ERRATA

Pg 2	Line 17	'Such manifold, if exists' should be 'Such a manifold, if it exists'.
Pg 8	Line 14	'by 1/2' should be 'by half'.
Pg 8	Line 15	'a one-parameter family solution' should be 'a one-parameter family of solutions'.
Pg 14	Line 10	'extra care shown' should be 'extra care as shown'.
Pg 16	Line 1	'kernal' should be 'kernel'.
Pg 18	Line 12	'kernal' should be 'kernel'.
Pg 19	Line 13	'There is necessity' should be 'There is a necessity'.
Pg 21	Line 14	'relation to give real' should be 'relation gives real'.
Pg 23	Line 12	'The form of it is' should be 'The form of v_w is'.
Pg 24	Line 8	'From (2.33) and (*)' should be 'From (2.33) and (2.36)'.
Pg 27	Line 3	'Boyd casted doubt' should be 'Boyd cast doubt'.
Pg 27	Line 6	Omit 'Bear this in mind'.
Pg 27	Line 7	'HMP may still be applicable to other two forcings' should be 'HPM may still be applicable to two other forcings, $\operatorname{sech} x$ and $\exp(x^2)$.'
Pg 38		'u (order of magnitude) vs μ^2' should be '$\log u$ vs μ^2'.
Pg 39	Line 4	'was first known' should be 'was first noticed'.
Pg 40	Line 15	'the accuary' should be 'the accuracy'.
Pg 48	Line 14	'This can be' should be 'It can be'.
Pg 48	Line 18	'this can be readily' should be 'it can be readily'.
Pg 52	Line 4	'$+V_2 z_1 V_2 W_2$' should be '$+V_2 z_1 - V_2 W_2$'.
Pg 54	Line 3	'In practical situation' should be 'In practical situations'.
Pg 65	Line 19	'a well behave' should be 'a well behaved'.
Pg 66	Line 19	Omit 'one of the ways is to'.
Pg 72	Line 22	'system is that the persistence' should be 'system is the persistence'.
Pg 76	Item 6	The year should be '1997' not '97'.

ADDENDUM

Add an entry in **Bibliography**, on page 79, ' [39] Camassa, R. and Tin, S.-K.: The global geometry of the slow manifold in the Lorenz-Krishnamurthy model, *J. Atmos. Sci.* 53:3251-3264.'

On page 45, the integral of Eq.(4.7) has a lower limit at $s = 0$. This specific lower limit ensures that the compatibility condition Eq.(4.14) for W_2 is satisfied. Otherwise, for any other values of lower limit, the compatibility condition Eq.(4.14) will be violated in general and hence W_2 will contain exponential growth term.

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THESIS ACCEPTED IN SATISFACTION OF THE
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DOCTOR OF PHILOSOPHY

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for Sec. Research Graduate School Committee

Estimating the Inevitability of Fast Oscillations in Model Systems with Two Timescales

This thesis is submitted in fulfillment of
the requirements for the Degree of Doctor of Philosophy.

January 2001

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Statement

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university or other institution.

To the best of my knowledge, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

[redacted]

Vivian Choy

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The material appears in Chapter 5 of this thesis was obtained during my visit to the Department of Mathematical Sciences, Loughborough University, UK. Their warm hospitality and help during my visit is very much appreciated.

Abstract

This thesis contains a study on two model systems which exhibit two-timescale oscillations. In both models, the amplitudes of the fast oscillations are much weaker than the slow oscillations, the fast motions are exponentially small. This smallness makes the detection and calculation of the amplitude of such exponentially small oscillatory motions challenging.

The existence of the fast motions in these two models is inevitable and is inseparable from the slower motions, despite their weakness, and hence are of great importance in the sense that researchers must take into account the long term dynamical effect of such fast oscillations in order to have a realistic understanding of the models.

Contents

Chapter 1

Introduction

This thesis contains a study on model systems which evolve with different space/time scales. We explicitly calculate the asymptotic solutions for two very different model systems. Both models are nonlinear. The first model system is a quadratic nonlinear harmonic oscillator driven by a localised forcing term. This system is derived from a forced KdV equation (fKdV) and only stationary solutions are sought with prescribed far field conditions. Thus the governing equation contains one spatial independent variable only. The forcing term has a space scale much larger than the natural scale of the linearised harmonic oscillator. It turns out that the forcing term determines the exact profile of the solution at far field. Before we present a detailed analysis on this forced nonlinear oscillator, we first gives a brief account on the development of exponential asymptotics in Chapter 2. Exponential asymptotics is a collection of asymptotic methods which leads to a more accurate asymptotic approximation than the conventional asymptotic expansions, the latter usually at best give approximation which has an exponentially small error. The exponential asymptotics, however, is designed to include exponentially small quantities in the approximation.

Then it is followed by a derivation of the forced nonlinear oscillator from the fKdV equation. The chapter ends with two examples to illustrate the procedures for the use of the complex-plane matched asymptotic method which is a member of exponential asymptotics.

We devote Chapter 3 to a detailed analysis on the forced nonlinear oscillator with a Gaussian function as a driving force. Gaussian function *per se* has no singularities in the finite complex-plane. A special treatment is required to bring out the hidden singularities from the governing equation due to such forcing term. Thus this makes the analysis stand out from the types of forcing used in the two prior examples.

As a model this forced nonlinear oscillator has its limitation. The main limitation is that it cannot reveal how the fast oscillatory motion affects the slower oscillator, there is no interaction between the slow and fast oscillators.

To gain insight of the effect of interaction between two oscillators with distinct timescale, we require a model which allows feedback effect via nonlinear coupling between the coupled oscillators. To serve this purpose we investigate a low order system which consists of five first-order ordinary differential equations (ODE) in Chapters 4-5.

This low order dynamical model has its own place in study of atmospheric dynamics. This relatively simple atmospheric model is derived by Edward Lorenz in 1986 to numerically investigate whether a slow manifold exists for the system [27]. A slow manifold is a subspace of the full phase space in the question. Such manifold, if exists, contains slow evolving motions which usually have days as a timescale. Fast oscillatory motions usually have hours as timescale, such as inertia-gravity waves, and do not appear in the slow manifold. The implication of the existence of slow manifold is that there exist certain relations between fast and slow motions and such relations could be made as accurate as one requires, at least in theory. Thus, one would only need to deal with the problem in a smaller phase space and hence simplify the question in hand.

In Chapter 4 we analyse a conservative version of this five-dimensional system and leave the detailed analysis on a more realistic version, which includes dissipation and forcing in the model, until Chapter 5. The key to the analysis in Chapter 5 is to recognise that there is a timescale at work for dissipation other than a timescale separation between fast and slow motions. Hence the conservative version is a limiting case of the non-conservative model.

Finally, a chapter of concluding remarks brings an end to this thesis.

Chapter 2

A Review of Exponential Asymptotics

In this chapter we present a brief account on the development of a collection of asymptotic methods generally known as exponential asymptotics or asymptotics beyond all orders. We also include a method outline with two examples to illustrate the analysis on a boundary-value problem

$$\mu^2 u_{xx} + u - \epsilon u^2 = f(x), \qquad x \in R$$

in Chapter 3.

2.1 Introduction

Exponential asymptotics is a term for a set of techniques used to analyse asymptotics beyond all orders. To understand what asymptotics beyond all orders means and why one needs to go beyond all orders in the asymptotic solution, we first have to review the definition of an asymptotic expansion.

An asymptotic expansion is a series that provides a sequence of increasingly accurate approximations to a function in a particular limit. The formal definition , given by Poincare (1886, Acta Math. 8:295) is as follows. Given a function, $\phi(\epsilon)$, the series $\sum_{n=0}^{\infty} \phi_n \epsilon^n$ is said to be asymptotic to $\phi(\epsilon)$ as $\epsilon \longrightarrow 0$ for every nonnegative integer N,

$$\lim_{\epsilon \longrightarrow 0} \left[\frac{\phi(\epsilon) - \sum_{n=0}^{N} \phi_n \epsilon^n}{\epsilon^N} \right] = 0. \tag{2.1}$$

Note:

1. ϕ might also depend on another parameter, in the form $\phi(x, \epsilon)$. Then ϕ_n should be replaced by $\phi_n(x)$, and one tests the asymptoticity of the series at each fixed x.

2. Asymptotic series can be more complicated than simple power series in ϵ, but they are sufficient to illustrate our main points.

At the simplest level, $N = 0$, (2.1) implies that $\phi(\epsilon) \longrightarrow \phi_0$ as $\epsilon \longrightarrow 0$. A more accurate approximation is obtained at $N = 1$:

$$\frac{\phi(\epsilon) - \phi_0}{\epsilon} \longrightarrow \phi_1 \text{ as } \epsilon \longrightarrow 0,$$

and so on. If the series is asymptotic to $\phi(\epsilon)$, we write

$$\phi(\epsilon) \sim \sum_{n=0}^{\infty} \phi_n \epsilon^n. \tag{2.2}$$

Here we note that the limit in deciding whether the series is asymptotic is $\epsilon \longrightarrow 0, N$ fixed. In contrast, the limit to test the convergence of a series is $N \longrightarrow \infty$, ϵ fixed. This difference tells us that an asymptotic series need not converge for $\epsilon \neq 0$. In fact one advantage of asymptotic analysis is that one can accurately approximate a function, using a few terms of its asymptotic series.

An important feature of an asymptotic series like $\sum \phi_n \epsilon^n$ is that every term in the series is algebraic in ϵ. Transcendentally small terms like $\exp(-1/\epsilon^2)$ are smaller than every term in the series as $\epsilon \longrightarrow 0$, and are not captured by it. Therefore if (2.2) is valid, then

$$\phi(\epsilon) + \exp\left(\frac{-1}{\epsilon^2}\right) \sim \sum_{n=0}^{\infty} \phi_0 \epsilon^n \tag{2.3}$$

is also valid. Such transcendentally small terms are said to lie beyond all orders of the asymptotic expansion.

In most applications, these tiny corrections are insignificant and they can be safely neglected. However, exceptional problems in which these very small terms have great practical interest are known in many branches of science, such as non-linear waves, viscous fluid flow, dendritic crystal growth, quantum tunnelling and others. For these exceptional problems, conventional asymptotic analysis is simply inadequate. These problems require improved methods, designed to obtain meaningful corrections that lie beyond all orders of a conventional asymptotic expansion. Exponential asymptotics provides a means to capture these exponentially small terms.

The reader might wonder how a transcendentally small term, hiding behind all orders of a (divergent) asymptotic series, could have any effect to the problem being studied. Here we mention one example as an illustration.

A singularly perturbed fifth order KdV equation,

$$u_t + 6uu_x + u_{xxx} + \epsilon^2 u_{xxxxx} = 0 \tag{2.4}$$

has been proposed by Hunter and Scheurle in 1988 [23] as a model equation for solitary water waves of small amplitude when the Bond number is close to but just less than 1/3. They established the existence of non-local solitary wave solutions of (2.4) with co-propagating oscillatory tails. In the strict sense, local solitary waves are nonlinear waves and there exists a reference frame moving with the wave where the waveform is permanent and decays rapidly in their tail regions. However, a weakly non-local solitary wave consists of a central core resembling classical solitary waves but are accompanied by co-propagating oscillatory tails which extend indefinitely far from the core with non-zero constant exponentially small amplitude. The oscillations arise physically due to a phase speed resonance with the central core. In this problem conventional asymptotic approach cannot reveal the existence of these oscillatory tails. This is because the information in these tails lie beyond all orders of the asymptotic expansion in the form of (2.2). By the use of exponential asymptotics the amplitude of these tails in addition to the relationship with the phase shifts are established analytically [33, 21].

2.2 The Development

The development of the exponential asymptotics for nonlinear problems can be dated back to 1987. In this year Segur and Kruskal [35] proposed an idea called 'asymptotics beyond all orders'. This is a nonlinear WKB technique that focuses on the singularities of the conventional power series expansion (long-wave expansion) in the power of a small parameter say ϵ in the complex-plane, that is,

$$\sum_{n=0}^{\infty} \epsilon^n \theta(x;\epsilon).$$

Their analysis involved the construction of a power series which is valid close to each singularity then matched the asymptotic expansion with the long-wave expansion, imposing appropriate boundary conditions. Their technique required numerical integrations to solve the resulting differential equations in the intermediate stage. Hence their approach is not generally applicable. The reason is that for many problems the differential equations resulting from defining the *inner problem* of the original equation could be highly nonlinear and dispersive partial differential equations. The time required to devote to solve these resulting differential equations would be as much as to solve the original problem numerically.

The fully analytic approach was not devised until Pomeau *et al.* in 1988 [33] considered the singularly perturbed fifth order KdV equation (2.4). The KdV equation, famous for having been solved by inverse scattering methods, appears in various physical contexts and is generic in the sense that it is a nonlinear equation obtained by balancing nonlinearity with dispersion for weakly dispersive waves in shallow water. Indeed it is possible to continue the expansion beyond the order where Korteweg and de Vries stopped.

Following the ideas of Kruskal and Segur, but instead of using numerical methods, they used Borel summation to sum a divergent series which was supposed to be a solution of the leading order differential equation resulting from the inner expansion. This approach was based on the Borel summability of the algebraic asymptotic power series say $U(X)$. If the coefficient of X^n grows only as $n!$, by Watson's lemma [5] we can sum the series through a Borel summation. The Borel transformed series has a finite convergence radius and the nature of the singularity closest to the origin can be found. Then an exponentially small term associated to the closest singularity can be found by formally inverting the Borel summation. In turn, this exponentially small term can be related to the behaviour at infinity of the physical problem. This procedure is essentially Watson's lemma applied in reverse; they sought an integral expression for a function that has the known asymptotic expansion. Eventually they calculated the amplitude of the oscillatory tails but their result quantitatively differed from the result obtained by Grimshaw and Joshi [21] by 1/2. Also they failed to established a one-parameter family solution of the fifth order KdV equation whose existence was proved in 1992 by Amick and Toland [4].

The use of exponential asymptotics in the context which is relevant to the work shown in Chapter 3 can also be found in Grimshaw [19], Akylas & Grimshaw [1], Grimshaw [20], Grimshaw & Joshi [21], Akylas & Yang [2] and Boyd [11].

In [19], Grimshaw re-examined the problem of the nonexistence of certain travelling wave solutions of the Kuramoto-Sivashinsky equation using Borel summation. The aim of his paper is to show that Borel summation can lead to the main result in a simpler and more robust manner. The procedures of Borel summation were set up systematically and formed the layout for the use of this method. In [1], the equation to be solved by Akylas and Grimshaw is a partial differential equation, hence, the actual procedures of the calculation were very difficult. In [20], the main calculation involved solving two coupled ordinary differential equations. Grimshaw and Joshi [21] re-examined the fifth order KdV equation(2.4). This time the amplitude of the oscillatory tails was found explicitly and the one-parameter family solution was established which relates the phase shift and the amplitude of the tails. The amplitude found agreed well with the numerical results of Boyd [8]. Since the analysis of [1, 20, 21] was done in the complex-plane, their approach is called complex-plane matched asymptotics.

A somewhat different method was taken by Akylas & Yang [2]. They used a forced KdV equation with zero-wave condition at far downstream with three different forcing terms;

1. $f(x) = \operatorname{sech}^2 x$,
2. $f(x) = \operatorname{sech} x$
3. $f(x) = \exp(-x^2)$

as an example to show that the idea suggested by Karpman [25] and Milewski [31] is not generally valid. In [25] and [31] these authors suggested, using the fifth-order KdV equation as an illustration, by linearising about the long wave solution the information of the tails of weakly nonlocal solitary waves could be uncovered. In fact the structure of the tails depends very much on the nature of the equation. In this case, the fKdV equation, the forcing term determines the nature of the equation. More specifically, the nature of the singularities of the forcing term determines the trailing tail's features. Note that the singularity of the above forcing terms are a double pole, a single pole and no singularity in the finite complex-plane respectively. Surprisingly at first, the difficulty accompanying the analysis in [2] grows as the strength of the singularity of each forcing diminishes. Furthermore in [2] the authors claimed that using Fourier transform the analysis can be done without matching asymptotic equations in the complex plane. They also showed that their wavenumber-domain approach is suitable for determining the amplitude of the tails of weakly nonlocal solitary waves. The method used in [2] is called here a Fourier transform matched asymptotics because the analysis is done in a wavenumber plane via Fourier transform.

