MONASH UNIVERSITY
THESIS ACCEPTED IN SATISFACTION OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
ON.................7 March 2003.................
.....................[redacted].....
for Sec. Research Graduate School Committee

ERRATA

p iii line 21: "sufficiently" for "sufficient"
p iv line 13: "wave" for "waves"
p 2 line 16: "a case" for "case"
p 3 line 14: "which" for "for which"
p 8 equation (2.1.1): " H_2 " for " $H_2+\zeta$ " and " 0 " for " η "
p 9 line 17: " $z=H_2+\zeta(x,y,t)$ " for " $z=\zeta(x,y,t)$ "
p 11 equation (2.1.12): " $\frac{d^2F_i}{dz^2}-m^2F_i=0$ " for " $\frac{d^2F_i}{dz^2}+m^2F_i=0$ "
p 13 line 8: "parameter" for "parameters"
p 18 line 5: "already" for "which already"
p 19 line 2: "implementing" for "implementation"
p 29 line 3: "... of (2.3.29) and (2.3.30) ..." for "... of (2.3.30) and (2.3.31) ..."
p 50 equation (3.1.1): " H_2 " for " $H_2+\zeta$ " and " 0 " for " η "
p 51 line 6: "then be" for "be then"
p 51 equation (3.1.2): " $\mathbf{V}\cdot\nabla$ " for " $\mathbf{V}\nabla$ "
p 51 equation (3.1.3): " $\mathbf{V}\cdot\nabla$ " for " $\mathbf{V}\nabla$ "
p 62 equation (3.2.21): " $E_X(X,T)$ " for " $E_X(X.T)$ "
p 65 line 9: "transformation" for "transforming"
p 68 line 2: " $B_0\approx 0.95$ " for " $B_0\approx 1.18$ "
p 68 line 17: "... confined to linear ..." for "... assumed to be linear ..."
p 68 line 18: "... $O(B_0^2)$ " for "... $O(B_0)$ "
p 72 line 3: "corresponds" for "correspondent"
p 72 line 17: "parameters" for "parameter"
p 73 line 1: " $V\approx 0.3344$ for $B_0=0.01$ and $V\approx 0.3444$ for $B_0=0.1$ " for " $V\approx 1.0303$ for $B_0=0.01$ and $V\approx 1.2121$ for $B_0=0.1$ "
p 73 line 4: " $B_0=0.00969$ and $B_0=0.10966$ " for " $B_0=0.00995$ and $B_0=0.08773$ "
p 74 line 1: "... (3.2.35), (3.2.36, ..." for "... (3.2.25), (3.2.26, ..."
p 76 equation (3.4.5): " $\int_{-\infty}^{+\infty}\mathbf{R}^T\mathbf{q}_t dx=\int_{-\infty}^{+\infty}\mathbf{R}^T\mathbf{J}\mathbf{R}_x dx=\int_{-\infty}^{+\infty}(\mathbf{R}^T\mathbf{J}\mathbf{R})_x dx=0$ " for
" $\int_{-\infty}^{+\infty}\mathbf{R}\mathbf{q}_t dx=\int_{-\infty}^{+\infty}\mathbf{R}\mathbf{J}\mathbf{R}_x dx=\int_{-\infty}^{+\infty}(\mathbf{R}\mathbf{J}\mathbf{R})_x dx=0$ "
p 77 line 2: "formulating" for "formulation"
p 83 line 4: "... in equations (3.5.6) ..." for "... into equations (3.5.6) ..."
p 83 line 14: "due to" for "due"
p 87 equation (3.5.42): " $v_z(\ldots,z=\pm 0)\eta+\ldots\}\eta_y$ " for " $v_z(\ldots,z=\pm 0)\eta+\ldots\}\eta_x$ "

ADDENDUM

p 17 line 1: Delete "shallow water" and read "...this approach..."

p 21 line 4: Replace ε with $\hat{\varepsilon}$ and read "... $\kappa_{1,2}=O(\hat{\varepsilon})$, where $\hat{\varepsilon}$ ..."

p 44 line 1: Add after "... and *y*-direction." "Domain is doubly-periodic with $-32<x<32$ and $0<y<8$. Number of modes in *x*- and *y*-directions were 384 and 64 respectively; the time-step equalled 0.5×10^{-4}. The results have been checked with respect to spatial resolution of 256×32."

Addendum cont'd

p 44 line 5: Add after "...noise)." "The white noise was a random field superimposed on the solitary waves with magnitude of 0.01 (maximum and minimum values of 0.01 and –0.01 respectively)."

p 49 line 6: Replace the sentence "The evolution ..." with "The difference in the evolution of the two modes displayed in Figures 2 – 4 may be explained by the fact that the linear instability becomes quickly saturated by nonlinear instabilities, such that the η-mode becomes nonlinearly stable whereas the ζ-mode is left nonlinearly unstable."

p 63: Add after equation (3.2.29) "Coefficients C_2 and D_2 vanished due to the extremely weak long-wave dispersion determined by the resonance (3.1.33), (3.1.34) that eventually results in the absence of a dispersion term in the final equation (3.2.35)."

p 65 line 3: Add after the word "derivatives" "(see also comments on the equivalent conditions (1.9) and (1.14))"

p 76 line 15: Delete "for" and read "... the canonical Hamiltonian ..."

p 88 line 12: Add after the word "... orders.": "This jump of the vertical velocity does not have a physical reason and is caused only by the discontinuity of the z-derivate of the basic flow (3.5.20) across both interfaces, as $\Omega_1 \neq \Omega_2$ and $\Omega_2 \neq \Omega_3$."

NONLINEAR COUPLED WAVES IN STRATIFIED FLOWS

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Doctor of Philosophy

by

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CONTENTS

ABSTRACT

The object of this thesis is the investigation of coupled nonlinear waves propagating in three-layer inviscid fluid flows. After first discussing the concept of resonance conditions admitting coupled waves, we consider two different models of such flows. In the first model (model I) both the basic flow and density are assumed to be constant within each layer. In model II the basic flow is taken to be of the form of a piecewise linear function which is continuous across the unperturbed interfaces, while the density is piecewise constant. For both models the appropriate resonance conditions, under which the system admits coupled waves, are obtained. In a small vicinity of these resonances different sets of coupled equations are derived. The properties of these equations are then explored both analytically and numerically.

Coupled waves in model I are investigated in Chapter 2. Three-dimensional linear perturbations are initially considered, from which appropriate resonance conditions are obtained. On this basis a pair of coupled nonlinear Kadomtsev-Petviashvili (KP) equations are derived to describe the evolution of two-dimensional interfacial waves. The stability of two KdV solitons is then considered. The coupled KP equations are used to derive equations governing the evolution of the amplitudes and phases of these solitons due to both coupling and transversity. The stability of the system of two KdV solitons with respect to transverse perturbations is explored in section 2.4. In the case of *no radiation* a linear stability criterion is obtained, which demonstrates that the system is always unstable with respect to any transverse perturbations of sufficient wide spectrum. These analytical results are then confirmed numerically. The important role of coupling in the generation of two-dimensional solitary-like patterns is demonstrated, namely that the coupled KP equations can describe the formation of two-dimensional

solitary-like waves, while KP equations cannot admit two-dimensional solitary-like solutions under the same conditions.

Model II is investigated in Chapter 3. Initially the resonance conditions for two-dimensional perturbations of the flow II are considered. A pair of one-dimensional coupled KdV-like nonlinear equations is then derived in a small vicinity of these resonance conditions, following which some approximate (numerical and asymptotic) solutions of the coupled equations are described. It is then shown that these one-dimensional equations can be rewritten in canonical Hamiltonian form, and it is proven that it is a Hamiltonian system. Four invariants of the system are found. Three-dimensional perturbations of model II are then considered. The appropriate resonance conditions are explored, and then a set of two-dimensional coupled nonlinear waves are rigorously derived. Some results for the numerical simulation of the evolution of a solitary waves are finally given.

STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university or other institution and, to the best of my knowledge, contains no material previously published or written by another person, except where due reference is made in the text.

[redacted]

Yuri Skrynnikov

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CHAPTER 1

INTRODUCTION

The study of complex wave systems is the basis of modern Meteorology and Oceanography. Physically such systems often consist of solitary, large-scale wave-like formations travelling in a stratified fluid. The modelling and simulation of such systems and forecasting their evolution and behaviour is a complex mathematical problem requiring the implementation of modern methods of nonlinear dynamics and powerful computer modelling. This thesis is devoted to the investigation of the properties of a system of two coupled nonlinear interfacial waves arising in two different models of three-layer fluids and due to a special form of resonance between two wave modes.

It is well known that instability of some fluid flows can arise due to resonance between two waves when their phase speeds coincide under certain conditions (see, for example, *Chandrasekhar* 1961, *Drazin and Reid* 1981, *Craik* 1985, *Baines and Mitsudera* 1994). Such a resonance condition can be determined by a specific value of some external parameter of the system. Deviation of the external parameter from its resonance value may then lead to the coupling of wave modes.

It was *Grimshaw* (2000) who first determined those resonance conditions under which two wave modes must be coupled. Without loss of generality we can use the following long-wave dispersion relation

$$c^2 = \Delta^2 + \delta \tag{1.1}$$

to instantiate his concept. Here $c = \omega/k$ is the phase speed, Δ is an external parameter (it may be some intrinsic speed in the system) and δ is an unfolding parameter ($\delta << 1$). Plainly, such system in the resonance when $\delta = 0$, while $\delta > 0$ ($\delta < 0$) represents stable (unstable) conditions. The dispersion relation (1.1) can come from the following linear algebraic equations

$$\begin{cases} (c-\Delta)A - \kappa_1 B = 0, \\ -\kappa_2 A + (c+\Delta)B = 0, \end{cases} \tag{1.2}$$

describing a system of two waves in terms of Fourier harmonics. Here A, B are amplitudes of modes and $\kappa_1\kappa_2 = \delta$. At resonance $\delta = 0$, and at least one of these factors (κ_1 or κ_2) equals zero. If one of them is nonzero, say $\kappa_2 \neq 0$, then we must put $c = \Delta$ as neither A nor B can be zero. The first equation in (1.2) then becomes identically zero, while the second one states the relationship between the amplitudes A, B such that one of them can be expressed in terms of another, for instance, $A = \dfrac{2\Delta}{\kappa_2}B$. Thus in such a case there is actually only one independent mode (A or B). The canonical model for long waves in such case is described by two separate Boussinesq equations without coupling, as follows

$$A_{tt} - \Delta^2 A_{xx} + \frac{1}{2}\mu\left(A^2\right)_{xx} + \lambda A_{xxxx} = \delta A_{xx}, \tag{1.3}$$

$$B_{tt} - \Delta^2 B_{xx} + \frac{1}{2}\mu\left(B^2\right)_{xx} + \lambda B_{xxxx} = \delta B_{xx}. \tag{1.4}$$

Here μ and λ are the nonlinear and dispersive coefficients respectively. Equations of this form have been derived by *Hickernell* (1983a,b) for the Kelvin-Helmholtz instability, and by *Helfrich and Pedlosky* (1993) and *Mitsudera* (1994) for certain geophysical flows.

These modes have different behaviour if both $\kappa_1 = 0$ and $\kappa_2 = 0$. In this case other parameters must be equated to zero, e.g. $c = \Delta = 0$, as the amplitudes A, B must be nonzero. Accordingly, there is no relationship between A and B at resonance, as result they cannot be described by separate equations such as (1.3), (1.4). A suitable canonical model for such modes consists of two coupled Korteweg-de-Vries (KdV) equations,

$$A_t + \Delta A_x + \mu_1 A A_x + \lambda_1 A_{xxx} + \kappa_1 B_x = 0, \tag{1.5}$$

$$B_t - \Delta B_x + \mu_2 B B_x + \lambda_2 B_{xxx} + \kappa_2 A_x = 0. \tag{1.6}$$

Here $\kappa_{1,2}$ are coupling parameters. Equations of the form (1.5), (1.6) have been derived by *Mitsudera* (1994) and *Gottwald and Grimshaw* (1999) for certain geophysical flows, and by *Grimshaw* (2000) for a certain three-layer stratified shear flow.

In the general case (*Grimshaw and Skrynnikov* 2002[1]), when the linearised problem for a system of two waves, for which solutions are proportional to $\exp\{ik(x-ct)\}$, can be reduced to the following algebraic problem for the wave amplitudes A, B

$$\begin{cases} D_1 A + EB = 0, \\ EA + D_2 B = 0, \end{cases} \tag{1.7}$$

with coefficients $D_{1,2}$, E dependent on k, c and a set of parameters $\Delta_1, \Delta_2, \ldots$ which for the sake of convenience will be denoted below as a vector parameter $\mathbf{\Delta} = (\Delta_1, \Delta_2, \ldots)$. The dispersion relation

$$D \equiv D_1 D_2 - E^2 = 0 \tag{1.8}$$

[1] All ideas expounded hereafter in this chapter are due to Prof. R. Grimshaw.

defines the phase speed as a function of k and Δ. As in non-dissipative systems instability arises whenever $\mathrm{Im}(c) \neq 0$. Thus we must add to (1.8) the following necessary condition of instability

$$D_c = 0, \qquad D_{cc} \neq 0 \tag{1.9}$$

so as not to exclude the scenario of unstable behaviour for this system. Here the subscript[1] denotes the derivative with respect to c. Then taking the limit $k \to 0$ suppresses the k-dependence for further consideration. Now conditions (1.8), (1.9) define critical values of c and Δ for the onset of instability (resonance). Without loss of generality we can suppose that these are $c = 0$ and $\Delta = \mathbf{0}$.

First, let us suppose that $E \neq 0$ at criticality. Then from (1.8) it follows that $D_{1,2} \neq 0$ also, and the system (1.7) can be reduced to a single equation

$$D(c, \Delta, k)A = 0 \tag{1.10}$$

or $D(c, \Delta, k)B = 0$, as either of amplitudes A, B can be expressed in terms of other. Then expansion of D as an operator in c, Δ and k, followed by incorporation of a weakly nonlinear term, leads the Boussinesq equations (1.3), (1.4). Next, if $E = 0$ at criticality, then so does $D_1 D_2 = 0$. However, if either of $D_{1,2} \neq 0$, say $D_2 \neq 0$, then the system (1.7) can be again reduced to (1.10), and the Boussinesq equations (1.3), (1.4) are again the outcome.

The case of most interest to us here is when

$$E = D_1 = D_2 = 0 \tag{1.11}$$

at criticality, from which we see immediately that A and B are independent at resonance. It is this case which leads to two coupled equations of the form (1.5), (1.6).

---

[1] Throughout this text we will keep the suffixal denotation for derivatives, and the comma will separate differentiation subscripts from others if the latter takes place.

So, the general consideration based on (1.7) leads to the same conclusion obtained for the particular case (1.2). Namely, in order to find the resonance conditions, under which two-wave system may be described by coupled equations, we must resolve the simultaneous equations (1.11) at the long-wave limit.

Let us now show that equations (1.7) are reducible to the form of (1.2). After expanding D_1, D_2 and E in powers of c and Δ we can obtain

$$\begin{cases} \left(cD_{1,c} + \Delta D_{1,\Delta}\right)A + \left(cE_c + \Delta E_\Delta\right)B = 0, \\ \left(cE_c + \Delta E_\Delta\right)A + \left(cD_{2,c} + \Delta D_{2,\Delta}\right)B = 0. \end{cases} \tag{1.12}$$

We can change variables as follows

$$\begin{cases} \widetilde{A} = D_{1,c}A + E_c B, \\ \widetilde{B} = D_{2,c}B + E_c A. \end{cases} \tag{1.13}$$

Note that the above transformation is not singular as according to the second condition at (1.9)

$$K = \frac{1}{2}D_{cc} = D_{1,c}D_{2,c} - E_c^2 \neq 0\,. \tag{1.14}$$

Then equations (1.12) adopt the form

$$\begin{cases} \left(c - \Delta_1\right)\widetilde{A} - \kappa_1 \widetilde{B} = 0, \\ -\kappa_2 \widetilde{A} + \left(c - \Delta_2\right)\widetilde{B} = 0, \end{cases} \tag{1.15}$$

which is very similar to (1.2). Here

$$
\begin{aligned}
K\Delta_1 &= \left(-D_{2,c}D_{1,\Delta} + E_c E_\Delta\right)\Delta, \\
K\Delta_2 &= \left(-D_{1,c}D_{2,\Delta} + E_c E_\Delta\right)\Delta, \\
K\kappa_1 &= \left(-D_{1,c}E_\Delta + E_c D_{1,\Delta}\right)\Delta, \\
K\kappa_2 &= \left(-D_{2,c}E_\Delta + E_c D_{2,\Delta}\right)\Delta.
\end{aligned}
\tag{1.16}
$$

An important consequence of the obtained equations (1.15) is a simple criterion for this system to be unstable, that is instability occurs whenever

$$
(\Delta_1 - \Delta_2)^2 + 4\kappa_1\kappa_2 < 0. \tag{1.17}
$$

Finally, let us summarise results described in this chapter to outline a general procedure of how to construct equations governing coupled modes. First, the linear problem for a system of two waves should be reduced to the form of (1.7). Then all the coefficients in (1.7) should be taken in the limit $k \to 0$ to compose a set of simultaneous equations (1.11). The solution of these equations gives critical values of the model parameters that determine the critical (resonance) state of the system. At the final stage the basic equation of the system should be unfolded in a small vicinity of the critical state to obtain coupled equations.

The above approach is employed in the following chapters for the derivation of one- and two-dimensional nonlinear equations governing coupled interfacial waves in two different models of three-layer fluids. The three-layer fluids were chosen for investigation since these are the simplest models of stratified fluids, which give opportunity to focus on coupling of two waves only. Although this is admittedly a special case, the method of analysis described above indicates that the same outcome may be expected in more general cases.

Two different three-layer models have been considered in this thesis: layers with constant flow and density (I) and layers with constant density but with flow described by a continuous linear function with different gradient in each layer (II). For the model I, which is considered in Chapter 2, a set of nonlinear Kadomtsev-Petviashvili equations (2.2.21) and (2.2.22) has been obtained. These equations are a natural extension of the coupled KdV equations (1.5), (1.6) to two dimensions: all of them contain only linear coupling terms.

Equations of a different structure describe the interfacial waves of model II, which is considered in Chapter 3. In this case a set of KdV-like equations (3.2.35), (3.2.36) has been derived. The main feature of these equations is that they are coupled due to both linear and nonlinear terms. Thus linear analysis is not sufficient to define the coupling structure of this two-wave system. Another feature of equations (3.2.35), (3.2.36) is the absence of a dispersive term in one of the equations. Extension of these equations to two dimensions yields the set (3.5.64), (3.5.65), the first equation of which contains neither dispersive nor transverse terms.

In Chapter 4 all the main results of this thesis are summarised and discussed.

CHAPTER 2

COUPLED WAVES IN A THREE-LAYERED FLUID WITH A PIECEWISE CONSTANT BASIC FLOW

2.1. LINEAR APPROXIMATION

Let us consider the three-layered fluid with configuration shown schematically in Figure 1. The basic horizontal flow $U(x,y,z,t)$ and density $\rho(x,y,z,t)$ are piecewise constant functions given by

$$U = \begin{cases} U_3, & H_2+\zeta < z < H_2+H_3, \\ U_2, & \eta < z < H_2+\zeta, \\ U_1, & -H_1 < z < \eta, \end{cases} \tag{2.1.1}$$

$$\rho = \begin{cases} \rho_3, & H_2 + \zeta < z < H_2 + H_3, \\ \rho_2, & \eta < z < H_2 + \zeta, \\ \rho_1, & -H_1 < z < \eta, \end{cases} \tag{2.1.2}$$

where H_i is a height of the unperturbed *i-th* layer $i = 1, 2, 3$ (counting from the bottom), $\eta = \eta(x, y, t)$ and $\zeta = \zeta(x, y, t)$ are perturbations of the lower and upper interfaces respectively (see Figure 1).

The flow is assumed to be inviscid and incompressible so that it may be considered in each layer to be irrotational with a velocity potential ϕ_i, such that incompressibility condition takes the form of the Laplace's equation as follows

$$\phi_{i,xx} + \phi_{i,yy} + \phi_{i,zz} = 0. \tag{2.1.3}$$

The x, y and z components of perturbation velocities are $u_i = \phi_{i,x}$, $v_i = \phi_{i,y}$, $w_i = \phi_{i,z}$ respectively. Here subscript $i = 1, 2, 3$ denotes the index of a layer and $f_{i,x} \equiv \partial f_i / \partial x$ [1].

To describe this model the Laplace's equation should be supplemented by appropriate boundary conditions. First we will use the rigid plane condition at both upper and lower fixed boundaries putting vertical velocities $\phi_{3,z} = 0$ at $z = H_2 + H_3$ and $\phi_{1,z} = 0$ at $z = -H_1$ respectively. Another set of boundary conditions are given at the lower $z = \eta(x, y, t)$ and upper $z = \zeta(x, y, t)$ interfaces comprises both kinematic boundary conditions

$$\eta_t + (U_i + \phi_{i,x})\eta_x + \phi_{i,y}\eta_y = \phi_{i,z}, \quad i = 1, 2, \tag{2.1.4}$$

$$\zeta_t + (U_i + \phi_{i,x})\zeta_x + \phi_{i,y}\zeta_y = \phi_{i,z}, \quad i = 2, 3, \tag{2.1.5}$$

---

[1] Throughout this text we will keep the suffixal denotation for derivatives and the comma will separate differentiation subscripts from others if the latter takes place.

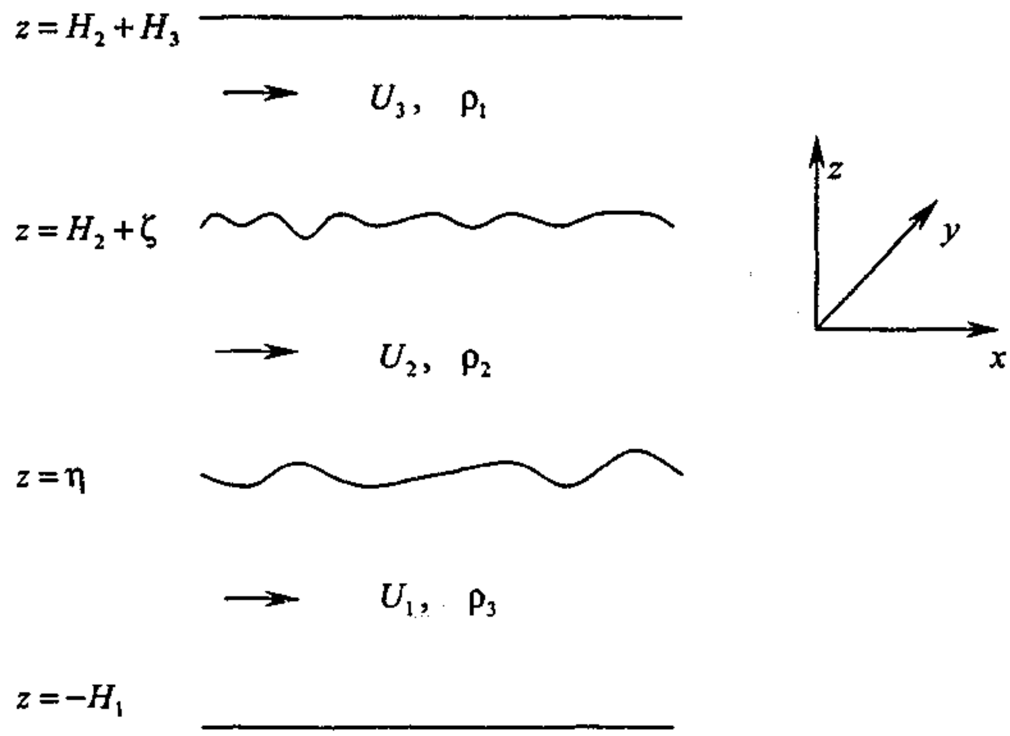

Figure 1. Configuration of three-layered fluid model.

and the pressure boundary conditions

$$\left[\rho\left(\phi_t + U\phi_x + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 + \frac{1}{2}\phi_z^2 + gz\right)\right]_{z=\eta} = 0, \tag{2.1.6}$$

$$\left[\rho\left(\phi_t + U\phi_x + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 + \frac{1}{2}\phi_z^2 + gz\right)\right]_{z=\zeta} = 0, \tag{2.1.7}$$

where

$$\left[f(x,y,z,t)\right]_{z=z_0} \equiv \lim_{\varepsilon\to 0^+}\{f(x,y,z_0+\varepsilon,t) - f(x,y,z_0-\varepsilon,t)\} \tag{2.1.8}$$

is a jump of a function $f(x,y,z,t)$ across a plane $z = z_0$.

After linearising equations (2.1.4) to (2.1.8) we can seek their solutions of the form

$$\eta = A\exp(ikx + ily - i\omega t) + c.c., \tag{2.1.9}$$

$$\zeta = B\exp(ikx + ily - i\omega t) + c.c., \tag{2.1.10}$$

$$\phi_i = F_i(z)\exp(ikx + ily - i\omega t) + c.c., \tag{2.1.11}$$

where *c.c.* denotes complex conjugation. Substituting (2.1.11) into (2.1.3) results in ordinary differential equations of the form

$$\frac{d^2F_i}{dz^2} + m^2F_i = 0, \quad m^2 = k^2 + l^2, \quad i = 1,2,3, \tag{2.1.12}$$

with solutions

$$F_1 = K\cosh(m(z + H_1)), \tag{2.1.13}$$

$$F_2 = L\cosh(mz) + M\sinh(mz), \tag{2.1.14}$$

$$F_3 = N\cosh(m(z - H_2 - H_3)), \tag{2.1.15}$$

Functions (2.1.11) with F_i given by expressions (2.1.13) – (2.1.15) automatically satisfy the rigid plane condition at both rigid boundaries. Substitution of them into the linearised version of boundary conditions (2.1.4) – (2.1.7) given at the unperturbed interfaces at $z = 0$ and $z = H_2$ followed by eliminating constants K, L, M and N yields the following equations for the amplitudes A and B of the interfaces disturbances

$$\begin{cases} D_1 A + EB = 0, \\ EA + D_2 B = 0, \end{cases} \tag{2.1.16}$$

where

$$D_1 = g(\rho_1 - \rho_2) - \frac{m\rho_1(c - \kappa_x U_1)^2}{\tanh(mH_1)} - \frac{m\rho_2(c - \kappa_x U_2)^2}{\tanh(mH_2)}, \tag{2.1.17}$$

$$D_2 = g(\rho_2 - \rho_3) - \frac{m\rho_2(c - \kappa_x U_2)^2}{\tanh(mH_2)} - \frac{m\rho_3(c - \kappa_x U_3)^2}{\tanh(mH_3)}, \tag{2.1.18}$$

$$E = \frac{m\rho_2(c - \kappa_x U_2)^2}{\sinh(mH_2)}, \tag{2.1.19}$$

$c = \omega/m$ is the phase velocity; $\kappa_x = k/m = k/\sqrt{k^2 + l^2}$. So if $\lim_{m \to 0} \kappa_x$ exists and is equal to α then $0 \le \alpha \le 1$ as $0 < \kappa_x < 1$ and $\alpha = 0$ if $k = o(l)$ and $\alpha = 1$ if $l = o(k)$.

