MONASH UNIVERSITY
THESIS ACCEPTED IN SATISFACTION OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

ON.....[redacted] 24 June 2003 ....

...........[redacted]....................

Sec. Research Graduate School Committee

ERRATA AND ADDENDUM

p. 24 The following paragraph should be inserted immediately following the statement of the Clifford-McLean theorem for bands (Theorem 1.3.14):
For any band, the relations $\mathcal{L}$ and $\mathcal{R}$ commute [111, Proposition 2.1.3]. From [111, Corollary 1.5.12] it follows that $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \vee \mathcal{R}$. We remark that these equalities are fundamental when manipulating $\mathcal{D}$ on any band.

p. 28 Paragraph 1.4.3, Line 7. The sentence:
Leech has shown that the class of quasilattices is a variety [145, Section 1], ...
should read:
Laslo and Leech have shown that the class of quasilattices is a variety [145, Section 1], ...

p. 28 Paragraph 1.4.3, Line 10. The sentence:
Leech has also shown that quasilattices satisfy a modified form of the Clifford-McLean theorem: ...
should read:
Laslo and Leech have also shown that quasilattices satisfy a modified form of the Clifford-McLean theorem: ...

p. 28 Paragraph 1.4.3, Line 12. The sentence:
Further information about quasilattices may be found in Leech [145, Sections 1, 2, 3, 4 and 6].
should read:
Further information about quasilattices may be found in Laslo and Leech [145, Sections 1, 2, 3, 4 and 6].

p. 28 Paragraph 1.4.4, Line 6. The sentence:
Leech has observed that the class of paralattices is a variety [145, Section 1], ...
should read:
Laslo and Leech have observed that the class of paralattices is a variety [145, Section 1], ...

p. 31 The following paragraph should be inserted immediately following the statement of the Clifford-McLean theorem for skew lattices (Theorem 1.4.10):
The Clifford-McLean theorem for skew lattices is also known as the *first decomposition theorem for skew lattices* in the literature.

p. 60 Identity (1.34) should read:
$$(x \dot{-} y) \dot{-} \big((x \dot{-} (y \dot{-} z)\big) \approx 0 \tag{1.34}$$

p. 105 Line 4. The sentence:
This is shown in the following result, which may be understood as a kind of a 'Clifford-McLean theorem' for pre-BCK-algebras.
should read:
This is shown by the following 'maximal image' result, which may be understood as a pre-cursor to a Clifford-McLean theorem for pre-BCK-algebras. In the sequel we shall in fact see that, under appropriate conditions, Theorem 2.1.14 *merges* with the usual assertion of the Clifford-McLean theorem for bands (Theorem 1.3.14).

p. 135 The first two sentences of Paragraph 2.2.16 should read:
2.2.16. Positive Implicative Pre-BCK-Algebras. A *positive implicative pre-BCK-algebra* is a pre-BCK-algebra $\mathbf{A}$ such that $\mathbf{A}/\Xi \cong B$ for some positive implicative BCK-algebra $\mathbf{B}$.

p. 299 The first sentence of Remark 3.3.9 should read:
One-sided non-commutative lattices were introduced by Laslo and Leech in [145, Section 4] under the name *flat* non-commutative lattices, in conformance with standard non-commutative lattice theory terminology.

p. 305 The first sentence of the paragraph immediately following Theorem 3.3.15 should read:
An *upper implicative BCS band* is an algebra $\langle A;\ \vee, \backslash, 0\rangle$ of type $\langle 2,2,0\rangle$ such that: (i) the reduct $\langle A;\ \vee, 0\rangle$ is an upper band with zero; ...

p. 306 Identity (3.72) should read:

$$x\backslash(x\backslash(x \vee y \vee x)) \approx x \tag{3.72}$$

p. 307 Top line. The statement:

- $\mathsf{PQ}_{C'}$ denote the variety of upper pre-BCK-bands when $C' = \{\wedge, \backslash, 0\}$;

should read:

- $\mathsf{PQ}_{C'}$ denote the variety of upper pre-BCK-bands when $C' = \{\vee, \backslash, 0\}$;

p. 308 The statement of Corollary 3.3.19 should read:
Corollary 3.3.19. *Let* $\wedge \in C'$. *For any* $\mathbf{A} \in \mathsf{IQ}_{C'}$, *the principal subalgebra* $(\mathbf{a}]_{\mathbf{A}}$ *generated by* $a \in A$ *is a Boolean lattice....*

p. 309 The statement of Corollary 3.3.20 should read:
Corollary 3.3.20. *Let* $\{\wedge, \vee\} \subseteq C'$ *and let* $\mathbf{A} \in \mathsf{IQ}_{C'}$....