Further Boyd [11] suggested a technique which incorporates the use of 'quasi-Newton' numerical algorithms and the idea of 'Hyperasymptotic Perturbative Method' (HPM) to calculate the amplitude of the fKdV equations induced oscillatory tail with the forcing $f(x) = \operatorname{sech}^2 x$.

HPM is a modified version of an *optimally truncated* asymptotic series. An optimally truncated series is a series which is truncated at the $N+1$ th order of the regular asymptotic expansion exclusively and has an error term $O\left[\exp(-1/\epsilon)\right]$ where ϵ is a perturbation parameter much smaller than unity. This truncated expansion represents an approximation with a smallest error. Including more terms in the expansion beyond this point will give an approximation with a larger error [5]. This intrinsic limitation of such series forbids any exponentially small quantities uncovered from the calculation. By applying the regular expansion to the fKdV equation, the second order derivative and the nonlinear term are dropped out from the calculation of such optimally truncated asymptotic series because these term are $O(\epsilon^2)$ smaller than the u. Boyd then pointed out when the Fourier transformed optimally-truncated error term peaks at wavenumber $k = \pm 1$, u_{xx}^{N+1} is as large as u^{N+1} but the nonlinear term is still small. This implies that neglecting the derivative term is no longer justifiable beyond where the series is truncated. Then HPM increases the accuracy of the approximation by retaining the second order derivative term in the expansion which is carried out beyond where the optimally truncated series stops.

Without going through all the details, instead we refer the interested reader to Boyd's book [13], we outline the principle of his method via the fKdV equation with $f = \operatorname{sech}^2 x$ and $\mu^2 = \epsilon$. We first write

$$u(x;\epsilon) = \Delta(x;\epsilon) + \sum_{j=1}^{N} \epsilon^j u^{(j)}(x) \tag{2.5}$$

where $\sum$ represents an optimal truncated long wave expansion, N is the number of terms in such expansion, u is exact solution and Δ is a measure of difference between the other two terms. One would obtain an exact equation for Δ in terms of u, u_j and f by inserting (2.5) into the fKdV equation,

$$\begin{aligned} r\left(\sum\right) &\equiv \epsilon\Delta_{xx} + \Delta - \epsilon\Delta^2 - 2\epsilon\Delta\sum \\ &= \operatorname{sech}^2 x - \epsilon\left(\sum\right)_{xx} - \sum + \epsilon\left(\sum\right)^2. \end{aligned} \tag{2.6}$$

where $r\left(\sum\right)$ is called by Boyd as residual function of the 'basic state' $\sum$ [12]. By scaling argument, when the long wave expansion has terms less than N then one can safely neglect the second derivative and the nonlinear term in the RHS of (2.6). As explained above the Fourier transform of r peaks at $k = \pm 1$ at N, then one must use different scaling and retain the second derivative in (2.6) for expansion beyond the Nth term. To solve for Δ one can use the quasi-Newton iterative scheme suggested by Boyd.

Basically the matched asymptotics method and HPM both share the same fundamental. That is one needs to rescale the original problem when the independent variable is getting close to the singularity or in the language of HPM when the residual function peaks at the certain wavenumbers. After the rescaling one then obtains a new equation to solve. In the literature for matched asymptotics and hyperasymptotics, this new equation is called inner problem or hyperasymptotic approximation respectively.

2.3 Background

A forced KdV equation (fKdV) can be written as

$$U_T + \delta U_X + \nu 2 U U_X + \lambda U_{XXX} = F_X\left(X\right) \tag{2.7}$$

where T, X, δ, ν and λ are temporal coordinate, spatial coordinate, linear long-wave phase speed, measures of nonlinearity and dispersion respectively. The forcing term F_X is a function of space only representing, for example in the context of water waves, a local topography. Thus,

$$F \longrightarrow 0 \quad \text{as} \quad |X| \longrightarrow \infty.$$

We assume the forcing is weak and has a large length scale, the three parameters are all order of unity and we assume $\lambda > 0$ without loss of generality.

We seek stationary solution hence from now on we drop the temporal derivative. That is

$$\lambda U_{XXX} + \delta U_X + \nu 2 U U_X = F_X(X).$$

In order that an *oscillatory* solution exists in the far field we require

$$\lambda\delta > 0.$$

The above requirement indicates the model is to be subcritical. If the inequality sign is reversed the model is termed supercritical and implies nonexistence of *oscillatory* solution for the fKdV equation.

We then integrate the above once and get

$$\lambda U_{XX} + \delta U + \nu U^2 = F(X).$$

We note that a transformation

$$(U, F) \longrightarrow (-U, -F)$$

gives us

$$\lambda U_{XX} + \delta U - \nu U^2 = F(X). \tag{2.8}$$

This transformation tells us that changing the sign of ν will change the sign of the solution. Effectively this makes solving the fKdV equation with a negative nonlinearity the same as solving with a positive nonlinearity. Since F is small with a long length scale, the dominant balance is between δU and F, after some rescaling on (2.8), it yields

$$\mu^2 u_{xx} + u - \epsilon u^2 = f(x), \qquad x \in R \tag{2.9}$$

where

$$0 < \epsilon, \mu << 1$$

with boundary conditions—symmetric or one-sided. This is interesting to note that now (2.9) can be viewed as a forced simple harmonic oscillator with the lowest degree of nonlinearity. The forcing terms are chosen so that this small (length) scale nonlinear harmonic oscillator is driven by a much larger (length) scale oscillator.

2.4 Method Outline

We present a brief method outline on complex-plane matched asymptotics in this section and two examples subject to two types of boundary conditions respectively.

1. For $0 < \epsilon$, $\mu << 1$, develop a formal asymptotic expansion (long wave expansion or outer problem) for the core region in the power of ϵ or μ:

$$u_s \sim \sum_{n=0}^{\infty} \epsilon^n u_n(x).$$

2. Note the singularities x_{sing} in the expansion u_s in the complex x-plane. Focus on the singularity which is closest to the real axis in the upper x-plane.

3. Define an inner problem by rescaling the independent variable

$$x = x_{sing} + \lambda q$$

and the dependent variable

$$u = \epsilon^r v$$

where λ and r are to be determined according to the problem in hand. They are to transform the originally singular perturbation problem to regular perturbation problem [32]. Also, need to generalise an appropriate boundary condition from real line to the complex q-plane and set a matching condition.

4. Sum the divergent series in inverse powers of q representing the leading order term, v_0, of the inner expansion in the form of a Laplace transform

$$v_0 = \int_0^{\infty} \exp(-sq)V'(s)ds.$$

This step is motivated by the aim to use Borel summation.

5. Identify the singularities of the integrand in the transformed s-plane. Work out the residue at the singularity which is the one closest to the real s-axis.

6. Impose the symmetry condition on v_0 by adding appropriate terms to v_0 to make the asymptotic expansion complete.

7. Match the solution back to the real x-axis.

The aim of steps 4-6 is to generate the exponentially small corrections on the imaginary x-axis then relate them to the oscillatory tails of $u(x)$ on the real axis. These two steps are not necessary if the inner problem is linear which is exactly the case with the Gaussian forcing. Also the step of matching can be subtle and requires extra care shown in the second example below.

2.4.1 Example 1: $f(x) = \operatorname{sech}^2 x$

The analysis for this forcing starts by setting $\mu^2 = \epsilon$ to balance the effect of nonlinearity and of dispersion. The boundary condition for this example is

$$u(x) \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \infty.$$

2.4.1.1 Long Wave Expansion

Upon substituting

$$u_s \sim \sum_{n=0}^{\infty} \epsilon^n u_n$$

into (2.9),

$$u_0 = \operatorname{sech}^2 x$$
$$u_1 = u_0^2 - u_{0xx}$$
$$\vdots$$

We find that u_s contains singularities in the complex x-plane at

$$x_{sing} = \pm(2n+1)i\pi/2 \qquad n \in J.$$

Then we proceed to step 3 to define an inner problem by assuming the most influential singularity is the one closest the real axis. There are two such singularities

$$x_{sing} = \pm\frac{i\pi}{2}.$$

We note that the solution of (2.9) is analytic in a strip $|\text{Im } x < \pi/2|$.

2.4.1.2 Inner Problem

W only consider the singularity in the upper-half plane in the following. The effect of the one located in the lower-half plane merely adds a complex conjugate of its counterpart to the final result. Thus we set

$$x = \frac{i\pi}{2} + \sqrt{\epsilon}q$$

where q now is the inner variable. The inner problem then takes the form

$$v_{qq} + v - v^2 = -\epsilon \ \text{cosech}^2\sqrt{\epsilon}q$$

where

$$v = \epsilon u$$

Next we let $v_s \sim \sum_{n=0}^{\infty} \epsilon^n v_n(q)$ and expand the right hand side of the inner problem in terms of power series as well. At leading order this yields then

$$v_{0qq} + v_0 - v_0^2 = \frac{-1}{q^2} \tag{2.10}$$

and a matching condition

$$v_0 \sim \left(\frac{-1}{q^2} + \frac{7}{q^4} + \cdots\right) \text{ as } |q| \to \infty \text{ in Re } q \geq 0, \ \text{Im } q < 0. \tag{2.11}$$

To solve the inner problem we seek solution of v_0 in the form of a Laplace transform

$$v_0 = \int_0^{\infty} \exp(-sq)V'(s)ds. \tag{2.12}$$

This step is motivated by the aim to use Borel summation.

2.4.1.3 Borel Summation

Upon substitution of the integral into (2.10) we require the kernal to satisfy the integral equation

$$(s^2+1)V'(s) - \int_0^s V'(\lambda)V'(s-\lambda)d\lambda = -s \tag{2.13}$$

By assuming

$$V'(s) = \sum_{n=0}^{\infty} a_n s^{2n+1}$$

and substitute it back to (2.12), we then get

$$v_0 \sim \sum_{n=0}^{\infty} \frac{a_n(2n+1)!}{q^{2(n+1)}}. \tag{2.14}$$

It can be shown by putting (2.14) into the (2.10), a_n satisfies the recurrence relation

$$a_n + a_{n-1} - \sum_{i=0}^{n-1} \frac{a_i a_{n-1-i}(2i+1)!(2n-2i-1)!}{(2n+1)!} = 0, \qquad n \geq 1 \tag{2.15}$$

where $a_0 = -1$ from (2.11). As $n \longrightarrow \infty$, the coefficient of the nonlinear term in (2.15) is $O(1/n^2)$. As a result

$$a_n \sim (-1)^n K$$

where K is found by computing the exact value of a_n up to some large value. This is found $K = -1.55\cdots$. Thus,

$$V'(s) \sim \frac{Ks}{1+s^2} \qquad \text{for } |s| < 1$$

Hence, there is a pole singularity at $s = \pm i$ in the complex s-plane. Only $s = i$ is in the allowed region. Thus in our subsequent development of the theory we focus on the contribution from the singularity at $s = i$.

There are two possible contours (see Fig.2.4.1.3), we denote them as γ_+ and γ_-. The corresponding solution v_0 is denoted by v_+ and v_- respectively. For the contour γ_-, as $|q| \longrightarrow \infty$ in Re $q < 0$ and Im $q < 0$, γ_- must be shifted across the imaginary

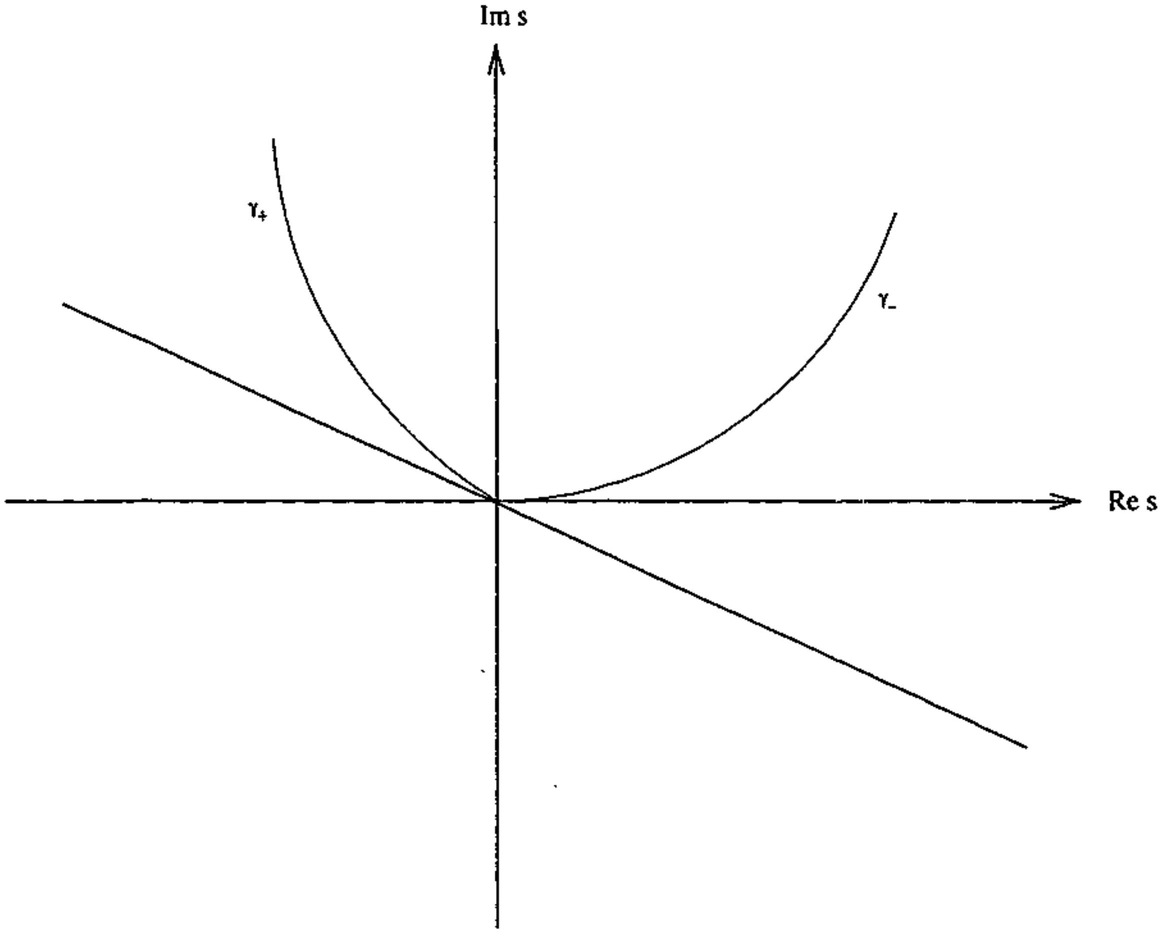

Figure 2.1: The contours in the transformed s-plane.

s-axis and thus generates a term proportional to $+\pi i K \exp(-iq)$. However, in the region Re $q > 0$ and Im $q < 0$, contour γ_- does not have to cross the imaginary s-axis. So v_- has no trailing oscillations as $x \longrightarrow \infty$ but it has a trailing oscillations as $x \longrightarrow -\infty$. By similar reasons, in Re $q > 0$ and Im $q < 0$, γ_+ must be shifted across the imaginary s-axis generating a term proportional to -$\pi i K \exp(-iq)$. In the region Re $q < 0$ and Im $q < 0$, γ_+ does not have to cross the imaginary s-axis hence v_+ contains trailing oscillations as $x \to \infty$ but not as $x \to -\infty$.