To obtain conditions for resonance between two modes A and B we must take the long-wave limit as $m \to 0$. Then the coefficients (2.1.17) – (2.1.19) become the following expressions

$$\widetilde{D}_1 = \frac{\rho_1}{H_1}\left\{c_1^2 - (\widetilde{c} - \alpha U_1)^2 - \frac{\rho_2 H_1}{\rho_1 H_2}(\widetilde{c} - \alpha U_2)^2\right\}, \tag{2.1.20}$$

$$\widetilde{D}_3 = \frac{\rho_3}{H_3}\left\{c_3^2 - (\widetilde{c} - \alpha U_3)^2 - \frac{\rho_2 H_3}{\rho_3 H_2}(\widetilde{c} - \alpha U_2)^2\right\}, \tag{2.1.21}$$

$$\widetilde{E} = \frac{\rho_2(\widetilde{c} - \alpha U_2)^2}{H_2}, \tag{2.1.22}$$

where

$$c_1^2 = \frac{g(\rho_1 - \rho_2)H_1}{\rho_1}, \qquad c_3^2 = \frac{g(\rho_2 - \rho_3)H_3}{\rho_3} \tag{2.1.23}$$

and $\widetilde{D}_{1,2}, \widetilde{E}, \widetilde{c} = \lim_{m\to 0}(D_{1,2}, E, c)$, i.e. the tilde indicates the value of a variable after taking the limit as $m \to 0$.

Simultaneously putting $\widetilde{D}_1 = \widetilde{D}_2 = \widetilde{E}_1 = 0$ we then obtain the following parameters restraints yielding resonance

$$\widetilde{c} = \alpha U_2 = \alpha U_1 \pm c_1 = \alpha U_3 \pm c_3. \tag{2.1.24}$$

If $\alpha = 0$ (i.e. $k = o(l)$) then the phase speed $\widetilde{c}$ should be equal to both c_1 and c_3 so that $c_1 = c_3$. This case is not of interest here as we are seeking the description of perturbations propagating primarily along the direction defined by that of the basic flow (i.e. along x-axis). If $\alpha = 1$ (i.e. $l = o(k)$) then we have the same conditions as those obtained by *Grimshaw* (2000) for the one-dimensional model. Without loss of generality we may set $U_2 = 0$ and take $U_3 > 0$ to obtain the following resonance conditions

$$\tilde{c} = 0, \qquad U_1 = \mp c_1, \qquad U_3 = c_3. \tag{2.1.25}$$

Note that in the case with $\alpha \neq 0$ after the transformation $\alpha U_i \to U_i$, $i = 1, 2, 3$ the resonance conditions become the same as for the case with $\alpha = 1$. So, taking $\alpha \neq 0$ does not yields new resonance states. Accordingly we take the case $\alpha = 1$ for further consideration.

Now suppose that the system is not exactly at the resonance state defined by (2.1.25), but located in a small neighbourhood of it. Its deviation from the resonance is then characterised by a small parameter ε, such that

$$U_1 = \mp c_1 + \delta_1, \qquad U_3 = c_3 + \delta_3 \tag{2.1.26}$$

with $\delta_{1,3} = O(\varepsilon^2)$, while

$$U_2 = O(\varepsilon), \qquad c = O(\varepsilon^2), \qquad k = O(\varepsilon), \qquad l = O(\varepsilon^2). \tag{2.1.27}$$

We now can unfold the coefficients of (2.1.16) as follows

$$D_1 = \frac{2\rho_1 U_1 (c - \delta_1)}{H_1} - \frac{\rho_2 U_2^2}{H_2} - \frac{1}{3} U_1^2 \rho_1 H_1 k^2 + \frac{\rho_1 U_1^2}{H_1} \frac{l^2}{k^2} + O(\varepsilon^3), \tag{2.1.28}$$

$$D_2 = \frac{2\rho_3 U_3 (c - \delta_3)}{H_3} - \frac{\rho_2 U_2^2}{H_2} - \frac{1}{3} U_3^2 \rho_3 H_3 k^2 + \frac{\rho_3 U_3^2}{H_3} \frac{l^2}{k^2} + O(\varepsilon^3), \tag{2.1.29}$$

$$E = \frac{\rho_2 U_2^2}{H_2} + O(\varepsilon^3). \tag{2.1.30}$$

After substituting (2.1.28) – (2.1.30) into (2.1.16) and replacing ik, il, $i\omega$ by $\partial/\partial x$, $\partial/\partial y$, $-\partial/\partial t$ respectively we obtain the following coupled linear Kadomtsev-Petviashvili (KP) equations

$$\left(\eta_t + \Delta_1\eta_x + \lambda_1\eta_{xxx}\right)_x + \gamma_1\eta_{yy} + \kappa_1\zeta_{xx} = 0\,, \tag{2.1.31}$$

$$\left(\zeta_t + \Delta_2\zeta_x + \lambda_2\zeta_{xxx}\right)_x + \gamma_2\zeta_{yy} + \kappa_2\eta_{xx} = 0\,, \tag{2.1.32}$$

with coefficients given by

$$\Delta_1 = \delta_1 + \frac{1}{2}\frac{U_2^2}{U_1}\frac{\rho_2 H_1}{\rho_1 H_2}, \qquad \Delta_2 = \delta_3 + \frac{1}{2}\frac{U_2^2}{U_3}\frac{\rho_2 H_3}{\rho_3 H_2}, \tag{2.1.33}$$

$$\lambda_1 = -\frac{1}{6}U_1 H_1^2, \qquad \lambda_2 = -\frac{1}{6}U_3 H_3^2, \tag{2.1.34}$$

$$\gamma_1 = -\frac{U_1}{2}, \qquad \gamma_2 = -\frac{U_3}{2}, \tag{2.1.35}$$

$$\kappa_1 = -\frac{1}{2}\frac{U_2^2}{U_1}\frac{\rho_2 H_1}{\rho_1 H_2}, \qquad \kappa_2 = -\frac{1}{2}\frac{U_2^2}{U_3}\frac{\rho_2 H_3}{\rho_3 H_2}, \tag{2.1.36}$$

Equations (2.1.31) and (2.1.32) describe the propagation of two coupled interface perturbations. The interaction between these modes is characterised by second derivative terms with $\kappa_{1,2}$ coefficients in each equation. If these coupling parameters equal zero, the equations separate into two linear KP equations without any interaction between perturbations. In the next section weak nonlinearity will be taken into account to obtain coupled nonlinear KP equations.

2.2. DERIVATION OF NONLINEAR COUPLED 2D-EQUATIONS

In the previous section two linear KP-like equations were derived to describe two coupled interfacial perturbations. That derivation was made on basis of solving Laplace's equation (2.1.3) in each layer and then substitution of the solutions obtained into linearised version of the kinematic and dynamic boundary conditions (2.1.4) – (2.1.7) given on both interfaces. To derive nonlinear equations we must solve Laplace's equation (2.1.3) as well, but in contradistinction to the linear case the Fourier approach (2.1.11) now is not fruitful for further satisfying the nonlinear boundary conditions (2.1.4) – (2.1.7).

An alternative approach applicable to the nonlinear boundary conditions is based on the assumption that we are interested only in long-wave perturbations, as this follows from the assumptions (2.1.27) made at the final stage of obtaining the linear KP equations. Consequently we may consider the velocity potential ϕ along with the interface perturbations η and ζ to be a slowly varying function of x, y and t. Therefore if ε is a small parameter which characterises such a slow dependence then in accordance with assumption (2.1.27), we introduce the scaling

$$X = \varepsilon x, \qquad Y = \varepsilon^2 y, \qquad T = \varepsilon^3 t. \tag{2.2.1}$$

In this case Laplace's equation (2.1.3) becomes

$$\varepsilon^2 \phi_{i,XX} + \varepsilon^4 \phi_{i,YY} + \phi_{i,zz} = 0. \tag{2.2.2}$$

A further consequence of this shallow water approach is that we can assume that the characteristic length of the perturbations is much longer than the height of each layer. Accordingly, with respect to such large scaled perturbations each layer may be considered to be "thin", so that it will be sufficient to consider only the leading terms of a power series to describe the dependence on the vertical variable z. Thus we let

$$\phi_i = \sum_{n=0}^{\infty} \Phi_i^{(n)}(X,Y,T)(z-a_i)^n, \qquad i = 1, 2, 3, \tag{2.2.3}$$

where a_i are some constants. Such an approach has been applied to the derivation of the KdV equation for various stratified media (see, for example, *Dodd et al.* 1984). We now follow this approach to derive coupled KP equations.

Substituting (2.2.3) into Laplace's equation (2.2.2) yields the following recursion relation

$$\Phi_i^{(n+2)} = -\frac{\varepsilon^2}{(n+1)(n+2)}\Phi_{i,XX}^{(n)} - \frac{\varepsilon^4}{(n+1)(n+2)}\Phi_{i,YY}^{(n)}. \tag{2.2.4}$$

It follows from (2.2.4) that we only require the first two functions $\Phi_i^{(0)}$ and $\Phi_i^{(1)}$ to obtain all others for the series (2.2.3). Furthermore, it is easy to see another consequence of the relation (2.2.4); the higher the order n of $\Phi_i^{(n)}$ the smaller it is in terms of ε. Therefore, we really need only the first few terms in series (2.2.3).

Omitting details the following expressions for the velocity potential in each of the three layers can be obtained

$$\phi_1 = C - \frac{1}{2}\varepsilon^2(z+H_1)^2 C_{XX} - \frac{1}{2}\varepsilon^4(z+H_1)^2 C_{YY} + \frac{1}{24}\varepsilon^4(z+H_1)^4 C_{XXXX} + O(\varepsilon^6), \tag{2.2.5}$$

$$\begin{aligned}\phi_2 = D + zE - \frac{1}{2}\varepsilon^2 z^2 D_{XX} - \frac{1}{6}\varepsilon^2 z^3 E_{XX} - \frac{1}{2}\varepsilon^4 z^2 D_{YY} - \frac{1}{6}\varepsilon^4 z^3 E_{YY} \\ + \frac{1}{24}\varepsilon^4 z^4 D_{XXXX} + \frac{1}{120}\varepsilon^4 z^5 E_{XXXX} + O(\varepsilon^6),\end{aligned} \tag{2.2.6}$$

$$\phi_3 = F - \frac{1}{2}\varepsilon^2 (z - H_2 - H_3)^2 F_{XX} - \frac{1}{2}\varepsilon^4 (z - H_2 - H_3)^2 F_{YY}$$
$$+ \frac{1}{24}\varepsilon^4 (z - H_2 - H_3)^4 F_{XXXX} + O(\varepsilon^6), \tag{2.2.7}$$

where

$$C = \Phi_1^{(0)}(X,Y,T), D = \Phi_2^{(0)}(X,Y,T), E = \Phi_2^{(1)}(X,Y,T), F = \Phi_3^{(0)}(X,Y,T). \tag{2.2.8}$$

The series (2.2.5) and (2.2.7) which already satisfy the rigid plane conditions at $z = -H_1$ and $z = H_2 + H_3$ respectively, which is why they contain only one unknown function (C and F respectively).

In turn the functions (2.2.8) can be expanded in asymptotic series as follows

$$C = \varepsilon C_1 + \varepsilon^3 C_3 + O(\varepsilon^5), \qquad D = \varepsilon^2 D_2 + \varepsilon^3 D_3 + \varepsilon^4 D_4 + O(\varepsilon^5), \tag{2.2.9}$$

$$E = \varepsilon^4 E_4 + \varepsilon^5 E_3 + O(\varepsilon^6), \qquad F = \varepsilon F_1 + \varepsilon^3 F_3 + O(\varepsilon^5). \tag{2.2.10}$$

Also we will assume that the resonance conditions (2.1.26), (2.1.27) obtained from the linear, long-wave analysis apply. Thus

$$U_1 = \widetilde{U}_1 + \varepsilon^2 \widetilde{\delta}_1, \qquad U_2 = \varepsilon \widetilde{U}_2, \qquad U_3 = \widetilde{U}_3 + \varepsilon^2 \widetilde{\delta}_3, \tag{2.2.11}$$

where $\widetilde{U}_1 = \mp c_1$ and $\widetilde{U}_3 = c_3$. And finally the interfacial perturbations must be represented as asymptotic expansions in powers of ε as follows

$$\eta = \varepsilon^2 A_2 + \varepsilon^4 A_4 + O(\varepsilon^6), \qquad \zeta = \varepsilon^2 B_2 + \varepsilon^4 B_4 + O(\varepsilon^6). \tag{2.2.12}$$

At this stage, now that all the necessary asymptotic expansions are defined, we can insert the solutions (2.2.5) – (2.2.7) of Laplace's equation into the boundary conditions

(2.1.4) – (2.1.7) together with the asymptotic expansions (2.2.9) – (2.2.12). After implementation this procedure and grouping of terms in powers of ε we obtain the following equations.

At *leading order* the kinematic boundary conditions (2.1.4), (2.1.5) yield

$$\widetilde{U}_1 A_{2,X} + H_1 C_{1,XY} = 0, \qquad E_4 = \widetilde{U}_2 A_{2,X}, \tag{2.2.13}$$

$$\widetilde{U}_2 B_{2,X} = E_4 - H_2 D_{2,XY}, \quad \widetilde{U}_3 B_{2,X} - H_3 F_{1,XY} = 0; \tag{2.2.14}$$

the pressure boundary condition (2.1.6) yields

$$g(\rho_1 - \rho_2)A_2 + \rho_1 \widetilde{U}_1 C_{1,X} = 0; \tag{2.2.15}$$

while the pressure boundary condition (2.1.7) yields

$$g(\rho_3 - \rho_2)B_2 + \rho_3 \widetilde{U}_3 F_{1,X} = 0. \tag{2.2.16}$$

Equations (2.2.13) – (2.2.16) simply describe linear effects without dispersion, transverse dependence and nonlinearity. All these effects will be taken in account at the *next order* of asymptotic expansion with respect to ε. At this order the kinematic boundary conditions yield

$$\widetilde{U}_1 A_{4,X} + H_1 C_{3,XY} = -A_{2,T} - \widetilde{\delta}_1 A_{2,X} - \left(A_2 C_{1,X}\right)_X - H_1 C_{1,YY} + \frac{1}{6} H_1^3 C_{1,XXXY}, \tag{2.2.17}$$

$$\widetilde{U}_3 B_{4,X} - H_3 F_{3,XY} = -B_{2,T} - \widetilde{\delta}_3 B_{2,X} - \left(B_2 F_{1,X}\right)_X + H_3 F_{1,YY} - \frac{1}{6} H_3^3 F_{1,XXXY}; \tag{2.2.18}$$

the pressure boundary condition (2.1.6) yields

$$g(\rho_1-\rho_2)A_4+\rho_1\widetilde{U}_1C_{3,X}+\rho_1C_{1,T}+\rho_1\widetilde{\delta}_1C_{1,X}-\rho_2\widetilde{U}_2D_{2,X}$$
$$-\frac{1}{2}\rho_1\widetilde{U}_1H_1^2C_{1,XXX}+\frac{1}{2}\rho_1\left(C_{1,X}\right)^2=0, \tag{2.2.19}$$

and the pressure boundary condition (2.1.7) yields

$$g(\rho_2-\rho_3)B_4-\rho_3\widetilde{U}_3F_{3,X}-\rho_3F_{1,T}-\rho_3\widetilde{\delta}_3F_{1,X}+\rho_2\widetilde{U}_2D_{2,X}$$
$$+\frac{1}{2}\rho_3\widetilde{U}_3H_3^2F_{1,XXX}-\frac{1}{2}\rho_3\left(F_{1,X}\right)^2=0. \tag{2.2.20}$$

After eliminating all variables except A_2, B_2 and rescaling all variables and parameters to their original forms we can obtain two coupled nonlinear KP equations of the form

$$(\eta_t+\Delta_1\eta_x+\mu_1\eta\eta_x+\lambda_1\eta_{xxx})_x+\gamma_1\eta_{yy}+\kappa_1\zeta_{xx}=0, \tag{2.2.21}$$

$$(\zeta_t+\Delta_2\zeta_x+\mu_2\zeta\zeta_x+\lambda_2\zeta_{xxx})_x+\gamma_2\zeta_{yy}+\kappa_2\eta_{xx}=0. \tag{2.2.22}$$

where

$$\mu_1=-\frac{3}{2}\frac{U_1}{H_1},\qquad \mu_2=\frac{3}{2}\frac{U_3}{H_3} \tag{2.2.23}$$

and other coefficients are given by (2.1.33) – (2.1.36).

We note that taking into account weak nonlinearity does not result in the appearance of new coupling terms. So, within the framework of this model coupling of two distinctive two-dimensional nonlinear modes is described by linear terms, as for the one-dimensional case (*Gottwald and Grimshaw* 1998, *Grimshaw* 2000). This differs from another considered later in Chapter 3, where the two modes are coupled not only due linear but also to nonlinear terms.

With no coupling (i.e. $\kappa_1 = \kappa_2 = 0$) we have two separate KP equations describing two waves, which do not interact with each other. This property can be used for the investigation of the interaction between two exact solutions of KP equations in the case of weak coupling (i.e. assuming $\kappa_{1,2} = O(\varepsilon)$, where ε is some small parameter characterising small coupling). For example, we can use equations (2.2.21) and (2.2.22) with small coupling parameters for asymptotically describing the behaviour of a wave system consisting of two KdV solitons or two lump solitons, both of which are exact solutions of a separate KP equation.

In the next sections we will obtain equations describing the dependence of the amplitudes and phases of two KdV solitons weakly coupled to each other, and then explore the stability of such a system with respect to transverse perturbations. Earlier the same problem, but in one-dimensional context, was investigated by *Gottwald and Grimshaw* (1998).

Putting $\gamma_1 = \gamma_2 = 0$ we can exclude all terms containing derivatives with respect to the transverse variable y from the coupled KP equations, thereby transforming them into the coupled KdV equations investigated by *Gottwald and Grimshaw* (1998), *Grimshaw* (2000). Thus, any solution of the coupled KdV equations will also be a solution of the coupled KP equations. We note that the functions

$$\eta = \tilde{a}_1 \operatorname{sech}^2(w(x-ct)), \qquad \zeta = \tilde{a}_2 \operatorname{sech}^2(w(x-ct)) \tag{2.2.24}$$

solve equations (2.2.21) and (2.2.22) exactly (*Gottwald and Grimshaw* 1998) provided

$$\tilde{a}_1 = \frac{12\lambda_1 w^2}{\mu_1}, \qquad \tilde{a}_2 = \frac{12\lambda_2 w^2}{\mu_2}, \tag{2.2.25}$$

$$c = \Delta_1 + \frac{1}{3}\mu_1\tilde{a}_1 + \kappa_1\frac{\lambda_2\mu_1}{\lambda_1\mu_2} = \Delta_2 + \frac{1}{3}\mu_2\tilde{a}_2 + \kappa_2\frac{\lambda_1\mu_2}{\lambda_2\mu_1}. \tag{2.2.26}$$

The following sections of this chapter are devoted to an investigation of the evolution of this two soliton solution in the two dimensions.

2.3. ASYMPTOTIC DESCRIPTION OF TWO COUPLED KdV SOLITONS

As stated at the end of the previous section we now investigate the evolution of the coupled KdV solitons (2.2.24) – (2.2.26), disturbed by a small perturbation. The principal idea of modelling a perturbed KdV soliton by using a multiple scale perturbation analysis to derive evolution equations for the amplitude and phase was introduced by *Johnson* (1973). This method was further developed by *Karpman and Maslov* (1978) and *Kaup and Newell* (1978) whose procedure was based on the inverse scattering technique. An extension of this work was undertaken by *Grimshaw and Mitsudera* (1993), taking into account higher-order terms when applying a multiple scale asymptotic expansion. *Gottwald and Grimshaw* (1998) then applied this method to describe the evolution of a pair of one-dimensional, coupled KdV solitons. We will follow this analysis to describe the evolution of two coupled KdV solitons in two dimensions.

We assume that the coupling between solitons propagating along two interfaces is weak and their shape slowly depends on the transverse variable, so that $\kappa_1 = \varepsilon\tilde{\kappa}_1$, $\kappa_2 = \varepsilon\tilde{\kappa}_2$, where ε is some small parameter. Further, it is assumed that both η and ζ slowly depend on the transverse variable y, so that they actually depend on $Y = \varepsilon y$. Under such assumptions equations (2.2.21), (2.2.22) become

$$(\eta_t + \Delta_1\eta_x + \mu_1\eta\eta_x + \lambda_1\eta_{xxx})_x + \varepsilon^2\gamma_1\eta_{YY} + \varepsilon\tilde{\kappa}_1\zeta_{xx} = 0, \tag{2.3.1}$$

$$(\zeta_t + \Delta_2\zeta_x + \mu_2\zeta\zeta_x + \lambda_2\zeta_{xxx})_x + \varepsilon^2\gamma_2\zeta_{YY} + \varepsilon\tilde{\kappa}_2\eta_{xx} = 0. \tag{2.3.2}$$

Note that both these equations transform to separate KdV equations as $\varepsilon \to 0$. Therefore we can seek a solution of equations (2.3.1), (2.3.2) in the form of asymptotic expansion

$$\eta = \eta^{(0)} + \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} + O(\varepsilon^3), \tag{2.3.3}$$

$$\zeta = \zeta^{(0)} + \varepsilon\zeta^{(1)} + \varepsilon^2\zeta^{(2)} + O(\varepsilon^3) \tag{2.3.4}$$

with $\eta^{(0)}$ and $\zeta^{(0)}$ approaching appropriate solutions of separate KdV equations as $\varepsilon \to 0$. We then take these functions in the form of KdV solitons with parameters depending on Y and on a slow timescale $T = \varepsilon t$ as follows

$$\eta^{(0)} = a_1(T,Y)\operatorname{sech}^2(\theta_1), \qquad \zeta^{(0)} = a_2(T,Y)\operatorname{sech}^2(\theta_2), \tag{2.3.5}$$

where

$$\theta_i = w_i(T,Y)(x - \Phi_i), \qquad \Phi_i = \frac{1}{\varepsilon}\int_0^T c_i(T',Y)dT', \qquad i = 1, 2, \tag{2.3.6}$$

$$c_i = c_i^{(0)} + \varepsilon c_i^{(1)} + \varepsilon^2 c_i^{(2)} + O(\varepsilon^3), \qquad i = 1, 2. \tag{2.3.7}$$

As can be seen the amplitudes $a_{1,2}$ of these solitons and their phases $\Phi_{1,2}$ depend on the two slow variables T and Y. Our task is now to derive equations governing this dependence.

After substitution of the expansions (2.3.3), (2.3.4) into equations (2.3.1), (2.3.2) we have at leading order the following equations

$$\left(\Delta_1 - c_1^{(0)}\right)\eta_x^{(0)} + \mu_1\eta^{(0)}\eta_x^{(0)} + \lambda_1\eta_{xxx}^{(0)} = 0, \tag{2.3.8}$$

$$\left(\Delta_2 - c_2^{(0)}\right)\zeta_x^{(0)} + \mu_2\zeta^{(0)}\zeta_x^{(0)} + \lambda_2\zeta_{xxx}^{(0)} = 0. \tag{2.3.9}$$

It is easy to check that the functions (2.3.5) solve the above equations provided

$$c_i^{(0)} = \Delta_i + \frac{1}{3}\mu_i a_i, \qquad a_i = \frac{12 w_i^2 \lambda_i}{\mu_i}, \qquad i = 1, 2. \tag{2.3.10}$$

At second order we have the following linear inhomogeneous equations for $\eta^{(1)}$ and $\zeta^{(1)}$

$$\left(\Delta_1 - c_1^{(0)}\right)\eta_x^{(1)} + \mu_1\left(\eta^{(0)}\eta^{(1)}\right)_x + \lambda_1\eta_{xxx}^{(1)} = -\eta_T^{(0)} + c_1^{(1)}\eta_x^{(0)} - \widetilde{\kappa}_1\zeta_x^{(0)}, \tag{2.3.11}$$

$$\left(\Delta_2 - c_2^{(0)}\right)\zeta_x^{(1)} + \mu_2\left(\zeta^{(0)}\zeta^{(1)}\right)_x + \lambda_2\zeta_{xxx}^{(1)} = -\zeta_T^{(0)} + c_2^{(1)}\zeta_x^{(0)} - \widetilde{\kappa}_2\eta_x^{(0)}. \tag{2.3.12}$$

To obtain solvability conditions we introduce two linear operators

$$H_1 = \left(\Delta_1 - c_1^{(0)}\right) + \mu_1\eta^{(0)} + \lambda_1\frac{\partial^2}{\partial x^2}, \tag{2.3.13}$$

$$H_2 = \left(\Delta_2 - c_2^{(0)}\right) + \mu_2\zeta^{(0)} + \lambda_2\frac{\partial^2}{\partial x^2}. \tag{2.3.14}$$

As these operators contain neither complex valued functions or constants, nor derivatives of an odd order they are selfadjoint. Note that the leading order equations may be expressed as

$$H_1\eta_x^{(0)} = 0, \qquad H_2\zeta_x^{(0)} = 0. \tag{2.3.15}$$

Then the adjoint homogeneous equation corresponding to (2.3.11) has the form

$$H_1 f_x = 0. \tag{2.3.16}$$

From (2.3.15) this equation has a solution $f_x = \eta_x^{(0)}$, while another linearly independent solution has a form

$$f_x = \theta_1 \operatorname{sech}^2(\theta_1)\tanh(\theta_1) + \frac{2}{15}\cosh^2(\theta_1) + \frac{1}{3} - \operatorname{sech}^2(\theta_1). \tag{2.3.17}$$

Due to the unboundedness of the solution (2.3.17) the first solution ($f = \eta^{(0)}$) should be used to construct the following solvability condition for the equation (2.3.11),

$$\int_{-\infty}^{\infty} \left(-\eta_T^{(0)} + c_1^{(1)}\eta_x^{(0)} - \tilde{\kappa}_1\zeta_x^{(0)}\right)\eta^{(0)} d\theta_1 = 0. \tag{2.3.18}$$

The same manipulations lead to a similar solvability condition for the equation (2.3.12),

$$\int_{-\infty}^{\infty} \left(-\zeta_T^{(0)} + c_2^{(1)}\zeta_x^{(0)} - \tilde{\kappa}_2\eta_x^{(0)}\right)\zeta^{(0)} d\theta_2 = 0. \tag{2.3.19}$$

After substitution of the expressions (2.3.5) into equations (2.3.18), (2.3.19), and after some simplification, we obtain the following equations governing the solitons' amplitude variation

$$\frac{\partial a_1}{\partial T} = 2\tilde{\kappa}_1 w_2 a_2 \int_{-\infty}^{\infty} \operatorname{sech}^2(\theta)\operatorname{sech}^2\left(\frac{w_2}{w_1}\theta - w_2\Delta\Phi\right)\tanh\left(\frac{w_2}{w_1}\theta - w_2\Delta\Phi\right)d\theta, \tag{2.3.20}$$

$$\frac{\partial a_2}{\partial T} = 2\tilde{\kappa}_2 w_1 a_1 \int_{-\infty}^{\infty} \operatorname{sech}^2(\theta)\operatorname{sech}^2\left(\frac{w_1}{w_2}\theta + w_1\Delta\Phi\right)\tanh\left(\frac{w_1}{w_2}\theta + w_1\Delta\Phi\right)d\theta, \tag{2.3.21}$$

where $\Delta\Phi = \Phi_2 - \Phi_1$.