p. 319 The first sentence of the statement of Proposition 3.3.35 should read:
Corollary 3.3.35. *For any* $\mathbf{A} \in \mathsf{BP}_C$ *the following assertions hold:* ...

p. 320 The proof of Corollary 3.3.36 should be deleted.
Since $\mathcal{L} \vee \mathcal{R} = \mathcal{D}$ holds for either $\wedge$ or $\vee$, Corollary 3.3.36 is an immediate consequence of Corollary 3.3.35, and as such does not require any proof.

p. 328 The statement of Theorem 3.3.49 should read:
Theorem 3.3.49. *An algebra* $\mathbf{A} := \langle A;\ \wedge, \vee, /, 0\rangle$ *of type* $\langle 2,2,2,0\rangle$ *is an implicative BCK* $\leq_0$*-BCK local paralattice iff the reduct* $\langle A;\ \wedge, \vee, 0\rangle$ *is a local paralattice with zero,* ...

p. 375 Reference [6] should read:

[6] P. Agliano, *Congruence quasi-orederability in subtractive varieties*, J. Austral. Math. Soc. **71** (2001), 421–445.

p. 375 Reference [7] should read:

[7] P. Aglianò, *Fregean subtractive varieties with definable congruences*, J. Austral. Math. Soc. **71** (2001), 353–366.

p. 387 Reference [145] should read:

[145] G. Laslo and J. Leech, *Green's equivalences on noncommutative lattices*, Acta Sci. Math. (Szeged) **68** (2002), 501–533.

CONTRIBUTIONS TO THE THEORY OF PRE-BCK-ALGEBRAS

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Doctor of Philosophy

by
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March 2002

For the Bee

1988–1997

In loving memory

For Martin Seymour-Smith

1928–1998

In memoriam

Contents

Abstract

The class of BCK-algebras (hereafter BCK) is a relatively point-regular quasivariety that arises naturally both in algebraic logic and universal algebra. In algebraic logic, BCK arises as the equivalent algebraic semantics of Meredith's BCK logic, an important substructural logic with applications to proof theory. In universal algebra, BCK arises as the class of all residuation subreducts of partially ordered commutative residuated integral monoids (briefly, pocrims); the class of pocrims is a quasivariety whose relative subvarieties include the varieties of hoops and dual Brouwerian semilattices. The class of pre-BCK-algebras (hereafter PBCK) is a subtractive but not point-regular variety, the members of which naturally generalise BCK-algebras. The theory of pre-BCK-algebras and the applications of this theory to universal algebra and algebraic logic are the subject of this thesis.

Chapter 1 provides a structured account of the theory relevant to the study of pre-BCK-algebras, including: Laslo and Leech's theory of quasilattices, paralattices and skew lattices; Blok and Pigozzi's hierarchy of varieties with Equationally Definable Principal Congruences (briefly, EDPC); the theory of BCK-algebras and BCK-lattices due to Iséki, Idziak and others; Agliano and Ursini's theory of ideals and subtractive varieties; and the theory of algebraisable and assertional logics due to Blok, Pigozzi, Raftery and others. The main new results concern distributivity in skew lattices. A counterexample is presented showing that the middle distributive identities for skew lattices are independent; and a theorem asserting the interderivability of the middle distributive identities for symmetric skew lattices is stated. The results obtained answer two questions of Leech.

Chapter 2 is devoted to a study of the theory of pre-BCK-algebras. The elementary theory of pre-BCK-algebras is considered in Section 1 of Chapter 2. Some results relating PBCK to existing classes of algebras generalising BCK to the subtractive but not (relatively) point-regular case are presented. In one of two key results of the section, a 'Clifford-McLean'-type theorem for

pre-BCK-algebras, it is shown that the equivalence Ξ induced by the natural quasiordering $\preceq$ on a pre-BCK-algebra $\mathbf{A}$ is a congruence on $\mathbf{A}$ such that the quotient algebra $\mathbf{A}/\Xi$ is the maximal BCK-algebra homomorphic image of $\mathbf{A}$. For an appropriate notion of ideal, the ideal theory of pre-BCK-algebras is investigated. The relationship between ideals and congruences on pre-BCK-algebras is also briefly explored. In the other major result of the section, it is proved that the assertional logics of the variety of pre-BCK-algebras and the quasivariety of BCK-algebras coincide, and hence that a quasi-identity of the form $\underset{i \leq n}{\&} t_i(\vec{x}) \approx \mathbf{0} \supset t(\vec{x}) \approx \mathbf{0}$ is satisfied by PBCK iff it satisfied by BCK. Collectively, the results indicate that much of the first-order theory of BCK-algebras extends to pre-BCK-algebras.