For the reasons stated above we choose v_- since it satisfies the far field condition i. e. $v_0 \to 0$ as $|q| \to \infty$ in Re $q > 0$, Im $q < 0$. However as $|q| \longrightarrow \infty$ in Re $q < 0$, Im $q < 0$ the contour γ_- must be moved across the imaginary $s-$axis generating the residue $+i\pi K \exp(-iq)$. As a result,

$$v_0 \sim v_- + i\pi K \exp(-iq). \tag{2.16}$$

2.4.1.4 Matching

We have to ensure (2.16) indeed matches (2.11) as $|q| \to \infty$, Im $q < 0$. To see that, we deform the contour γ_- into Re $s > 0$, away from the singularities on the imaginary $s-$axis. Then v_- satisfies the matching condition (2.11). Now we change the variable back to x from q and let Re $x \to -\infty$ and Im $x \to 0$. Simultaneously we collect the corresponding contribution from the singularity at $x = -i\pi/2$. We then obtain, as $x \to -\infty$,

$$u \sim u_s + \frac{i\pi K}{\epsilon} \exp\left(-\frac{\pi}{2\sqrt{\epsilon}} - \frac{ix}{\sqrt{\epsilon}}\right) - \frac{i\pi K}{\sqrt{\epsilon}} \exp\left(-\frac{\pi}{2\sqrt{\epsilon}} + \frac{ix}{\sqrt{\epsilon}}\right)$$

$$\sim \frac{2\pi K}{\epsilon} \exp\left(-\frac{\pi}{2\sqrt{\epsilon}}\right) \sin\left(\frac{x}{\sqrt{\epsilon}}\right).$$

As mentioned in the preceding sections, the nature of the singularity of the forcing determines the details of the solution. This point now becomes apparent. The double pole belonging to the forcing term generates a single pole belonging to V', the kernal of (2.12). This single pole singularity of V' is the reason the calculation for this forcing term being easier than the calculation in the next example. The next example shows that if the forcing term contains a single pole, V' then possesses a double pole. The increase of pole order in Laplace transform space leads to a more involved calculation.

2.4.2 Example 2: $f(x) =$ sech x

We replace sech$^2 x$ in the forcing term by sech$x$. Consequently, to bring the effects between the nonlinear term and the dispersive term to balance, we set $\epsilon = \mu$. Then we have

$$\epsilon^2 u_{xx} + u - \epsilon u^2 = \text{sech} x \tag{2.17}$$

with a symmetry condition

$$u(x) = u(-x) \quad \text{for} \quad x \in R. \tag{2.18}$$

Despite the simple look of this forcing, it actually requires more attention on the stage of fulfilling a symmetry condition and of matching the solution to the real axis.

2.4.2.1 Long Wave Expansion

We proceed as in the previous section, assuming

$$u_s \sim \sum_{n=0}^{\infty} \epsilon^n u_n \tag{2.19}$$

where u_s denotes core region. Substituting (2.19) into (2.17) then gets

$$\begin{aligned} u_0 &= \operatorname{sech} x \\ u_1 &= u_0^2 \\ &\vdots \end{aligned} \tag{2.20}$$

Again, as in the preceding example, u_s is singular in the complex x-plane at

$$x_{sing} = \pm(2n+1)i\pi/2$$

where $n \in I$. There is necessity to consider the structure of the solution near these points. To do this, for the singularity closest to the real x-axis in the upper-half plane we put

$$x = \frac{i\pi}{2} + \epsilon q. \tag{2.21}$$

Note that the singularity is a single pole.

Then from (2.20) and (2.21) we find as $\epsilon q \longrightarrow 0$,

$$u_s \sim \epsilon^{-1}\left(\frac{-i}{q} - \frac{1}{q^2} + \cdots\right) + \epsilon\left(\frac{iq}{6} - \frac{1}{3} + \cdots\right) + O(\epsilon^3). \tag{2.22}$$

Now (2.18) is replaced by

$$\text{Im } \{u(x)\} = 0 \text{ Re } x = 0, \ |\text{Im } x| < \frac{\pi}{2}. \tag{2.23}$$

Next we investigate the inner problem.

2.4.2.2 Inner Problem

To define an inner problem, put (2.21) and $v = \epsilon u$ into (2.17) then we have

$$v_{qq} + v - v^2 = i\epsilon \operatorname{cosech} \epsilon q. \tag{2.24}$$

Assuming

$$v \sim \sum_{n=0}^{\infty} \epsilon^n v_n(q). \tag{2.25}$$

Substituting (2.25) into (2.24), yields

$$v_{0qq} + v_0 - v_0^2 = \frac{-i}{q}. \tag{2.26}$$

From (2.22), we obtain the matching condition

$$v_0 \sim \frac{-i}{q} - \frac{1}{q^2} + \cdots, \text{ as } |q| \longrightarrow \infty \text{ in Re } q \geq 0 \text{ and Im } q < 0. \tag{2.27}$$

In general,

$$v_0 \sim \sum_{n=1}^{\infty} b_n q^{-n}, \text{ as } |q| \longrightarrow \infty \text{ in Re } q \geq 0 \text{ and Im } q < 0. \tag{2.28}$$

Also the symmetry condition becomes

$$\text{Im } \{v_0(q)\} = 0 \text{ on Re } q = 0, \text{Im } q < 0. \tag{2.29}$$

Now substituting (2.28) into (2.26) then gets

$$\sum_{n=3}^{\infty} \left[(n-2)(n-1)b_{n-2} - \sum_{i=1}^{n-1} b_i b_{n-i} + b_n \right] q^{-n}$$
$$+ b_1 q^{-1} + (b_2 - b_1^2) q^{-2} = -iq^{-1}.$$

Hence $b_1 = -i$ and $b_2 = -1$. This is consistent with (2.27). Also

$$(n-2)(n-1)b_{n-2} - \sum_{i=1}^{n-1} b_i b_{n-i} + b_n = 0, \; n \geq 3. \tag{2.30}$$

Now we are ready to move to the next stage — Borel-summation.

2.4.2.3 Borel Summation

We seek a solution of (2.26) in the form of a Laplace transform

$$v_o = \int_\gamma \exp(-sq)V'(s)ds \tag{2.31}$$

where γ runs from $s = 0$ to infinity in the upper-half plane provided Re $(sq > 0)$ in order (2.28) to be bounded. Substituting (2.31) into (2.26), implies

$$(1+s^2)V'(s) - \int_0^s V'(\lambda)V'(s-\lambda)d\lambda = -i. \tag{2.32}$$

We seek solution of (2.32) as a power in s,

$$V'(s) = \sum_{n=0}^{\infty} a_n s^n. \tag{2.33}$$

Putting then (2.33) into (2.31) gets

$$v_0 \sim \sum_{n=0}^{\infty} \frac{b_{n+1}}{q^{n+1}}$$

So we have

$$a_n = \frac{b_{n+1}}{n!} \qquad n \geq 0. \tag{2.34}$$

Using (2.34), (2.30) becomes

$$a_{n-1} = \sum_{k=1}^{n-1} \frac{a_{k-1}a_{n-1-k}(k-1)!(n-1-k)!}{(n-1)!} - a_{n-3}, \qquad n \geq 3 \tag{2.35}$$

where $a_0 = -i$ and $a_1 = -1$. To ensure the recurrence relation to give real values, we let

$$a_n = (-i)^{n+1}\alpha_n \qquad \alpha_n \in R \tag{2.36}$$

then (2.35) becomes

$$\alpha_{n-1} = \sum_{j=1}^{n-1} \frac{\alpha_{j-1}\alpha_{n-1-j}(j-1)!(n-1-j)!}{(n-1)!} + \alpha_{n-3} \tag{2.37}$$

where

$$\alpha_0 = 1 = \alpha_1.$$

As $n \longrightarrow \infty$, the nonlinear term in (2.37) becomes less important, we find that

$$\alpha_n \sim Cn + D + O\left(\frac{1}{n}\right) \quad \text{as} \quad n \longrightarrow \infty. \tag{2.38}$$

Note that D can be determined as follows. First expand (2.37) then we have

$$-\alpha_{n-3} + \alpha_{n-1} - \frac{2\alpha_0\alpha_{n-2}}{n-1} - \frac{2\alpha_1\alpha_{n-3}}{(n-1)(n-2)} + O\left(\frac{1}{n^3}\right) = 0. \tag{2.39}$$

Using (2.38) with a next order term included

$$\alpha_n \sim Cn + D + \frac{E}{n} + O\left(\frac{1}{n^2}\right)$$

the left hand side of (2.39) becomes

$$\sim \; -\frac{2D}{n-1} + O\left(\frac{1}{n^2}\right)$$

We can conclude that $D = 0$. Therefore

$$\alpha_n \sim Cn + O\left(\frac{1}{n}\right).$$

C is found approximately equal to 0.94 by computing the exact value of α_n in (2.37) up to some large value of n.

Recall

$$V'(s) = \sum_{n=0}^{\infty} a_n s^n \quad \text{as} \quad n \longrightarrow \infty$$

hence

$$V'(s) = \frac{-iC}{(1+is)^2} + \frac{iC}{1+is}. \tag{2.40}$$

Equation (2.40) indicates $s = i$ is a double pole of $V'(s)$. So

$$v_0 = \int_\gamma iC\left[\frac{1}{1+is} - \frac{1}{(1+is)^2}\right]\exp(-sq)ds$$

where the residue of the integrand at $s = i$ is

$$2\pi Cq\exp(-iq) + 2\pi iC\exp(-iq). \tag{2.41}$$

Note that the above expression contains a secular term as a result of $s = i$ being a double pole.

2.4.2.4 Imposing Symmetry Condition

At this stage we know that $v_0 = \int_\gamma \exp(-sq)V'(s)ds$ cannot satisfy the symmetry condition (2.29). Part of the reason is that the asymptotic expression for the solution v_0 is not complete because of the exponentially small imaginary term on the Im s-axis. In addition to this reason, we also have a secular term which although still subdominant to v_s, is $O(q)$ relative to $\exp(-iq)$ as $|q| \longrightarrow \infty$. So to see what appropriate terms we should add to balance the effect of this secular term we need to rewrite v_0 as

$$v_0 = v_s + v_w \tag{2.42}$$

where v_s represents the outer expansion which has the form, from (2.28),

$$v_s \sim \sum_{n=1}^{\infty} b_n q^{-n}$$

and v_w represents the trailing oscillations. The form of it is to be determined. Putting (2.42) into (2.26) and neglecting the nonlinear term then we get

$$v_{wqq} + v_w - 2v_w v_s \approx 0. \tag{2.43}$$

Let

$$\begin{aligned} v_w &\sim \sum_{n=-1}^{\infty} \beta_n q^{-n} \exp(-iq) \\ &= W \exp(-iq). \end{aligned}$$

Note that in the above expression n is summed from $n = -1$. This is to balance the secular term. Substituting the above into (2.43) then yields

$$-2iW_q + W_{qq} - 2Wv_s = 0. \tag{2.44}$$

$$\Longrightarrow \beta_0 = i\beta_{-1}.$$

We observe from the above that each β_n is proportional to β_{-1} thus

$$v_w \sim \beta_{-1}(q + i + \cdots)\exp(-iq).$$

Hence we add v_w to $\int_\gamma \exp(-sq)V'(s)ds$ to form a complete asymptotic solution

$$v_0 \sim \int_\gamma \exp(-sq)V'(s)ds + \beta_{-1}(q + i + \cdots)\exp(-iq) \tag{2.45}$$

which can now be made satisfy the symmetry condition.

To impose the symmetry condition (2.28), we let $q = -iy$, $y \in R^+$ and put the contour γ onto the imaginary s-axis where $s = i\sigma$, $\sigma \in R$. Then (2.45) now becomes

$$\begin{aligned} v_0 \sim i -\!\!\!\!\!\!\int_0^\infty \exp(-\sigma y)V'(i\sigma)d\sigma - i\pi C y \exp(-y) \\ + i\pi C \exp(-y) + i\beta_{-1}(1 - y + \cdots)\exp(-y), \end{aligned}$$

where the integral is a principal value integral. From (2.33) and (*),

$$iV'(i\sigma) = \sum_{n=0}^{\infty}(-1)^{n+1}\alpha_n \sigma^n (i)^{2n+2}.$$

So we can infer that

$$i -\!\!\!\!\!\!\int_0^\infty \exp(-\sigma y)V'(i\sigma)d\sigma$$

gives a real value.

To ensure the other terms to be real-valued we put

$$-\pi C = \text{ Re } \{\beta_{-1}\}.$$

We write β_{-1} in the polar form

$$\beta_{-1} = \tau \exp(i\delta)$$

thus

$$\tau \cos\delta = -\pi C.$$

Hence

$$v_w \sim \tau(q + i + \cdots)\exp(-iq + i\delta). \tag{2.46}$$

Eventually we established an asymptotic solution of (2.26) and this asymptotic solution satisfies the symmetry condition. Next step, is to match the solution back to the real x-axis.

2.4.2.5 Matching

Using $v = \epsilon u$, (2.25) and (2.42), we have

$$u \sim u_s + u_w \tag{2.47}$$

Putting (2.47) into (2.17) then yields for x on the real x-axis,

$$\epsilon^2 u_{wxx} + u_w - 2\epsilon u_s u_w \approx 0. \tag{2.48}$$

From (2.19) we know $u_s \sim \operatorname{sech} x + O(\epsilon)$ and u_w^2 is neglected.

Next we try

$$u_w \sim A \exp(-i\phi)$$

where A and ϕ are real. Substituting this into (2.48) then from the imaginary part gives

$$\phi_x A^2 = \text{const.}$$

From the real part,

$$\phi_x^2 = \frac{(1 - 2\epsilon u_s)}{\epsilon^2} + \frac{A_{xx}}{A},$$

where $A_{xx}/A = O(\epsilon)$ so neglected in the following. Thus

$$\phi = \frac{x}{\epsilon} - 2\tan^{-1}[\exp(x)] + \phi_0 + O(\epsilon)$$

where ϕ_0 is an integration constant and implies

$$A = A_0(1 - \epsilon u_s)^{-\frac{1}{2}}$$

where A_0 is a constant. Using (2.21) and recall $\operatorname{Re} q \geq 0$ then

$$\phi \sim \frac{i\pi}{2\epsilon} + q - \pi - i\ln\left[\coth\left(\frac{\epsilon q}{2}\right)\right] + \phi_0 + \cdots .$$

Thus, as $|q| \to \infty$

$$A\exp(-i\phi) \sim \frac{-A\epsilon q}{2} \exp\left(\frac{\pi}{2\epsilon} - iq - i\phi_0\right).$$

From $v = \epsilon u$, (2.25), (2.42), (2.46) and (2.47), matching at leading order, gives

$$\frac{-A_0 \epsilon q}{2} \exp\left(\frac{\pi}{2\epsilon}\right) \exp\left(-iq - i\phi_0\right) = \frac{\tau q}{\epsilon} \exp\left(-iq + i\delta\right)$$

$$\iff \qquad A_0 = \frac{-2\tau}{\epsilon^2} \exp\left[-\frac{\pi}{2\epsilon} + i(\delta + \phi_0)\right].$$

Since A_0 is real

$$\Longrightarrow \phi_0 = -\delta.$$

Therefore as $x \longrightarrow \infty$

$$\phi \sim \frac{x}{\epsilon} - \delta - \pi$$

and

$$A_0 \exp\left(-i\phi\right) \sim \frac{2\tau}{\epsilon^2} \exp\left[-\frac{\pi}{2\epsilon} - i\left(\frac{x}{\epsilon} - \delta\right)\right]$$

to the leading order.

