined above. The `Zb` command in the Fast Zeilberger Mathematica package produces the following recursion for F_n.

$$n(a-b-n-1)(a-b+n+1)\, F_n(a,b) + (n+1)(2n+3)(a+b)\, F_{n+1}(a,b) + (n+1)(n+2)(n+3)\, F_{n+2}(a,b) = 0 \tag{7.9}$$

It also produces the following recursion for G_n.

$$(n-1)(n+1)\, G_n(a,b) + (2n+1)(b-2a)\, G_{n+1}(a,b) - (b-n-1)(b+n+1)\, G_{n+2}(a,b) = 0 \tag{7.10}$$

To finish the proof, we substitute $L_n(a,b) = F_n(a, a-b)$ in equation (7.9) to obtain a recursion for the left side of equation (7.8). Similarly, we substitute $R_n(a,b) = (-1)^{n+1}\binom{2n+1}{n}\binom{b+n}{2n+1} G_{n+1}(a,b)$ in equation (7.10) to obtain a recursion for the right side of equation (7.8). It turns out that these two recursions are identical. Since once can check that $L_1(a,b) = R_1(a,b)$ and $L_2(a,b) = R_2(a,b)$ explicitly, it follows by induction that $L_n(a,b) = R_n(a,b)$ for all positive integers n. Therefore, equation (7.8) holds and this completes the proof. □

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