Varieties of pre-BCK-algebras are investigated in Section 2 of Chapt : 2. For a variety V of BCK-algebras, the natural pre-BCK-algebraic counterpart $\mathsf{V}_\flat$ of V is the class $\{\mathbf{A} \in \mathsf{PBCK} : \mathbf{A}/\Xi \cong \mathbf{B}$ for some $\mathbf{B} \in \mathsf{V}\}$. In the main result of the section, it is shown that the natural pre-BCK-algebraic counterpart of any variety of BCK-algebras is itself a variety. The varieties of commutative, positive implicative and implicative BCK-algebras are important classes of BCK-algebras; their natural pre-BCK-algebraic counterparts are the varieties of commutative, positive implicative and implicative pre-BCK-algebras respectively. An order-theoretic characterisation of commutative pre-BCK-algebras is provided, and an ideal-theoretic characterisation of positive implicative pre-BCK-algebras is presented. For a suitable notion of prime ideal, it is also shown that an ideal of an implicative pre-BCK-algebra is prime iff it is maximal iff it is irreducible. The results obtained suggest that, for any variety V of BCK-algebras, the first-order theory of $\mathsf{V}_\flat$ stands in relation to V as the first-order theory of pre-BCK-algebras stands in relation to the first-order theory of BCK-algebras.

The variety of implicative BCS-algebras, a class of pointed groupoids, is studied in Section 3 of Chapter 2. It is shown that the variety of implicative BCS-algebras is a subvariety of the variety of implicative pre-BCK-algebras. Some examples are presented showing that implicative BCS-algebras arise naturally in several contexts in universal algebra and algebraic logic, including binary discriminator varieties (in particular, pseudocomplemented semilattices) and fixedpoint discriminator varieties (in particular, certain varieties of n-potent BCK-algebras). It is shown that an implicative pre-BCK-algebra is an implicative BCS-algebra iff it has a certain left normal band with zero polynomial reduct whose underlying partial ordering respects implicative pre-BCK difference in a precise sense. A representation theorem is proved showing that the category of implicative BCS-algebras is isomorphic to the category of left handed locally Boolean bands for suitable choices of objects and morphisms. The subdirectly irreducible implicative BCS-algebras are characterised (with R. J. Bignall): they are the 2-element implicative BCK-algebra and the algebras $\hat{\mathbf{B}}$ obtained from the non-trivial Boolean algebras $\mathbf{B}$ upon replacing the unit element of each $\mathbf{B}$ with a two-element clique. It is shown that the class of implicative BCK-algebras is generated (as a variety) by a certain 3-element pre-BCK-algebra $\mathbf{B}_2$, and hence that the lattice of varieties of implicative BCS-algebras is a three-element chain; the only non-trivial subvariety of the variety of implicative BCS-algebras is the variety of implicative BCK-algebras. Collectively, the results attest that implicative BCS-algebras are a 'non-commutative' analogue of implicative BCK-algebras, and as such, more closely resemble implicative BCK-algebras than do implicative pre-BCK-algebras.