Finally, taking into account the singularity at $x = -i\pi/2$ gives

$$\begin{aligned} u_w \sim & A_0 \exp(-i\phi) + A_0 \exp(i\phi) \\ = & \frac{4\tau}{\epsilon^2} \exp\left(-\frac{\pi}{2\epsilon}\right) \cos\left(\frac{x}{\epsilon} - \delta\right) \end{aligned}$$

to the leading order, for $x \longrightarrow \infty$. Since the solution is symmetric,

$$u_w \sim \frac{4\tau}{\epsilon^2} \exp\left(-\frac{\pi}{2\epsilon}\right) \cos\left(\frac{|x|}{\epsilon} - \delta\right)$$

for all x.

2.5 Discussion

We have seen how the complex-matched asymptotics can be applied to two qualitatively different forcing terms. Our method yields the same results as obtained by Akylas & Yang [2].

Applying HPM to the case of $f(x) = \operatorname{sech}^2$, Boyd [11] also found the magnitude of K without specifying what boundary condition that the fKdV equation is subject to. Boyd casted doubt on HPM's feasibility for the other forcings, especially for the Gaussian forcing $f(x) = \exp(-x^2)$ [11]. We believe that our examples shed light on this issue. As shown in §2.4.1 - 2.4.2, the type of the singularity of V' determines the details of the calculations. Bear this in mind we believe that the principle of HMP may still be applicable to other two forcings however the algorithm, which is shown suitable for the forcing $\operatorname{sech}^2 x$ only, requires modifications which take into account the qualitative difference between different forcings.

In next chapter we solve the fKdV equation with a forcing term $f = \exp(-x^2)$. This forcing is different from the previous two because it has no singularity in a finite complex plane. This property has an important effect on the final result and makes the analysis stand out from the previous two examples.

Chapter 3

A Forced Nonlinear Oscillator

We present a detailed asymptotic analysis on

$$\mu^2 u_{xx} + u - \epsilon u^2 = f(x) \tag{3.1}$$

where

$$f(x) = \exp(-x^2), \tag{3.2}$$

subject to two types of boundary conditions

$$u(x) = u(-x) \quad \text{for} \quad x \in R \tag{3.3a}$$

or

$$u(x) \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \infty \tag{3.3b}$$

This forcing term is special because it has no singularities in the finite complex-plane and this makes the analysis very different from the examples shown in the preceding chapter.

3.1 Long Wave Expansion

We first follow [2] setting $\epsilon = \mu^2$ to bring the effect of nonlinearity and dispersion into balance. We get

$$\mu^2 u_{xx} + u - \mu^2 u^2 = \exp(-x^2) \tag{3.4}$$

Then we expand u as a power series in ϵ,

$$u = \sum_{n=0}^{\infty} \mu^{2n} u_n.$$

We obtain, correct up to $O(\mu^4)$,

$$\begin{aligned} u \;\sim\; & \exp\left(-x^2\right)\left[1 - \mu^2(4x^2 - 2) + \mu^4(16x^4 - 48x^2 + 12) + \cdots\right] \\ & + \exp\left(-2x^2\right)\left[\mu^2 - \mu^4(20x^2 - 6) + \cdots\right] \\ & + \exp\left(-3x^2\right)\left(2\mu^4 + \cdots\right) + \sum_{m=4}^{\infty} \exp(-mx^2) S_{m-4}(x;\mu). \end{aligned}$$

The above expression contains a nonuniformity caused by secular terms. To handle this nonuniformity we set, suggested by the above expression,

$$u = \sum_{n=1}^{\infty} \mu^{2(n-1)} \exp(-nx^2) P_n(X) \quad \text{where } X = \mu x.$$

Upon substitution of the above into (3.4), we have for $n = 1$,

$$\mu^4 P_1'' - \mu^2 2\left(2XP_1' + P_1\right) + \left(1 + 4X^2\right) P_1 = 1. \tag{3.5}$$

and for $n \geq 2$,

$$\mu^{2(n+1)} P_n'' - \mu^{2n} 2n\left(2XP_n' + P_n\right) + \mu^{2(n-1)}\left(1 + 4n^2X^2\right) P_n = \mu^{2(n-1)} \sum_{i=1}^{n-1} P_i P_{n-i}. \tag{3.6}$$

Note that (3.5) is equivalent to solving a linear version of (3.4) by setting

$$u = P_1(X)\exp(-x^2) \quad \text{where } X = \mu x. \tag{3.7}$$

We devote the next section to this linear case and the nonlinear case will be dealt with in the section after.

3.2 An Associated Linear Problem

As mentioned above solving (3.5) is equivalent to solving the linear version of (3.4) in the interest of the asymptotic behaviour of u. That is solving

$$\mu^2 u_{1xx} + u_1 = \exp(-x^2), \tag{3.8}$$

subject to specified boundary condition (3.3a). Then

$$u = A\cos\left(\frac{x}{\mu}\right) + \frac{1}{\mu}\int_0^x \exp(-t^2)\sin\left(\frac{x-t}{\mu}\right)\,dt$$

which is an even function of x and A is an arbitrary constant.

As $|x| \to \infty$,

$$\begin{aligned} u &\sim B\cos\left(\frac{x}{\mu}\right) + \left[\frac{1}{\mu}\int_0^\infty \exp(-t^2)\cos\left(\frac{t}{\mu}\right)\,dt\right]\sin\left(\frac{x}{\mu}\right) \\ &= B\cos\left(\frac{x}{\mu}\right) + \frac{\sqrt{\pi}}{2\mu}\exp\left(-\frac{1}{4\mu^2}\right)\sin\left(\frac{x}{\mu}\right) \end{aligned}$$

where

$$B = A - \int_0^\infty \exp\left(-t^2\right)\sin\left(\frac{t}{\mu}\right)\,dt.$$

However in the spirit of exploring the use of asymptotic method, we tackle this linear problem in complex plane.

3.2.1 Outer Expansion

Having substituted (3.7) with the replacement of u by u_1 into (3.8), we obtained (3.5) which is our outer problem arising from long wave expansion. At leading order balance,

$$P_1 \sim \frac{1}{1+4X^2},$$

therefore P_1 is singular at

$$X_1 = \pm\frac{i}{2}.$$

These are the only singularities for P_1. Thus we set

$$X = \frac{i}{2} + \mu\xi. \tag{3.9}$$

where $\operatorname{Im}\xi < 0$. The inner limit of the outer expansion as $\mu\xi \to 0$ in $\operatorname{Re}\xi \geq 0$ is our matching condition

$$P_1 \sim \frac{1}{4i\mu\xi} + \cdots .$$

3.2.2 Inner Expansion

In terms of the inner variable ξ, (3.5) becomes

$$\mu^2 \left(P_1'' - 4\xi P_1' - 2P_1 + 4\xi^2 P_1\right) + \mu \left(4i\xi P_1 - 2iP_1'\right) = 1$$

which is our inner problem. At leading order,

$$P_1 \sim \frac{\exp(\xi^2)}{2i\mu} \int_\xi^\infty \exp(-\lambda^2)\, d\lambda + C_1 \exp(\xi^2) \tag{3.10}$$

where C_1 is to be determined by the boundary condition.

3.2.3 Symmetry Condition

At this stage we are to impose boundary condition to determine the constant C_1. We first apply the symmetry condition (3.3a). The case of radiation condition, (3.3b), is treated in §3.5.

We first generalise (3.3a) from real line to complex plane. That becomes

$$\operatorname{Im}\left[P_1(\xi)\right] = 0, \text{ on } \operatorname{Re}\xi = 0.$$

Then we let $\xi = -iy$ where $y > 0$,

$$P_1 \sim \exp(-y^2) \left[\frac{1}{2i\mu} \left(\int_{-iy}^0 \exp(-\lambda^2)\, d\lambda + \int_0^\infty \exp(-\lambda^2)\, d\lambda\right) + C_1\right].$$

Since the first integral gives real value, this implies

$$-\frac{1}{2i\mu} \int_0^\infty \exp(-\lambda^2)\, d\lambda = \operatorname{Im} C_1$$

$$\Longrightarrow \operatorname{Im} C_1 = -\frac{\sqrt{\pi}}{4i\mu}.$$

Indeed we can be more general by setting

$$C_1 = \frac{\tilde{a}\sqrt{\pi}}{4\mu} + \frac{i\sqrt{\pi}}{4\mu}$$

where $\tilde{a}$ is an arbitrary real number. By virtue of the above, (3.10) becomes

$$P_1 \sim M_1(\xi) - \frac{\sqrt{\pi}}{4i\mu}\exp(\xi^2) + \frac{\tilde{a}\sqrt{\pi}}{4\mu}\exp(\xi^2)$$

where

$$M_1(\xi) = \frac{\exp(\xi^2)}{2i\mu}\int_\xi^\infty \exp(-\lambda^2)\,d\lambda.$$

3.2.4 Matching

Now we have to match our solution back to the real axis ($\operatorname{Im} x \to 0$). Using (3.7) and (3.9) we have

$$u_1 \sim \frac{\exp(-x^2)}{2(1+i2X)} - \left(\frac{\sqrt{\pi}}{4i\mu} - \frac{\tilde{a}\sqrt{\pi}}{4\mu}\right)\exp\left(-\frac{1}{4\mu^2} - \frac{ix}{\mu}\right). \tag{3.11}$$

We must include the corresponding contribution from the singularity at $X = -i/2$. This simply adds a complex conjugate to (3.11). We then obtain for all x, as $|x| \to \infty$

$$u_1 \sim \frac{\exp(-x^2)}{1+4X^2} + \frac{\sqrt{\pi}}{2\mu}\exp\left(-\frac{1}{4\mu^2}\right) D_1 \sin\left(\frac{|x|}{\mu} + \delta\right) \tag{3.12}$$

where

$$D_1 \sin\delta = \tilde{a}.$$

We have established the first order asymptotic solution for (3.4) satisfying the symmetry condition (3.3a). Eq.(3.12) agrees with the solution obtained from the exact linear theory shown in §3.2. Note that (3.12) forms an one-parameter family solution. This parameter relates the phase shift to the amplitude of the tails.

In the next section we establish an asymptotic solution for (3.6) which takes into account the nonlinear effect of (3.1).

3.3 A Nonlinear Problem

In regard of (3.6) the leading order balance is

$$P_n \sim \frac{\sum_{i=1}^{n-1} P_i P_{n-i}}{1+4n^2X^2} \quad \text{where} \quad n \geq 2\ .$$

Note that for each P_n the singularity closest to the real axis is at

$$X_n = \pm\frac{i}{2n} \quad \text{where} \quad n \geq 2.$$

These singularities, for each P_n, contribute the dominant effect to the tails of the wave so we only include these points in the following analysis and the singularities which are further away from the real axis are not relevant. Following the same route of §3.2.4 we set

$$X = \frac{i}{2n} + \mu\xi$$

and

$$R_n(X) = \sum_{i=1}^{n-1} P_i(X)P_{n-i}(X).$$

The matching condition as $\mu\xi \to 0$, in $\mathrm{Re}\,\xi \geq 0$ and $\mathrm{Im}\,\xi < 0$ is

$$P_n \sim \frac{\tilde{R}_n}{4in\mu\xi}$$

where

$$\tilde{R}_n = R_n\left(\frac{i}{2n}\right).$$

3.3.1 Inner Expansion

In terms of the inner variable, (3.6) then becomes

$$\mu^{2n}\left(P_n'' - 4n\xi P_n' - 2nP_n + 4n^2\xi^2 P_n\right) + \mu^{2n-1}\left(4in\xi P_n - 2iP_n'\right)$$

$$= \mu^{2(n-1)}R_n(\xi).$$

At leading order balance,

$$P_n' - 2\xi n P_n \sim -\frac{\tilde{R}_n}{2i\mu}.$$

$$\Rightarrow P_n \sim \frac{\tilde{R}_n \exp(n\xi^2)}{2i\mu} \int_\xi^\infty \exp(-n\lambda^2)\, d\lambda + A \exp(-n\xi^2) \tag{3.13}$$

Following the same line of reason as in §3.2.3 by imposing a symmetry condition on P_n in the complex plane, we find

$$A = \hat{a}\frac{\tilde{R}_n\sqrt{\pi}}{4\mu\sqrt{n}} + \frac{i\tilde{R}_n\sqrt{\pi}}{4\mu\sqrt{n}}.$$

Hence

$$P_n \sim M_n(X) - \frac{\tilde{R}_n\sqrt{\pi}}{4i\mu\sqrt{n}}\exp(n\xi^2) + \frac{\hat{c}\tilde{R}_n\sqrt{\pi}}{4\mu\sqrt{n}}\exp(n\xi^2).$$

where

$$M_n(\xi) = \frac{\tilde{R}_n \exp(n\xi^2)}{2i\mu} \int_\xi^\infty \exp(-n\lambda^2)\, d\lambda$$

Next stage is to match P_n to real axis. At the same time we have to include the contribution from the neighbourhood of $X = -i/2n$. We get, for all x as $|x| \to \infty$ and $n \geq 2$,

$$u_n \sim \frac{\tilde{R}_n \exp(-nx^2)}{1 + 4n^2X^2} + \frac{\tilde{R}_n\sqrt{\pi}}{2\mu\sqrt{n}} \exp\left(-\frac{1}{4n\mu^2}\right) D_1 \sin\left(\frac{|x|}{\mu} + \delta\right). \tag{3.14}$$

3.4 The Full Asymptotic Solution

From the results obtained in §3.2.4 and §3.3.1, we have established a solution for large x. That is combining (3.12) and (3.14) we have

$$u \sim A_{mp} D_1 \sin\left(\frac{|x|}{\mu} + \delta\right) \tag{3.15}$$

where

$$A_{mp} = \left[\frac{\sqrt{\pi}}{2\mu}\exp\left(-\frac{1}{4\mu^2}\right) + \frac{\sqrt{\pi}}{2\mu^3}\sum_{n=2}^{\infty}\frac{\tilde{R}_n}{\sqrt{n}}\mu^{2n}\exp\left(-\frac{1}{4n\mu^2}\right)\right].$$

3.4.1 The Amplitude of the Tails

Adopting the principle of the stationary phase method [32] we can evaluate the infinite sum in (3.15) provided $\mu << 1$. Note that when

$$n = \hat{N}$$

where

$$\hat{N} = \frac{1}{8\ln\mu} - \frac{\sqrt{\mu^2 - 8\ln\mu}}{8\mu\ln\mu}.$$

the phase of the terms in the infinite sum (i. e. $\mu^{2n}, \sqrt{n}$ and the exponential term) attains a stationary point. Since, however, n can only take integer and μ is much smaller than unity we then neglect the first term and the quadratic term of μ above and conclude the terms of the sum peak sharply around $n = N$ where N is the integer closest to as, $\mu \to 0$,

$$\frac{1}{2\mu\sqrt{-2\ln\mu}}.$$

So the main contribution to the sum comes from the neighbourhood $n = N + q$ where $|q|/N << 1$ and

$$\frac{\mu^{2n}}{\sqrt{n}} \exp\left(-\frac{1}{4n\mu^2}\right) \approx \frac{\exp\left(-\frac{1}{2\mu^2 N}\right)}{\sqrt{N}} \exp\left(-\frac{|1-\mu^2 N|q^2}{4N^3\mu^2}\right).$$

Note that $\mu^2 N << 1$, we neglect this term from the above expression from now on. Accordingly,

$$A_{mp} \sim \frac{\sqrt{\pi}\tilde{R_N}}{2\mu^3\sqrt{N}} \exp\left(-\frac{\sqrt{-2\ln\mu}}{\mu}\right) \sum_{q=-\infty}^{\infty} \exp\left(-\frac{q^2}{4\mu^2 N^3}\right)$$

$$= \frac{\pi\tilde{R_N}}{2^{9/4}\mu^{7/2}\sqrt{N}(-\ln\mu)^{3/4}} \exp\left(-\frac{\sqrt{-2\ln\mu}}{\mu}\right). \quad (3.16)$$