To obtain equations describing the variation of the solitons' phases, $\Phi_{1,2}$, we must use solvability conditions for the equations arising at the next, third, order of asymptotic expansion of the basic equations (2.3.1), (2.3.2). However, to complete that we need a solution of the equations (2.3.11) and (2.3.12). Following the procedure developed by *Grimshaw and Mitsudera* (1993) we first integrate (2.3.11) once with respect to x to obtain a second order differential equation

$$
\begin{aligned}
&\left(\Delta_1 - c_1^{(0)}\right)\eta^{(1)} + \mu_1\eta^{(0)}\eta^{(1)} + \lambda_1\eta_{xx}^{(1)} \\
&\qquad = \frac{\partial}{\partial T}\int_x^{\infty}\eta^{(0)}dx + c_1^{(1)}\eta^{(0)} - \widetilde{\kappa}_1\zeta^{(0)} + \left(\Delta_1 - c_1^{(0)}\right)\eta^{+}.
\end{aligned}
\tag{2.3.22}
$$

Hereafter $\eta^{\pm} = \lim_{x\to\pm\infty}\eta^{(1)}$ which, in general, is non-zero as slowly varying solitary waves generate a tail (see, for instance, *Johnson* 1973, *Karpman and Maslov* 1978, *Grimshaw* 1979). Such a radiative tail is essentially a linear wave and propagates behind or ahead of the solitary wave depending on whether $\lambda_1 > 0$ or $\lambda_1 < 0$ respectively. So, we must set $\eta^{+} = 0$ in the case $\lambda_1 > 0$ and $\eta^{-} = 0$ if $\lambda_1 < 0$. Then taking the limit of (2.3.22) as $x \to -\infty$, and assuming $\lim_{x\to\pm\infty}\eta_x^{(1)} = 0$ we will have

$$
\left(\Delta_1 - c_1^{(0)}\right)\left(\eta^{+} - \eta^{-}\right) = -\frac{\partial}{\partial T}\int_{-\infty}^{\infty}\eta^{(0)}dx. \tag{2.3.23}
$$

It is relatively easy to check that the function

$$
\nu^{(1)} = \eta^{(1)} - \frac{1}{2}\left(\eta^{+} + \eta^{-}\right) \tag{2.3.24}
$$

solves the following equation

$$
\begin{aligned}
&\left(\Delta_1 - c_1^{(0)}\right)\nu^{(1)} + \mu_1\eta^{(0)}\nu^{(1)} + \lambda_1\nu_{xx}^{(1)} \\
&\qquad = \left(c_1^{(1)} - \frac{1}{2}\mu_1\left(\eta^{+} + \eta^{-}\right)\right)\eta^{(0)} - \frac{\partial}{\partial T}\int_0^x\eta^{(0)}dx - \widetilde{\kappa}_1\zeta^{(0)}
\end{aligned}
\tag{2.3.25}
$$

and $\nu^{(1)} \to \pm\frac{1}{2}\left(\eta^{+} - \eta^{-}\right)$ as $x \to \pm\infty$. As stated above the left hand side of the equation (2.3.25) is equated to zero since $\nu_1^{(1)} = \eta_x^{(0)}$ and $\nu_2^{(1)}$ is defined by (2.3.17). The Wronskian of these linearly independent solutions is then

$$W_\eta = v_{1,x}^{(1)} v_2^{(1)} - v_1^{(1)} v_{2,x}^{(1)} = \frac{16 a_1 w_1^2}{15} \tag{2.3.26}$$

Then the general solution of the inhomogeneous equation (2.3.25) can be expressed as

$$\begin{aligned} v^{(1)} = {} & \frac{3}{2\mu_1 a_1}\left(c_1^{(1)} - \frac{1}{2}\mu_1\left(\eta^+ + \eta^-\right)\right)\left(2\eta^{(0)} + \theta_1 \eta_x^{(0)}\right) + A_1 \eta_x^{(0)} + A_2 v_2^{(1)} \\ & - \frac{1}{W_\eta \lambda_1} \eta_x^{(0)} \int_0^x v_2^{(1)} G dx - \frac{1}{W_\eta \lambda_1} v_2^{(1)} \int_x^\infty \eta_x^{(0)} G dx \end{aligned} \tag{2.3.27}$$

where A_1 and A_2 are arbitrary constants and

$$G = \frac{\partial}{\partial T} \int_0^x \eta^{(0)} dx + \breve{\kappa}_1 \zeta^{(0)} . \tag{2.3.28}$$

Finally after reverting to the variable $\eta^{(1)}$, defined by (2.3.24), we have the solution of equation (2.3.11) in the form

$$\begin{aligned} \eta^{(1)} = {} & \frac{1}{2}\left(\eta^+ + \eta^-\right) + \frac{3}{2\mu_1 a_1}\left(c_1^{(1)} - \frac{1}{2}\mu_1\left(\eta^+ + \eta^-\right)\right)\left(2\eta^{(0)} + \theta_1 \eta_x^{(0)}\right) \\ & + A_1 \eta_x^{(0)} + A_2 v_2^{(1)} - \frac{1}{W_\eta \lambda_1} \eta_x^{(0)} \int_0^x v_2^{(1)} G dx - \frac{1}{W_\eta \lambda_1} v_2^{(1)} \int_x^\infty \eta_x^{(0)} G dx . \end{aligned} \tag{2.3.29}$$

The solution of equation (2.3.12) can be similarly obtained in the form

$$\begin{aligned} \zeta^{(1)} = {} & \frac{1}{2}\left(\zeta^+ + \zeta^-\right) + \frac{3}{2\mu_2 a_2}\left(c_2^{(1)} - \frac{1}{2}\mu_2\left(\zeta^+ + \zeta^-\right)\right)\left(2\zeta^{(0)} + \theta_2 \zeta_x^{(0)}\right) \\ & + B_1 \zeta_x^{(0)} + B_2 u_2^{(1)} - \frac{1}{W_\zeta \lambda_2} \zeta_x^{(0)} \int_0^x u_2^{(1)} R dx - \frac{1}{W_\zeta \lambda_2} u_2^{(1)} \int_x^\infty \zeta_x^{(0)} R dx , \end{aligned} \tag{2.3.30}$$

where

$$\zeta^{\pm} = \lim_{x\to\pm\infty} \zeta^{(1)}, \qquad R = \frac{\partial}{\partial T}\int_0^x \zeta^{(0)} dx + \widetilde{\kappa}_2 \eta^{(0)}, \tag{2.3.31}$$

$$W_\zeta = u_{1,x}^{(1)} u_2^{(1)} - u_1^{(1)} u_{2,x}^{(1)} = \frac{16 a_2 w_2^2}{15} \tag{2.3.32}$$

with $u_1^{(1)} = \zeta_x^{(0)}$ and $u_2^{(1)}$ defined by the same expression (2.3.17) but after substitution of θ_2 for θ_1.

The third order terms of the asymptotic expansion of equations (2.3.1), (2.3.2) result in the following inhomogeneous equations

$$\begin{aligned}
&\left(\Delta_1 - c_1^{(0)}\right)\eta_x^{(2)} + \mu_1\left(\eta^{(0)}\eta^{(2)}\right)_x + \lambda_1 \eta_{xxx}^{(2)} \\
&\quad = -\eta_T^{(1)} + c_1^{(1)}\eta_x^{(1)} + c_1^{(2)}\eta_x^{(0)} - \mu_1 \eta^{(1)}\eta_x^{(1)} - \gamma_1\left(\Phi_{1,Y}\right)^2 \eta_x^{(0)} \\
&\quad + \gamma_1 \Phi_{1,YY}\eta^{(0)} - \widetilde{\kappa}_1 \zeta_x^{(1)},
\end{aligned} \tag{2.3.33}$$

$$\begin{aligned}
&\left(\Delta_2 - c_2^{(0)}\right)\zeta_x^{(2)} + \mu_2\left(\zeta^{(0)}\zeta^{(2)}\right)_x + \lambda_2 \zeta_{xxx}^{(2)} \\
&\quad = -\zeta_T^{(1)} + c_2^{(1)}\zeta_x^{(1)} + c_2^{(2)}\zeta_x^{(0)} - \mu_2 \zeta^{(1)}\zeta_x^{(1)} - \gamma_2\left(\Phi_{2,Y}\right)^2 \zeta_x^{(0)} \\
&\quad + \gamma_2 \Phi_{2,YY}\zeta^{(0)} - \widetilde{\kappa}_2 \eta_x^{(1)}.
\end{aligned} \tag{2.3.34}$$

Note, that both equations obtained contain the same operators on their left hand sides as for the equations (2.3.11) and (2.3.12) respectively. So, the procedure of obtaining the solvability conditions for equations (2.3.33) and (2.3.34) is similar to that used for the equations of the second order. Namely, we must multiply the right hand side of equation (2.3.33) by $\eta^{(0)}$ and the right hand side of equation (2.3.34) by $\zeta^{(0)}$, and then integrate the products with respect to x over the whole spatial domain. For example, after some simplification the solvability condition for the equation (2.3.33) has the form

$$\frac{\partial}{\partial T}\int_{-\infty}^{\infty}\eta^{(0)}\eta^{(1)}dx+\frac{1}{2}\left(\Delta_1-c_1^{(0)}\right)\left(\left(\eta^+\right)^2-\left(\eta^-\right)^2\right)$$

$$-\gamma_1\Phi_{1,rr}\int_{-\infty}^{\infty}\left(\eta^{(0)}\right)^2dx+\tilde{\kappa}_1\int_{-\infty}^{\infty}\left(\eta^{(1)}\zeta_x^{(0)}+\eta^{(0)}\zeta_x^{(1)}\right)dx=0. \tag{2.3.35}$$

After substitution of (2.3.30) and (2.3.31) into equation (2.3.35) and working through the necessary simplifications (for details see *Grimshaw and Mitsudera* 1993), followed by rescaling of all variables and parameters to restore them to their original values, we obtain an equation describing the evolution of the phase Φ_1,

$$\frac{36w_1\lambda_1}{\mu_1^2}\frac{\partial}{\partial t}\left[\frac{\partial\Phi_1}{\partial t}-\Delta_1-\frac{1}{3}\mu_1 a_1\right]$$

$$-\frac{3\kappa_1 a_2 w_2}{\mu_1 w_1^2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\left[\Psi\operatorname{sech}^2(\Psi)+\tanh(\Psi)-\operatorname{sgn}(\lambda_1)\operatorname{sech}^2(\Psi)\right] \tag{2.3.36}$$

$$\times\operatorname{sech}^2\left(\frac{w_2}{w_1}\Psi-w_2\Delta\Phi\right)\tanh\left(\frac{w_2}{w_1}\Psi-w_2\Delta\Phi\right)d\Psi-\frac{4a_1^2\gamma_1}{3w_1}\frac{\partial^2\Phi_1}{\partial y^2}=0.$$

Similarly the equation for Φ_2 can be derived in the form

$$\frac{36w_2\lambda_2}{\mu_2^2}\frac{\partial}{\partial t}\left[\frac{\partial\Phi_2}{\partial t}-\Delta_2-\frac{1}{3}\mu_2 a_2\right]$$

$$-\frac{3\kappa_2 a_1 w_1}{\mu_2 w_2^2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\left[\Psi\operatorname{sech}^2(\Psi)+\tanh(\Psi)-\operatorname{sgn}(\lambda_2)\operatorname{sech}^2(\Psi)\right] \tag{2.3.37}$$

$$\times\operatorname{sech}^2\left(\frac{w_1}{w_2}\Psi+w_1\Delta\Phi\right)\tanh\left(\frac{w_1}{w_2}\Psi+w_1\Delta\Phi\right)d\Psi-\frac{4a_2^2\gamma_2}{3w_2}\frac{\partial^2\Phi_2}{\partial y^2}=0.$$

Here $\Delta\Phi=\Phi_2-\Phi_1$. Thus, we now have equations (2.3.20), (2.3.21), (2.3.36) and (2.3.37) which describe the evolution of the amplitudes and phases of two coupled KdV solitons. Extension of the problem to two-dimensions effects only the equations for the phases. These differ from those obtained by *Gottwald and Grimshaw* (1998) in the term containing second order derivative with respect to the transverse spatial variable y. It is not difficult to check that this term can be identically transferred to similar terms obtained for a single KdV soliton using the KP equation (see, for example, *Ablowitz and Segur* 1981). The amplitude equations (2.3.20), (2.3.21) do not change from those obtained by *Gottwald and Grimshaw* (1998).

The next section is devoted to examining the linear stability of the exact solutions (2.2.24) – (2.2.26) of the coupled KP equations (2.2.21), (2.2.22) with respect to transverse perturbations. The basis of this is the solution of the linearised version of the evolution equations obtained in the current section. Although consideration of transverse effects results in the appearance of only one additional term in the phase equations, they essentially change the stability properties of the two coupled KdV solitons system considered, as will be seen.

2.4. ANALYSIS OF STABILITY OF TWO COUPLED KdV SOLITONS WITH RESPECT TO TRANSVERSE PERTURBATIONS

As stated at the end of the previous section the aim of the current section is to examine stability properties of the exact solution (2.2.24) – (2.2.26) of the coupled KP equations (2.2.21), (2.2.22) with respect to transverse perturbations. To complete this task we must linearise equations (2.3.20), (2.3.21), (2.3.36) and (2.3.37) describing the evolution of the amplitudes $a_{1,2}$ and phases $\Phi_{1,2}$ of each KdV soliton considered.

Suppose that all parameters of the system of two KdV solitons (2.2.24) are slightly perturbed from their exact values defined by the expressions (2.2.25) and (2.2.26) i.e. they can be expressed as

$$a_{1,2} = \widetilde{a}_{1,2} + \delta a_{1,2}, \qquad w_{1,2} = w + \delta w_{1,2}, \qquad \Phi_{1,2} = ct + \delta\Phi_{1,2} \tag{2.4.1}$$

so that

$$\left|\delta a_{1,2}\right| << \widetilde{a}_{1,2}, \qquad \left|\delta w_{1,2}\right| << w, \qquad \left|\delta\Phi_{1,2}\right| << ct. \tag{2.4.2}$$

Then after substitution of (2.4.1) into equations (2.3.20), (2.3.21), (2.3.36) and (2.3.37) and subsequent simplification we can obtain the linearised version of those equations,

$$\frac{\partial(\delta a_1)}{\partial t} = -b_1\delta\Phi_1 + b_1\delta\Phi_2, \tag{2.4.3}$$

$$\frac{\partial(\delta a_2)}{\partial t} = -b_2\delta\Phi_1 + b_2\delta\Phi_2\,, \tag{2.4.4}$$

$$\frac{\partial^2(\delta\Phi_1)}{\partial t^2} = d_1\frac{\partial(\delta a_1)}{\partial t} + f_1\frac{\partial(\delta a_2)}{\partial t} - h_1\frac{\partial(\delta\Phi_1)}{\partial t} + h_1\frac{\partial(\delta\Phi_2)}{\partial t} + g_1\frac{\partial^2(\delta\Phi_1)}{\partial Y^2}\,, \tag{2.4.5}$$

$$\frac{\partial^2(\delta\Phi_2)}{\partial t^2} = f_2\frac{\partial(\delta a_1)}{\partial t} + d_2\frac{\partial(\delta a_2)}{\partial t} - h_2\frac{\partial(\delta\Phi_1)}{\partial t} + h_2\frac{\partial(\delta\Phi_2)}{\partial t} + g_2\frac{\partial^2(\delta\Phi_2)}{\partial Y^2}\,. \tag{2.4.6}$$

Here

$$b_1 = -\frac{4\mu_2\kappa_1 a_2^2}{45\lambda_2}, \qquad b_2 = \frac{4\mu_1\kappa_2 a_1^2}{45\lambda_1}, \tag{2.4.7}$$

$$d_1 = \frac{\mu_1}{3} - \left(\frac{2}{3} + \frac{\pi^2}{45}\right)\frac{\kappa_1}{a_1}, \qquad d_2 = \frac{\mu_2}{3} - \left(\frac{2}{3} + \frac{\pi^2}{45}\right)\frac{\kappa_2}{a_2}, \tag{2.4.8}$$

$$f_1 = \left(\frac{2}{3} + \frac{\pi^2}{45}\right)\frac{\kappa_1}{a_1}, \qquad f_2 = \left(\frac{2}{3} + \frac{\pi^2}{45}\right)\frac{\kappa_2}{a_2}, \tag{2.4.9}$$

$$h_1 = -\frac{8\kappa_1\lambda_2\mu_1 w}{15|\lambda_1|\mu_2}, \qquad h_2 = \frac{8\kappa_2\lambda_1\mu_2 w}{15|\lambda_2|\mu_1}, \tag{2.4.10}$$

$$g_1 = \frac{4}{9}\gamma_1\mu_1 a_1, \qquad g_2 = \frac{4}{9}\gamma_2\mu_2 a_2\,. \tag{2.4.11}$$

With $g_1 = g_2 = 0$ the equations above coincide with those obtained by *Gottwald and Grimshaw* (1998).

After substituting $\delta a_{1,2}(T,Y) = \delta\overline{a}_{1,2}\exp(\gamma T + \nu Y)$, $\delta\Phi_{1,2}(T,Y) = \delta\overline{\Phi}_{1,2}\exp(\gamma T + \nu Y)$ into equations (2.4.3) – (2.4.6), they reduce to the following system of linear equations

$$
\begin{pmatrix}
-\gamma & 0 & -b_1 & b_1 \\
0 & -\gamma & -b_2 & b_2 \\
d_1\gamma & f_1\gamma & -h_1\gamma-\gamma^2+g_1\nu^2 & h_1\gamma \\
f_2\gamma & d_2\gamma & -h_2\gamma & h_2\gamma-\gamma^2+g_2\nu^2
\end{pmatrix}
\begin{pmatrix}
\delta\overline{a}_1 \\ \delta\overline{a}_2 \\ \delta\overline{\Phi}_1 \\ \delta\overline{\Phi}_2
\end{pmatrix} = 0. \qquad (2.4.12)
$$

The solvability condition for the system of equations (2.4.12) has the form

$$
\begin{gathered}
\gamma^6 + (h_1 - h_2)\gamma^5 + \left\{b_1(d_1 - f_2) + b_2(f_1 - d_2) - (g_1 + g_2)\nu^2\right\}\gamma^4 + (g_1 h_2 - g_2 h_1)\nu^2\gamma^3 \\
+\left(-\nu^2 d_1 b_1 g_2 - f_1 b_2 \nu^2 g_2 + f_2 b_1 \nu^2 g_1 + b_2 d_2 \nu^2 g_1 + g_1 g_2 \nu^4\right)\gamma^2 = 0.
\end{gathered} \qquad (2.4.13)
$$

Let us examine roots of the above equation in the case of *no radiation*, where $h_1 = h_2 = 0$. Then, omitting the trivial double root $\gamma = 0$, equation (2.4.13) reduces to the following biquadratic equation

$$
\gamma^4 + \left(\delta_1 - \delta_2 - (g_1 + g_2)\nu^2\right)\gamma^2 - \nu^2\delta_1 g_2 + \delta_2\nu^2 g_1 + g_1 g_2 \nu^4 = 0, \qquad (2.4.14)
$$

where $\delta_1 = d_1 b_1 + b_2 f_1$, $\delta_2 = f_2 b_1 + b_2 d_2$. It follows from (2.4.14) that if γ_0 is a root of this equation then so must be $-\gamma_0$. Consequently this system is stable if and only if $\mathrm{Re}\,\gamma = 0$ or $\gamma^2 < 0$. Then using simple relationship between roots and coefficients of a quadratic equation the stability criterion can be formulated in the form of three simultaneous inequalities,

$$
\begin{cases}
\left(\delta_1 - \delta_2 - (g_1 + g_2)\nu^2\right)^2 + 4\left(\delta_1\nu^2 g_2 - \delta_2\nu^2 g_1 - g_1 g_2 \nu^4\right) > 0, \\
\delta_1 - \delta_2 - (g_1 + g_2)\nu^2 > 0, \\
\delta_1\nu^2 g_2 - \delta_2\nu^2 g_1 - g_1 g_2 \nu^4 < 0.
\end{cases} \qquad (2.4.15)
$$

In the absence of y-dependence ($\nu = 0$) the set (2.4.15) reduces to the following single inequality

$$\delta_1 - \delta_2 > 0, \tag{2.4.16}$$

which coincides with the stability criterion obtained by *Gottwald and Grimshaw* (1998). To have only bounded perturbations in the *y*-direction, thus we require $\nu^2 < 0$, and so, everywhere in this section we consider ν to be a pure imaginary number.

It is easy to show that the first inequality of the set (2.4.15) can be expressed as one of two equivalent inequalities,

$$\left(\delta_1 - \delta_2 - (g_1 - g_2)\nu^2\right)^2 - 4\delta_2 (g_1 - g_2)\nu^2 > 0, \tag{2.4.17}$$

or

$$\left(\delta_1 - \delta_2 + (g_1 - g_2)\nu^2\right)^2 - 4\delta_1 (g_1 - g_2)\nu^2 > 0. \tag{2.4.18}$$

These versions of the inequality will be used in further detailed analysis of the consistency of the set (2.4.15).

It is obvious that the transverse terms can affect the stability properties of the system. For example, for $g_1 > 0$ and $g_2 > 0$ there always exist $\nu_0 > 0$ such that every inequality from (2.4.15) is valid for all pure imaginary ν with $|\nu| > \nu_0$. Besides, it follows from (2.4.17), (2.4.18) that the first inequality from (2.4.15) is satisfied by all pure imaginary ν provided

$$\delta_{1,2}(g_1 - g_2) > 0. \tag{2.4.19}$$

Thus, it is possible for the transverse terms to stabilise this system in the case where the one-dimensional system is unstable (i.e. $\delta_1 - \delta_2 < 0$). So, the influence of the transverse terms on the stability of the system is nontrivial and is worthy of detailed consideration.

Before we commence analysing the consistency of the set (2.4.15) it should be noted that in various terms of the three-layered model under consideration the sign of $g_{1,2}$ are strongly definite, since

$$g_1 = \frac{4}{9}\gamma_1\mu_1 a_1 = \frac{1}{3}\frac{U_1^2}{H_1} > 0, \qquad g_2 = \frac{4}{9}\gamma_2\mu_2 a_2 = -\frac{1}{3}\frac{U_3^2}{H_3} < 0. \tag{2.4.20}$$

Thus, hereafter we confine our analysis to the case $g_1 > 0$, $g_2 < 0$ only. For further simplification let us also put $s = -\nu^2$, so then $s > 0$ for all pure imaginary ν. Hence, the initial set of inequalities is modified to either of the equivalent sets,

$$\begin{cases} \left(\delta_1 - \delta_2 + (g_1 + |g_2|)s\right)^2 + 4\delta_2 (g_1 + |g_2|)s > 0, \\ (g_1 - |g_2|)(s - s_1) > 0, \\ 0 < s < s_2, \end{cases} \tag{2.4.21}$$

or

$$\begin{cases} \left(\delta_1 - \delta_2 - (g_1 + |g_2|)s\right)^2 + 4\delta_1 (g_1 + |g_2|)s > 0, \\ (g_1 - |g_2|)(s - s_1) > 0, \\ 0 < s < s_2, \end{cases} \tag{2.4.22}$$

where

$$s_1 = \frac{\delta_2 - \delta_1}{g_1 - |g_2|}, \qquad s_2 = -\frac{\delta_2 g_1 + \delta_1 |g_2|}{g_1 |g_2|}. \tag{2.4.23}$$

To determine the relative location of s_1 and s_2 on the real axis the sign of their difference

$$s_2 - s_1 = -\frac{\delta_2 g_1^2 - \delta_1 g_2^2}{g_1 |g_2| (g_1 - |g_2|)} \tag{2.4.24}$$

should be examined.

I. First let $\delta_2 > 0$. Under this condition the set (2.4.21) is preferred for analysis, as in this case the first inequality of (2.4.21) is identically satisfied by all positive s, and so, need no longer be considered. Let us then consider all possible subcases:

(i) Let $g_1 > |g_2|$ and $-\frac{g_1}{|g_2|}\delta_2 \le \delta_1$. Then the third inequality of (2.4.21) is inconsistent because

$$s_2 = -\frac{|\delta_1| + \delta_2 \frac{g_1}{|g_2|}}{g_1} \le 0. \qquad (2.4.25)$$

The set of inequalities (2.4.21) is thus inconsistent (and, consequently, the system of solitons is unstable).

(ii) Let $g_1 > |g_2|$, but $\delta_1 < -\frac{g_1}{|g_2|}\delta_2 < 0$. Then

$$s_1 = \frac{\delta_2 + |\delta_1|}{g_1 - |g_2|} > 0, \qquad s_2 = -\frac{|\delta_1| + \delta_2 \frac{g_1}{|g_2|}}{g_1} > 0, \qquad (2.4.26)$$

and the set (2.4.21) reduces to the following one

$$\begin{cases} s > s_1, \\ 0 < s < s_2. \end{cases} \qquad (2.4.27)$$

which is inconsistent because

$$s_2 - s_1 = -\frac{\delta_2 g_1^2 + |\delta_1| g_2^2}{g_1 |g_2| (g_1 - |g_2|)} < 0. \qquad (2.4.28)$$

(iii) Let $0 < g_1 < |g_2|$ and $\delta_1 < \delta_2$. The second inequality becomes $0 < s < s_1$ with

$$s_1 = -\frac{\delta_2 - \delta_1}{|g_2| - g_1} < 0. \tag{2.4.29}$$

So, the set (2.4.21) is inconsistent again. If $g_1 = |g_2|$ the second inequality is also invalid because (see the second inequality at (2.4.15)) $\delta_1 - \delta_2 = \delta_1 + |\delta_2| > 0$.

(iv) Let $0 < g_1 \le |g_2|$ and $0 < \delta_2 < \delta_1$. In this case

$$s_2 = -\frac{\delta_2 g_1 + \delta_1 |g_2|}{g_1 |g_2|} < 0 \tag{2.4.30}$$

and the third inequality of (2.4.21) is invalid.

Thus the system turns out to be unstable in the case $\delta_2 > 0$ regardless of any other relationships between parameters.

II. Now let $\delta_2 < 0$ but $\delta_1 > 0$. Under these new conditions the first inequality of (2.4.21) is no longer satisfied for all $s > 0$. In this case the second version (2.4.18) of this inequality is preferred, as in this form it is still satisfied for all positive s. Therefore, the application of the set (2.4.22) in this case reduces the analysis to solving just the two last inequalities.