In Chapter 3 the theory of pre-BCK-algebras is applied to the study of certain classes of algebras arising naturally in universal algebra and algebraic logic. In Section 1 of Chapter 3 subtractive varieties with Equationally Definable Principal Ideals (briefly, EDPI) are considered. For subtractive varieties, equational definability of principal ideals is the ideal-theoretic analogue of equationally definable principal congruences. It is shown that the variety of positive implicative pre-BCK-algebras is termwise definitionally equivalent to the variety of MINI-algebras introduced recently by Agliano and Ursini. A result is proved showing that a variety V is subtractive with EDPI iff every $\mathbf{A} \in \mathsf{V}$ has a MINI-algebra polynomial reduct whose ideals coincide with those of $\mathbf{A}$. A structure theorem for MINI-algebras is also proved: for a suitable notion of weakly compatible operation, it is shown that a variety is termwise definitionally equivalent to a variety of MINI-algebras with weakly compatible operations iff it is subtractive, weakly congruence orderable with EDPI. Subtractive weak Brouwerian algebras with filter-preserving operations (briefly, subtractive WBSO varieties) are an important class of subtractive varieties with EDPI that arise in the first instance from algebraic logic. A natural example of a subtractive WBSO variety is the variety of Nelson algebras, which arises from the algebraisation of constructive logic with strong negation. It is shown that the variety of Nelson algebras has a commutative (but not regular) Ternary Deductive term and is congruence permutable. An explicit Quaternary Deductive term is also given. The results obtained answer a question of Blok and Pigozzi.

Binary discriminator and dual binary discriminator varieties are studied in Section 2 of Chapter 3. The binary discriminator and dual binary discriminator were recently introduced by Chajda, Halaš and Rosenberg in an attempt to generalise the ternary discriminator and dual ternary discriminator to varieties with $\mathbf{0}$ exhibiting congruence permutability and congruence distributivity only locally at $\mathbf{0}$ respectively. It is shown that the variety generated by the class of all algebras $\langle A;\ h, 0\rangle$, where h is the dual binary discriminator on A and 0 is a nullary operation, is precisely the variety of left normal bands with zero. A semigroup-theoretic characterisation of dual binary discriminator varieties is also provided. It is shown that the variety generated by the class of all algebras $\langle A;\ b, 0\rangle$, where b is the binary discriminator on A and 0 is a nullary operation, is exactly the variety of implicative BCS-algebras. A natural characterisation of binary discriminator varieties is presented: a pointed variety is a binary discriminator variety iff it is subtractive with EDPI and is generated by a class of ideal simple algebras. Point-regular binary discriminator varieties are an important subclass of binary discriminator varieties; two results are proved that together show a point-regular variety is a binary discriminator variety iff it is a 'pointed' fixedpoint discriminator variety. In the major result of the section, the 'pointed' fixedpoint discriminator varieties are characterised: they are precisely the varieties that are ideal determined, semisimple with EDPC. Some theorems connecting 'pointed' fixedpoint discriminator varieties with pointed ternary discriminator varieties are also presented. The results answer in part a question of Blok and Pigozzi.

Pre-BCK-algebras structurally enriched with band or skew lattice operations are studied in Section 3 of Chapter 3. Bands and skew lattices structurally enriched with difference operations arise naturally in pointed discriminator varieties as skew Boolean algebras and skew Boolean intersection algebras (briefly, skew Boolean $\cap$-algebras). Skew Boolean algebras arise as distributive skew lattices structurally enriched with a relative complementation operation; skew Boolean $\cap$-algebras are skew Boolean algebras for which finite meets exist with respect to the natural skew lattice partial order. A theory of pre-BCK bands and pre-BCK quasilattices that parallels Laslo and Leech's theory of quasilattices is briefly outlined. In one of the two main results of the section, the skew Boolean algebras are characterised among the pre-BCK quasilattices: they are precisely the pre-BCK quasilattices for which the quasilattice with zero reduct is a join symmetric skew lattice with zero and the pre-BCK-algebra reduct is an implicative BCS-algebra. A theory of $\leq_0$-BCK bands and $\leq_0$-BCK paralattices that parallels Idziak's theory of BCK-semilattices and BCK-lattices is briefly outlined. In the other main result of the section, the skew Boolean $\cap$-algebras are characterised among the $\leq_0$-BCK paralattices: they are precisely the $\leq_0$-BCK paralattices for which the paralattice with zero reduct is a join symmetric local skew lattice with zero and the BCK-algebra reduct is an implicative BCK-algebra. A theory of double-pointed skew Boolean $\cap$-algebras akin to that of skew Boolean $\cap$-algebras is also presented, together with an axiomatisation of the assertional logic of the variety of (left handed) double-pointed skew Boolean $\cap$-algebras. Collectively, the results obtained intimate that pre-BCK algebras structurally enriched with band or skew lattice operations may provide a unifying framework for the study of several classes of 'generalised Boolean structures' arising naturally in universal algebra and algebraic logic.

In Chapter 4 the research undertaken in this dissertation is briefly reviewed, and some potential avenues for future research are presented.