3.5 Radiation Condition

In this section we consider the fKdV equation with boundary condition

$$u \to 0 \text{ as } x \to \infty.$$

In the complex plane the above condition is equivalent to

$$u \longrightarrow 0 \text{ as } \operatorname{Re}\xi \longrightarrow \infty \text{ and } \operatorname{Im}\xi < 0.$$

For $n = 1$, C_1 in (3.10) must be zero in order to satisfy the radiation condition in complex plane. Hence

$$C_1 = 0.$$

Therefore

$$P_1 = \frac{\exp(\xi^2)}{2i\mu} \int_{\xi}^{\infty} \exp(-\lambda^2)\, d\lambda. \tag{3.17}$$

Then as $\operatorname{Re}\xi \to -\infty$ and $\operatorname{Im}\xi < 0$, (3.17) becomes

$$P_1 \sim \frac{\exp(\xi^2)}{2i\mu}\left[\int_{-\infty}^{\infty} - \int_{-\infty}^{\xi}\right]$$

$$\sim \frac{\exp(\xi^2)}{2i\mu}\int_{-\infty}^{\infty} \exp(-\lambda^2)\, d\lambda + \frac{1}{4i\mu\xi^2} + \cdots .$$

Having done the above and included the contribution from the singularity at $X = -i/2n$ we then get, after matching the solution to the real axis,

$$u_1 \sim \exp(-x^2)P_1(X) - \frac{\sqrt{\pi}}{\mu}\exp\left(-\frac{1}{4\mu^2}\right)\sin\left(\frac{x}{\mu}\right). \tag{3.18}$$

Similarly, for $n \geq 2$, we have

$$u_n \sim \exp(-nx^2)P_n(X) - \frac{\tilde{R}_n\sqrt{\pi}}{\mu\sqrt{n}}\exp\left(-\frac{1}{4n\mu^2}\right)\sin\left(\frac{x}{\mu}\right). \tag{3.19}$$

The asymptotic behaviour of u for $x \to -\infty$ is, combining (3.18) and (3.19),

$$u \sim \left[-\frac{\sqrt{\pi}}{\mu}\exp\left(-\frac{1}{4\mu^2}\right) - \frac{\sqrt{\pi}}{\mu^3}\sum_{n=2}^{\infty}\frac{\tilde{R}_n}{\sqrt{n}}\mu^{2n}\exp\left(-\frac{1}{4n\mu^2}\right)\right]\sin\left(\frac{x}{\mu}\right). \tag{3.20}$$

The amplitude,

$$A_{mp} = -\frac{\sqrt{\pi}}{\mu}\exp\left(-\frac{1}{4\mu^2}\right) - \frac{\sqrt{\pi}}{\mu^3}\sum_{n=2}^{\infty}\frac{\tilde{R}_n}{\sqrt{n}}\mu^{2n}\exp\left(-\frac{1}{4n\mu^2}\right),$$

is evaluated using the same strategy as in 3.4.1

$$A_{mp} \sim -\frac{\pi\tilde{R}_N \exp\left(-\frac{\sqrt{-2\ln\mu}}{\mu}\right)}{2^{5/4}\mu^{7/2}\sqrt{N}\,(-\ln\,\mu)^{3/4}}. \tag{3.21}$$

Eq.(3.21) has the same form as its counterpart obtained by Fourier transformed matched asymptotics in Akylas and Yang [2] except the recurrence relation $\tilde{R_N}$. Despite this apparent difference between the two expressions, the plot of (3.21) shows no appreciated distinctions between the two expressions for the amplitude of u at far field. The plot of (3.21) is shown in Fig.3.1.

3.6 Discussion

From the previous sections, we see that the tail's amplitude depends on the non-linearity. The solution of the associated linear problem does not have significant contribution to the amplitude of the tail. The solution of (3.1) also reveals that a fast oscillator driven by a much slower oscillator is still capable of producing fast oscillatory behaviour. Adding nonlinearity to this fast oscillator cannot suppress the fast oscillatory motion although this nonlinearity term is very important for an accurate estimate on the solution at the far field.

In next chapter we investigate a model system which consists of two oscillators with different timescales nonlinearly coupled together.

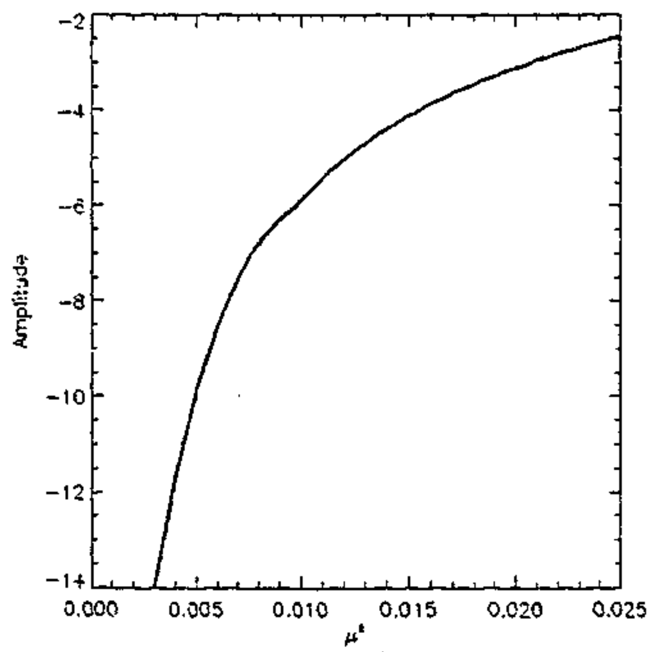

Figure 3.1: The plot of eq.(3.21); u (order of magnitude) vs μ^2.

Chapter 4

A Low Order Conservative Model

4.1 Introduction

In the study of numerical weather forecast, raw field data cannot be used as initial conditions of a primitive-equation (PE) model. Otherwise unrealistically large high-frequency oscillations occur after a short time of numerical integration. This numerical phenomena was first known by Lewis Richardson [34].

Later, Charney realized that this anomaly is associated with free gravity waves which mainly come from measurement errors [15]. He then devised a new set of equations known as 'quasi-geostrophic theory' as a remedy. In this theory sound waves and gravity waves are filtered out and only low frequency Rossby waves prevail.

In fact, quasi-geostrophic model is just a member of family of balanced models. A balanced model is derived from its parent model, usually the PE, as a reduced set of equations by invoking some approximate relations such that the reduced set of equations represents the slow time behaviour of the flow on a lower dimensional manifold in phase space and the high frequency oscillations are filtered out. A balanced model then consists of balance dynamics and balance conditions. Balance dynamics defines a prognostic equation for a chosen slowly evolving variable, e.g. pontential vorticity (PV). Balance conditions consist of the invoked approximate conditions, define diagnostic equations as constraints between the slow time variable, described in the prognostic equation, and the other state variables of the flow .

For example, in the quasi-geostrophic theory mentioned above, the prognostic equation for potential vorticity and a set of diagnostic equations denote a balance dynamics and balance conditions respectively. Once the PV is found by integrating the PV-evolution equation forward in time, one then reconstructs other state variables, such as pressure field and wind fields from the balance conditions via PV inversion [22].

Balanced models, as an approximation of PE, provide qualitative insight of our geophysical fluid system due to the reduced dynamics. However as a tool of weather prediction balanced models succumb to the failure of capturing the so called spontaneous emission of inertia-gravity waves by the vortical flow in an unsteady stratified, rotating vortical flow [16]. This emission affects the mass, energy and momentum budgets of the flow and hence has profound implication on the accuracy of balanced models compared to the PE. (For detail discussion on the issue over the accuarcy of some balanced models versus a PE model see McIntyre *et al.*[30].)

An other approach to minimise the effect of high frequency waves is to project the initial data onto a balanced state before feeding it into the PE. The projection of the initial data onto the balanced state is known as initialization. In 1980 Leith [26] introduced a concept of slow manifold as an idealized stage of any good initialization scheme. A slow manifold is a hypothetical subspace of the dynamical variables' phase-space. Onto this slow manifold, which is devoid of high frequency gravity waves, slow oscillations loosely called Rossby waves, remain slow. The fast oscillations should never be able to penetrate into the vicinity of slow manifold.

Accordingly, initialization schemes are to attempt to purify the field data so that the purified data can force the primitive equations to evolve entirely on the slow manifold, which is a slow-mode-only subspace of the full phase space of the model. On the other hand a balanced model is to describe the dynamics of the flow totally onto a slow manifold where balanced conditions define the slow manifold by relating the fast variables to the slow variables.

Since the introduction of the slow manifold concept, debate over its definition and hence its existence led to a stream of publications (see Warn 1997 [37]; Lorenz 1986 [27]; Lorenz and Krishnamurthy 1987 [29]; Jacobs 1991 [24] Lorenz 1992 [28]; Boyd 1994, 1995 [9, 10]; Camassa 1995 [14]; Bokhove and Shepherd 1996 [7]) via theoretical and numerical studies of some simplified low order models. As a result, instead of this highly idealized concept, people now tend to employ a refined concept called *fuzzy slow manifold* as first postulated by Warn [37]. A fuzzy slow manifold or a slow quasi manifold [17] can be thought of a stochastic layer having varying thickness in the full phase space. This stochastic layer is expected to be very thin in the *fast* direction in which some fast oscillations will eventually occur but the amplitudes could be minimized up to exponentially small [10].

The most studied model in this area is a highly truncated five-mode model devised by Lorenz [27] and its extension [29]. Various methods have been exploited to understand the qualitative properties of these models and hence the large time behaviour of the system *viz* dynamical analysis [14, 7]; asymptotic perturbation [9, 18]; and numerical methods [27, 29].

In this chapter we obtain explicit results for the large time behaviour of this model using asymptotic expansion by expanding the dynamical variables to the fourth order of the small coupling parameter in the problem. We discuss the effect of feedback by the first order fast modes to the slow modes, the frequency shift and aspects in long term averaging. Compatibility conditions are established in order to prevent a growing system which is not possible for the model because of the conservation of energy of this system.

4.2 The Lorenz Model

The model being studied in this chapter was developed by Lorenz in 1986 [27] later extended by Lorenz and Krishnamurthy [29] by including dissipation and forcing into the model. Here the former model will be referred as L86 and the latter will be referred as LK87 and is a subject of next chapter. The model, LK87, is a set of five ordinary differential equations which couples three slow mode amplitudes, representing Rossby modes, and two fast mode amplitudes, representing gravity modes, including dissipation and forcing via two non dimensional parameters shown below,

$$
\begin{aligned}
U_t &= -VW + \epsilon Vz - aU \\
V_t &= UW - \epsilon Uz - aV + F \\
W_t &= -UV - aW \\
x_t &= -z - ax \\
z_t &= x + \epsilon UV - az,
\end{aligned}
\tag{4.1}
$$

where a is the damping coefficient, ϵ is a coupling parameter and F is the forcing which is a constant. L86 is obtained by putting $a = 0$ and $F = 0$. Hence this a logical first step to investigate L86 before we turn to a more realistic model, LK87. L86 can be understood as a nonlinear pendulum, denoted by the slow modes, coupled to a simple harmonic oscillator, denoted by the fast modes, via the parameter ϵ. The parameter ϵ can be regarded as rotational Froude number [7]. Another parameter will appear from the following analysis. For the record we write L86 explicitly

$$U_t = -VW + \epsilon Vz \tag{4.2a}$$

$$V_t = UW - \epsilon Uz \tag{4.2b}$$

$$W_t = -UV \tag{4.2c}$$

$$x_t = -z \tag{4.2d}$$

$$z_t = x + \epsilon UV. \tag{4.2e}$$

Note that there are two integrals of motion for this model

$$U^2 + V^2 = E \tag{4.3}$$

$$-U^2 + W^2 + x^2 + z^2 = K, \tag{4.4}$$

hence all variables stay bounded. It can be expressed in Hamiltonian form as

$$\mathbf{u}_t = J\frac{\delta H}{\delta \mathbf{u}}$$

or

$$\begin{pmatrix} U_t \\ V_t \\ W_t \\ x_t \\ z_t \end{pmatrix} = \begin{pmatrix} 0 & W & 0 & 0 & \epsilon V \\ -W & 0 & -V & 0 & -\epsilon U \\ 0 & U & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -\epsilon V & \epsilon U & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -2U \\ -V \\ W \\ x \\ z \end{pmatrix}$$

where $H = (K - E)/2$ represents the Hamiltonian of the system.

In the following analysis we set the dynamical variable

$$\mathbf{u} = \sum_{n=0}^{\infty} \epsilon^n \mathbf{u}_n$$

and expand it up to $n = 4$. Also (4.2) possesses symmetries. If (U, V, W, x, z) is a solution of (4.2) so are $(-U, -V, W, x, z)$, $(U, -V, -W, -x, -z)$ and $(-U, V, -W, -x, z)$.

4.3 The Leading Order

Consider the zeroth order system, we see that equations for x_0 and z_0 are uncoupled from the slow variables U_0, V_0 and W_0, so the fast variables form a homogeneous system. Since minimum fast oscillations are desirable, we set

$$x_0 = 0 = z_0.$$

Then we are left with

$$\begin{aligned} U_0' &= -V_0 W_0 \\ V_0' &= U_0 W_0 \\ W_0' &= -U_0 V_0. \end{aligned} \tag{4.5}$$

At this order, the slow oscillations come with two integrals of motion

$$U_0^2 + V_0^2 = E_0 \tag{4.6a}$$

$$U_0^2 - W_0^2 = K_0 \tag{4.6b}$$

where (4.6) can be obtained from (4.3) and (4.4) respectively. Therefore

$$U_0 = R\ \mathrm{sech}(Rt)$$

$$V_0 = R\tanh(Rt)$$

$$W_0 = R\ \mathrm{sech}(Rt)$$

is one of the solution sets of (4.5) where $R = \sqrt{E_0}$.

Here R is the other parameter which can characterize the time-scale separation between the slow modes and the fast modes. The time-scale and the amplitudes of the slow modes at the leading order is $O(1/R)$. This implies R needs to be small to get a meaningful time-scale separation. Indeed, R plays a role as a Rossby number here. To simplify the following analysis, we can choose $E = R^2$ so that $E_n = 0$ for $n \neq 0$ which can be done with no loss of generality.

4.4 The Existence of Fast Oscillations

To the first order, $O(\epsilon)$, we observe that x_0 and z_0 are zero and this in turn makes U_1, V_1 and W_1 form a homogeneous system. Thus the solutions are determined up to the integration constants. We choose the constants so that the slow modes are zero at this order. Therefore

$$U_1 = V_1 = W_1 = 0.$$

On the other hand the gravity modes receive excitation from the slow modes of the previous order due to the coupling effect. We get

$$\begin{aligned} x_1' &= -z_1 \\ z_1' &= x_1 + U_0 V_0. \end{aligned}$$

The above can be rewritten as

$$x_1'' + x_1 = -R^2 \operatorname{sech}(Rt)\tanh(Rt)$$

$$\Longrightarrow x_1 = C_1 \cos t + C_2 \sin t - \int_0^t U_0(s)V_0(s)\sin(t-s)ds \tag{4.7}$$

Note that the integral is an odd function of t. As $t \to -\infty$

$$x_1 = \left(C_1 + D_1^-\right)\cos t + \left(C_2 + D_2^-\right)\sin t \tag{4.8}$$

where

$$\begin{aligned} D_1^- &= -\int_{-\infty}^0 U_0 V_0 \sin s\, ds, \\ D_2^- &= \int_{-\infty}^0 U_0 V_0 \cos s\, ds. \end{aligned}$$

Similarly as $t \to \infty$,

$$x_1 = \left(C_1 + D_1^+\right)\cos t + \left(C_2 + D_2^+\right)\sin t \tag{4.9}$$

where

$$\begin{aligned} D_1^+ &= \int_0^\infty U_0 V_0 \sin s\, ds \\ D_2^+ &= -\int_0^\infty U_0 V_0 \cos s\, ds. \end{aligned}$$

Since these two integrands are even and odd functions respectively, we have

$$D_1^+ = -D_1^- \text{ and } D_2^+ = D_2^-.$$

$$\begin{aligned} D_1^+ &= R^2 \int_0^\infty \operatorname{sech}(Rs)\tanh(Rs)\sin s\, ds \\ &= \frac{R}{2}\int_{-\infty}^\infty \operatorname{sech}(Rs)\cos s\, ds \\ &= \frac{\pi}{2}\operatorname{sech}\left(\frac{\pi}{2R}\right). \end{aligned}$$

As $R \longrightarrow 0$,

$$D_1^+ \sim \pi \exp\left(-\pi/2R\right).$$

Note that, there is no closed form for D_2^+, a Fourier sine transform of $\operatorname{sech}(Rx)$. Nevertheless as $R \to 0$, an asymptotic approximation for it is obtained by applying integration by parts twice and found that

$$D_2^+ \sim R^3 + O(R^5)$$

which is not exponentially small. Since, however, D_2^+ is always in combination of C_2, a free constant, it is not of great importance.