(i) If $g_1 > |g_2|$ then $g_1 - |g_2| > 0$ and $s_1 = -\frac{\delta_1 + |\delta_2|}{g_1 - |g_2|} < 0$. So the second inequality is also satisfied by any positive s and the set (2.4.22) thus reduces to a single inequality $0 < s < s_2$. This inequality defines a nonempty number set as long as the parameter $s_2 = \frac{|\delta_2| g_1 - \delta_1 |g_2|}{g_1 |g_2|}$ is positive (for $|\delta_2| > \delta_1 \frac{|g_2|}{g_1}$). Otherwise (for $|\delta_2| \le \delta_1 \frac{|g_2|}{g_1}$) the solution of this inequality and consequently that of set (2.4.22) are empty.

(ii) If $g_1 < |g_2|$ then $g_1 - |g_2| < 0$ and $s_1 = \frac{|\delta_2| + \delta_1}{|g_2| - g_1} > 0$. So, the second inequality reduces to $0 < s < s_1$. But because of $s_2 - s_1 = -\frac{|\delta_2| g_1^2 + \delta_1 g_2^2}{g_1 |g_2| (|g_2| - g_1)} < 0$ our problem again reduces to the same single inequality as in the previous case.

(iii) If $g_1 = |g_2|$ the second inequality is satisfied identically as $\delta_1 - \delta_2 = \delta_1 + |\delta_2| > 0$ and as above we have only the third inequality left to determine the stability/instability criterion.

So, in the case $\delta_1 > 0$ and $\delta_2 < 0$ regardless of any additional relationships between $g_1 > 0$ and $g_2 < 0$ the system is stable for $0 < s < s_2$ provided $|\delta_2| \geq \delta_1 \dfrac{|g_2|}{g_1}$ and unstable otherwise.

III. Now let us consider the most complicated case when both δ_1 and δ_2 are negative. Now either the first inequality of (2.4.21) or (2.4.22) is not necessarily satisfied for all positive s. It is not difficult to show that this inequality is satisfied by $s \in (0, s_3) \cup (s_4, \infty)$ with

$$s_3 = \frac{\left(\sqrt{|\delta_1|} - \sqrt{|\delta_2|}\right)^2}{g_1 + |g_2|} > 0, \quad s_4 = \frac{\left(\sqrt{|\delta_1|} + \sqrt{|\delta_2|}\right)^2}{g_1 + |g_2|} > 0, \tag{2.4.31}$$

so that the basic set of inequalities then can be rewritten as follows

$$\begin{cases} s \in (0, s_3) \cup (s_4, \infty), \\ (g_1 - |g_2|)(s - s_1) > 0, \\ 0 < s < s_2. \end{cases} \tag{2.4.32}$$

Consider then the relative location of all four parameters s_i, $i = 1 \ldots 4$ on the real axis. First it is easy to show that

$$s_4 - s_2 = -\frac{\left(g_1\sqrt{|\delta_2|} - |g_2|\sqrt{|\delta_1|}\right)^2}{g_1 |g_2| (g_1 + |g_2|)} < 0, \tag{2.4.33}$$

so the s_2, s_3 and s_4 are always ordered as

$$0 < s_3 < s_4 < s_2. \tag{2.4.34}$$

The location of s_1 with respect to the other s_i depends on the relationship between other parameters of the model. To determine this order the following expressions are introduced

$$s_3 - s_1 = -2\frac{\left(\sqrt{|\delta_1|} - \sqrt{|\delta_2|}\right)\left(|g_2|\sqrt{|\delta_1|} + g_1\sqrt{|\delta_2|}\right)}{g_1^2 - g_2^2}, \tag{2.4.35}$$

$$s_4 - s_1 = 2\frac{\left(\sqrt{|\delta_1|} + \sqrt{|\delta_2|}\right)\left(g_1\sqrt{|\delta_2|} - |g_2|\sqrt{|\delta_1|}\right)}{g_1^2 - g_2^2}. \tag{2.4.36}$$

(i) First let $g_1 > |g_2|$ and $|\delta_1| < |\delta_2|$. In this case $g_1 - |g_2| \geq 0$ and

$$s_1 = -\frac{|\delta_2| - |\delta_1|}{g_1 - |g_2|} < 0, \tag{2.4.37}$$

so the second inequality of (2.4.32) is valid for all positive s. Then due to (2.4.34) the set of inequalities (2.4.32) is valid for

$$s \in (0, s_3) \cup (s_4, s_2). \tag{2.4.38}$$

(ii) Now let $g_1 > |g_2|$ and $|\delta_2| < |\delta_1| \leq \frac{g_1^2}{g_2^2}|\delta_2|$. In this case $0 < s_3 < s_1 < s_4 < s_2$, since

$$s_3 - s_1 = -2\frac{\left(\sqrt{|\delta_1|} - \sqrt{|\delta_2|}\right)\left(|g_2|\sqrt{|\delta_1|} + g_1\sqrt{|\delta_2|}\right)}{g_1^2 - g_2^2} < 0, \tag{2.4.39}$$

$$s_4 - s_1 = 2\frac{|g_2|\left(\sqrt{|\delta_1|} + \sqrt{|\delta_2|}\right)\left(\frac{g_1}{|g_2|}\sqrt{|\delta_2|} - \sqrt{|\delta_1|}\right)}{\left(g_1^2 - g_2^2\right)} > 0 \tag{2.4.40}$$

and the set (2.4.32) reduces to $s_4 < s < s_2$.

(iii) If $g_1 > |g_2|$ and $|\delta_1| > \frac{g_1^2}{g_2^2}|\delta_2|$ then $g_1 - |g_2|$ is still positive but

$$s_2 - s_1 = -\frac{\left(|\delta_1| - |\delta_2|\frac{g_1^2}{g_2^2}\right)|g_2|}{g_1\left(g_1 - |g_2|\right)} < 0, \tag{2.4.41}$$

so in this case $0 < s_3 < s_4 < s_2 < s_1$ and the second inequality at (2.4.32) is inconsistent to others.

(iv) Under the conditions $g_1 < |g_2|$, $|\delta_1| > |\delta_2|$ the second inequality of the set (2.4.32) is not valid for any positive s as in this case $g_1 - |g_2| < 0$ and

$$s_1 = -\frac{|\delta_1| - |\delta_2|}{|g_2| - g_1} < 0. \tag{2.4.42}$$

Therefore the set of inequalities (2.4.32) cannot be satisfied by any positive s.

(v) If we let $g_1 < |g_2|$ and $|\delta_1| \le |\delta_2| < \frac{g_2^2}{g_1^2}|\delta_1|$ then $0 < s_3 < s_1 < s_4 < s_2$ since

$$s_3 - s_1 = -2\frac{\left(\sqrt{|\delta_2|} - \sqrt{|\delta_1|}\right)\left(|g_2|\sqrt{|\delta_1|} + g_1\sqrt{|\delta_2|}\right)}{g_2^2 - g_1^2} < 0, \tag{2.4.43}$$

$$s_4 - s_1 = 2\frac{|g_1|\left(\sqrt{|\delta_1|} + \sqrt{|\delta_2|}\right)\left(\frac{g_2}{|g_1|}\sqrt{|\delta_1|} - \sqrt{|\delta_2|}\right)}{\left(g_2^2 - g_1^2\right)} > 0. \tag{2.4.44}$$

Thus the set (2.4.32) is equivalent to $0 < s < s_3$.

(vi) For the conditions $g_1 < |g_2|$ and $|\delta_2| \ge \frac{g_2^2}{g_1^2}|\delta_1|$ we have

$$s_2 - s_1 = -\frac{\left(|\delta_2| - |\delta_1|\frac{g_2^2}{g_1^2}\right)g_1}{|g_2|\left(|g_2| - g_1\right)} < 0 \qquad (2.4.45)$$

resulting in $0 < s_3 < s_4 < s_2 < s_1$. Therefore, in this case the set of inequalities (2.4.32) reduces to $s_2 < s < s_1$.

(vii) Finally let us consider the case $g_1 = |g_2|$. We cannot use the second inequality in terms of s_i as it is singular. Putting $g_1 = |g_2|$ in the second inequality of (2.4.15) gives $\delta_1 - \delta_2 > 0$. So, if $\delta_1 < \delta_2$ the set (2.4.15) is inconsistent, otherwise (i.e. for $\delta_1 > \delta_2$) it reduces to the inequality $s \in (0, s_3) \cup (s_4, \infty)$.

Summarising the results obtained above we can make the following conclusion about the stability/instability of the system of two KdV solitons. So, the system is ***stable***:

- for $0 < s < s_2$ provided $\delta_2 \leq \frac{g_2}{g_1}\delta_1 < 0$ and $\delta_1 > 0$;
- for $0 < s < s_3$ or $s_4 < s < s_2$ provided $0 < -g_2 \leq g_1$ and $\delta_2 < \delta_1 < 0$;
- for $s_4 < s < s_2$ provided $0 < -g_2 < g_1$ and $\frac{g_1^2}{g_2^2}\delta_2 \leq \delta_1 < \delta_2 < 0$;
- for $0 < s < s_3$ provided $0 < g_1 \leq -g_2$ and $\frac{g_2^2}{g_1^2}\delta_1 < \delta_2 \leq \delta_1 < 0$;
- for $s_2 < s < s_1$ provided $0 < g_1 < -g_2$ and $\delta_2 \leq \frac{g_2^2}{g_1^2}\delta_1 < 0$;

and the system is ***unstable***:

- for all $s > 0$ provided $\delta_2 > 0$;

  or $0 < \delta_2 \frac{g_1}{g_2} < \delta_1$ and $\delta_2 < 0$;

  or $0 < -g_2 \leq g_1$ and $\delta_1 < \frac{g_1^2}{g_2^2}\delta_2 < 0$;

  or $0 < g_1 \leq -g_2$ and $\delta_1 < \delta_2 < 0$;

- for $s > s_2$ provided $\delta_2 \le \frac{g_2}{g_1}\delta_1 < 0$, $\delta_1 > 0$;
- for $s_3 < s < s_4$ or $s_2 < s < \infty$ provided $0 < -g_2 < g_1$ and $\delta_2 < \delta_1 < 0$;
- for $0 < s < s_4$ or $s_2 < s < \infty$ provided $0 < -g_2 \le g_1$ and $\frac{g_1^2}{g_2^2}\delta_2 \le \delta_1 < \delta_2 < 0$;
- for $s_3 < s < \infty$ provided $0 < g_1 < -g_2$ and $\frac{g_2^2}{g_1^2}\delta_1 < \delta_2 \le \delta_1 < 0$;
- for $0 < s < s_2$ or $s_1 < s < \infty$ provided $0 < g_1 < -g_2$ and $\delta_2 \le \frac{g_2^2}{g_1^2}\delta_1 < 0$.

It follows that there is no conditions under which the system may be stable for all positive s. Consequently, for a perturbation with a sufficiently wide spectrum the system is unstable. In particular the system perturbed by the white noise is always unstable (given that the conditions $g_1 > 0$ and $g_2 < 0$ apply).

2.5. NUMERICAL SIMULATION OF TWO COUPLED KDV SOLITONS EVOLUTION

In the previous section we investigated the linear stability of a solution of the coupled two-dimensional equations (2.2.21), (2.2.22) in the form of a pair of the KdV solitons (2.2.24) – (2.2.26). As stated this solution is unstable to any perturbations, which have a sufficiently wide spatial spectrum relative to the transverse direction. In particular, this system is also unstable when perturbed with two-dimensional white noise.

The aim of this section is to confirm numerically the previously obtained results and determine the possible evolution of this system where the initial profile is given by (2.2.24) at $t = 0$, perturbed by white noise. Based on the properties of a single KP equation, such perturbations would be expected to evolve to either a single two-dimensional solitary wave or a system of such waves. As known (see, for example, *Kadomtsev and Petviashvili* 1970, *Akylas* 1994) the KdV soliton is stable (unstable) to two-dimensional perturbations if it is described by a KP equation (say (2.2.21) with $\kappa_1 = 0$) with positive (negative) dispersive coefficient λ. In the case of stability (instability) of the KdV soliton there does not (does) exist a two-dimensional solitary solution of the KP equation. Thus, there is a link between stability of the KdV solitons and existence of a two-dimensional solitary solution for the uncoupled KP equation.

The coupled KP equations (2.2.21), (2.2.22) were integrated numerically[1] by means of a pseudospectral technique (*Canuto et al* 1988, *Fornberg* 1998). Periodic boundary

[1] The code written and maintained by my supervisor Dr. S. Clarke has been used for these integrations.

conditions were imposed both in the x-direction and y-direction. The initial values were taken of the form,

$$\eta(x,y,t=0)=a_1\,\mathrm{sech}^2 x\,, \qquad \zeta(x,y,t=0)=a_2\,\mathrm{sech}^2 x\,. \tag{2.5.1}$$

These initial wave profiles were perturbed with a random function of x and y with uniform spatial spectrum (white noise).

The amplitude of (2.5.1) and parameters of equations (2.2.21), (2.2.22) are chosen to satisfy relationships (2.2.25), (2.2.26), so that in the case of no dependence on the transverse coordinate (i.e. when $\gamma_1=\gamma_2=0$) such perturbations would propagate along the x-direction with no change. We have used the following 4 sets of parameters for modelling the evolution of the initial profiles (2.5.1):

I. $\Delta_1=8$, $\Delta_2=-8$, $\mu_1=\mu_2=6$, $\lambda_1=1$, $\lambda_2=4$, $\gamma_1=2$, $\gamma_2=-1$, $\kappa_1=0.5$, $\kappa_2=24$, $a_1=2$, $a_2=8$.

II. $\Delta_1=7$, $\Delta_2=-7$, $\mu_1=\mu_2=6$, $\lambda_1=1$, $\lambda_2=4$, $\gamma_1=2$, $\gamma_2=-1$, $\kappa_1=0.5$, $\kappa_2=16$, $a_1=2$, $a_2=8$.

III. $\Delta_1=6$, $\Delta_2=-6$, $\mu_1=\mu_2=6$, $\lambda_1=1$, $\lambda_2=4$, $\gamma_1=2$, $\gamma_2=-1$, $\kappa_1=0.5$, $\kappa_2=8$, $a_1=2$, $a_2=8$.

IV. $\Delta_1=-22$, $\Delta_2=22.05$, $\mu_1=\mu_2=6$, $\lambda_1=5$, $\lambda_2=-1$, $\gamma_1=1$, $\gamma_2=2$, $\kappa_1=-0.5$, $\kappa_2=4$, $a_1=10$, $a_2=-2$.

Results of the modelling for sets I – III are represented in Figures 2 – 4 respectively in the form of a series of contour plots taken at five consecutive moments of time $t=0,\ 0.5,\ 1.0,\ 1.5,\ 2.0$ as indicated on top of each plot. The left column describes the evolution of η perturbation, the right one of the ζ perturbation evolution. For each plot horizontal and vertical axes are for x- and y-dependence respectively. Figure 5 represents evolution of the system, which parameters defined by set III as well as Figure 4, but in the form of series of three-dimensional surface plots taken at three consecutive

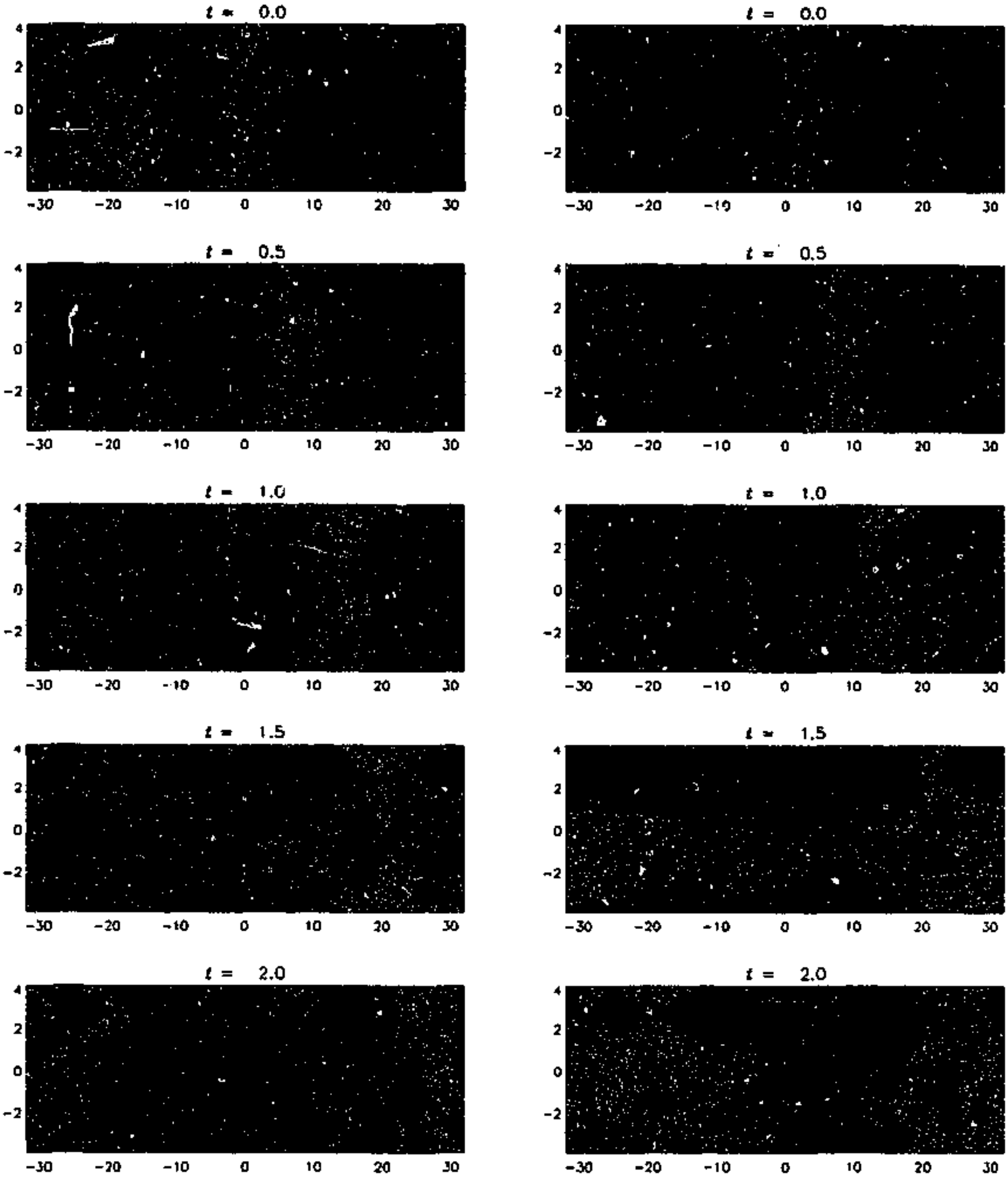

Figure 2. Evolution of two coupled KdV solitons with parameters of set I.

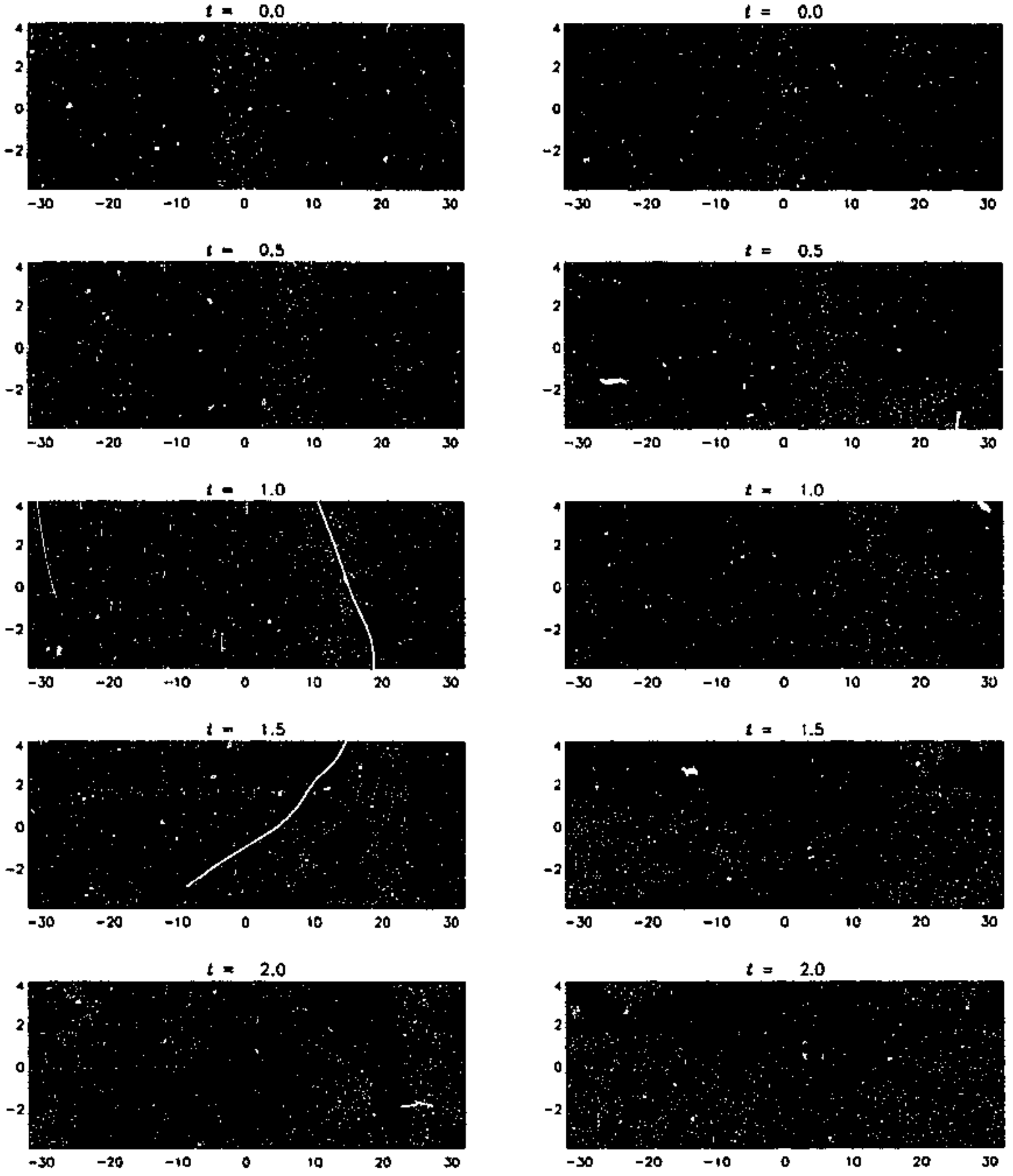

Figure 3. Evolution of two coupled KdV solitons with parameters of set II.

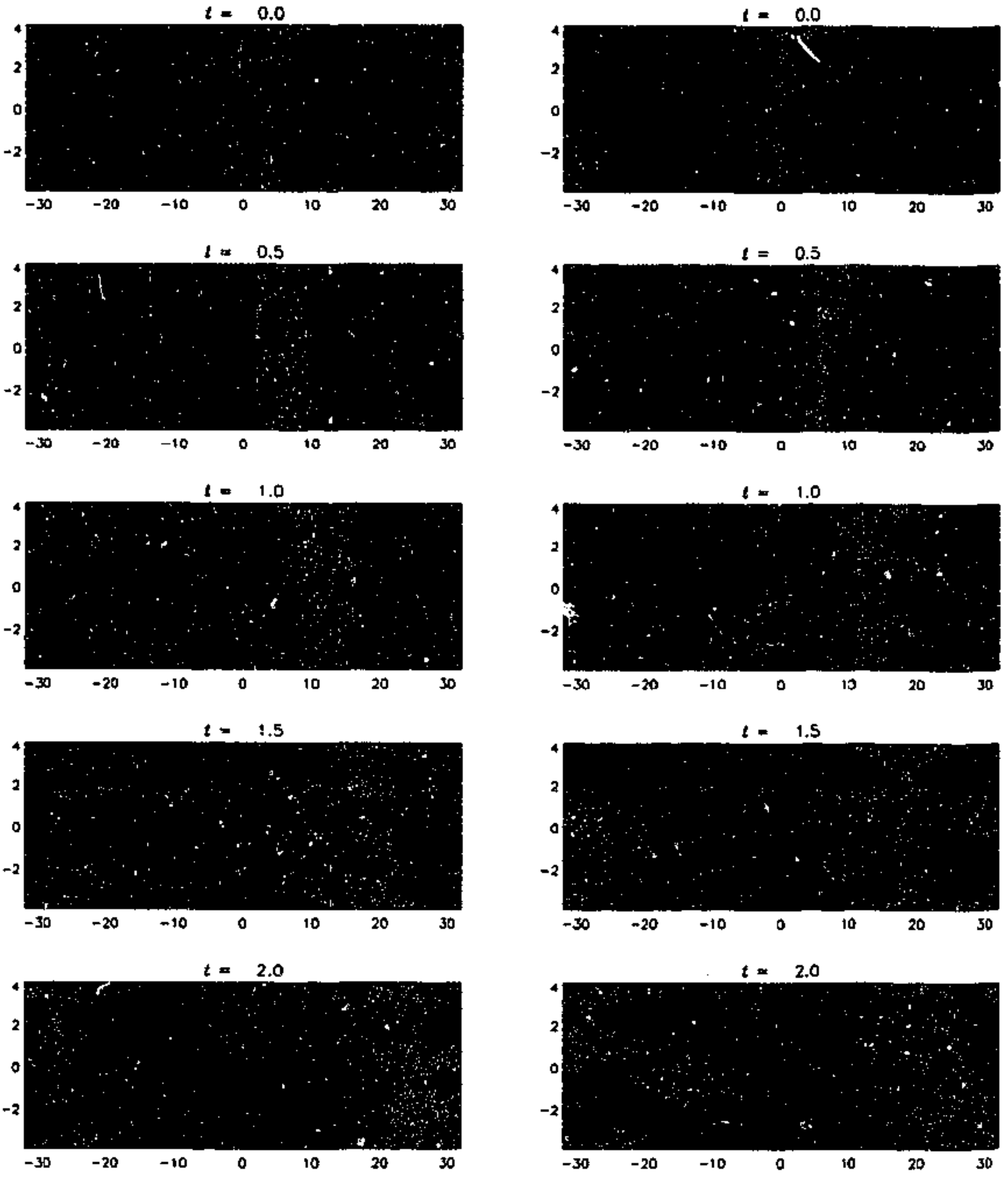

Figure 4. Evolution of two coupled KdV solitons with parameters of set III.