Declaration

This thesis contains no material accepted for the award of any other degree or diploma at any university or other institution. To the best of my knowledge this thesis contains no material written or previously published by another person except where due reference is made within the text.

[redacted]

Matthew Spinks

Acknowledgements

First and foremost among those I would like to thank is my supervisor, Professor Robert Bignall. The amount of time and energy he has invested over the past seven years to see this thesis through to completion has been quite unprecedented. He has been, at all times over the long period of my Ph.D. candidature, the best possible supervisor I could have envisaged: his continual support and guidance, enthusiasm and seemingly endless patience have been the mainstay of my Ph.D. candidature. Above all, his tremendous ability to *do* mathematics and the extraordinary depth and elegance of his mathematical ideas and insights have been absolutely formative in the development of my mathematical and intellectual life as a graduate student. I consider myself very fortunate to have worked under a mathematician of his calibre.

During my time at Monash University, I was in receipt of a Faculty Scholarship from the Gippsland School of Computing and Information Technology; the financial support so provided is gratefully acknowledged. Professor John Crossley, Head of the Department of Computer Science in the Faculty of Information Technology, was kind enough to act as associate supervisor throughout my candidature, in spite of his myriad other duties. Lloyd Humberstone, of the Department of Philosophy in the Faculty of Arts, generously shared his encyclopaedic knowledge of logic over the course of several years, while Dr. John Jeavons, of the Department of Mathematics in the Faculty of Science, assisted with general mathematical issues over the same period.

I owe a huge debt to the academic community at large. Much of the initial research for this thesis was conducted using computer-based tools, most notably Dr. Jiang Zhang and Dr. Hantao Zhang's finite model enumerator SEM and Dr. William McCune's automated theorem-prover OTTER. In particular, the proof of Proposition 2.3.21 is a manual simplification of an automated proof initially obtained using OTTER; it seems unlikely this proof would have been obtained without the use of automated theorem-proving software. For many helpful discussions about automated theorem-proving, universal algebra and algebraic logic I am indebted to Prof. George Grätzer, Prof. Kiyoshi Iséki, Dr. William McCune, Prof. Don Pigozzi, Prof. James Raftery, Dr. Greg Restall and Prof. Aldo Ursini. I would like in particular to thank Dr. McCune: he always answered my innumerable questions about automated theorem-proving, no matter how trivial, with unfailing courtesy. For the provision of papers, preprints, and other manuscripts I am grateful to Dr. Paolo Agliano, Dr. Clint van Alten, Prof. Martin Bunder, Dr. Isabel Ferreirim, Lloyd Humberstone, Dr. Tomasz Kowalski, Prof. Raftery, Prof. Ursini and Prof. Andrzej Wroński. I would especially like to thank Dr. van Alten, who kindly made his doctoral dissertation available for study at a critical juncture in my PhD candidature; the influence of his work on this thesis is quite obvious.

My sincere thanks go also to various of my family and friends. My parents and my brother Jeremy have always afforded me their care and support, never more so than during the difficult last fifteen months of my candidature. Both Konrad Talmont-Kaminski and Anthony Romeo have remained steadfast in their friendship over two decades; here I would like to acknowledge the hospitality shown me by the former during my stay in Poland in 1999, and the emergency accommodation provided me by the latter over the summer of 1999-2000. Over the past seven years the Rev. Judy Redman, her husband Bruce and children Hugh and Ceiridwen have always been ready to furnish me with a home away from home; their generosity of spirit in so doing is gratefully acknowledged.

Others who should specially be mentioned include Rhondda Curtis, who provided much-needed counsel at various points throughout my Ph.D. candidature; to her I am deeply indebted. During my recent serious illness I was attended by Dr. David Harding-Smith, who treated me with great competency and professionalism; it is not really an exaggeration to say I owe my life to his care. Finally, it would be an omission not to acknowledge the almost subliminal influence of Larry Prosser: in effect, this dissertation witnesses the culmination of advice given me by him some twenty years ago.