4.4.1 Three Choices

At this stage there are three choices for us to prescribe C_1 and C_2

1. Eliminate oscillation as $t \to -\infty$:
   By setting $C_1 = -D_1^-$ and $C_2 = -D_2^-$ the left side of (4.8) becomes zero. This implies, from (4.9),

   $$x_1 \sim 2D_1^+ \cos t \quad \text{as} \quad t \to +\infty.$$

   Fast oscillations persist as the system evolves.

2. Eliminate oscillation as $t \to \infty$:
   By setting $C_1 = -D_1^+$ and $C_2 = -D_2^+$ the left side of (4.9) becomes zero. This implies, from (4.9),

   $$x_1 \sim -2D_1^+ \cos t \quad \text{as} \quad t \to -\infty.$$

   Fast oscillations must exist at the beginning.

3. x_1 is an odd function of t:
   To satisfy a compatibility condition which appears in next section, we put

$C_1 = 0$ to make x_1 an odd function of t. This implies fast oscillations exist as $t \to \pm\infty$. I.e.

$$x_1 \sim \pm D_1^+ \cos t + \left(C_2 + D_2^+\right) \sin t \quad \text{as} \quad t \to \pm\infty.$$

This is necessary to go higher order in the expansion to understand how these exponentially small fast oscillations affect the slow oscillations via nonlinear coupling.

4.5 The Second Order

At second order, the fast modes are identical to zero so the system reduces to

$$\begin{aligned} U_2' &= -W_0 V_2 - V_0 W_2 + V_0 z_1 \\ V_2' &= W_0 U_2 + U_0 W_2 - U_0 z_1 \\ W_2' &= -V_0 U_2 - U_0 V_2 \end{aligned}$$

where

$$z_1 = -x_1'$$

with two invariants from (4.3) and (4.4) respectively

$$E_2 = U_0 U_2 + V_0 V_2 = 0 \tag{4.10}$$

$$K_2 = x_1^2 + z_1^2 + 2W_0 W_2 - 2U_0 U_2 \tag{4.11}$$

Now we are in the position to show that the first two choices shown in §4.4.1 are not compatible with (4.11).

4.5.1 The Persistence of Fast Oscillations

Suppose there is no oscillation as $t \to -\infty$, i.e. choice one. As $t \to |\infty|$, (4.11) becomes $x_1^2 + z_1^2 = K_2$. That means

$$\lim_{t \to -\infty} (x_1^2 + z_1^2) = \lim_{t \to \infty} (x_1^2 + z_1^2)$$

but this contradicts our assumption. The same argument applies to the second choice. Hence only the third choice is compatible with (4.11) and

$$K_2 = D_1^{+2} + \left(C_2 + D_2^{+}\right)^2.$$

Also from (4.10), we have

$$V_2 = \frac{-U_0 U_2}{V_0} \to 0 \quad \text{as } t \to \pm\infty.$$

The asymptotic behaviour of V_2 is determined.

4.5.2 The Feedback Effect

To determine the asymptotic behaviour of W_2 and U_2 and to obtain explicit result of the feedback effect on the slow modes by the fast modes we need to solve the governing set of DEs. This is more convenient to work with if the above system of differential equations is reduced to a single DE. We choose to get a single equation for W_2.

Therefore, after some algebraic manipulations,

$$W_2'' - \frac{4U_0^2 V_0}{V_0^2 - U_0^2} W_2' - \left(V_0^2 - U_0^2\right) W_2 = -\left(V_0^2 - U_0^2\right) z_1 \tag{4.12}$$

This can be verified that one of the solutions for the homogeneous equation of (4.12) is

$$\omega_1 = W_0' = -R^2 \operatorname{sech}(Rt) \tanh(Rt).$$

To find the other solution base, we set $\omega_2 = \omega_1 S$ then substitute this into the homogeneous equation. By virtue of this substitution, this can be readily shown that

$$\begin{aligned} \omega_2 =& \frac{R}{2} \sinh(Rt) \tanh(Rt) + R \operatorname{sech}(Rt) \\ & - \frac{R^2}{2} t \operatorname{sech}(Rt) \tanh(Rt). \end{aligned}$$

ω_1 and ω_2 is an odd function and an even function of t respectively as $|t| \to \infty$,

$$\omega_2 \sim \frac{R}{4} \exp\left(Rt\right).$$

The solution of (4.12) is given by

$$W_2 = \alpha\omega_1 + \beta\omega_2 + \omega_1 \int_0^t \frac{f\omega_2}{D}\, d\hat{t} - \omega_2 \int_0^t \frac{f\omega_1}{D}\, d\hat{t}$$

where

$$D = \omega_1\omega_2' - \omega_1'\omega_2 = -\frac{R^2}{2}\left(V_0^2 - U_0^2\right) = \text{Wronskian},$$

$$f(t) = -\left(V_0^2 - U_0^2\right) z_1$$

and α and β are two integration constants.

Recall that $x_1' = -z_1$,

$$W_2 = \alpha\omega_1 + \beta\omega_2 - \frac{2\omega_1}{R^2} \int_0^t \omega_2 x_1'\, d\hat{t} + \frac{2\omega_2}{R^2} \int_0^t \omega_1 x_1'\, d\hat{t} \tag{4.13}$$

To keep W bounded we need to put the second term and the last term to balance as $|t| \longrightarrow \infty$. For large t the first term is zero and the third term is a bounded oscillatory function. So

$$\beta + \frac{2}{R^2} \int_0^\infty \omega_1 x_1'\, d\hat{t} = 0 \text{ as } t \longrightarrow \infty.$$

Similarly,

$$\beta + \frac{2}{R^2} \int_0^{-\infty} \omega_1 x_1'\, d\hat{t} = 0 \text{ as } t \longrightarrow -\infty.$$

This implies the compatibility condition

$$\int_{-\infty}^{\infty} \omega_1 x_1'\, d\hat{t} = 0. \tag{4.14}$$

The above can be satisfied by setting x_1' to be an even function. I.e. x_1 is to be an odd function by choosing C_1 to zero. Then as $|t| \to \infty$,

$$\begin{aligned} W_2 &\sim -R\exp\left(R|t|\right) \int_\infty^{|t|} z_1 \exp\left(-R\hat{t}\right) d\hat{t} + R\exp\left(-R|t|\right) \int_0^{|t|} z_1 \exp\left(R\hat{t}\right) d\hat{t} \\ &\sim -\frac{2R^2 D_1^+}{R^2+1} \sin|t| + \frac{2R^2\left(C_2 + D_2^+\right)}{R^2+1} \cos t. \end{aligned}$$

To obtain the above expression for W_2 we first split up the second integral in (4.13) into $\int_0^\infty + \int_\infty^{|t|}$ then observe that at large time the contribution only comes from the second part. This results the first integral shown in the first line for W_2 at large time. From

$$U_2 \sim -\frac{W_2'}{R} \quad \text{as } |t| \to \infty, \tag{4.15}$$

we also have

$$U_2 \sim \frac{2RD_1^+}{R^2+1}\cos t + \frac{2R\left(C_2 + D_2^+\right)}{R^2+1}\sin|t| \tag{4.16}$$

as $|t| \to \infty$.

4.6 The Third Order

At $O(\epsilon^3)$,

$$\begin{aligned} x_3' &= -z_3 \\ z_3' &= x_3 + (U_0V_2 + U_2V_0) \\ \implies \quad x_3'' + x_3 &= -(U_0V_2 + U_2V_0)\,. \end{aligned}$$

As $t \to \pm\infty$,

$$\begin{aligned} x_3'' + x_3 &\sim \frac{R^2}{R^2+1}\left[\mp D_1^+\cos t - \left(C_2 + D_2^+\right)\sin t\right] \\ &= -\frac{R^2}{R^2+1}x_1, \end{aligned}$$

so resonant growth is expected. To prevent growth, we have to renormalise the independent variable t by setting

$$\tau = \left[1 + \epsilon^2\omega_2 + O(\epsilon^{-4})\right]t.$$

Next is to substitute the above into

$$x \sim \epsilon x_1 + \epsilon^3 x_3$$

where

$$x_1 \sim D_1^+ \cos t + (C_2 + D_2^+) \sin t$$

$$x_3 \sim \frac{R^2(C_2 + D_2^+)}{2(R^2+1)} t \cos t - \frac{R^2 D_1^+}{2(R^2+1)} t \sin t.$$

Then Taylor-expand the sin and cos functions around τ and equate the sum of square of coefficients assoicated with secular terms to zero. This gives

$$\omega_2 = \frac{R^2}{2\,(R^2+1)}.$$

Hence the frequency shift is

$$1 \longrightarrow 1 + \frac{\epsilon^2 R^2}{2\,(R^2+1)} + O(\epsilon^4).$$

The solution for x_3 can now be readily written

$$x_3 = C_3 \cos t + D_3 \sin t + \bar{R} \int_0^t x_1 \sin(t-s) ds$$

where

$$\bar{R} = -\frac{R^2}{2(R^2+1)},$$
$$x_1 = D_1^+ \cos\left[\left(1+\epsilon^2 \bar{R}\right) t\right] + \left(C_2 + D_2^+\right) \sin\left[\left(1+\epsilon^2 \bar{R}\right) t\right],$$

C_3 and D_3 are integration constants.

Recall that $z_3 = -x_3'$ and in next section we require z_3 to be an even function so we put C_3 equal to zero. Hence

$$x_3 = D_3 \sin t + \bar{R} \int_0^t x_1 \sin(t-s) ds \tag{4.17}$$

4.7 The Fourth Order

From previous sections, we have shown that the existence of fast oscillations and its effect on the slow oscillations due to the coupling. One might argue that using averaging process one can average out the effect of fast modes on the slow modes at least on the theoretical basis. To prove or disprove this statement, we arrive at the fourth order expansion.

$$U_4' = -W_0V_4 - V_0W_4 + V_0z_3 + V_2z_1V_2W_2$$
$$V_4' = W_0U_4 + U_0W_4 - U_0z_3 - U_2z_1 + U_2W_2$$
$$W_4' = -V_0U_4 - U_0V_4 - U_2V_2$$

with

$$2U_0U_4 + U_2^2 + V_2^2 + 2V_0V_4 = E_4 = 0 \tag{4.18}$$

As $t \to \pm\infty$ the first and third terms of (4.18) go to zero. We can write

$$V_4 = \mp\frac{U_2^2}{2R}.$$

Immediately, we can deduce that the average value of V_4 at $t \longrightarrow \pm\infty$ is a non-zero constant. This non-zero constant is a consequence of non vanishing fast modes x and z. Hence even on the sense of averaging, the effect of fast-mode behaviour must be felt by the slow modes indicating the slow modes are under the influence of fast time-scale motions.

To gain more insight we will find asymptotic expressions for the other two slow variables. We choose to work with a single equation for W_4.

$$W_4'' - \frac{4U_0^2V_0}{V_0^2-U_0^2}W_4' - \left(V_0^2 - U_0^2\right)W_4 = I(t)\left(\frac{V_0^2-U_0^2}{U_0}\right) \tag{4.19}$$

where

$$I(t) = \left\{U_2W_2 - \frac{U_2^2+V_2^2}{2} - U_2z_1 + \frac{V_0^2}{2}\left(\frac{U_2^2+V_2^2}{V_0^2-U_0^2}\right) + \left[\frac{V_0}{2}\left(\frac{U_2^2+V_2^2}{V_0^2-U_0^2}\right)\right]' \right.$$

$$\left. -U_0z_3 - \frac{U_0V_0U_2V_2}{V_0^2-U_0^2} - \left(\frac{U_0U_2V_2}{V_0^2-U_0^2}\right)'\right\}.$$

Close examination of terms in $I(t)$ reveals that I is an even function. The solution of (4.19) is

$$W_4 = \gamma_1\omega_1 + \gamma_2\omega_2 - \frac{2\omega_1}{R^2}\int_0^t \frac{I\omega_2}{U_0}\,ds + \frac{2\omega_2}{R^2}\int_0^t \frac{I\omega_1}{U_0}\,ds \tag{4.20}$$

since (4.19) has the same homogeneous part as (4.12).

Armed with (4.10) and $U_2' \sim Rz_1 - RW_2$, we show in appendix A that I is a function of order at least $O(U_0)$ asymptotically. Therefore I/U_0 are bounded. We can now proceed as at $O(\epsilon^2)$. As $|t| \to \infty$ the first term and the third term of (4.20) is zero and oscillatory respectively. We, therefore, require

$$W_4 \sim \gamma_2\omega_2 - \frac{2\omega_2}{R^2}\int_0^{\pm\infty} \frac{I\omega_1}{U_0}\,ds$$

to be finite.

$$\Longrightarrow \gamma_2 = \frac{2}{R^2}\int_0^{\pm\infty} \frac{I\omega_1}{U_0}\,ds$$

$$\Longleftrightarrow \int_{-\infty}^{\infty} \frac{I\omega_1}{U_0}\,ds = 0. \tag{4.21}$$

(4.21) is identically satisfied since the integrand is an odd function.

By virtue of the above analysis, only the third term in (4.20) contributes the residual effect to W_4. In fact W_4 contains constant term and fast oscillations at no higher harmonic than those detected in W_2 as $t \longrightarrow \pm\infty$. To have higher harmonic oscillations we require I to have terms like U_2^2 at order $O(U_0)$ only. However all the terms in I which could have contributed higher harmonic are either at least of $O(U_0^2)$ or they are cancelled out by each other. Hence we infer that there is no higher harmonic oscillations in W_4. In next section the implication of the constant terms found at this order is revealed.

4.8 Discussion

We establish explicit asymptotic expressions for the Lorenz model of atmosphere up to $O(\epsilon^4)$. The model is a simple model which captures the essence of the interaction between the fast and slow modes. The sense of fast and slow is characterised by the other parameter, Rossby number R, which we choose $E_0 = R^2 = E$ in (4.6a) where E is one of the two integrals of motion of (4.2). In practical situation $R << 1$.

At leading order, the system decoupled as two oscillators. The fast modes and the slow modes correspond to a linear oscillator and a nonlinear oscillator respectively. At this order we can set the fast modes to zero leaving the slow modes oscillate in a much slower period. The time-scale for the slow modes is $\tau = Rt$ also the amplitude is proportional to R.

For the fast modes their effect kicks in at $O(\epsilon)$. The existence of these fast modes owe directly to the slow-mode-forcing from previous order, see (4.7). The generation of fast modes due to slow-mode-forcing in this model is analogous to the 'spontaneous radiation of gravity waves' by the vortical flow found by Ford *et al* [17]. This fact illustrates that the method of slaving cannot prevent the generation of fast oscillatory modes. In addition to the above, the elimination of these exponentially small, i.e. $O\left[\exp\left(-R^{-1}\right)\right]$, fast modes at large t by imposing condition of no fast oscillations at $t = \infty$ is shown in § 5.1 not compatible with this system.