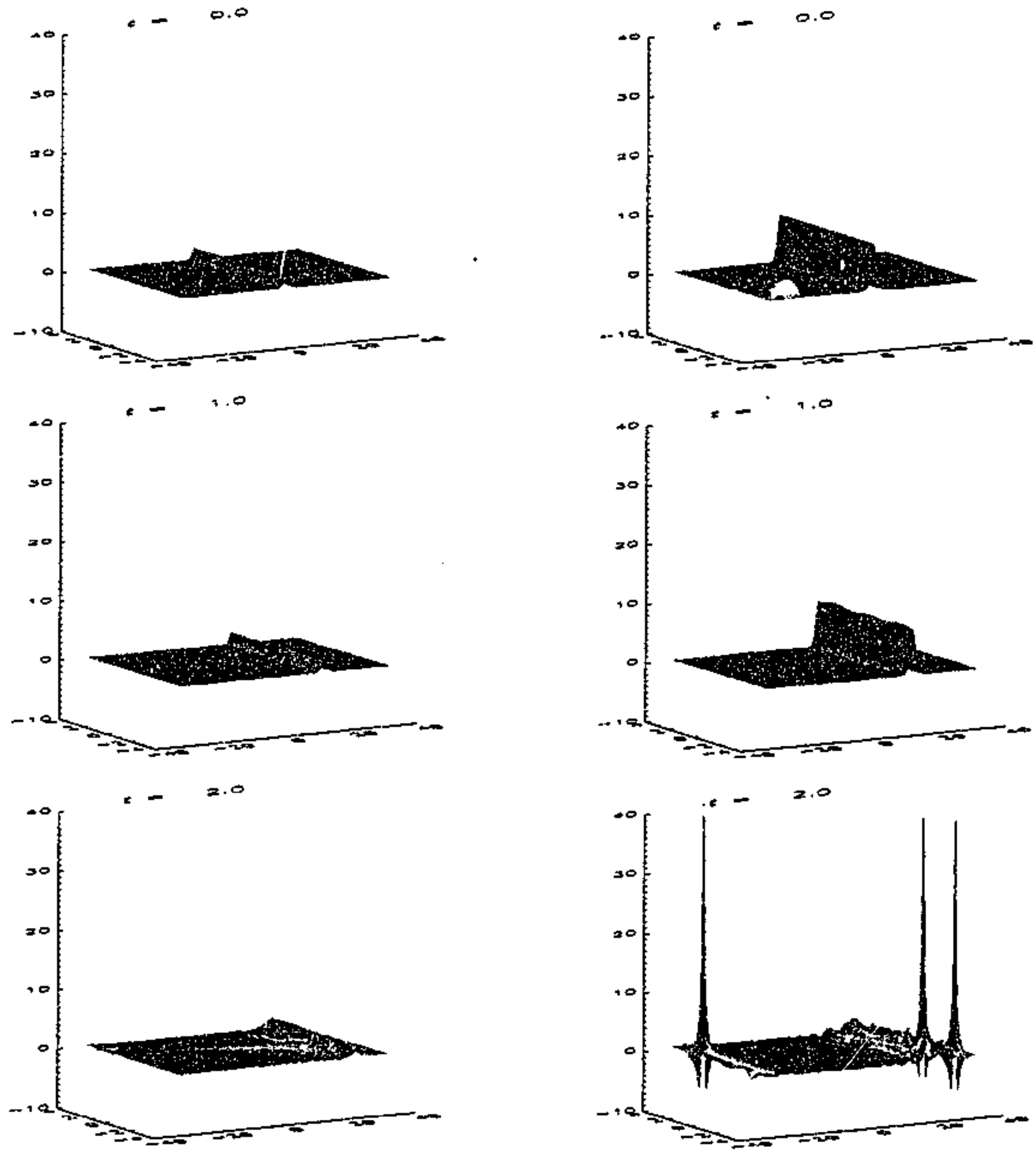

Figure 5. The same as Figure 4 but in a three-dimensional format.

moments of time $t = 0, \quad 1.0, \quad 2.0$. The results of the evolution for the set IV system are not represented graphically here since this system happened to be stable.

As seen from the various figures, sets I – III describe unstable evolution of the system. Instability results in the initial ζ profile breaking into two (set I, II) and three (set III) compacted peaks (light points in the contour plots), which may be associated with two-dimensional solitary waves. The evolution of another profile (perturbation of the η interface) is also unstable, but its instability is much slower, so that its leave into compact peaks takes much more time and was not tracked, as it requires much more computer resources. Note, that separate KP equations (with $\kappa_1 = \kappa_2 = 0$) in these cases describe stable KdV solitons as $\lambda_{1,2} > 0$ and cannot generate two-dimensional solitary waves. Therefore, unstable behaviour of these systems is a consequence of the coupling.

One of these (set I) corresponds to the stable one-dimensional system as $\delta_1 - \delta_2 > 0$ in this case. This illustrates the destabilisation property of the transverse perturbations.

The set IV satisfies the linear instability criteria formulated in the previous section as well as the stability criteria for one-dimensional system ($\delta_1 - \delta_2 < 0$). Hence, the system would have been expected to evolve unstably. We can assume that linear instability generated at the initial period of evolution was then suppressed by nonlinear stability effects. Thus, this fact shows that nonlinear stability analysis must be employed to state the final stability of this system.

Thus the numerical modelling of two coupled KdV solitons shows that both coupling and transversity affect the evolution of this system.

CHAPTER 3

COUPLED WAVES IN A THREE-LAYERED FLUID WITH A PIECEWISE LINEAR SHEAR FLOW

3.1. LINEAR APPROXIMATION

We now consider some modifications to the three-layered fluid model explored in Chapter 2. With the same geometrical configuration as shown in the Figure 1 we still let the density of the fluid be piecewise constant and described by expression (2.1.2), but the horizontal basic flow is now assumed to become piecewise linear as follows

$$U = \begin{cases} U_3 + \Omega_3(z - H_2), & H_2 + \zeta < z < H_2 + H_3, \\ U_1 + \Omega_2 z, & \eta < z < H_2 + \zeta, \\ U_1 + \Omega_1 z, & -H_1 < z < \eta. \end{cases} \tag{3.1.1}$$

Here $U_3 = U_1 + \Omega_2 H_2$ and η, ζ are still the displacements of the lower and the upper interfaces respectively. The basic flow defined by expression (3.1.1) has constant shear in each layer and is continuous across each undisturbed interfaces (i.e. when $\eta = \zeta = 0$). The continuity of the undisturbed basic flow is enforced to prevent Kelvin-Helmholtz instability (see, for instance, *Craik* 1985). Thus taking the basic flow in the form of (3.1.1) can be regarded as an improvement over the model discussed in Chapter 2[1].

In contrast with Chapter 2 we commence here by considering two-dimensional flow that will be then extended to three dimensions. To describe evolution of the flow in such a model we use the Euler equation

$$\rho \frac{\partial \mathbf{V}}{\partial t} + \rho(\mathbf{V}\nabla)\mathbf{V} = -\nabla P - \rho g \mathbf{k} \tag{3.1.2}$$

along with the equation of mass conservation

$$\frac{\partial \rho}{\partial t} + \mathbf{V}\nabla\rho = 0 \tag{3.1.3}$$

and the incompressibility equation

$$\operatorname{div} \mathbf{V} = 0 . \tag{3.1.4}$$

Here ρ is the mass density of fluid and P is the pressure therein, $\mathbf{k}$ is the unit vector along z-axis; $\mathbf{V} = \overline{\mathbf{V}} + \widetilde{\mathbf{V}}$ where $\overline{\mathbf{V}} = (\overline{U}, 0)$ is an undisturbed basic flow with functional dependence on the vertical variable $\overline{U} = \overline{U}(z)$ obtained from (3.1.1) by letting $\eta = \zeta = 0$:

$$\overline{U} = \begin{cases} U_3 + \Omega_3(z - H_2), & H_2 < z < H_2 + H_3, \\ U_1 + \Omega_2 z, & 0 < z < H_2, \\ U_1 + \Omega_1 z, & -H_1 < z < 0, \end{cases} \tag{3.1.5}$$

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[1] Although the previous model cannot be obtained from the one just formulated here as $\Omega_i \to 0$, $i = 1, 2, 3$.

and $\widetilde{\mathbf{V}} = (u, w)$ is the velocity perturbation, such that equation (3.1.4) can be rewritten in a form

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{3.1.6}$$

We should note that the equation (3.1.3) does not contradict the fact that the density ρ of this fluid is a constant within a layer, as this follows from (2.1.2). Here, in equation (3.1.3), the density is implied to be a function defined at fixed (not moving) point (with coordinates x, z) in contrast to the function (2.1.2) defined at points moving along with the flow. So, in terms of definition (2.1.2) the density is indeed constant at any points "stuck" to streamlines within a layer. However the layer boundaries are variable as the interfaces may be disturbed, in such a case some fixed points located in a small vicinity of the undisturbed layers boundaries ($z = 0$ or $z = H_2$) may alternately belong either to the lower or to the upper layer. It means that density becomes a function of all spatial variables and time, since the displacement of both interfaces η and ζ also is dependent on these. This fact becomes obvious after rewriting the expression (2.1.2) in terms of the unit step function $\Theta(z-a)$ (equal to 1 for $z > a$ and 0 otherwise) as follows

$$\begin{aligned}\rho = {} & \rho_1 \Theta(z + H_1)\Theta(\eta(x,t) - z) + \rho_2 \Theta(z - \eta(x,t))\Theta(H_2 + \zeta(x,t) - z) \\ & + \rho_3 \Theta(z - H_2 - \zeta(x,t))\Theta(H_2 + H_3 - z).\end{aligned} \tag{3.1.7}$$

Both descriptions mentioned above are equivalent. The first one is based on a consideration of a layer as a set of streamlines with the density constant within such a layer. This approach is useful when a condition of continuity across a disturbed interface such as (2.1.6) or (2.1.7) is known, and it has been used in the Chapter 2 to derive the evolution equations for the coupled waves. The second description implies that geometrical configuration of a layer is fixed and formally not disturbed, as a result the density becomes a function of the spatial variables and time. This approach does not require knowledge of any invariants such as (2.1.6) or (2.1.7). This description will be used below for deriving two-dimensional evolution equations for coupled waves propagating in the three-layered model with piecewise continuous shear flow. In this section this description is also applied to linear waves.

Putting $P=\overline{p}+\widetilde{p}$ and $\rho=\overline{\rho}+\widetilde{\rho}$ we can linearise the equations (3.1.2), (3.1.3) respectively with respect to small perturbations u, w, $\widetilde{p}$ and $\widetilde{\rho}$ as follows

$$\overline{\rho}\left(u_t+\overline{U}u_x\right)+\overline{\rho}\Omega w+\widetilde{p}_x=0\,, \tag{3.1.8}$$

$$\zeta_t+\left(U_i+\phi_{i,x}\right)\zeta_x+\phi_{i,y}\zeta_y=\phi_{i,z}\,, \tag{3.1.9}$$

$$\widetilde{\rho}_t+\overline{U}\widetilde{\rho}_x+w\overline{\rho}_z=0\,, \tag{3.1.10}$$

where $\overline{\rho}$ is basic density given by

$$\overline{\rho}=\begin{cases}\rho_3, & H_2<z<H_2+H_3,\\ \rho_2, & 0<z<H_2,\\ \rho_1, & -H_1<z<0,\end{cases} \tag{3.1.11}$$

and $\overline{p}$ is unperturbed pressure such that $\overline{p}_z=-g\overline{\rho}$; $\overline{U}=\overline{U}(z)$ is a shear flow defined by (3.1.5) and $\Omega=\overline{U}_z$ is piecewise shear gradient.

Now let $\xi=\xi(x,z,t)$ be a vertical displacement of a fluid particle given by

$$\xi_t+\overline{U}\xi_x=w\,, \tag{3.1.12}$$

so that the equation (3.1.10) becomes

$$\widetilde{\rho}=-\overline{\rho}_z\xi\,. \tag{3.1.13}$$

Then after substitution of (3.1.12), (3.1.13) into equations (3.1.8), (3.1.9), (3.1.6) and subsequently assuming u, ξ, $\widetilde{p}$ to be proportional to $\exp(ikx-i\omega t)$ we can rewrite these equations in a form

$$\overline{\rho}(\omega-k\overline{U})u+\overline{\rho}\Omega(\omega-k\overline{U})\xi-k\tilde{p}=0\,, \tag{3.1.14}$$

$$\overline{\rho}(\omega-k\overline{U})^2\xi+g\overline{\rho}_z\xi-\tilde{p}_z=0\,, \tag{3.1.15}$$

$$ku-(\omega-k\overline{U})\xi_z+k\Omega\xi=0\,. \tag{3.1.16}$$

Eliminating u and $\tilde{p}$ from the above equations yields

$$\left(\overline{\rho}(\omega-k\overline{U})^2\xi_z\right)_z-\overline{\rho}k^2(\omega-k\overline{U})^2\xi-gk^2\overline{\rho}_z\xi=0\,. \tag{3.1.17}$$

The vertical displacement of a fluid particle (3.1.12) must be continuous across an interface, therefore

$$[\xi]_{z=0}=[\xi]_{z=H_2}=0\,. \tag{3.1.18}$$

Then it follows from (3.1.17) the following function must also be continuous at the interface:

$$\left[\overline{\rho}(\omega-k\overline{U})^2\xi_z-gk^2\overline{\rho}\xi\right]_{z=0}=\left[\overline{\rho}(\omega-k\overline{U})^2\xi_z-gk^2\overline{\rho}\xi\right]_{z=H_2}=0\,. \tag{3.1.19}$$

Since $\overline{\rho}$ is piecewise constant equation (3.1.17) can be rewritten as a function $\psi=(\omega-k\overline{U})\xi$ within each layer, of the form

$$\psi_{zz}=k^2\psi\,, \tag{3.1.20}$$

so that an explicit expression for the vertical displacement can be readily obtained:

$$\xi_1=\frac{\omega-kU_1}{\omega-k(U_1+\Omega_1 z)}A\frac{\sinh(k(z+H_1))}{\sinh(kH_1)}\qquad \text{for } -H_1<z<0\,; \tag{3.1.21}$$

$$\xi_2 = \frac{\omega - kU_1}{\omega - k(U_1 + \Omega_2 z)} A \frac{\sinh(k(H_2 - z))}{\sinh(kH_2)} + \frac{\omega - kU_3}{\omega - k(U_1 + \Omega_2 z)} B \frac{\sinh(kz)}{\sinh(kH_2)}$$

$$\text{for } 0 < z < H_2; \qquad (3.1.22)$$

$$\xi_3 = \frac{\omega - kU_3}{\omega - k(U_3 + \Omega_3 (z - H_2))} B \frac{\sinh(k(H_2 + H_3 - z))}{\sinh(kH_3)}$$

$$\text{for } H_2 < z < H_2 + H_3. \qquad (3.1.23)$$

Functions (3.1.21) – (3.1.23) are constructed to be continuous across the interfaces at $z = 0$ and $z = H_2$. Consequently the conditions (3.1.18) are satisfied and $\xi(0) = A$, $\xi(H_2) = B$, where A and B are the amplitudes of the interfaces defined as

$$(\eta, \zeta) = (A, B)\exp(ikx - i\omega t). \qquad (3.1.24)$$

After substituting functions (3.1.21) – (3.1.23) into the continuity conditions (3.1.19) we obtain the following algebraic equations

$$D_1 A + EB = 0, \qquad EA + D_2 B = 0, \qquad (3.1.25)$$

where

$$D_1 = g(\rho_1 - \rho_2) + (c - U_1)(\rho_2 \Omega_2 - \rho_1 \Omega_1) - k(c - U_1)^2 [\rho_1 \coth(kH_1) + \rho_2 \coth(kH_2)], \qquad (3.1.26)$$

$$D_2 = g(\rho_2 - \rho_3) + (c - U_3)(\rho_3 \Omega_3 - \rho_2 \Omega_2) - k(c - U_3)^2 [\rho_2 \coth(kH_2) + \rho_3 \coth(kH_3)], \qquad (3.1.27)$$

$$E = \frac{k\rho_2 (c - U_1)(c - U_3)}{\sinh(kH_2)}, \qquad (3.1.28)$$

and $c = \omega / k$ is the phase speed. In the long-wave limit as $k \to 0$ expressions (3.1.26) – (3.1.28) become

$$D_1 \to g(\rho_1 - \rho_2) + (c - U_1)(\rho_2\Omega_2 - \rho_1\Omega_1) - (c - U_1)^2\left(\frac{\rho_1}{H_1} + \frac{\rho_2}{H_2}\right), \tag{3.1.29}$$

$$D_2 \to g(\rho_2 - \rho_3) + (c - U_3)(\rho_3\Omega_3 - \rho_2\Omega_2) - (c - U_3)^2\left(\frac{\rho_2}{H_2} + \frac{\rho_3}{H_3}\right), \tag{3.1.30}$$

$$E \to \frac{\rho_2(c - U_1)(c - U_3)}{H_2}. \tag{3.1.31}$$

Then following general theory (see Chapter 1) we can determine the resonance conditions by equating expressions (3.1.29) – (3.1.31) to zero simultaneously

$$E = D_1 = D_2 = 0. \tag{3.1.32}$$

The first equation ($E = 0$) yields either $c = U_1$ or $c = U_3$. Without loss of generality we may set $c = 0$. Then we have two sets of resonance conditions

$$c = U_1 = 0, \qquad \rho_1 = \rho_2, \tag{3.1.33}$$

$$\frac{g(\rho_2 - \rho_3)H_3}{\rho_3} = \Omega_2\Omega_3 H_2 H_3 + \Omega_2^2 H_2^2, \tag{3.1.34}$$

and

$$c = U_3 = 0, \qquad \rho_2 = \rho_3, \tag{3.1.35}$$

$$\frac{g(\rho_1 - \rho_2)H_1}{\rho_1} = \Omega_1\Omega_2 H_1 H_2 + \Omega_2^2 H_2^2. \tag{3.1.36}$$

The second set of resonance conditions can be obtained from the first one after the substitution of subscript 1 for 3. Hence, we can confine our consideration to conditions (3.1.33), (3.1.34) only.

In the presence of weak dispersion characterised by a small parameter $\varepsilon << 1$ the above resonance conditions should be rewritten as follows

$$A, B = O(\varepsilon^2), \qquad c = O(\varepsilon^2), \qquad k = O(\varepsilon), \tag{3.1.37}$$

$$\rho_2 = \rho_1 + \varepsilon^2 \delta\rho_2, \qquad \rho_3 = \rho_c + \varepsilon^2 \delta\rho_3, \qquad U_1 \equiv 0, \tag{3.1.38}$$

where ρ_c is a critical value of the third layer density such that condition (3.1.34) is satisfied, i.e.

$$\rho_c = \rho_1 \frac{gH_3}{gH_3 + \Omega_2^2 H_2^2 + \Omega_2 \Omega_3 H_2 H_3}. \tag{3.1.39}$$

Then coefficients (3.1.26) – (3.1.28) can be unfolded in the form

$$D_1 = -k^2 g(\rho_2 - \rho_1) + \omega k \rho_1 (\Omega_2 - \Omega_1) + (\varepsilon^6), \tag{3.1.40}$$

$$\begin{aligned} D_2 = {} & \left(\rho_c \Omega_3 + \rho_1 \Omega_2 + 2\rho_c \frac{H_2}{H_3} \Omega_2 \right) \omega k + g \frac{\rho_c \rho_2 - \rho_3 \rho_1}{\rho_c} k^2 \\ & - \frac{1}{3} (\rho_c H_3 + \rho_1 H_2) \Omega_2^2 H_2^2 k^4 + O(\varepsilon^6), \end{aligned} \tag{3.1.41}$$

$$E = -\rho_1 \Omega_2 \omega k + O(\varepsilon^6). \tag{3.1.42}$$

Now substituting the above expressions (3.1.40) – (3.1.41) into (3.1.25) and replacing ik and $i\omega$ for $\partial/\partial x$ and $-\partial/\partial t$ respectively (as in section 2.1) we can obtain two coupled linear equations

$$\alpha_1 A_t + \sigma B_t + \Delta_1 A_x = 0, \tag{3.1.43}$$

$$\sigma A_t + \alpha_2 B_t + \Delta_2 B_x + \lambda B_{xxx} = 0, \tag{3.1.44}$$

where

$$\alpha_1 = \rho_1(\Omega_1 - \Omega_2), \qquad \alpha_2 = -\rho_c\Omega_3 - \rho_1\Omega_2 - 2\rho_c\frac{H_2}{H_3}\Omega_2, \tag{3.1.45}$$

$$\Delta_1 = g(\rho_1 - \rho_2), \qquad \Delta_2 = g\frac{\rho_c\rho_2 - \rho_3\rho_1}{\rho_c}, \tag{3.1.46}$$

$$\sigma = \rho_1\Omega_2, \qquad \lambda = \frac{1}{3}\Omega_2^2 H_2^2(\rho_3 H_3 + \rho_1 H_2). \tag{3.1.47}$$

The above coupled linear equations have important differences from those obtained for the model with constant basic flow (*Grimshaw* 2000). First, they have a different structure as the dispersive term is contained only in one equation. Another distinction is in the form of coupling terms. Now the equations are coupled through time derivative terms, not to spatial derivative terms as previously, and the linear coupling coefficient σ is the same for each equation. Through a linear transformation of the variables A and B we can exclude one of the above mentioned distinctions, however we cannot fully transform these equations to those describing linear coupled waves in a model with piecewise constant basic flow.

3.2. DERIVATION OF NONLINEAR COUPLED 1D-EQUATIONS

In this section we extend the coupled equations derived in the previous section to the nonlinear case. Confining our attention to two-dimensional flow we can assume that the perturbed flow in each layer is irrotational (i.e. $\operatorname{curl}\widetilde{\mathbf{V}} = \mathbf{0}$), so that we can introduce a velocity potential ϕ to satisfy identically the incompressibility equation (3.1.6)

$$u = \phi_x, \qquad w = \phi_z, \tag{3.2.1}$$

provided the velocity potential satisfies Laplace's equation $\phi_{xx} + \phi_{zz} = 0$. It is also useful to introduce a stream function ψ

$$u = -\psi_z, \quad w = \psi_x, \tag{3.2.2}$$

which identically satisfies equation (3.1.6) as well. There is a simple relationship between functions ϕ and ψ emanating from (3.2.1) and (3.2.2), namely $\phi_x = -\psi_z$, $\phi_z = \psi_x$. Thus ψ also satisfies Laplace's equation, as $\psi_{xx} + \psi_{zz} = \phi_{zx} - \phi_{xz} = 0$.

Then the Euler equation (3.1.2) can be rewritten in a form

$$\frac{\partial u}{\partial t} + U(z)\frac{\partial u}{\partial x} + w\Omega + \frac{\partial}{\partial x}\left(\frac{1}{2}\left(u^2 + w^2\right)\right) + \frac{\partial}{\partial x}\left(\frac{P}{\rho}\right) = 0, \tag{3.2.3}$$

$$\frac{\partial w}{\partial t} + U(z)\frac{\partial w}{\partial x} + \frac{\partial}{\partial z}\left(\frac{1}{2}\left(u^2 + w^2\right)\right) + \frac{\partial}{\partial z}\left(\frac{P}{\rho}\right) + g = 0, \tag{3.2.4}$$

where U is supposed to be a linear function of z, $\Omega = \partial U/\partial z = const$ and the density ρ is also assumed to be constant. After rewriting the velocity components in terms of either ϕ or ψ both above equations can be expressed as a single vector equation of the form

$$\nabla\left(\phi_t + U(z)\phi_x + \Omega\psi + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + \frac{P}{\rho} + gz\right) = 0, \tag{3.2.5}$$

from which immediately follows an equation for pressure:

$$p_0 - p = -\rho\left(\phi_t + U(z)\phi_x + \Omega\psi + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + gz\right), \tag{3.2.6}$$

where p_0 is pressure on some level, say at an interface between any two layers. Then due to continuity of pressure the following dynamic boundary condition applies at an interface between two layers with a constant shear

$$\left[\rho\left(\phi_t + U(z)\phi_x + \Omega\psi + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + gz\right)\right] = 0. \tag{3.2.7}$$

Thus the former pressure condition (2.1.6), (2.1.7), used for the model with a piecewise constant basic flow, must be replaced in this section by the above boundary condition applied at both interfaces. Further the kinematic boundary conditions (2.1.4), (2.1.5) must be modified as follows

$$\eta_t + \left(\Omega_i\eta + \phi_{i,x}\right)\eta_x = \phi_{i,z}, \quad i = 1,2, \tag{3.2.8}$$

$$\zeta_t + \left(\Omega_2 H_2 + \Omega_i\zeta + \phi_{i,x}\right)\zeta_x = \phi_{i,z}, \quad i = 2,3 \tag{3.2.9}$$

after taking into account the basic flow structure of the form (3.1.1) with $U_1 = 0$ in accordance with one of the resonance conditions (3.1.33). Here the subscript i indicates a layer index counting from the bottom, another subscript following the comma is used for designation of a partial derivative.

To satisfy the other resonance conditions (3.1.33), (3.1.34) we must introduce the following scaling

$$X = \varepsilon x, \qquad T = \varepsilon^3 t, \tag{3.2.10}$$

and represent the density in the form

$$\rho = \rho^{(0)} + \varepsilon^2 \rho^{(2)}, \tag{3.2.11}$$

where

$$\rho^{(0)} = \begin{cases} \rho_c, & H_2 + \zeta < z < H_2 + H_3, \\ \rho_{\cdot}, & -H_1 < z < H_2 + \zeta, \end{cases} \tag{3.2.12}$$

$$\rho^{(2)} = \begin{cases} \delta\rho_3, & H_2 + \zeta < z < H_2 + H_3, \\ \delta\rho_2, & \zeta < z < H_2 + \eta, \\ 0, & -H_1 < z < \eta. \end{cases} \tag{3.2.13}$$

Here $\delta\rho_{2,3}$ and ρ_c are defined by expressions (3.1.38) and (3.1.39) respectively. Under the scaling (3.1.10), Laplace's equation, say, for the stream function ψ_i inside the i-th layer, becomes

$$\psi_{i,zz} + \varepsilon^2 \psi_{i,XX} = 0\,. \tag{3.2.14}$$

Then following the procedure of section 2.2, we can seek a solution of equation (3.2.14) in the form of power series in z

$$\psi = \sum_{n=0}^{\infty} \Psi_i^{(n)}(X,T)(z - a_i)^n\,, \qquad i = 1,2,3, \tag{3.2.15}$$

where a_i is some constant depending on the index of a layer i. Substituting (3.2.15) into (3.2.14) yields the following recursive relationship

$$\Psi_i^{(n+2)} = -\frac{\varepsilon^2}{(n+1)(n+2)}\Psi_{i,XX}^{(n)} \tag{3.2.16}$$

that in its turn allows us to state

$$\psi_1 = C(X,T)(z+H_1) - \frac{1}{6}\varepsilon^2 C_{XX}(X,T)(z+H_1)^3 + O(\varepsilon^4), \tag{3.2.17}$$

$$\psi_2 = D(X,T) + E(X,T)z - \frac{1}{2}\varepsilon^2 D_{XX}(X,T)z^2 - \frac{1}{6}\varepsilon^2 E_{XX}(X,T) + O(\varepsilon^4), \tag{3.2.18}$$

$$\psi_3 = F(X,T)(z-H_2-H_3) - \frac{1}{6}\varepsilon^2 F_{XX}(X,T)(z-H_2-H_3)^3 + O(\varepsilon^4). \tag{3.2.19}$$

Since $\phi_z = \varepsilon\psi_X$ and $\psi_z = -\varepsilon\phi_X$ the velocity potential in each layer can be represented as follows

$$\phi_1 = -\frac{1}{\varepsilon}\int C(\xi,\mathrm{T})d\xi + \frac{1}{2}\varepsilon C_X(X,T)(z+H_1)^2 + O(\varepsilon^3), \tag{3.2.20}$$

$$\phi_2 = -\frac{1}{\varepsilon}\int E(\xi,\mathrm{T})d\xi + \varepsilon D_X(X,T)z + \frac{1}{2}\varepsilon E_X(X.T)z^2 + O(\varepsilon^3), \tag{3.2.21}$$

$$\phi_3 = -\frac{1}{\varepsilon}\int F(\xi,\mathrm{T})d\xi + \frac{1}{2}\varepsilon F_X(X,T)(z-H_2-H_3)^2 + O(\varepsilon^3). \tag{3.2.22}$$

Finally, we should unfold all unknown previously introduced functions

$$C(X,T) = \varepsilon^2 C_2(X,T) + \varepsilon^4 C_4(X,T) + O(\varepsilon^6), \tag{3.2.23}$$

$$D(X,T)=\varepsilon^2 D_2(X,T)+\varepsilon^4 D_4(X,T)+O(\varepsilon^6), \tag{3.2.24}$$

$$E(X,T)=\varepsilon^2 E_2(X,T)+\varepsilon^4 E_4(X,T)+O(\varepsilon^6), \tag{3.2.25}$$

$$F(X,T)=\varepsilon^2 F_2(X,T)+\varepsilon^4 F_4(X,T)+O(\varepsilon^6), \tag{3.2.26}$$

$$\eta(X,T)=\varepsilon^2 A_2(X,T)+\varepsilon^4 A_4(X,T)+O(\varepsilon^6), \tag{3.2.27}$$

$$\zeta(X,T)=\varepsilon^2 B_2(X,T)+\varepsilon^4 B_4(X,T)+O(\varepsilon^6). \tag{3.2.28}$$

After substitution of (3.2.17) – (3.2.28) into the boundary conditions (3.2.7) – (3.2.9) at both interfaces, taking into account (3.2.10) – (3.2.13) and grouping terms with the same power of ε we can obtain sets of equations governing the above functions.