Carey St
25 March 2002

Chapter 1

Introduction

1.1 Introduction

1.1.1. About This Thesis. Beginning with the papers of Krull [143] and Ward and Dilworth [235] residuated algebraic structures have been the subject of study for over half a century. Nonetheless, it is only recently that the residuated algebraic structures associated with logical systems have been seriously investigated [34, p. 597], even though residuation in such structures is typically the algebraic counterpart of implication in the associated logic. Much of the recent work concerning residuated algebraic structures in logic has focussed on two related classes of algebras: the class of partially ordered commutative residuated integral monoids (briefly, pocrims) [109, 176, 39]; and the class of BCK-algebras [126, 70, 38]. Let $\langle A; \leq\rangle$ be a poset such that: (i) $\langle A; \leq\rangle$ is integral in the sense that there exists a least element $0 \in A$ which acts as an identity element for an order compatible commutative associative binary multiplication $\oplus$ on A; and (ii) $\langle A; \leq\rangle$ is residuated in the sense that there exists a binary operation $\dot{-}$ on A such that, for any $a, b \in A$, $a \dot{-} b = \min\{c \in A : a \leq c \oplus b\}$. Then $\langle A; \leq\rangle$ is first-order definitionally equivalent to an *algebra* $\langle A; \oplus, \dot{-}, 0\rangle$ of type $\langle 2,2,0\rangle$; such an algebra is a *pocrim*. The class of all pocrims is a quasivariety [121, 123] but is not a variety [109]. A *BCK-algebra* is a $\langle \dot{-}, 0\rangle$-subreduct of a pocrim; equivalently, by results of Wroński [242], Ono and Komori [176], Fleischer [90] and Pałasiński [178], an algebra $\langle A; \dot{-}, 0\rangle$ of type $\langle 2,0\rangle$ is a BCK-algebra iff the following identities and quasi-identity are satisfied:

$$((x \dot{-} y) \dot{-} (x \dot{-} z)) \dot{-} (z \dot{-} y) \approx \mathbf{0} \tag{1.1}$$

$$(x \dot{-} (x \dot{-} y)) \dot{-} y \approx \mathbf{0} \tag{1.2}$$

$$x \dot{-} x \approx \mathbf{0} \tag{1.3}$$

$$\mathbf{0} \dot{-} x \approx \mathbf{0} \tag{1.4}$$

$$x \dot{-} y \approx \mathbf{0} \,\&\, y \dot{-} x \approx \mathbf{0} \supset x \approx y. \tag{1.5}$$

The class of all BCK-algebras is thus a quasivariety; it is not a variety [240]. By results of Blok and Pigozzi [31], Blok and Jònsson [28] and Raftery and van Alten [192] the class of BCK-algebras [pocrims] is termwise definitionally equivalent to the equivalent algebraic semantics of Meredith's BCK logic [165], [186, p. 316] [BCK logic with 'fusion' [28]]. BCK logic [with fusion] is an important substructural logic which arises in the first instance from proof theory; see [80, Section 4] for a discussion and references.

A significant body of work now exists showing pocrims and BCK-algebras play a central role in the theory of the residuated structures associated with logical systems (for an extended discussion and references see Blok and Pigozzi [34, Section 6]). This is most readily seen from the theory of hoops, which are pocrims that are 'naturally ordered' in the sense that for any $a, b \in A$, $a \leq b$ implies there exists $c \in A$ such that $a \oplus c = b$. The study of hoops and their residuation subreducts is due variously to Büchi and Owens [48], Ferreirim [88], Blok and Pigozzi [34] and Blok and Ferreirim [26, 27] (see also Bosbach [44, 45, 46]), while the study of hoops with 'normal multiplicative operators' is due to Blok and Pigozzi [34]. 'Hoop logics' are deductive systems whose equivalent algebraic semantics are termwise definitionally equivalent to quasivarieties of (structurally enriched) hoops. Included among the hoop logics are the classical and intuitionistic sentential calculi; all the normal modal extensions of classical and intuitionistic propositional logic; the ω-valued propositional calculus of Lukasiewicz [155]; and the logics $\mathbb{L}_{\mathsf{BCK}}$ and $\mathbb{LJ}^*$ of Ono and Komori [176]. Respectively, the equivalent algebraic semantics of these deductive systems are termwise definitionally equivalent to: the varieties of Boolean and Heyting algebras; the varieties of Boolean and Heyting algebras with 'normal multiplicative operators'; the variety of Wajsberg algebras (see Font, Rodriguez and Torrens [92], Chang [59] and Mundici [171]); and certain varieties of residuated lattices (see Blount [43]). See Agliano [5, Section 4.6].