At second order, the effect on the slow modes by fast-mode oscillations shows up. W_2 exhibits oscillations with two different time-scales, $O(1)$ and $O(R)$. The $O(R)$ oscillations can be made decay but the $O(1)$ oscillation cannot be removed by the choice of integration constants. Hence this shows that an invariant slow manifold does not exist for this system. At this order, a compatibility condition is established to prevent W_2 from growing without bound. This is reasonable, at this order, to suggest that the effect of this fast oscillations can be averaged out by some averaging process, that is

$$< U_2 >, < W_2 > \ \to 0 \text{ as } |t| \to \infty.$$

To determine whether the same is true at higher order we need to extend the analysis to higher orders.

At third order, the analysis reduces to a classical problem of prevention of secular growth. After renormalisation, we found that the frequency of the fast mode oscillations is shifted. The shifted frequency is

$$1+\frac{\epsilon^2 R^2}{2\left(R^2+1\right)}.$$

Since the correction term is of $O(\epsilon^2 R^2)$, there is no adverse correction on our previous results and we believe this shift has no qualitative effect on the evolution of the system.

So far the asymptotic analysis of (4.2) only shows that the long term dynamics of slow modes, U, V and W, must be modified by the existence of fast modes hence exhibits fast oscillations, albeit the response is exponentially small. One might still hope that averaging process can smooth out the effect of the fast modes. However at $O\left(\epsilon^4\right)$ even the averaging process fails to take away the effect of fast modes on the slow modes. The important consequence of nonvanishing gravity waves at $t=\pm\infty$ turns up at the fourth order expansion. The expansion of (4.3) to $O(\epsilon^4)$ together with (4.16) shows that

$$\begin{aligned} < V_4 > &= \mp\frac{< U_2^2 >}{R} \text{ as } t \to \pm\infty \\ \Longrightarrow \quad < V_4 > &\neq 0 \text{ as } t \to \pm\infty. \end{aligned}$$

Hence

$$< V > \longrightarrow \text{a non zero correction term.}$$

The size of this correction term depends on the amount of fast oscillations in the system. So fast and slow motions are mutual and inseparable. To conclude, the slow manifold of (4.2) which does not contain any fast oscillations does not exist. However, as pointed out by Boyd [9, 10], a manifold which contains fast oscillations whose amplitude is an exponential function of the reciprocal of the Rossby number can be constructed. This type of manifolds resembles the definition of fuzzy slow manifold.

Chapter 5

A Low Order Non-Conservative Model

5.1 Introduction

In Chapter 4 we established analytic solutions up to the fourth order of ϵ which is a Froude number. The results show that the generation of fast oscillations are inevitable meaning no any set of initial conditions can lead the system to a, so called, slow manifold. In other word, nonexistence of slow manifold for this model. In addition, the influence of these fast oscillations on slow modes can't be eliminated by the process of averaging. The strength of fast oscillatory influence is of $O\left[\exp\left(-1/R\right)\right]$ where R is Rossby number. This implies that in principle, any type of balanced model can only be viewed as an approximation to the parent model with irreducible error. This irreducible error represents the mutual existence and the inseparable of fast and slow waves when there is an absence of dissipation and external force acting on it.

Since this is rare to observe in reality that a dynamical system not being subject to any amount of external force and dissipation, we extend the analysis to include forcing and dissipation in the model.

The main aim of this chapter is to understand quantitatively how dissipation and forcing modulate the fast and slow activities and, especially, the influence of fast gravity waves on the slow Rossby waves. We find that dissipation brings in a longer time scale than R^{-1} and this longer time scale is at $O(a)$ where a is a coefficient of dissipation which is smaller than unity to be realistic. The existence of such dissipation does not affect the generation of gravity waves due to the coupling between the fast modes and the slow modes. The only function of such dissipation is to dissipate these generated gravity waves in the longer time scale. To bring out this two-time-scale behaviour of the model the method of matched asymptotics is employed.

5.2 The Lorenz-Kristnamurthy Model

The dynamical system is

$$U_t = -VW + \epsilon Vz - aU \tag{5.1a}$$

$$V_t = UW - \epsilon Uz - aV + F \tag{5.1b}$$

$$W_t = -UV - aW \tag{5.1c}$$

$$x_t = -z - ax \tag{5.1d}$$

$$z_t = x + \epsilon UV - az \tag{5.1e}$$

where ϵ is a Froude number much smaller than unity, a is a dissipation parameter and F is a positive constant forcing term. The solution of (5.1) possesses a set of symmetries

- $U \longrightarrow -U$, $V \longrightarrow -V$, $F \longrightarrow -F$
- $V \longrightarrow -V$, $W \longrightarrow -W$, $x \longrightarrow -x$, $z \longrightarrow -z$, $F \longrightarrow -F$
- $U \longrightarrow -U$, $W \longrightarrow -W$, $x \longrightarrow -x$, $z \longrightarrow -z$

A point $H = (0, F/a, 0, 0, 0)$ is a critical point of (5.1). A linear stability analysis shows that H is hyperbolic (i. e. no pure imaginary eigenvalues) and a stability criteria is given by Lorenz and Krishnamurthy [29]. If $|F/a| > F_c$ where

$$F_c = \sqrt{\frac{a^2(1+a^2)}{1+a^2+a^2\epsilon^2}} \tag{5.2}$$

there is an one dimensional unstable manifold and a four dimensional stable manifold. In such case there are other two critical points which are located at

$$(U, V, W, x, z) = (\pm\mathcal{R}, F_c, \mp\frac{F_c\mathcal{R}}{a}, \mp\frac{\epsilon F_c\mathcal{R}}{1+a^2}, \pm\frac{a\epsilon F_c\mathcal{R}}{1+a^2})$$

where $\mathcal{R} = \sqrt{F_c(F/a - F_c)}$.

Next we set up two energy-type relations for the system. They are

$$\frac{dK}{dt} = -2aK \tag{5.3a}$$

$$\frac{dE}{dt} = -2aE + 2FV \tag{5.3b}$$

where $E = U^2 + V^2$ and $K = x^2 + z^2 + W^2 - U^2$. Immediately the solutions of (5.3) can be written down as

$$K = K_{in} \exp(-2at) \tag{5.4a}$$

$$E = E_{in} \exp(-2at) + 2F \int_0^t V(s) \exp[2a(s-t)]\, ds \tag{5.4b}$$

where K_{in} and E_{in} are the values of K and E at $t = 0$ respectively. Clearly, the above expressions indicate that (5.1) evolves with time scales $T = O(at)$ as well as t.

5.3 Scaling

With the intention of treating (5.1) as a perturbation of a heteroclinic orbit in the case of zero dissipation, zero forcing and zero coupling, i. e. an orbit in a U-V-W phase space which originates from a fixed point and ends at another fixed point, we let $a = \epsilon^2\gamma, F = \epsilon^2 f$ and $T = \epsilon^2 t$ where γ and $f = O(1)$ relative to ϵ. We then transform (5.2) in terms of γ and f, to

$$f > \gamma^2\epsilon^2\sqrt{\frac{1+\gamma^2\epsilon^4}{1+\gamma^2\epsilon^4+\gamma^2\epsilon^6}}.$$

This scaling effectively means the time domain is divided into three regimes, a short-time regime t and two long-time regimes $T_{\pm}$. We define a short-time regime, t, as a region of the time domain where the solutions of (5.1) exhibit only one time-scale behaviour, the short time scale t. On the contrary, in the long-time regimes $T_{\pm}$ multiple time-scales will show up in the solutions where the subscripts represent positive time and negative time. In our study we only have two time scales at work t and T where T is termed as long time scale.

The thickness of the t-regime is $O(\epsilon^{-2})$ on T-scale. The solutions valid in the t-regime and $T_{\pm}$-regimes are called inner-solutions and outer-solutions respectively. The matching between these two types of solutions is done by treating the inner-solutions as initial conditions of the governing equations valid in the T-regimes. Mathematically, we require an intermediate region to exist where both solutions are valid In this region, as $\epsilon \longrightarrow 0$

$$f(t \to \pm\infty) \sim f(T \to 0^{\pm}).$$

This justifies the use of inner-solutions as initial conditions of outer governing equations. The converse is also true that if the inner-solution can match with the outer-solution then there is an intermediate region. This idea is depicted in fig. 5.1

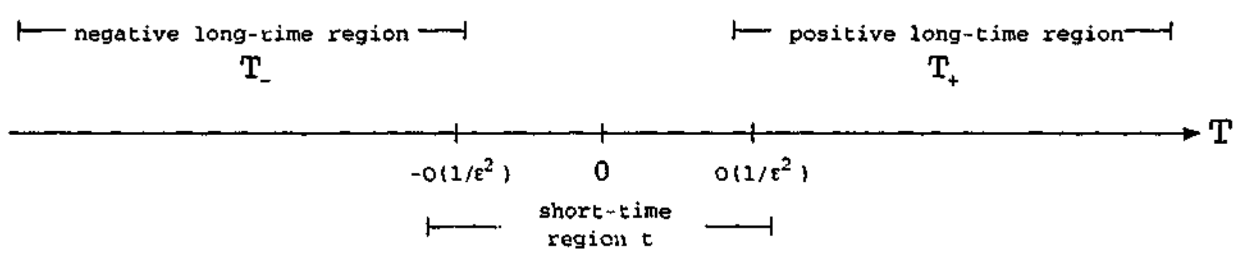

Figure 5.1: Schematic figure of 'short' and 'long' time regions.

Next we have to emphasize that, *a priori*, in the T-regions the leading order of slow modes, U_0, W_0 and V_0, are functions of T only and only the leading order of slow modes have this property. This argument makes sense because the leading order of the slow modes are not subject to any influence which is a function of t through coupling with the fast modes nor through dissipation. On the contrary, we expect x_1 and z_1 will, when they enter into the T-region, inherit some fast-time influence from the coupling with lower order slow modes. Therefore in the $T_\pm$-regimes, (5.1) is transformed to

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \epsilon^2 \frac{\partial}{\partial T} \tag{5.5}$$

with

$$\begin{aligned} U &= U_0(T) + \epsilon^2 U_2(t,T) + \text{ h.o.t.} \\ W &= W_0(T) + \epsilon^2 W_2(t,T) + \text{ h.o.t.} \\ V &= V_0(T) + \epsilon^2 V_2(t,T) + \text{ h.o.t.} \\ x &= \epsilon x_1(t,T) + \epsilon^3 x_3(t,T) + \text{ h.o.t.} \\ z &= \epsilon z_1(t,T) + \epsilon^3 z_3(t,T) + \text{h.o.t.} \end{aligned}$$

where h.o.t. represents higher order terms.

Using the above scaling and (5.4a), we infer that K is a function of T only to all orders, i.e.

$$K = K(T).$$

5.3.1 List of Initial Conditions

We list, for the reader's easy reference, the initial conditions we need to use in the forthcoming analysis. These expressions are obtained from Chapter 4 and thus only valid in the t-regime. The asymptotic sign, $\sim$, is used to indicate that the expressions are valid in the intermediate region i. e. $t \to \pm\infty$. At leading order

$$\begin{aligned} U_0 &= \pm R \operatorname{sech}(Rt) \\ V_0 &= +R \tanh(Rt) \\ W_0 &= \pm R \operatorname{sech}(Rt) \end{aligned} \tag{5.6a}$$

or

$$\begin{aligned} U_0 &= \pm R \operatorname{sech}(Rt) \\ V_0 &= -R \tanh(Rt) \\ W_0 &= \mp R \operatorname{sech}(Rt) \end{aligned} \tag{5.6b}$$

where R is a constant amplitude of slow modes at this order which also characterizes the time scale separation between fast and slow modes to all orders. In the following analysis we choose only to work with the positive branch of (5.6a), i. e.

$$(U_0, V_0, W_0) = [R \operatorname{sech}(Rt), R \tanh(Rt), R \operatorname{sech}(Rt)],$$

since the analysis is the same to the negative branch and (5.6b). The above implies that in the intermediate region

$$U_0 \to 0 \tag{5.7a}$$

$$W_0 \to 0 \tag{5.7b}$$

$$V_0 \sim \pm R \tag{5.7c}$$

as $t \to \pm\infty$.

For the fast modes valid in the fast-time regime, we have

$$x_1 = A \sin t - \int_0^t U_0(s) V_0(s) \sin(t-s) ds \tag{5.8a}$$

$$z_1 = -A \cos t + \int_0^t U_0(s) V_0(s) \cos(t-s) ds \tag{5.8b}$$

where A is an arbitrary constant and x_1 and z_1 are related by the relation

$$\frac{dx_1}{dt} = -z_1.$$

In the intermediate region, $|t| \to \infty$,

$$x_1 \sim D_1^+ \cos t + (A + D_2^+) \sin t \tag{5.9a}$$

where

$$D_1^+ = \frac{\pi}{2}\,\mathrm{sech}\left(\frac{\pi}{2R}\right) \tag{5.9b}$$

$$D_2^+ = -R^2\int_0^\infty \mathrm{sech}\,(Rs)\tanh\,(Rs)\sin s\,ds. \tag{5.9c}$$

At second order, we have

$$V_2 \longrightarrow 0, \tag{5.10a}$$

$$U_2 \sim \frac{2RD_1^+}{R^2+1}\cos t + \frac{2R(A+D_2^+)}{R^2+1}\sin|t|, \tag{5.10b}$$

$$W_2 \sim -\frac{2R^2D_1^+}{R^2+1}\sin|t| + \frac{2R^2(A+D_2^+)}{R^2+1}\cos t. \tag{5.10c}$$

We omit x_3 since its analytic form is not required in the following analysis instead we state

$$V_4 \sim -\frac{U_2^2}{2R}. \tag{5.11}$$

This completed our list of initial conditions.

5.4 Solutions in the Long-Time Regimes

In $T_\pm$-regimes, we immediately see that at $O(1)$

$$-V_0W_0 = 0$$
$$U_0W_0 = 0$$
$$-U_0V_0 = 0.$$

We rule out the trivial solutions and deduce from their initial conditions,

$$U_0 \equiv 0 \equiv W_0$$

in the $T_\pm$ regimes.

The asymptotic behaviour of V_0 can be discovered by considering (5.3b) at $O(\epsilon^2)$. Using (5.5) and equating same order terms, we get

$$\frac{\partial}{\partial t}E_2(t,T) + \frac{d}{dT}E_0(T) = -2\gamma E_0(T) + 2fV_0(T).$$

To prevent a secularity in E_2 we need to set

$$\frac{dE_0}{dT} + 2\gamma E_0 - 2fV_0 = 0. \tag{5.12}$$

In the T-regimes, $E_0 = V_0^2$ due to $U_0 \equiv 0$. Substituting this into (5.12) gives an equation that governs V_0 in $T_\pm$ regions,

$$\frac{dV_0}{dT} + \gamma V_0 = f \tag{5.13}$$

$$\Longrightarrow \quad V_0 = C_0 \exp(-\gamma T) + \frac{f}{\gamma}. \tag{5.14}$$

Using (5.7c) thus yields,

$$V_0 = \begin{cases} \frac{f}{\gamma}\left[1 - \exp(-\gamma T)\right] - R\exp(-\gamma T) & T < 0 \\ \\ \frac{f}{\gamma}\left[1 - \exp(-\gamma T)\right] + R\exp(-\gamma T) & T > 0. \end{cases} \tag{5.15}$$

Clearly on T-scale, V_0 approaches a positive value f/γ from negative infinity, as T increases from $-\infty$ to ∞, with a rapid change of value from $-R$ to $+R$ in a region which has thickness $O(\epsilon^{-2})$. The unboundedness of V_0 at infinite negative time is caused by moving a dissipative system backward in time. On the other hand, (5.13) is just a linearized version of (5.1b), hence $V_0 \to f/\gamma$ is expected. In the following analysis we will consider solutions in the T_+-regime only.