At the *leading* order these equations result in

$$C_2 = D_2 = 0, \qquad E_2 = \Omega_2 B_2, \qquad F_2 = -\frac{\Omega_2 H_2}{H_3} B_2. \tag{3.2.29}$$

At the *next* order the kinematic condition (3.2.8) gives two equations

$$H_1 C_{4,X} = A_{2,T} + \Omega_1 A_2 A_{2,X}, \qquad D_{4,X} = A_{2,T} + \Omega_2 A_2 A_{2,X} - \Omega_2 (A_2 B_2)_X, \tag{3.2.30}$$

the next two equations result from another pair of the kinematic conditions (3.2.9)

$$H_2 E_{4,X} - \Omega_2 H_2 B_{4,X} = -A_{2,T} + B_{2,T} - \Omega_2 A_2 A_{2,X} - \Omega_2 B_2 B_{2,X} + \Omega_2 (A_2 B_2)_X, \tag{3.2.31}$$

$$H_3 F_{4,X} + \Omega_2 H_2 B_{4,X} = -B_{2,T} - \frac{2\Omega_2 H_2}{H_3} B_2 B_{2,X} - \Omega_3 B_2 B_{2,X}. \tag{3.2.32}$$

The pressure condition (3.2.7) at the interface $z = \eta$ gives the following equation

$$\rho_1 \int E_{2,T} d\xi - \frac{1}{2}\rho_1 E_2^2 - \delta\rho_2 g A_2 - \rho_1 \Omega_2 D_4 + \rho_1 \Omega_1 H_1 C_4 = 0\,, \tag{3.2.33}$$

and the pressure condition (3.2.7) at the other interface yields the equation

$$\begin{aligned} &-\rho_1 \int E_{2,T} d\xi + \rho_c \int F_{2,T} d\xi - \frac{1}{2}\rho_c F_2^2 + \frac{1}{2}\rho_1 E_2^2 + \rho_c \Omega_3 H_3 F_4 \\ &-\frac{1}{6}\rho_c \Omega_3 F_{2,XX} H_3^3 + \rho_1 \Omega_2 D_4 + \rho_c \Omega_2 H_2 F_4 - \rho_c g B_4 + \delta\rho_2 g B_2 + \frac{1}{3}\rho_1 \Omega_2 H_2^3 E_{2,XX} \\ &+\delta\rho_3 \Omega_2 H_2 F_2 - \frac{1}{2}\rho_c \Omega_2 H_2 H_3^2 F_{2,XX} + \delta\rho_3 \Omega_3 H_3 F_2 + \rho_1 g B_4 - \delta\rho_3 g B_2 = 0. \end{aligned} \tag{3.2.34}$$

Making allowance for the resonance condition (3.1.34) we can eliminate all unknown functions from the set of equations (3.2.29) – (3.2.34) except A_2 and B_2. Then after rescaling obtain the following set of coupled nonlinear equations

$$\alpha_1 A_t + \sigma B_t + \Delta_1 A_x + \mu_1 A A_x + \nu (AB)_x - \nu B B_x = 0\,, \tag{3.2.35}$$

$$\sigma A_t + \alpha_2 B_t + \Delta_2 B_x + \lambda B_{xxx} + \mu_2 B B_x - \nu (AB)_x + \nu A A_x = 0\,, \tag{3.2.36}$$

where

$$A = \varepsilon^2 A_2 = \eta + O(\varepsilon^4)\,, \qquad B = \varepsilon^2 B_2 = \zeta + O(\varepsilon^4) \tag{3.2.37}$$

are the leading order magnitude of the displacements of the interfaces and

$$\mu_1 = \rho_1 \left(\Omega_1^2 - \Omega_2^2\right), \qquad \nu = \rho_1 \Omega_2^2\,, \tag{3.2.38}$$

$$\mu_2 = \rho_1 \Omega_2^2 - \rho_c \Omega_3^2 - 3\rho_c \Omega_2 \Omega_3 \frac{H_2}{H_3} - 3\rho_c \Omega_2^2 \frac{H_2^2}{H_3^2}\,. \tag{3.2.39}$$

Other coefficients are defined by (3.1.45) – (3.1.47). Hereafter it is also assumed that $\alpha_1\alpha_2 - \sigma^2 \neq 0$, otherwise these equations could not be resolved with respect to temporal derivatives.

The obtained equations (3.2.35), (3.2.36) differ from the previous set of coupled equation (2.2.21), (2.2.22) derived for the model with piecewise constant basic flow. The main difference is again in the form of coupling. The above equations have both nonlinear and linear coupling terms whereas the equations (2.2.21), (2.2.22) have only linear terms. As for the linear analysis of the previous section, equation (3.2.35) also lacks a linear dispersive term. It is now clear that a linear transforming of the variables A, B can introduce dispersive terms in both equations, but this will cause the nonlinear terms to become significantly more complex. Although both equations do not have typical KdV form, their solutions in the form of solitary waves, as it will be shown below, are very similar to KdV solitons in a small amplitude approximation.

3.3. SOME APPROXIMATE SOLUTIONS OF THE COUPLED 1D-EQUATIONS

Here we seek a solution of equations (3.2.35), (3.2.36) in the form of a stationary wave, i.e. that depends on a single variable $\xi = x - Vt$, $V = const$. Such solutions and their derivatives are also assumed to vanish as $\xi \to \infty$. After taking into account the dependence of A, B only on ξ equation (3.2.35) can be immediately integrated and thereby reduced to the following polynomial equation

$$G(A,B) \equiv 2\alpha_1 VA + 2\sigma VB - 2\Delta_1 A - \mu_1 A^2 - 2\nu AB + \nu B^2 = 0 . \tag{3.3.1}$$

Equation (3.2.26) can be readily integrated once. Then the resultant equation obtained is multiplied by B_ξ and, following some standard manipulations involving equation (3.3.1), can be integrated once more to give

$$B_\xi^2 = F(A,B)/\lambda , \tag{3.3.2}$$

where

$$F(A,B) = \sigma VAB + \frac{1}{2}\nu AB^2 + \frac{1}{6}\mu_1 A^3 + \alpha_2 VB^2 - \Delta_2 B^2 - \frac{1}{3}\mu_2 B^3 . \tag{3.3.3}$$

We can now use equation (3.3.1) to represent A as an explicit function of B:

$$A = \frac{V\alpha_1 - B\nu - \Delta_1 - \mathrm{sgn}(V\alpha_1 - \Delta_1)\sqrt{(V\alpha_1 - B\nu - \Delta_1)^2 + B\mu_1(B\nu + 2V\sigma)}}{\mu_1}. \tag{3.3.4}$$

It is important that the expression above is defined such that A vanishes as $B \to 0$. Accordingly this relationship between A and B can be employed for describing coupled solitary waves with decaying tails. After substituting (3.3.4) into the polynomial $F(A,B)$ for A we will have the following equation

$$B_\xi^2 = \widetilde{F}(B), \tag{3.3.5}$$

where $\widetilde{F}(B) \equiv F(A(B), B)/\lambda$. It is not hard to extract the leading term of the right hand side of the above equation as $B \to 0$

$$\widetilde{F}(B) = \frac{(\alpha_1\alpha_2 - \sigma^2)V^2 - (\alpha_1\Delta_2 + \alpha_2\Delta_1)V + \Delta_1\Delta_2}{(V\alpha_1 - \Delta_1)\lambda} B^2 + O(B^3). \tag{3.3.6}$$

Therefore the following condition must be imposed on parameters of the model

$$\frac{(\alpha_1\alpha_2 - \sigma^2)V^2 - (\alpha_1\Delta_2 + \alpha_2\Delta_1)V + \Delta_1\Delta_2}{(V\alpha_1 - \Delta_1)\lambda} \geq 0, \tag{3.3.7}$$

otherwise equation (3.3.5) will be undefined for small B. The fact that the main term in expansion (3.3.6) is proportional to B^2 ensures the presence of exponential tails at a solitary solution of equation (3.3.5). In the case $V = 0.4$, $\sigma = 2$ and all other parameters equal to unity the graph of the polynomial $\widetilde{F}(B)$ is displayed in Figure 6. It is clear that polynomial shape is similar to the cubic that appears in the case of the KdV soliton. So we can now set up the initial value problem as follows

$$B_\xi = -\mathrm{sgn}(\xi)\sqrt{\widetilde{F}(B)}, \qquad B(0) = B_0 \tag{3.3.8}$$

to describe a solitary wave solution. Here B_0 is the root of the polynomial $\widetilde{F}(B)$, which is the closest to the origin. For the set of parameters mentioned above $B_0 \approx 1.18$ and the solution of the problem (3.3.8) is displayed in Figure 7.

It is important to note that in some cases condition (3.3.7) may impose a restriction on the wave amplitude B_0. For instance, when $\alpha_1 < 0$, $\alpha_2 < 0$, $\Delta_1 > 0$, $\Delta_2 < 0$, $\lambda > 0$ and $\alpha_1\alpha_2 - \sigma^2 > 0$ (these may really follow from (3.1.45) – (3.1.47)), the numerator in the left hand side of (3.3.7) has two real roots of different signs:

$$V_0^{(1)} = \frac{-\alpha_1\Delta_2 - \alpha_2\Delta_1 - \sqrt{(\alpha_1\Delta_2 - \alpha_2\Delta_1)^2 + 4\Delta_1\Delta_2\sigma^2}}{2(\sigma^2 - \alpha_1\alpha_2)}, \tag{3.3.9}$$

$$V_0^{(2)} = \frac{-\alpha_1\Delta_2 - \alpha_2\Delta_1 + \sqrt{(\alpha_1\Delta_2 - \alpha_2\Delta_1)^2 + 4\Delta_1\Delta_2\sigma^2}}{2(\sigma^2 - \alpha_1\alpha_2)}, \tag{3.3.10}$$

and for $V > V_p > 0$, where V_p is either $V_0^{(1)}$ or $V_0^{(2)}$, whichever of these two is positive, condition (3.3.7) becomes invalid. Since the amplitude B_0 depends on V this may define an upper limit for the amplitude of a wave traveling with a positive velocity.

The graphs in the Figures 6 and 7 appear similar to those obtained for the KdV equation, which gives a reason to attempt to seek an approximate small-amplitude solution in a form similar to the KdV solitary wave. For this purpose we have to assume that the velocity of such a solitary wave V should depend on its amplitude B_0. For small amplitudes this dependence is assumed to be linear, i.e.

$$V = V_0 + V_1 B_0 + O(B_0) \tag{3.3.11}$$

Amplitudes of both waves A_0 and B_0 are roots of simultaneous equations

$$\begin{cases} F(A_0, B_0) = 0, \\ G(A_0, B_0) = 0. \end{cases} \tag{3.3.12}$$

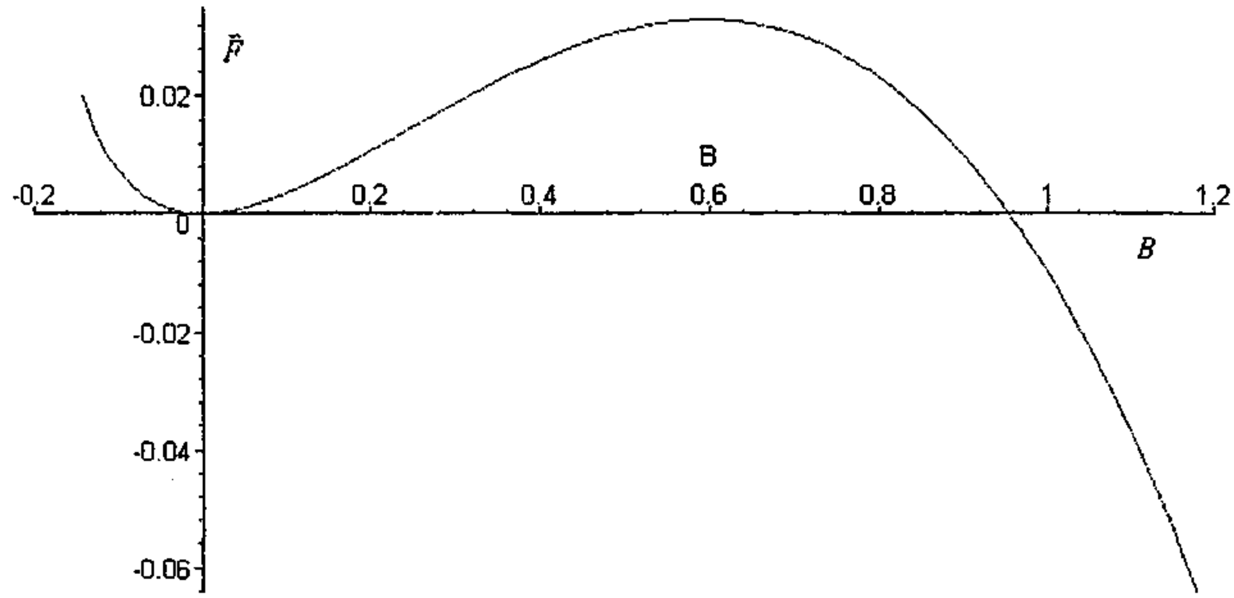


Figure 6. The typical graph of the polynomial $\widetilde{F}(B)$.

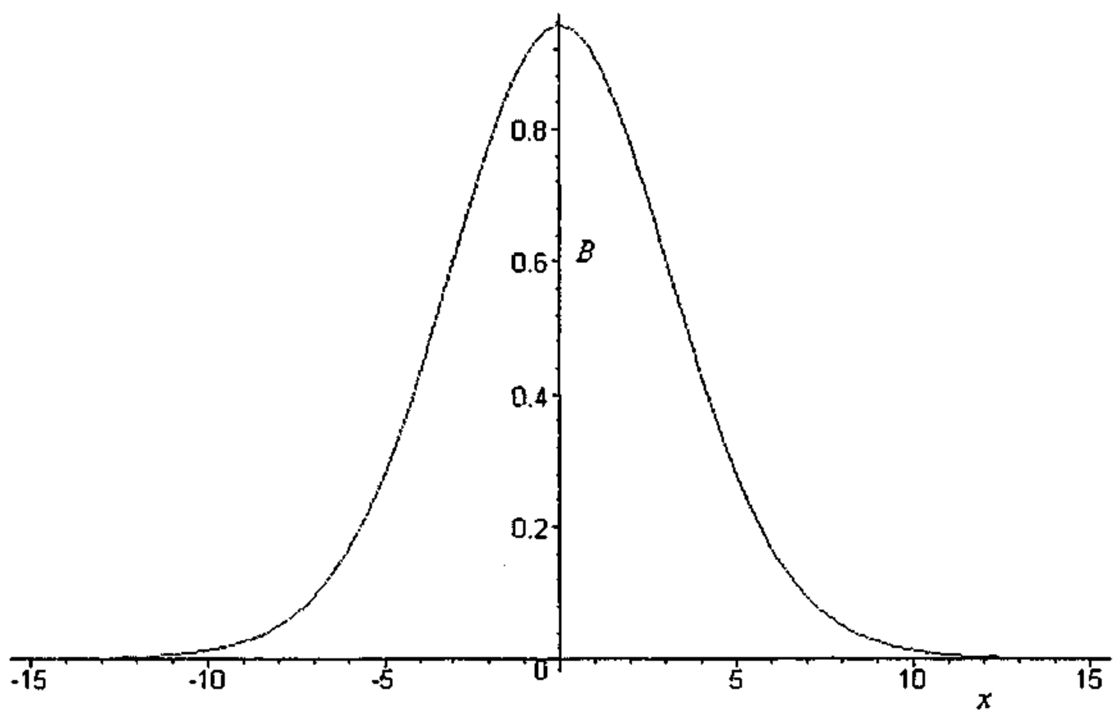


Figure 7. A solution of initial value problem (3.3.8).

The second equation of the set (3.3.12) may be employed to express A_0 in terms of B_0 as

$$A_0 = \frac{V\alpha_1 - B_0\nu - \Delta_1 - \mathrm{sgn}(V_0\alpha_1 - \Delta_1)\sqrt{(V\alpha_1 - B_0\nu - \Delta_1)^2 + B_0\mu_1(B_0\nu + 2V\sigma)}}{\mu_1}. \tag{3.3.13}$$

After substituting (3.3.11) into (3.3.13) and then expanding for small B_0 we have

$$\begin{aligned} A_0 &= \frac{\sigma V_0}{\Delta_1 - \alpha_1 V_0} B_0 \\ &- \frac{2V_0\Delta_1(\nu\sigma + \alpha_1(\nu + \sigma V_1)) - (\nu + 2\sigma V_1)\Delta_1^2 + V_0^2(\sigma^2\mu_1 - 2\nu\sigma\alpha_1 - \nu\alpha_1^2)}{2(\Delta_1 - \alpha_1 V_0)^3} B_0^2 \\ &+ O(B_0^3). \end{aligned} \tag{3.3.14}$$

Putting (3.3.14) into the first equation of the set (3.3.12) gives the following equation

$$\begin{aligned} &\left(\frac{V_0\alpha_2}{2} + \frac{\sigma^2 V_0^2}{2(\Delta_1 - V_0\alpha_1)} - \frac{\Delta_2}{2}\right) B_0^2 \\ &+ \left(\frac{\alpha_2 V_1}{2} + \frac{\nu\sigma V_0}{4(\Delta_1 - \alpha_1 V_0)} + \frac{\sigma^2 V_0 V_1}{2(\Delta_1 - \alpha_1 V_0)} + \frac{\sigma^3 V_0^3 \mu_1}{12(\Delta_1 - \alpha_1 V_0)^3}\right. \\ &- \frac{\sigma V_0\left(2V_0\Delta_1(\nu\sigma + (\nu + \sigma V_1)\alpha_1) - (\nu + 2\sigma V_1)\Delta_1^2 + V_0^2(-2\nu\sigma\alpha_1 - \nu\alpha_1^2 + \sigma^2\mu_1)\right)}{4(\Delta_1 - \alpha_1 V_0)^3} \\ &\left. - \frac{\mu_2}{6}\right) B_0^3 + O(B_0^4) = 0. \end{aligned} \tag{3.3.15}$$

At the leading order we have

$$\frac{V_0\alpha_2}{2}+\frac{\sigma^2 V_0^2}{2(\Delta_1-V_0\alpha_1)}-\frac{\Delta_2}{2}=0, \tag{3.3.16}$$

which determines the principle value of the wave velocity V_0. The above quadratic equation has two values given by (3.3.9) and (3.3.10). These in general correspondent to two different pairs of coupled waves. At the next order we have the equation

$$\begin{aligned}&\frac{\alpha_2 V_1}{2}+\frac{\nu\sigma V_0}{4(\Delta_1-\alpha_1 V_0)}+\frac{\sigma^2 V_0 V_1}{2(\Delta_1-\alpha_1 V_0)}+\frac{\sigma^3 V_0^3\mu_1}{12(\Delta_1-\alpha_1 V_0)^3}\\ &-\frac{\sigma V_0\left(2V_0\Delta_1(\nu\sigma+(\nu+\sigma V_1)\alpha_1)-(\nu+2\sigma V_1)\Delta_1^2+V_0^2\left(-2\nu\sigma\alpha_1-\nu\alpha_1^2+\sigma^2\mu_1\right)\right)}{4(\Delta_1-\alpha_1 V_0)^3} \qquad (3.3.17)\\ &-\frac{\mu_2}{6}=0,\end{aligned}$$

which gives

$$V_1=\frac{R}{Q}, \tag{3.3.18}$$

where

$$\begin{aligned}R=&\Delta_1^3\mu_2-3V_0\Delta_1^2(\nu\sigma+\alpha_1\mu_2)+3V_0^2\Delta_1\left(\nu\sigma^2+2\nu\sigma\alpha_1+\alpha_1^2\mu_2\right)\\ &-V_0^3\left(3\nu\alpha_1\sigma^2+3\nu\alpha_1^2\sigma-\mu_1\sigma^3+\alpha_1^3\mu_2\right),\end{aligned} \tag{3.3.19}$$

$$\begin{aligned}Q=&V_0^3\left(3\alpha_1^2\sigma^2-3\alpha_1^3\alpha_2\right)-3V_0^2\Delta_1\left(3\alpha_1\sigma^2-3\alpha_1^2\alpha_2\right)\\ &+3V_0\Delta_1^2\left(2\sigma^2-3\alpha_1\alpha_2\right)+3\alpha_2\Delta_1^3.\end{aligned} \tag{3.3.20}$$

Thus for given amplitude B_0 of a solitary wave we can estimate its velocity in a form (3.3.11) using (3.3.9) or (3.3.10) along with (3.3.18) – (3.3.20). For example, using the same set of parameter (except the value for V) as used above for obtaining the numerical results displayed in the Figures 6 and 7, the above procedure yields

$V \approx 1.0303$ for $B_0 = 0.01$ and $V \approx 1.2121$ for $B_0 = 0.1$. Reversing this procedure and keeping the value for V as obtained in the last step, with the other parameters still the same we can estimate the roots of the simultaneous equations (3.3.12) to be $B_0 = 0.00995$ and $B_0 = 0.08773$. These are in good agreement with the values 0.01 and 0.1 given initially for the amplitude of the B wave.

As the above example has shown a KdV-like relationship between the amplitude of a wave and its velocity in a form of (3.3.11) has been derived if the wave amplitude is small. We now must show that the shape of a small amplitude solitary wave is KdV-like as well. For this purpose we first need to expand equation (3.3.4) with V given by (3.3.11) provided $|B| < |B_0| << 1$ and then substitute the result obtained for A in the expression for $F(A,B)$. This procedure results in the following approximation of $\widetilde{F}(B)$

$$\widetilde{F}(B) = aB^2(B_0 - B) + O(B_0^4), \tag{3.3.21}$$

where

$$a = \left(\frac{\alpha_2}{\lambda} + \frac{2c_0\sigma^2}{\lambda(\Delta_1 - c_0\alpha_1)} + \frac{\alpha_1\sigma^2c_0^2}{\lambda(\Delta_1 - c_0\alpha_1)^2} \right) c_1 . \tag{3.3.22}$$

Then the solution to equation (3.3.8) with $\widetilde{F}(B)$ in the form (3.3.21) can be expressed as

$$B = B_0 \operatorname{sech}^2\left(\frac{1}{2}\sqrt{aB_0}\, x \right) + O(B_0^3), \tag{3.3.23}$$

which is similar to a KdV soliton in the small amplitude approximation. As it follows from (3.3.23) $|B| \le |B_0|$ then the expression (3.3.23) can be obtained as a solution of (3.3.21) under the following conditions: either $a > 0$ and $0 < B < B_0$ or $a < 0$ and $B_0 < B < 0$. So, the sign of the solitary wave (3.3.23) coincides with a sign of expression (3.3.22). This is consequence of the constraint (3.3.7) with the substitution (3.3.11). Thus, there is a constraint on the parameters involved to have a solitary like solution of equations (3.2.25), (3.2.26), and it may be that a solution of type (3.3.23) does not exist for all set of parameters of the model.

An attempt[1] at seeking an exact solution by substitution of functions

$$A = a\,\mathrm{sech}^2(\alpha\xi), \qquad B = b\,\mathrm{sech}^2(\alpha\xi) \tag{3.3.24}$$

into equations (3.2.35) and (3.2.36) yields the following values for the amplitudes a, b and other parameters of this solution

$$a = \frac{12\alpha^2\beta\lambda}{\mu_2 - 2\nu\beta + \nu\beta^2}, \qquad b = \frac{12\alpha^2\lambda}{\mu_2 - 2\nu\beta + \nu\beta^2}, \tag{3.3.25}$$

$$V = \frac{\beta\Delta_1}{\alpha_1\beta + \sigma}, \qquad \alpha^2 = \frac{\beta\Delta_1}{4\lambda}\frac{\alpha_2 + \sigma\beta}{\alpha_1\beta + \sigma} - \frac{\Delta_2}{4\lambda}, \tag{3.3.26}$$

$$\beta = -\frac{\nu}{\mu_1} \pm \left(\left(\frac{\nu}{\mu_1}\right)^2 + \frac{\nu}{\mu_1}\right)^{1/2}. \tag{3.3.27}$$

The above solution (3.3.24) has a $sech^2$-like form, which is typical for solitons, however equations (3.3.25) – (3.3.27) do not define a family of solutions, as a set of the model parameters gives only two different sets of values for functions (3.3.24). Another feature of this solution is that the amplitudes of the waves (3.3.24) do not depend on the velocity V. Thus the approximate solution (3.3.23) cannot converge to the exact solution (3.3.24) – (3.3.27), and so, belongs to a separate branch.

[1] This solution has been obtained by my supervisor Dr S. Clarke.