Although the theory of hoops provides a unifying framework for many of the logical systems and associated classes of algebras traditionally considered the domain of algebraic logic, there exist important examples of algebraisable deductive systems whose equivalent algebraic semantics are classes of residuated algebraic structures that are not termwise definitionally equivalent to quasivarieties of (structurally enriched) pocrims or BCK-algebras. For example, linear logic [97] is algebraisable [3, Section 2.2]; its equivalent algebraic semantics is termwise definitionally equivalent to the variety of girales, a class of semilattice ordered residuated monoids that is not a variety of pocrims [5, Section 4.1]. Relevance logic [12] is algebraisable [93]; its equivalent algebraic semantics is termwise definitionally equivalent to the variety of De Morgan monoids, also a class of semilattice ordered residuated monoids that is not a variety of pocrims [5, Section 4.4]. Residuated structures that are not pocrims also arise naturally in universal algebra. Ordinals closed under addition provide natural examples of residuated ordered monoids for which the monoid operation is not commutative [229, Example 1.8]. Ideal lattices of rings (considered with ideal multiplication and set inclusion) also form residuated ordered monoids with a non-commutative monoid operation [229, Example 1.7]; notably, lattices of topologising filters on rings with identity [98] may be understood as such monoids [229, Appendix]. Besides being of independent interest in their own right, classes of residuated algebraic structures found in universal algebra are often a[illegible]f intrinsic interest from the perspective of algebraic logic in the sense that they may arise naturally as 'quasivarieties of logic': that is, they are termwise definitionally equivalent to the equivalent algebraic semantics of some algebraisable deductive system. See Blok and Raftery [40, Section 5] and Barbour and Raftery [16, Section 6].

Recent developments in algebraic logic and universal algebra have thus lead to the study of several classes of algebras generalising pocrims and BCK-algebras. The most prominent of these generalisations is that of pocrims to polrims and BCK-algebras to BCC- or left residuation algebras. A polrim is a partially ordered integral monoid that is residuated on the left; the residuation subreducts of polrims are the BCC- or left residuation algebras. The classes of polrims and left residuation algebras are both quasivarieties [229, Proposition 1.4] that are not varieties [229, Proposition 4.1]. Commutative polrims are precisely the pocrims, while left residuation algebras satisfying a certain principle of 'quasi-commutation' are exactly the BCK-algebras [229, Example 1.5]. Polrims and their residuation subreducts have been investigated by several authors, including Komori [138, 139], Ono and Komori [176], Raftery and van Alten [192], van Alten [229], and van Alten and Raftery [231, 230]. Polrims arise naturally both in universal algebra and algebraic logic; in particular, the quasivariety of polrims [left residuation algebras] is termwise definitionally equivalent to the equivalent algebraic semantics of the $\{\&, \supset\}$-fragment [$\{\supset\}$-fragment] of the logic $\mathbb{H}_{\mathsf{BCC}}$ of Ono and Komori [176]. Other classes of residuated monoids generalising pocrims and BCK-algebras to have been considered in the literature include the sircomonoids of Raftery and van Alten [193] and their residuation subreducts, the BCI-algebras of Iséki [122]; and the semilattice ordered residuated monoids of Agliano [5]. The study of all such monoids is part of a much larger theory of 'residuals without residuation' pioneered by several authors including Meyer and Routley [166], Dunn [85, 86], Ursini [223] and Agliano [5]. Of particular relevance to algebraic logic is Dunn's theory of 'partial gaggles' [86], which seeks to provide a uniform semantical approach to the study of substructural propositional logics, including: classical and intuitionistic logic; the various modal and relevance logics; linear logic; BCK logic; and the Lambek calculus [144]. See also Restall [198, Chapter II].

One principle common to pocrims, BCK-algebras and most of their generalisations is the existence of a finite set of binary terms $\{d_i(x, y) : i = 1, \ldots, n\}$ and a (definable) constant $\mathbf{0}$ such that the following identities and quasi-identity are satisfied:

$$d_i(x,x) \approx 0 \quad \text{and} \quad \underset{i \leq n}{\&} d_i(x,y) \approx 0 \supset x \approx y. \tag{1.6}$$