5.4.1 Solutions of the Fast Modes

To describe the evolutions of x_1 and z_1 we require to consider equations at $O(\epsilon)$ and $O(\epsilon^2)$. First at $O(\epsilon)$;

$$\begin{aligned}\frac{\partial x_1}{\partial t} &= -z_1 \\ \frac{\partial z_1}{\partial t} &= x_1.\end{aligned}$$

This gives

$$x_1 = C_1(T)\cos t + C_2(T)\sin t.$$

Using (5.9b) and (5.9c) as initial conditions for C_1 and C_2 we get

$$C_1(0^+) = \frac{\pi}{2}\,\mathrm{sech}\left(\frac{\pi}{2R}\right) \tag{5.16a}$$

$$C_2(0^+) = A + D_2^+. \tag{5.16b}$$

At $O(\epsilon^2)$;

$$\frac{\partial U_2}{\partial t} = V_0 z_1 - V_0 W_2 \tag{5.17a}$$

$$\frac{\partial V_2}{\partial t} = 0 \tag{5.17b}$$

$$\frac{\partial W_2}{\partial t} = -U_2 V_0 \tag{5.17c}$$

Thus,

$$V_2 = V_2(T).$$

Indeed we can determine V_2 from (5.3b) with the same argument used to find V_0 in §5.4. Since

$$E_2 = 2V_0V_2,$$

E_2 is a function of T only. Then at $O(\epsilon^4)$, we have

$$\frac{\partial E_4}{\partial t} + \frac{dE_2}{dT} + 2\gamma E_2 = 2fV_2.$$

Setting the inhomogeneous terms equal to zero to prevent any secularity we then get,

$$\frac{d}{dT}(2V_0V_2) + 4\gamma V_0V_2 = 2fV_2$$

$$\frac{V_0}{V_2}\frac{dV_2}{dT} + \frac{dV_0}{dT} + 2\gamma V_0 = f$$

Using (5.10a) as initial condition and with (5.13) the above becomes

$$\frac{dV_2}{dT} + \gamma V_2 = 0$$

$$\Longrightarrow \quad V_2 \equiv 0.$$

Moreover from (5.17a) and (5.17c) we get

$$\begin{aligned}\frac{\partial^2 U_2}{\partial t^2} - V_0^2 U_2 &= V_0\frac{\partial z_1}{\partial t} \\ &= V_0C_1\cos t + V_0C_2\sin t\end{aligned}$$

The general solution of U_2 is

$$U_2 = K(T)\exp\left[\Gamma(t)\right] + Q(T)\exp\left[-\Gamma(t)\right] - \frac{V_0C_1}{1+V_0^2}\cos t - \frac{V_0C_2}{1+V_0^2}\sin t$$

where

$$\Gamma(t) = \int^t V_0(t')dt',$$

upon substitution of $T = \epsilon^2 t$ in (5.15).

The first term above would lead to an unbounded solution in finite time. However we know from (5.10b) that U_2 is a well behave function in the t-regime, therefore we have to set K to identically zero. In theory Q can be determined by matching the outer solution to the inner solution, which requires a more accurate expression than we have in (5.10b). However, in practice, Q bears no significance since its association with an exponentially decaying term where Γ gives positive values only. We do not consider it in the rest of the analysis. Thus, W_2 can be determined from

$$W_2 = z_1 - \frac{1}{V_0}\frac{\partial U_2}{\partial t}.$$

Having done the above we can proceed to next order where C_1 and C_2 are to be determined.

At $O(\epsilon^3)$;

$$\begin{cases} \frac{\partial x_3}{\partial t} &= -z_3 - \gamma x_1 - \frac{\partial x_1}{\partial T} \\ \\ \frac{\partial z_3}{\partial t} &= x_3 - \gamma z_1 + U_2 V_0 - \frac{\partial z_1}{\partial T} \end{cases}$$

$$\Longrightarrow \quad \frac{\partial^2 x_3}{\partial t^2} + x_3^2 = 2\gamma z_1 + 2\frac{\partial z_1}{\partial T} - U_2 V_0.$$

Using the previous results the inhomogeneous terms can be rewritten as

$$\frac{1}{2}\left[C_{1T} + \gamma C_1 + \frac{V_0^2 C_2}{2\,(1+V_0^2)}\right]\sin t \qquad - \frac{1}{2}\left[C_{2T} + \gamma C_2 - \frac{V_0^2 C_1}{2\,(1+V_0^2)}\right]\cos t.$$

To prevent a secularity we then have to set the first two coefficients above to zero and get

$$\frac{d\boldsymbol{v}}{dT} = M\boldsymbol{v} \tag{5.18}$$

where

$$\boldsymbol{v} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, M = \begin{pmatrix} -\gamma & -\Theta_T \\ \Theta_T & -\gamma \end{pmatrix} \text{ and } \Theta_T = \frac{d\Theta}{dT} = \frac{V_0^2}{2(1+V_0^2)}.$$

To solve this set of first order variable coefficient linear ODEs, one of the ways is to diagonalise M to decouple the equations then transform the solutions back to the original coordinate system. The solutions of (5.18) are

$$v(T) = e^{-\gamma T} \begin{pmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{pmatrix} v_0$$

where v_0 is an initial condition. Indeed, asymptotically

$$\begin{aligned} C_1 &\sim \exp(-\gamma T)\left[C_1(0^+)\cos(sT) - C_2(0^+)\sin(sT)\right], && (5.19) \\ & && (5.20) \\ C_2 &\sim \exp(-\gamma T)\left[C_1(0^+)\sin(sT) + C_2(0^+)\cos(sT)\right] && (5.21) \end{aligned}$$

where

$$s = \frac{f^2}{2\sqrt{f^4 + 2f^2\gamma^2 + \gamma^2}}.$$

The above result implies that x_1 and z_1 decay due to the presence of dissipation but only on the T-scale that is much longer than the oscillatory frequency of these waves.

5.4.2 Higher Order

From (5.3b) V_4 can be found as follows. Recall that

$$\frac{\partial E_4}{\partial t} = 0.$$

Hence

$$\begin{aligned} E_4 &= E_4(T) \\ &= U_2^2 + 2V_0V_4 \\ &= V_0\left[\frac{V_0(C_1^2 + C_2^2)}{2(1+V_0^2)^2} - \frac{V_0C_1C_2}{(1+V_0^2)^2}\sin 2t + \frac{V_0(C_1^2 - C_2^2)}{2(1+V_0^2)^2}\cos 2t + 2V_4\right] \end{aligned}$$

From this we can construct the solution of V_4 so that E_4 is a function of T only, we get

$$V_4 = \frac{V_0C_1C_2}{2(1+V_0^2)^2}\sin 2t - \frac{V_0(C_1^2 - C_2^2)}{4(1+V_0^2)^2}\cos 2t + g(T) \qquad (5.22)$$

where $g(T)$ will be determined in the next section. Substituting (5.22) into the above yields,

$$E_4 = V_0 \left[\frac{V_0(C_1^2 + C_2^2)}{2(1+V_0^2)^2} + 2g(T) \right], \tag{5.23}$$

which is a function of T only.

5.4.2.1 Another Slow-Time Function

At $O(\epsilon^6)$ of (5.3b)

$$\frac{\partial E_6}{\partial t} + \frac{dE_4}{dT} = -2\gamma E_4 + 2fV_4(t,T). \tag{5.24}$$

Then we are required to set

$$\frac{dE_4}{dT} + 2\gamma E_4 - 2fg = 0 \tag{5.25}$$

after substituting (5.22) into (5.24) in order to prevent a secularity from happening to E_6.

Substituting (5.23) into (5.25) and using (5.12) we then get an equation for $g(T)$,

$$\frac{dg}{dT} + \gamma g = -\frac{\gamma V_0}{2(1+V_0^2)^2} - \frac{1}{V_0}\frac{d}{dT}\left[\frac{V_0(C_1^2+C_2^2)}{4(1+V_0^2)}\right] \tag{5.26}$$

$$\begin{aligned} \Longrightarrow g(T) &= C\exp(-\gamma T) \\ &\quad - \exp(-\gamma T)\int_0^T \exp(\gamma s)\left\{\frac{\gamma V_0}{2(1+V_0^2)^2}\right. \\ &\quad \left. + \frac{1}{V_0}\frac{d}{dT}\left[\frac{V_0(C_1^2+C_2^2)}{4(1+V_0^2)}\right]\right\} ds \end{aligned} \tag{5.27}$$

where C is a constant which can be determined from initial condition of V_4. However the exact value of C does not bear qualitative importance and we will not pursue its value. Eq.(5.27) can be evaluated since C_1, C_2 and V_0 are all known functions. Nevertheless the form of (5.27) above can serve the purpose of this study and we do not evaluate the integral. By inspection we infer that g's main contribution to V_4 at large T is adding a constant term to it.

Therefore V_4 contains second harmonic oscillations in the T_+-regime which is subject to exponential decay on T-scale represented by C_1 and C_2 and adds a constant term to V.

5.5 Discussion

The fate of (5.1) is different from the conservative system in the way that first we cannot consider the system in arbitrary large negative time because of the unboundedness occurrence at infinite negative time. Secondly, the fast oscillations will be dissipated in the time scale of $T = \epsilon^2 t$. However the generation of gravity waves is not eliminated by the presence of dissipation.The generation of such waves is totally control by the nonlinear coupling terms. The dissipation rate is at $O(\epsilon^2)$. This means the system will in theory eventually becomes gravity waves free but this stage takes a long time to arrive.

The orbit under our investigation shows tendency to move towards a fixed point $H = (0, F/a, 0, 0, 0)$. We know from linear stability analysis on H that H indeed is a hyperbolic fixed point, however, the fact of U_0 and W_0 being zero in the long-time regime masks the effect of instability of H at least up to the order four. We hence infer that the instability of the orbit may eventually show up from higher order terms of expansion. In fact any instability around H must be caused by the modes U and W at orders higher than we have considered in this chapter since from linear analysis V is a stable mode around H.

Despite our objective of this chapter is the quantitative effect of dissipation and constant forcing, we nevertheless conjecture the fate of the orbit without proof that the orbit will come close to H and then the instability of H will eventually show up and hence the orbit will move away from H.

By observation gravity waves are dissipated much quicker than the Rossby waves, so a modification to (5.1) will be to allow different strength of dissipations to these two types of waves. A smaller dissipative coefficient of Rossby waves will make the system more realistic.

Chapter 6

Concluding Remarks

In this thesis we have analysed two model systems and established their asymptotic solutions.

In the first case study, we revisit a model system which was first studied by Akylas and Yang [2] using the complex-matched asymptotics, a member of the collection of exponential asymptotics. This model is a slowly driven nonlinear oscillator satisfying either a symmetric condition or an one-sided radiation condition at far field. Three different types of functions were used as a forcing term,

1. $f(x) = \operatorname{sech}^2 x$
2. $f(x) = \operatorname{sech} x$
3. $f(x) = \exp(-x^2)$.

The analysis for each forcing term is modified according to the type of the singularities and their location. The first two forcing terms possess a double pole and a single pole respectively and the third one has no singularities in the finite complex-plane. A double(single) pole singularity in the forcing term induces a single(double) pole singularity in the Borel-transformed inner problem. Thus, the analysis for the first forcing is not adequate to handle the second forcing. Hence algorithms that are applicable to the first forcing is not necessary applicable to the second forcing

For the third forcing, the singularities of the problem are masked by the Gaussian forcing. To bring out the hidden singularities we are required to use a special transformation suggested by the expression resulted from the regular asymptotic expansion of the equation. Once the singularities of the transformed problem are located, the complex-matched asymptotics can be applied. It turns out that linear theory is enough to handle the transformed problem. Despite the fact that the transformed problem is linear, linearisation of the original equation cannot give useful estimate of the far field solution. The reason is that in the transformed problem, the forcing term appears in each order of expansion is a result of the nonlinearity from the original problem. If one neglects the nonlinear term in the original problem, a forcing term will be absent in each order of expansion of the transformed problem except at the leading order. As shown in §3.4.1, the contribution of the tail's amplitude comes mainly from the higher order expansion of the transformed problem, thus, an accurate estimate of the tail's amplitude must include the contribution from the nonlinearity of the original problem.

The analysis on these three different types of forcing indicates the robustness of the complex-plane matched asymptotics. As Boyd pointed out in his book [12], this method, and as many other members of exponential asymptotics, lack some mathematical rigours on the assumptions made in the procedures. One of such assumptions is the dominant role played by the singularity closest to the real axis in the outer problem over the other singularities which are further away from the real axis. Another point to note from this model system is that the persistence of fast oscillatory waves at far field despite they are exponentially small and the forcing has a much larger timescale. And this led us to consider a second model system in Chapters 4-5.

The second model system is a low order system called Lorenz model in this thesis. This model allows interactions between fast modes and slow modes and includes dissipation and forcing.

For this model, we found that the generation and persistence of fast oscillations are inevitable although these fast oscillations are exponentially small, i.e. $O[\exp(-1/2R)]$ where R is a timescale separating the fast modes from the slow modes. This implies there is no any point in the phase space where the orbit originates from there can be totally devoid of fast oscillatory motions. The slow modes must exhibit some fast oscillatory behaviour during their excursion in the phase space. The persistence of such fast behaviour in the slow modes also means even in the sense of averaging the slow modes cannot be totally freed from the influence of the fast modes. Therefore a slow manifold for this system does not existence but a fuzzy slow manifold does [37].

This is correct to claim that, at least up to the order considered in this thesis, the dissipation and the constant forcing cause the fast oscillations decay asymptotically with a slow timescale as shown in (5.19).

That would be interesting to consider a Lorenz model with a variable forcing and/or the dissipative coefficients associated with the fast and slow modes being scaled differently.

Appendix A

The Scale of $I(t)$ in Eq.(4.20)

Recall that

$$I(t) = \left\{ U_2 W_2 - \frac{U_2^2 + V_2^2}{2} - U_2 z_1 + \frac{V_0^2}{2}\left(\frac{U_2^2 + V_2^2}{V_0^2 - U_0^2}\right) + \left[\frac{V_0}{2}\left(\frac{U_2^2 + V_2^2}{V_0^2 - U_0^2}\right)\right]' \right.$$

$$\left. -U_0 z_3 - \frac{U_0 V_0 U_2 V_2}{V_0^2 - U_0^2} - \left(\frac{U_0 U_2 V_2}{V_0^2 - U_0^2}\right)' \right\}.$$

We require I to be at least $O(U_0)$ or higher as a necessary condition for the integrals in (4.20) to converge. The posssible candidates in I which are O(1) relative to U_0 are the first, the third and the fifth terms.

Let us expand the term

$$\left[\frac{V_0}{2}\left(\frac{U_2^2 + V_2^2}{V_0^2 - U_0^2}\right)\right]'$$

we then have

$$\frac{V_0 U_2 U_2'}{V_0^2 - U_0^2} + O(U_0).$$

Recall that as $|t| \to \infty$,

$$V_0 \sim R$$
$$U_2 \sim -\frac{W_2'}{R}$$
$$W_2'' \sim -W_2$$
$$-z_1 \sim \frac{W_2'' - R^2 W_2}{R^2}$$

where the last expression is obtained from

$$U_2' = -W_0 V_2 - V_0 W_2 + V_0 z_1.$$

Substitute the above into

$$U_2 W_2 - U - 2z_1 + \frac{U_2 V_0 U_2'}{V_0^2 - U_0^2}$$

$$= U_2 \left(W_2 - \frac{W_2}{R^2} - W_2 + \frac{W_2}{R^2} \right)$$

$$= 0$$

as $|t| \to \infty$.

Therefore the quantity I/U_0 in the integrands of (4.20) are bounded.

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