3.4. HAMILTONIAN STRUCTURE OF THE COUPLED 1D-EQUATIONS

The symmetry of the nonlinear coefficients in equations (3.2.35), (3.2.36) ensures that this system is Hamiltonian. To reduce these equations to canonical Hamiltonian form let us first rewrite them in vector form as follows

$$\frac{\partial \mathbf{q}}{\partial t} = \mathbf{J}\frac{\partial \mathbf{R}}{\partial x}, \tag{3.4.1}$$

where vectors $\mathbf{q}$ and $\mathbf{R}$ are given by

$$\mathbf{q} = \begin{pmatrix} A \\ B \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} -\Delta_1 A - \frac{1}{2}\mu_1 A^2 - \nu AB + \frac{1}{2}\nu B^2 \\ -\Delta_2 B - \lambda B_{xx} - \frac{1}{2}\mu_2 B^2 - \frac{1}{2}\nu A^2 + \nu AB \end{pmatrix}, \tag{3.4.2}$$

and $\mathbf{J}$ is the following symmetric matrix ($J_{ik} = J_{ki}$, $i, k = 1, 2$)

$$\mathbf{J} = \frac{1}{\alpha_1\alpha_2 - \sigma^2}\begin{bmatrix} \alpha_2 & -\sigma \\ -\sigma & \alpha_1 \end{bmatrix}, \tag{3.4.3}$$

provided $\alpha_1\alpha_2 - \sigma^2 \neq 0$. Let us now prove that the following functional

$$H=\int_{-\infty}^{\infty}\left\{-\frac{1}{2}\Delta_1 A^2-\frac{1}{2}\Delta_2 B^2+\frac{1}{2}\lambda B_x^2-\frac{1}{6}\mu_1 A^3-\frac{1}{6}\mu_2 B^3-\frac{1}{2}\nu A^2 B+\frac{1}{2}\nu AB^2\right\}dx \quad (3.4.4)$$

is the Hamiltonian of the system described by the coupled equations (3.2.35), (3.2.36). First of all the Hamiltonian (3.4.4) is indeed an invariant of the system as

$$\frac{dH}{dt}=\int_{-\infty}^{\infty}\left\{\left[-\Delta_1 A-\frac{1}{2}\mu_1 A^2-\nu AB+\frac{1}{2}\nu B^2\right]A_t\right.$$

$$\left.+\left[-\Delta_2 B-\lambda B_{xx}-\frac{1}{2}\mu_2 B^2-\frac{1}{2}\nu A^2+\nu AB\right]B_t\right\}dx$$

$$=\int_{-\infty}^{\infty}\mathbf{R}\mathbf{q}_t dx=\int_{-\infty}^{\infty}\mathbf{RJR}_x dx=\int_{-\infty}^{\infty}(\mathbf{RJR})_x dx=0 \quad (3.4.5)$$

under the assumption that A, B and all their derivatives vanish as $x\to\pm\infty$, which is assumed throughout this section. The symmetry of the matrix $\mathbf{J}$ has been taken into account as well.

It is also not difficult to show that the variational derivatives (see, for instance *Gelfand and Fomin* 1963, *Swaters* 2000) of the Hamiltonian H with respect to A and B have the following form

$$\frac{\delta H}{\delta A}=-\Delta_1 A-\frac{1}{2}\mu_1 A^2-\nu AB+\frac{1}{2}\nu B^2\equiv R_1. \quad (3.4.6)$$

$$\frac{\delta H}{\delta B}=-\Delta_2 B-\lambda B_{xx}-\frac{1}{2}\mu_2 B^2-\frac{1}{2}\nu A^2+\nu AB\equiv R_2, \quad (3.4.7)$$

or $\dfrac{\delta H}{\delta \mathbf{q}}=\mathbf{R}$, so that the equation (3.4.1) can be rewritten in the canonical for Hamiltonian formulation form

$$\frac{\partial \mathbf{q}}{\partial t}=\tilde{\mathbf{J}}\frac{\delta H}{\delta \mathbf{q}}, \quad (3.4.8)$$

where $\widetilde{\mathbf{J}} = \mathbf{J}\dfrac{\partial}{\partial x}$ is a matrix of differential operators. This operator $\widetilde{\mathbf{J}}$ involved in formulation our equations in the Hamiltonian form (3.4.8) must define the Poisson bracket for arbitrary smooth functionals F and G as follows

$$[F,G] \equiv \int_{-\infty}^{\infty} \frac{\delta F}{\delta \mathbf{q}} \widetilde{\mathbf{J}} \frac{\delta G}{\delta \mathbf{q}} dx \equiv \int_{-\infty}^{\infty} \frac{\delta F}{\delta q_i} J_{ik} \frac{\partial}{\partial x} \frac{\delta G}{\delta q_k} dx. \tag{3.4.9}$$

From hereon repeated subscripts denote summation. To accomplish the proof that the coupled equations (3.2.35), (3.2.36) have Hamiltonian structure we must prove (*Swaters* 2000) that the Poisson bracket (3.4.9) satisfies the following properties: *self-commutation, skew symmetry, distributive* and *associative* properties as well as *Jacobi identity*.

The bracket (3.4.9) meets the *self-commutation* property due to symmetry of the matrix $\mathbf{J}$ as

$$[F,F] = \int_{-\infty}^{\infty} \frac{\delta F}{\delta q_i} J_{ik} \frac{\partial}{\partial x} \frac{\delta F}{\delta q_k} dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left(\frac{1}{2} \frac{\delta F}{\delta q_i} J_{ik} \frac{\delta F}{\delta q_k} \right) dx = 0. \tag{3.4.10}$$

The *skew symmetry* property is also a consequence of the symmetry of the matrix $\mathbf{J}$ as

$$\begin{aligned} [F,G] &= \int_{-\infty}^{\infty} \frac{\delta F}{\delta q_i} J_{ik} \frac{\partial}{\partial x} \frac{\delta G}{\delta q_k} dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left(\frac{\delta F}{\delta q_i} J_{ik} \frac{\delta G}{\delta q_k} \right) dx - \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \frac{\delta F}{\delta q_i} J_{ik} \frac{\delta G}{\delta q_k} dx \\ &= -\int_{-\infty}^{\infty} \frac{\delta G}{\delta q_i} J_{ik} \frac{\partial}{\partial x} \frac{\delta F}{\delta q_k} dx = -[G,F]. \end{aligned} \tag{3.4.11}$$

The *distributive* and *associative* properties result from the linearity of the bracket as

$$[\alpha F + \beta G, Q] = \int_{-\infty}^{\infty} \left(\alpha \frac{\delta F}{\delta \mathbf{q}} + \beta \frac{\delta G}{\delta \mathbf{q}} \right) \mathbf{J} \frac{\partial}{\partial x} \frac{\delta Q}{\delta \mathbf{q}} dx = \alpha [F,Q] + \beta [G,Q], \tag{3.4.12}$$

$$[FG,Q]=\int_{-\infty}^{\infty}\frac{\delta(FG)}{\delta\mathbf{q}}\mathbf{J}\frac{\partial}{\partial x}\frac{\delta Q}{\delta\mathbf{q}}dx$$

$$=\int_{-\infty}^{\infty}\left(F\frac{\delta G}{\delta\mathbf{q}}+\frac{\delta F}{\delta\mathbf{q}}G\right)\mathbf{J}\frac{\partial}{\partial x}\frac{\delta Q}{\delta\mathbf{q}}dx=F[G,Q]+[F,Q]G. \tag{3.4.13}$$

For manipulating the latter bracket we use the fact that F and G, as functionals, are functions of time only and can therefore be taken outside of the spatial integrals.

The last property of the bracket (3.4.9) we must prove is that the *Jacobi identity*

$$[F,[G,Q]]+[G,[Q,F]]+[Q,[F,G]]\equiv 0 \tag{3.4.14}$$

is valid for all allowable functionals F, G and Q. Because of the symmetry of $\mathbf{J}$

$$\delta[G,Q]=\delta\int_{-\infty}^{\infty}\frac{\delta G}{\delta q_i}J_{ik}\frac{\partial}{\partial x}\frac{\delta Q}{\delta q_k}dx$$

$$=\int_{-\infty}^{\infty}\left\{\frac{\delta^2 G}{\delta q_i\delta q_l}\delta q_l J_{ik}\frac{\partial}{\partial x}\frac{\delta Q}{\delta q_k}+\frac{\delta G}{\delta q_i}J_{ik}\frac{\partial}{\partial x}\left(\frac{\delta^2 Q}{\delta q_k\delta q_l}\delta q_l\right)\right\}dx \tag{3.4.15}$$

$$=\int_{-\infty}^{\infty}\left\{\frac{\delta^2 G}{\delta q_i\delta q_l}J_{ik}\frac{\partial}{\partial x}\frac{\delta Q}{\delta q_k}-\frac{\delta^2 Q}{\delta q_i\delta q_l}J_{ik}\frac{\partial}{\partial x}\frac{\delta G}{\delta q_k}\right\}\delta q_l dx$$

the variational derivative of a bracket can be expressed as follows

$$\frac{\delta[G,Q]}{\delta q_l}=\frac{\delta^2 G}{\delta q_i\delta q_l}J_{ik}\frac{\partial}{\partial x}\frac{\delta Q}{\delta q_k}-\frac{\delta^2 Q}{\delta q_i\delta q_l}J_{ik}\frac{\partial}{\partial x}\frac{\delta G}{\delta q_k} \tag{3.4.16}$$

and then a compound bracket is given by

$$[F,[G,Q]] = -\int_{-\infty}^{\infty} \frac{\delta^2 G}{\delta q_i \delta q_l} J_{ik} J_{lm} \frac{\partial}{\partial x}\left(\frac{\delta Q}{\delta q_k}\right) \frac{\partial}{\partial x}\left(\frac{\delta F}{\delta q_m}\right) dx$$

$$+ \int_{-\infty}^{\infty} \frac{\delta^2 Q}{\delta q_i \delta q_l} J_{ik} J_{lm} \frac{\partial}{\partial x}\left(\frac{\delta G}{\delta q_k}\right) \frac{\partial}{\partial x}\left(\frac{\delta F}{\delta q_m}\right) dx. \tag{3.4.17}$$

Similar expressions can be obtained for the other brackets in (3.4.14)

$$[G,[Q,F]] = -\int_{-\infty}^{\infty} \frac{\delta^2 Q}{\delta q_i \delta q_l} J_{ik} J_{lm} \frac{\partial}{\partial x}\left(\frac{\delta G}{\delta q_k}\right) \frac{\partial}{\partial x}\left(\frac{\delta F}{\delta q_m}\right) dx$$

$$+ \int_{-\infty}^{\infty} \frac{\delta^2 F}{\delta q_i \delta q_l} J_{ik} J_{lm} \frac{\partial}{\partial x}\left(\frac{\delta Q}{\delta q_k}\right) \frac{\partial}{\partial x}\left(\frac{\delta G}{\delta q_m}\right) dx, \tag{3.4.18}$$

$$[Q,[F,G]] = -\int_{-\infty}^{\infty} \frac{\delta^2 F}{\delta q_i \delta q_l} J_{ik} J_{lm} \frac{\partial}{\partial x}\left(\frac{\delta Q}{\delta q_k}\right) \frac{\partial}{\partial x}\left(\frac{\delta G}{\delta q_m}\right) dx$$

$$+ \int_{-\infty}^{\infty} \frac{\delta^2 G}{\delta q_i \delta q_l} J_{ik} J_{lm} \frac{\partial}{\partial x}\left(\frac{\delta Q}{\delta q_k}\right) \frac{\partial}{\partial x}\left(\frac{\delta F}{\delta q_m}\right) dx. \tag{3.4.19}$$

Expressions (3.4.18), (3.4.19) have been obtained from (3.4.17) after appropriate swapping of the symbols F, G, Q over each other and renaming some repeated subscripts. Substitution of (3.4.17) – (3.4.19) into the left hand side of (3.4.14) identically yields zero, which proves the *Jacobi identity*.

Thus we have proved the fact that the system described by the coupled equations (3.2.35), (3.2.26) is Hamiltonian and can be formulated in the canonical form (3.4.8) with an operator $\tilde{\mathrm{J}}$ generating the Poisson bracket (3.4.9), which satisfies all five properties required. One of the most important consequences of this fact is an opportunity to construct an infinite set of invariants of the coupled equations, as a Poisson bracket of any two invariants is an invariant too. This would result in the proof of the integrability of the coupled equations (3.2.35), (3.2.36).

Integrating both equations (3.2.35), (3.2.36) over the infinite spatial domain provided $\alpha_1\alpha_2 - \sigma^2 \neq 0$ yields

$$\frac{dC_1}{dt}=0, \qquad \frac{dC_2}{dt}=0, \tag{3.4.20}$$

where

$$C_1=\int_{-\infty}^{\infty} A dx, \qquad C_2=\int_{-\infty}^{\infty} B dx \tag{3.4.21}$$

are two Casimirs as $\frac{\delta C_1}{\delta A}=\frac{\delta C_2}{\delta B}=1$ and $\frac{\delta C_1}{\delta B}=\frac{\delta C_2}{\delta A}=0$, so both C_1 and C_2 result in Poisson bracket (3.4.9) to be zero with any functional. Another invariant

$$P=\int_{-\infty}^{\infty}\left(\frac{1}{2}\alpha_1 A^2+\sigma AB+\frac{1}{2}\alpha_2 B^2\right)dx \tag{3.4.22}$$

can be obtained by integrating the sum of equations (3.2.35) and (3.2.36) multiplied by A and B respectively. The invariant (3.4.22) can be interpreted as a momentum invariant, while the Hamiltonian (3.4.4) can be considered as the energy of the system.

To construct an infinite set of invariant proving integrability of the system we need to have one more invariant of order higher than 3. Unfortunately, we could not find any more invariants of a higher order. Thus the integrability of the coupled equations (3.2.35), (3.2.36) is an open research question.

3.5. DERIVATION OF NONLINEAR COUPLED 2D-EQUATIONS

Here we generalise the three-layered fluid model described above to the case of three-dimensions to obtain a two-dimensional modification of the coupled nonlinear equations (3.2.35), (3.2.36). The geometrical configuration of the model is the same as for two-dimensional case with basic flow and basic density given by (3.1.1) and (2.1.2) respectively. The three-dimensional fluid is no longer irrotational so that a simple scalar invariant such as (2.1.6) or (3.2.7) is not applicable as a boundary condition on an interface. In this case we use the approach of section 3.1 for deriving the linear coupled 1D-equations, according to which the basic configuration is assumed to be fixed and described by (2.1.2) and (3.1.5).

Our starting equations are therefore the Euler equation (3.1.2), the equation of mass conservation (3.1.3) and the incompressibility equation (3.1.4) written for a three-dimensional fluid $\mathbf{V}=\overline{\mathbf{V}}+\widetilde{\mathbf{V}}$, where $\overline{\mathbf{V}}=\left(\overline{U},0,0\right)$ is the undisturbed flow with $\overline{U}$ given by (3.1.5) and $\widetilde{\mathbf{V}}=(u,v,w)$ is a velocity perturbation. Repeating the procedure described in the section 3.1, but now for three-dimensional fluid we can obtain the analogous linear equations (3.1.25) for the amplitudes A and B of the interface disturbances

$$(\eta,\zeta)=(A,B)\exp(ikx+ily-i\omega t)+c.c.,\tag{3.5.1}$$

with

$$D_1 = g(\rho_1 - \rho_2)\left(1 + \frac{l^2}{k^2}\right) + (c - U_1)(\rho_2\Omega_2 - \rho_1\Omega_1)$$
$$- (c - U_1)^2 [\rho_1 m \coth(mH_1) + \rho_2 m \coth(mH_2)], \tag{3.5.2}$$

$$D_2 = g(\rho_2 - \rho_3)\left(1 + \frac{l^2}{k^2}\right) + (c - U_3)(\rho_3\Omega_3 - \rho_2\Omega_2)$$
$$- (c - U_3)^2 [\rho_2 m \coth(mH_2) + \rho_3 m \coth(mH_3)], \tag{3.5.3}$$

$$E = \frac{m\rho_2(c - U_1)(c - U_3)}{\sinh(mH_2)}. \tag{3.5.4}$$

Here $m^2 = k^2 + l^2$. A similar analysis of the relationship between k and l, as has been done for the linear approximation of the previous three-dimensional model (see section 2.1), shows again that the only appropriate assumption is $l = o(k)$. Then in long-wave limit ($k \to 0$) we obtain the same expressions for $D_{1,2}$ and E as (3.1.29) – (3.1.30). As a result the same resonance conditions (3.1.33) – (3.1.36) are obtained for three-dimensional case as well.

It is now clear that to derive nonlinear two-dimensional equations describing resonant interaction of two coupled waves we must introduce the following scaling

$$T = \varepsilon^3 t, \qquad X = \varepsilon x, \qquad Y = \varepsilon^2 y, \tag{3.5.5}$$

where ε is a small parameter, and represents the density configuration in the form (3.2.11) – (3.2.13). Without loss of generality we can use (3.1.5) with $U_1 = 0$. The basic equations (3.1.2) – (3.1.4) can be represented in the form

$$(\rho + \overline{\rho})\left((u + \overline{U})_t + (u + \overline{U})(u + \overline{U})_x + v(u + \overline{U})_y + w(u + \overline{U})_z\right) + p_x = 0, \tag{3.5.6}$$

$$(\rho + \overline{\rho})\left(v_t + (u + \overline{U})v_x + vv_y + wv_z\right) + p_y = 0, \tag{3.5.7}$$

$$(\rho+\overline{\rho})(w_t+(u+\overline{U})w_x+vw_y+ww_z)+g\rho+p_z=0, \tag{3.5.8}$$

$$\rho_t+(u+\overline{U})\rho_x+v\rho_y+w(\overline{\rho}'+\rho_z)=0, \tag{3.5.9}$$

$$(u+\overline{U})_x+v_y+w_z=0. \tag{3.5.10}$$

All functions involved into equations (3.5.6) –(3.5.10) must be expanded in powers of ε as follows

$$u=\varepsilon^2u^{(0)}(T,X,Y,z)+\varepsilon^4u^{(1)}(T,X,Y,z)+O(\varepsilon^6), \tag{3.5.11}$$

$$v=\varepsilon^3v^{(0)}(T,X,Y,z)+\varepsilon^5v^{(1)}(T,X,Y,z)+O(\varepsilon^7), \tag{3.5.12}$$

$$w=\varepsilon^3w^{(0)}(T,X,Y,z)+\varepsilon^5w^{(1)}(T,X,Y,z)+O(\varepsilon^7), \tag{3.5.13}$$

$$\rho=\varepsilon^2\rho^{(0)}(T,X,Y,z)+\varepsilon^4\rho^{(1)}(T,X,Y,z)+O(\varepsilon^6), \tag{3.5.14}$$

$$p=\varepsilon^2p^{(0)}(T,X,Y,z)+\varepsilon^4p^{(1)}(T,X,Y,z)+O(\varepsilon^6), \tag{3.5.15}$$

$$\eta=\varepsilon^2A^{(0)}(T,X,Y)+\varepsilon^4A^{(1)}(T,X,Y)+O(\varepsilon^6), \tag{3.5.16}$$

$$\zeta=\varepsilon^2B^{(0)}(T,X,Y)+\varepsilon^4B^{(1)}(T,X,Y)+O(\varepsilon^6). \tag{3.5.17}$$

The equation of mass conservation (3.5.9) is among the basic equations which determine the variation of density due perturbations of the interfaces. It is also useful to estimate the variation of the basic flow. The basic flow given in the form (3.1.1) can be rewritten in term of the Heaviside step function $\Theta(z)$ (equal 1 for positive z and 0 otherwise) as follows

$$U = \Omega_1 z\Theta(\eta - z)\Theta(z + H_1) + \Omega_2 z\Theta(z - \eta)\Theta(H_2 + \zeta - z)$$
$$+ \{\Omega_2 H_2 + \Omega_3 (z - H_2)\}\Theta(z - H_2 - \zeta)\Theta(H_2 + H_3 - z). \tag{3.5.18}$$

As η and ζ are small with respect to the height of each layer we can formally expand (3.5.18) in powers of the interfaces displacement

$$U = \overline{U} + \frac{1}{2}((\Omega_1 - \Omega_2)\delta(z))\eta^2 + \frac{1}{2}((\Omega_2 - \Omega_3)\delta(z - H_2))\zeta^2 + O\left((\eta^2 + \zeta^2)^{3/2}\right), \tag{3.5.19}$$

where $\overline{U} = \overline{U}(z)$ is a basic flow given by

$$\overline{U} = \begin{cases} \Omega_2 H_2 + \Omega_3 (z - H_2), & H_2 < z < H_2 + H_3, \\ \Omega_2 z, & 0 < z < H_2, \\ \Omega_1 z, & -H_1 < z < 0. \end{cases} \tag{3.5.20}$$

Then substitution of (3.5.16), (3.5.17) into (3.5.19) yields

$$U = U^{(0)} + \varepsilon^4 U^{(1)} + O(\varepsilon^6). \tag{3.5.21}$$

Here

$$U^{(0)} = \overline{U}, \tag{3.5.22}$$

$$U^{(1)} = \overline{U} + \frac{1}{2}((\Omega_1 - \Omega_2)\delta(z))(A^{(0)})^2 + \frac{1}{2}((\Omega_2 - \Omega_3)\delta(z - H_2))(B^{(0)})^2. \tag{3.5.23}$$

It follows from the expansion (3.5.21) the perturbation of U, caused by interface displacements, is $O(\varepsilon^4)$. Consequently this perturbation does not contribute at the leading two orders of the expansions of the basic equations (3.5.6) – (3.5.10) used for the derivation procedure below. Hence we can place $\overline{U}$ in each equation of the set

(3.5.6) – (3.5.10) instead of U. Of course, the perturbation of the basic flow must be taken into account for higher orders.

After substitution of the expansions (3.5.11) – (3.5.17) into equations (3.5.6) – (3.5.10) they become at the leading order

$$\overline{\rho}^{(0)}\left(w^{(0)}\overline{U}'+u_X^{(0)}\overline{U}\right)+p_X^{(0)}=0, \tag{3.5.24}$$

$$g\rho^{(0)}+p_z^{(0)}=0, \tag{3.5.25}$$

$$\overline{U}\rho_X^{(0)}+w^{(0)}\overline{\rho}_z^{(0)}=0, \tag{3.5.26}$$

$$u_X^{(0)}+w_z^{(0)}=0. \tag{3.5.27}$$

The y-component of the Euler equation (3.5.7) does not contribute at this order. Eliminating all variables except $w^{(0)}$ this set can be reduced to a single equation

$$\left\{\overline{\rho}^{(0)}\left(w^{(0)}\overline{U}'-w_z^{(0)}\overline{U}\right)\right\}_z+\frac{g\overline{\rho}_z^{(0)}w^{(0)}}{\overline{U}}=0. \tag{3.5.28}$$

Within each layer equation (3.5.28) becomes

$$w_{zz}^{(0)}=0, \tag{3.5.29}$$

as $\rho^{(0)},\overline{U}'=const$ there. Thus the vertical velocity in each layer is just a linear function with respect to z.

The vertical velocity of the interface $z=\eta(t,x,y)$ is given by

$$w(\ldots,z=\eta\pm 0)=\eta_t+\left(U(\eta\pm 0)+u(\ldots,z=\eta\pm 0)\right)\eta_x+v(\ldots,z=\eta\pm 0)\eta_y, \tag{3.5.30}$$

where ellipsis denotes other independent variables. To leading order the right hand side is $O(\varepsilon^3)$, then it follows that

$$w^{(0)}(\dots, z=+0)=w^{(0)}(\dots, z=-0)=0 \tag{3.5.31}$$

since $\overline{U}(0)=0$. The same algebra carried out for the other interface $z=H_2+\zeta(t,x,y)$ yields a nonzero value of the vertical velocity on the interface

$$w^{(0)}(\dots, z=H_2+0)=w^{(0)}(\dots, z=H_2-0)=\Omega_2 H_2 B_X^{(0)}, \tag{3.5.32}$$

as $\overline{U}(H_2)=\Omega_2 H_2$. Thus $w^{(0)}$ is continuous across each interfaces

$$\left[w^{(0)}\right]_{z=0}=\left[w^{(0)}\right]_{z=H_2}=0. \tag{3.5.33}$$

As then follows from (3.5.28) there is another condition of continuity

$$\begin{aligned} & w^{(0)}(\dots,0)\left[\overline{\rho}^{(0)}\overline{U}'\right]_{z=0}-\left[w_z^{(0)}\overline{\rho}^{(0)}\right]_{z=0}\overline{U}(0)+\frac{gw^{(0)}(\dots,0)}{\overline{U}(0)}\left[\overline{\rho}^{(0)}\right]_{z=0} \\ & =w^{(0)}(\dots,H_2)\left[\overline{\rho}^{(0)}\overline{U}'\right]_{z=H_2}-\left[w_z^{(0)}\overline{\rho}^{(0)}\right]_{z=H_2}\overline{U}(H_2)+\frac{gw^{(0)}(\dots,H_2)}{\overline{U}(H_2)}\left[\overline{\rho}^{(0)}\right]_{z=H_2}=0. \end{aligned} \tag{3.5.34}$$

The solution of equation (3.5.29) satisfying all conditions of continuity (3.5.33) and (3.5.34) is given by

$$w^{(0)}=\begin{cases} \dfrac{(H_2+H_3-z)\Omega_2 H_2}{H_3}B_X^{(0)}, & H_2<z<H_2+H_3, \\ \Omega_2 z B_X^{(0)}, & 0<z<H_2, \\ 0, & -H_1<z<0. \end{cases} \tag{3.5.35}$$

Employing equations (3.5.24) - (3.5.27) we can find expressions for the other variables involved, namely

$$u^{(0)} = \begin{cases} \dfrac{\Omega_2 H_2 B_X^{(0)}}{H_3}, & H_2 < z < H_2 + H_3, \\ -\Omega_2 B^{(0)}, & 0 < z < H_2, \\ 0, & -H_1 < z < 0, \end{cases} \tag{3.5.36}$$

$$\rho^{(0)} = (\rho_1 - \rho_c) B^{(0)} \delta(z - H_2), \tag{3.5.37}$$

$$p^{(0)} = -g(\rho_1 - \rho_c) B^{(0)} \Theta(z - H_2) \Theta(H_2 + H_3 - z). \tag{3.5.38}$$

Before considering the next order basic equations let us determine if the vertical velocity is continuous across an interface at higher order or not. The exact equation for the vertical velocity value on both sides of the interface $z = \eta(t, x, y)$ is given by (3.5.30). After expansion of all components of velocity in a small vicinity of the undisturbed interface $z = 0$

$$u(\ldots, z = \eta \pm 0) = u(\ldots, z = \pm 0) + u_z(\ldots, z = \pm 0)\eta + O(\eta^2), \tag{3.5.39}$$

$$v(\ldots, z = \eta \pm 0) = v(\ldots, z = \pm 0) + v_z(\ldots, z = \pm 0)\eta + O(\eta^2), \tag{3.5.40}$$

$$w(\ldots, z = \eta \pm 0) = w(\ldots, z = \pm 0) + w_z(\ldots, z = \pm 0)\eta + O(\eta^2), \tag{3.5.41}$$

we can obtain the values of the vertical velocity on both sides of this interface as follows

$$\begin{aligned} w(\ldots, z = \pm 0) = {} & \eta_t + U(\eta \pm 0)\eta_x \\ & + \{u(\ldots, z = \pm 0) + u_z(\ldots, z = \pm 0)\eta + \ldots\}\eta_x \\ & + \{v(\ldots, z = \pm 0) + v_z(\ldots, z = \pm 0)\eta + \ldots\}\eta_x \\ & - \{w_z(\ldots, z = \pm 0)\eta + \ldots\}. \end{aligned} \tag{3.5.42}$$

Unfolding this equation after substitution of (3.5.11) – (3.5.17) gives us (3.5.31) at the leading order $O(\varepsilon^3)$ and the following equations

$$w^{(1)}(\ldots, z=+0) = A_T^{(0)} + \Omega_2 A^{(0)} A_X^{(0)} + \left(u^{(0)}(\ldots, z=+0) A^{(0)}\right)_X , \tag{3.5.43}$$

$$w^{(1)}(\ldots, z=-0) = A_T^{(0)} + \Omega_1 A^{(0)} A_X^{(0)} + \left(u^{(0)}(\ldots, z=-0) A^{(0)}\right)_X \tag{3.5.44}$$

at the next order $O(\varepsilon^5)$. The same procedure employed at the other interface $z = H_2$ yields (3.5.32) and

$$\begin{aligned} w^{(1)}(\ldots, z=H_2+0) = {} & B_T^{(0)} + \Omega_2 H_2 B_X^{(1)} + \Omega_2 B^{(0)} B_X^{(0)} \\ & + \left(u^{(0)}(\ldots, z=H_2+0) B^{(0)}\right)_X , \end{aligned} \tag{3.5.45}$$

$$\begin{aligned} w^{(1)}(\ldots, z=H_2-0) = {} & B_T^{(0)} + \Omega_2 H_2 B_X^{(1)} + \Omega_3 B^{(0)} B_X^{(0)} \\ & + \left(u^{(0)}(\ldots, z=H_2-0) B^{(0)}\right)_X \end{aligned} \tag{3.5.46}$$

at the leading and the higher order respectively. It follows from (3.5.43) – (3.5.46) the vertical velocity may not be continuous across both interfaces at higher orders. Therefore we must take into account the jump of $w^{(1)}$ across both interfaces at the next stage of the perturbation procedure.

At the next order of asymptotic expansion the basic equations (3.5.6) – (3.5.10) are given by

$$\begin{aligned} & \overline{\rho}^{(0)}\left\{w^{(0)} u_z^{(0)} + w^{(1)}\overline{U}' + \overline{U}\left(u_X^{(1)} + U_X^{(1)}\right) + u^{(0)} u_X^{(0)} + u_T^{(0)}\right\} \\ & + \left(\rho^{(0)} + \overline{\rho}^{(1)}\right)\left\{w^{(0)}\overline{U}' + u_X^{(0)}\overline{U}\right\} + p_X^{(1)} = 0 , \end{aligned} \tag{3.5.47}$$

$$\overline{\rho}^{(0)}\overline{U} v_X^{(0)} + p_Y^{(0)} = 0 , \tag{3.5.48}$$

$$\bar{\rho}^{(0)}\bar{U}w_X^{(0)}+g\rho^{(1)}+p_z^{(1)}=0\,, \tag{3.5.49}$$

$$\rho_T^{(0)}+\bar{U}\rho_X^{(1)}+u^{(0)}\rho_X^{(0)}+w^{(0)}\left(\rho_z^{(0)}+\bar{\rho}_z^{(0)}\right)+w^{(1)}\bar{\rho}_z^{(0)}=0\,, \tag{3.5.50}$$

$$u_X^{(1)}+U_X^{(1)}+v_Y^{(0)}+w_z^{(1)}=0\,. \tag{3.5.51}$$

As at the previous order we can derive the following single equation governing $w^{(1)}$ after eliminating all other variables of second order

$$\begin{aligned}
&\bar{U}\left\{\bar{\rho}^{(0)}\left(\bar{U}'\,w_X^{(1)}-\bar{U}w_{Xz}^{(1)}\right)\right\}_z+g\bar{\rho}_z^{(0)}w_X^{(1)}\\
&+\bar{U}\left\{\bar{\rho}^{(0)}\left(w^{(0)}u_z^{(0)}+u^{(0)}u_X^{(0)}+u_T^{(0)}\right)+\left(\rho^{(1)}-\frac{p_z^{(0)}}{g}\right)\left(\bar{U}'\,w^{(0)}-\bar{U}w_z^{(0)}\right)\right\}_{Xz}+\bar{U}p_{YYz}^{(0)}\\
&-\left\{p_{Tz}^{(0)}+u^{(0)}p_{Xz}^{(0)}+w^{(0)}\left(p_{zz}^{(0)}+g\bar{\rho}_z^{(1)}\right)\right\}_X-\bar{\rho}^{(0)}\bar{U}^2w_{XXY}^{(0)}=0\,.
\end{aligned} \tag{3.5.52}$$

We can integrate the left hand side of equation (3.5.52) with respect to z over the segment $(H_2-\tau, H_2+\tau)$ and after integrating take the limit as $\tau\to 0$. As a result the following condition of continuity across the interface $z=H_2$ applies

$$\left[\xi_{H_2}\right]_{z=H_2}=0\,, \tag{3.5.53}$$

where

$$\begin{aligned}
\xi_{H_2}=&\,\bar{U}\left\{\bar{\rho}^{(0)}\left(\bar{U}'\,w^{(1)}-\bar{U}w_z^{(1)}+w^{(0)}u_z^{(0)}+u^{(0)}u_X^{(0)}+u_T^{(0)}\right)_X+p_{YY}^{(0)}\right\}\\
&+\bar{U}\left(\left(\rho^{(1)}-\frac{p_z^{(0)}}{g}\right)\left(\bar{U}'\,w^{(0)}-\bar{U}w_z^{(0)}\right)\right)_X\\
&-\left\{p_T^{(0)}+u^{(0)}p_X^{(0)}+w^{(0)}\left(p_z^{(0)}+g\bar{\rho}^{(1)}\right)\right\}_X+g\bar{\rho}^{(0)}w_X^{(1)}.
\end{aligned} \tag{3.5.54}$$

Since $\overline{U}(0)=w^{(0)}(0)=0$ and $\overline{\rho}^{(0)}=const$, $p^{(0)}=\rho^{(0)}=0$ throughout the two lower layers including at the interface $z=0$. Employing the condition of continuity (3.5.53), (3.5.54) at the interface $z=0$ yields $\left[w_X^{(1)}\right]_{z=0}=0$, which contradicts (3.5.43), (3.5.44). This discrepancy is caused by degeneration of equation (3.5.52) at the lower interface. To circumvent such a difficulty one should set $\overline{\rho}^{(0)}=\rho_1=const$, $p^{(0)}=\rho^{(0)}=0$ in equations (3.5.47) – (3.5.51) before elimination procedure. Then the equation for $w^{(1)}$ within the two lower layers and the condition of continuity through the plane $z=0$ are respectively

$$\rho_1\left\{\overline{U}'w^{(1)}-\overline{U}w_z^{(1)}+u_T^{(0)}+u^{(0)}u_X^{(0)}+w^{(0)}u_z^{(0)}\right\}_z$$
$$+\left\{\overline{\rho}^{(1)}\left(\overline{U}'w^{(0)}-\overline{U}w_z^{(0)}\right)\right\}_z-\rho_1\overline{U}w_{XX}^{(0)}+\frac{g\overline{\rho}_z^{(1)}}{\overline{U}}w^{(0)}=0, \tag{3.5.55}$$

$$\left[\xi_0\right]_{z=0}=0, \tag{3.5.56}$$

where

$$\xi_0=\rho_1\left\{\overline{U}'w^{(1)}-\overline{U}w_z^{(1)}+u_T^{(0)}+u^{(0)}u_X^{(0)}+w^{(0)}u_z^{(0)}\right\}$$
$$+\left\{\overline{\rho}^{(1)}\left(\overline{U}'w^{(0)}-\overline{U}w_z^{(0)}\right)\right\}+\frac{g\overline{\rho}^{(1)}}{\overline{U}}w^{(0)}. \tag{3.5.57}$$

Within each layer both (3.5.52) and (3.5.55) can be reduced to the simple equation

$$w_{zz}^{(1)}=-w_{XX}^{(0)}, \tag{3.5.58}$$

and the function $w^{(1)}$ can be easily found in each layer, as follows:

i.If $-H_1<z<0$ then $w^{(0)}=0$ and

$$w^{(1)}=\frac{z+H_1}{H_1}a_X^{-(1)}. \tag{3.5.59}$$

ii. If $0 < z < H_2$ then $w^{(0)} = z\Omega_2 B_X^{(0)}$ and

$$w^{(1)} = -\frac{z^3}{6}\Omega_2 B_{XXX}^{(0)} + \frac{H_2 - z}{H_2} a_X^{+(1)} + \frac{z}{H_2}\left(b_X^{-(1)} + \frac{\Omega_2 H_2^3}{6} B_{XXX}^{(0)}\right). \qquad (3.5.60)$$

iii. If $H_2 < z < H_2 + H_3$ then $w^{(0)} = (H_2 + H_2 - z)\Omega_2 H_2 B_X^{(0)}/H_3$ and

$$w^{(1)} = \frac{(z - H_2 - H_3)^3 \Omega_2 H_2}{6H_3} B_{XXX}^{(0)} + \frac{H_2 + H_3 - z}{H_3}\left(b_X^{+(1)} + \frac{1}{6}\Omega_2 H_2 H_3^2 B_{XXX}^{(0)}\right), \qquad (3.5.61)$$

where

$$a_X^{\pm(1)} = w^{(1)}(\ldots, z = \pm 0), \qquad (3.5.62)$$

$$b_X^{\pm(1)} = w^{(1)}(\ldots, z = H_2 \pm 0) \qquad (3.5.63)$$

are defined by (3.5.43), (3.5.46).

Now after substitution of (3.5.59) – (3.5.61) into the conditions of continuity (3.5.53), (3.5.56) followed by appropriate manipulation including rescaling, we obtain the following coupled equations

$$\alpha_1 A_t + \sigma B_t + \Delta_1 A_x + \mu_1 A A_x + \nu (AB)_x - \nu B B_x = 0, \qquad (3.5.64)$$

$$(\sigma A_t + \alpha_2 B_t + \Delta_2 B_x + \lambda B_{xxx} + \mu_2 B B_x - \nu (AB)_x + \nu A A_x)_x + \gamma B_{yy} = 0, \qquad (3.5.65)$$

where

$$\gamma = g(\rho_1 - \rho_c), \qquad (3.5.66)$$

and the other coefficients are given by (3.1.45) – (3.1.47), (3.2.38) and (3.2.39).

As with the one-dimensional coupled equations (3.2.35), (3.2.36), the two-dimensional equations obtained here result in two equations of different structure. The first equation (3.5.64) remains the same as in the one-dimensional case. It does not contain either dispersive or any transverse term (A_{yy} or B_{yy}). The second equation (3.5.65) has a KP-like form, similar to those obtained for the three-dimensional model of Chapter 2 with piecewise constant basic flow, however with different coupling terms. It appear that the resonance conditions (3.1.33), (3.1.34) suppress the dispersive and transverse effects for the long-wave perturbation of the lower interface, and these effects can be only transferred from the higher interface perturbation due to coupling terms.

We have numerically[1] simulated the evolution of a system of two solitary waves of the form (3.3.24) with the following set of parameters: $\alpha_1 = \alpha_2 = 1$, $\mu_1 = \mu_2 = 1$, $\Delta_1 = 0.5$, $\Delta_2 = 1$, $\sigma = 2$, $\nu = 1$, $\lambda = -1$. Since any set of parameters gives two different values for β, depending on a sign which the square root in (3.3.27) is taken with, we have two pairs of waves to consider:

$$A = 7.5533\,\mathrm{sech}^2(1.7433x), \qquad B = -3.1287\,\mathrm{sech}^2(1.7433x) \tag{3.5.67}$$

when the negative sign was taken, and

$$A = -3.0533\,\mathrm{sech}^2(0.4591x), \qquad B = -7.3713\,\mathrm{sech}^2(0.4591x) \tag{3.5.68}$$

otherwise.

Functions (3.5.67), (3.5.68) define the form of solitary waves which are exact solutions to one-dimensional equations (3.2.35), (3.2.36). First, the stability of both solutions has been checked by numerical integration of these equations with initial value taken in the form of (3.5.67) and (3.5.68) perturbed by white noise. Evolution of the waves (3.5.67) and (3.5.68) in one dimension is displayed in Figures 8 and 9 respectively. As seen from this figures, the wave pair (3.5.67) is stable, while the waves (3.5.68) are not in this one-dimensional case. The stable modes (3.5.67) were chosen for further examination.

[1] The code written and maintained by my supervisor Dr. S. Clarke has again been used.

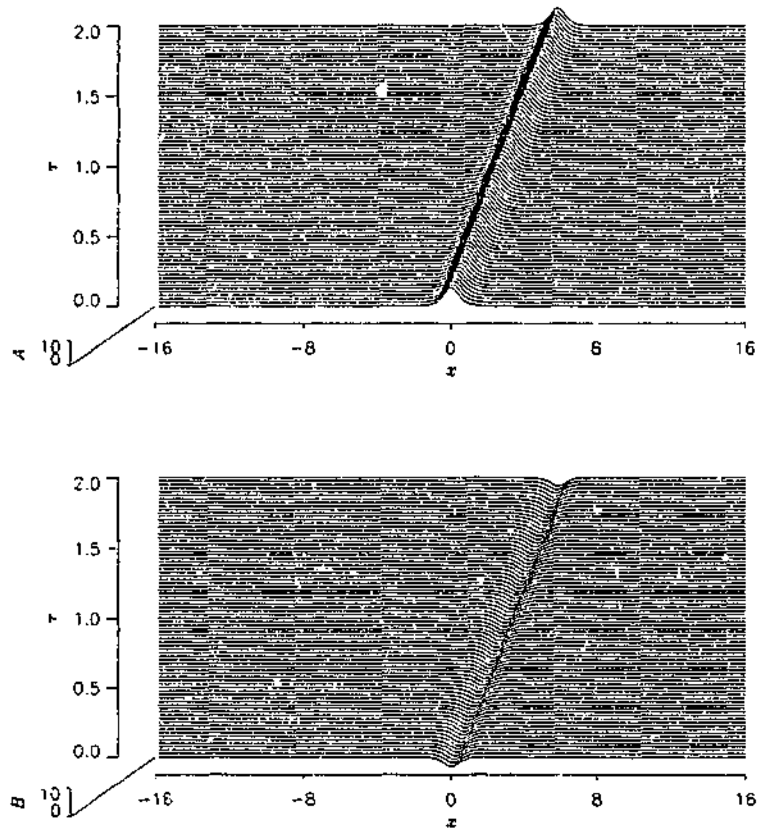
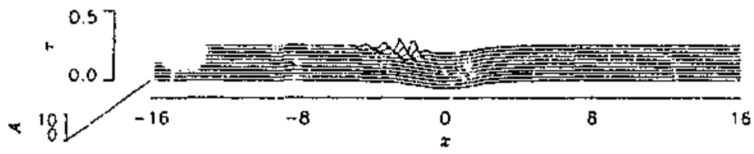


Figure 8. Evolution of A and B modes of wave (3.5.67) (the upper and the lower graphs respectively) perturbed by one-dimensional white noise.



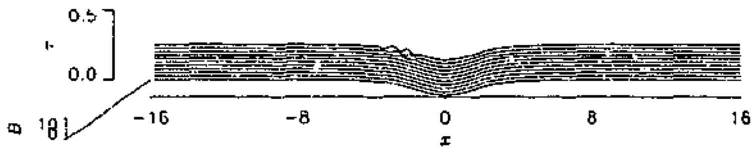


Figure 9. Evolution of A and B modes of wave (3.5.68) (the upper and the lower graphs respectively) perturbed by one-dimensional white noise.

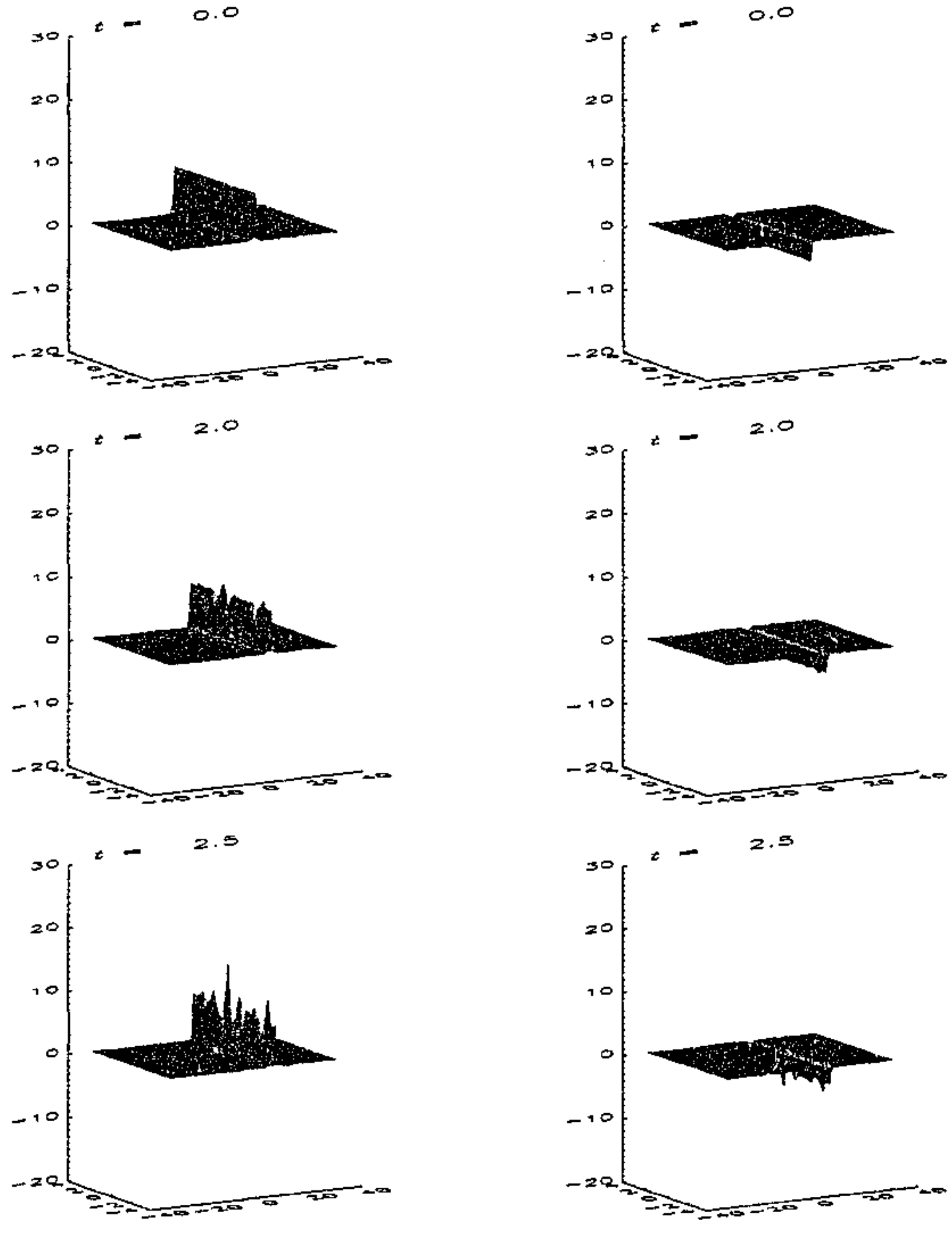


Figure 10. Evolution of A and B modes of wave (3.5.67) (the left and the right column of graphs respectively) perturbed by two-dimensional white noise; $\gamma = 0.5$

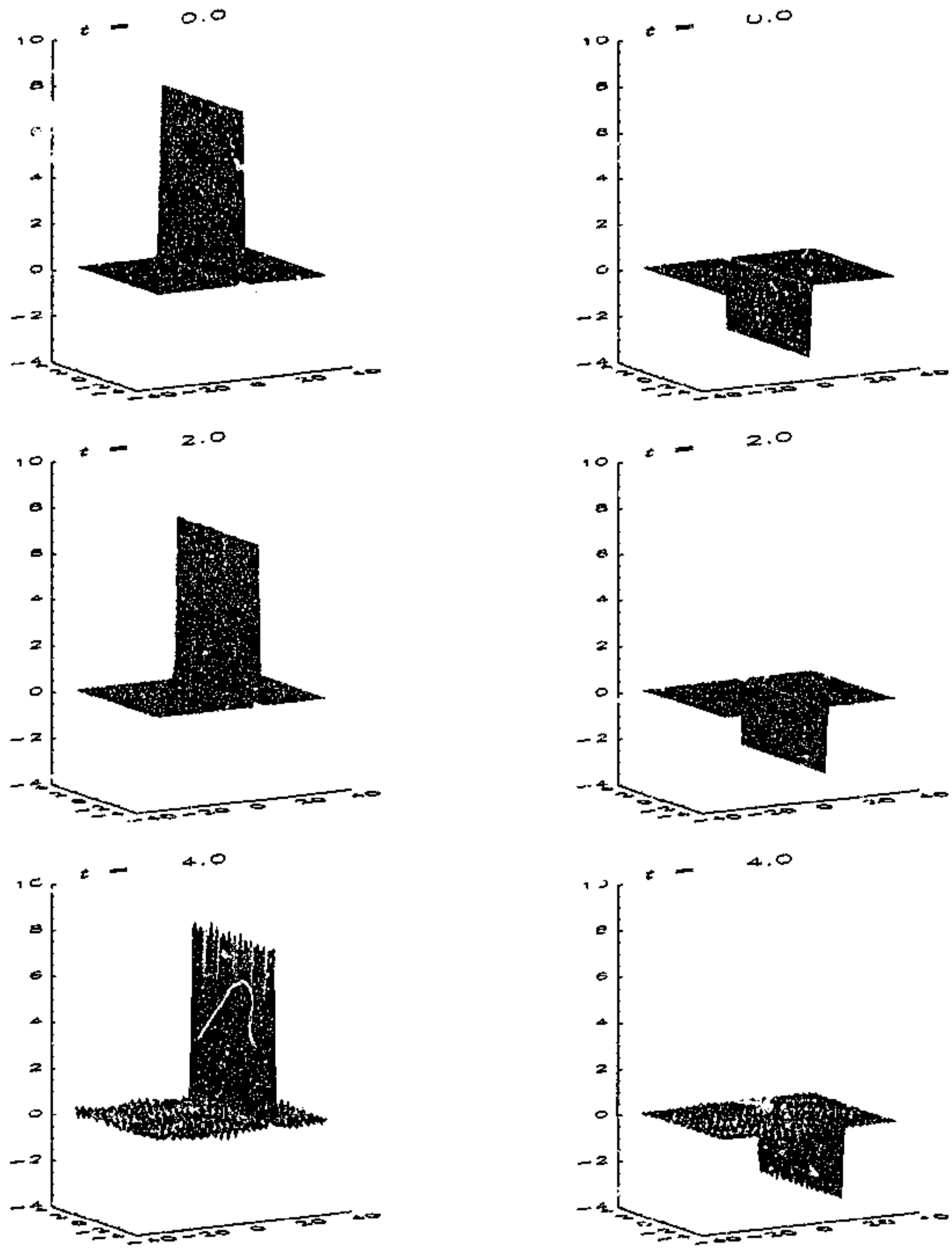


Figure 11. Evolution of A and B modes of wave (3.5.67) (the left and the right column of graphs respectively) perturbed by two-dimensional white noise; $\gamma = -0.5$

Equations (3.5.64), (3.5.65) have been then integrated with initial values of the form (3.5.67) perturbed by two-dimensional white noise. The evolution of waves (3.5.67) is displayed in Figures 10 and 11 for the two different values of γ (0.5 and –0.5). In the case of positive γ the instability causes the wave front to breakdown into separate coherent structures (see Figure 10) in a similar manner to waves governed by equations (2.2.21), (2.2.22). However, if γ is negative (see Figure 11) the waves appear to be only weakly unstable, with the instability only causing the white noise to be amplified. Such behaviour of the solitons may be explained by the fact that the KP equation defines unstable evolution of the KdV soliton when $\gamma\lambda < 0$, and stable evolution otherwise.

CHAPTER 4

CONCLUSION

In the Chapters 2,3 various problems describing coupled nonlinear interfacial waves in three-layer inviscid fluid flows have been considered. We have used the following two simple models of such flows. In the first model (model I) both the basic flow (2.1.1) and density (2.1.2) were assumed to be constant inside each layer. In model II the basic flow was taken to be of the form of a piecewise linear function (3.1.1) that ensures continuity of the basic flow in the case of unperturbed interfaces, while the density is piecewise constant as in model I (2.1.2). Chapter 2 was devoted to the investigation of coupled waves in model I, while the investigation of model II was carried out in Chapter 3.

Throughout this text all the evolution equations governing coupled waves have been derived in a small vicinity of the resonance found by means of the general procedure proposed by *Grimshaw* (2000) and described in Chapter 1. Here the properties of some exact solutions of the obtained two-dimensional evolution equation have been explored analytically and numerically.

In conclusion the new results, which have been obtained in the framework of the present research project, can be summarised as follows

I. The following results relate to investigation of the model I

- There has been derived a set of two coupled nonlinear Kadomtsev-Petviashvili (KP) equations (2.2.21), (2.2.22) for three-dimensional interfacial perturbations;

- A set of equations (2.3.20), (2.3.21), (2.3.36) and (2.3.37) have been derived by means of multiscale perturbation expansion to describe influence of both coupling and transverse effects on the amplitude and phase evolution for a system of two KdV solitons;

- The obtained equations (2.3.20), (2.3.21), (2.3.36) and (2.3.37) have been used to investigate the linear stability of the coupled KdV solitons (2.2.24) – (2.2.26) with respect to transverse perturbations;

- It has been shown that, unlike the results of the one-dimensional stability analysis of the same system carried out by *Gottwald and Grimshaw* (1998), there does not exist any conditions that can be imposed on the parameters of the model to ensure its stability with respect to transverse perturbations of any spatial spectrum. This means that this system is always unstable with respect to any transverse perturbation of sufficient wide spectrum (for example, white noise);

- The results obtained from the stability analysis of the system of two coupled KdV solitons have been confirmed by numerical simulation. In some cases the system evolved into a system consisting of two or three two-dimensional solitary waves. Some examples turned out to be stable as the linear instability was suppressed at a nonlinear stage of their evolution. The important role of coupling in the generation of two-dimensional solitary-like patterns has been demonstrated, as the coupled KP equations can describe the formation of two-dimensional solitary-like waves, while separate KP equations cannot admit two-dimensional solitary-like solutions under the same conditions (i.e. when both dispersion coefficients are positive).

II. The following results relate to investigation of the model II

- The resonance conditions (3.1.33) – (3.1.34) which ensure the existence of coupled interfacial waves have been derived;

- A set of two coupled KdV-like equations (3.2.35), (3.2.36) have been obtained;

- An approximate analytical solitary-like solution of equations (3.2.35), (3.2.36) has been constructed;

- The equations (3.2.35), (3.2.36) have been rewritten in canonical Hamiltonian form and it has been proven that it is a Hamiltonian system. Four invariants of the system have been found;

- A set of two nonlinear coupled two-dimensional equations (3.5.64), (3.5.65) have been derived to describe the influence of the transverse effects on the evolution of the interfacial perturbations;

- The evolution of the exact solitary-like solution (3.3.24) – (3.3.27) of the coupled one-dimensional equations (3.2.35), (3.2.36), perturbed with two-dimensional white noise, has been carried out numerically.

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