For example, the set of terms $\{x \dot{-} y, y \dot{-} x\}$ witnesses (1.6) for the quasivariety of BCK-algebras (and thus for pocrims); this follows immediately from (1.3) and (1.5). In algebraic terms, satisfaction of the identities and quasi-identity of (1.6) by a quasivariety K (with a (definable) constant term) is equivalent to *relative point* (or $\mathbf{0}$, if the constant term is specified as $\mathbf{0}$) *regularity*. Regularity conditions in universal algebra always demand congruences of algebras be determined by certain subsets of their universes; in particular, relative $\mathbf{0}$-regularity asserts that the K-congruences of any $\mathbf{A} \in \mathsf{K}$ are determined by their $\mathbf{0}^{\mathbf{A}}$-classes. While regularity conditions in universal algebra are well understood (see for instance [74, 89, 104, 78]), the metalogical significance of such conditions has been less clear. Recently the status of these conditions in algebraic logic has been clarified by Blok and Raftery [40], who have shown that a quasivariety K is a 'quasivariety of logic' precisely when the K-congruences of members of K are determined by suitably defined subsets of their universes. In logical terms, therefore, relative point regularity is a sufficient condition for a quasivariety to be a 'quasivariety of logic'; further, although not necessary, the condition of relative point regularity is satisfied by most familiar classes of algebras arising as the equivalent algebraic semantics of some algebraisable deductive system. Blok and Raftery's result has very recently engendered some interest in regularity conditions for algebraic logic, both syntactically (Blok and La Falce [25]) and from the perspective of full regularity (Barbour and Raftery [16]).

Perhaps because of the centrality of regularity conditions to algebraic logic, classes of algebras that are not relatively point regular but which naturally generalise pocrims, BCK-algebras or related structures in some sense have not been previously investigated in algebraic logic to any significant degree, bar two exceptions. The first of these exceptions is a certain variety V of algebras considered by Blok and Raftery in [38, Section 4] and by Agliano and Ursini in [10] (to within termwise definitional equivalence); members of V

(V-*algebras*) naturally generalise BCK-algebras but are not in general point regular. The role played by V-algebras in the theory of BCK-algebras has been briefly considered by Blok and Raftery in [38, Section 4]. In particular, Blok and Raftery have shown [38, Proposition 2, Theorem 8] that a certain 3-element algebra $\mathbf{B}_2 \in \mathsf{V}$, though not itself a BCK-algebra, plays an important role in the theory of BCK-algebras. The second exception is the variety of MINI-algebras considered by Agliano and Ursini in [11]. MINI-algebras naturally generalise positive implicative BCK-algebras (the $\langle \dot{-}, 0\rangle$-subreducts of dual Brouwerian semilattices, or, equivalently, hoops satisfying $x \oplus x \approx x$) but are not in general point regular. Results due to Agliano and Ursini show MINI-algebras play a fundamental role in the theory of subtractive varieties with equationally definable principal ideals: see in particular [11, Corollary 3.8]. Collectively, the work of Blok and Raftery and Agliano and Ursini suggests that an appropriate generalisation of BCK-algebras that subsumes both Blok and Raftery's variety V and Agliano and Ursini's variety of MINI-algebras (up to termwise definitional equivalence) may be of interest in algebraic logic.

Call an algebra $\langle A;\ \dot{-}, 0\rangle$ of type $\langle 2, 0\rangle$ a *pre-BCK-algebra* iff it satisfies the identities (1.1)–(1.4) and the identity:

$$x \dot{-} 0 \approx x. \tag{1.7}$$

The identity (1.7) in conjunction with (1.3) ensures that the variety of pre-BCK-algebras is *subtractive* in the sense of Agliano and Ursini [222, 10, 9, 11, 225] and hence that the variety of pre-BCK-algebras contains both Blok and Raftery's variety V and Agliano and Ursini's variety of MINI-algebras (up to termwise definitional equivalence). Since pre-BCK-algebras do not in general satisfy the quasi-identity (1.5), binary terms $d_i(x, y)$ satisfying (1.6) need *not* exist, whence the variety of pre-BCK-algebras is not $\mathbf{0}$-regular. On the other hand, by a result of Iséki [126, Theorem 2] the class of all BCK-algebras satisfies (1.7), and so is a subquasivariety of the variety of pre-BCK-algebras. Thus the variety of pre-BCK-algebras may be understood as a generalisation of BCK-algebras to the non-relatively point regular case that subsumes both the variety V of Blok and Raftery and (to within termwise definitional equivalence) the variety of MINI-algebras of Agliano and Ursini. In this thesis we offer and explore the theory of pre-BCK-algebras as a generalisation of the theory of BCK-algebras to the non-relatively point regular case. In particular, our program is to investigate the elementary theory of the variety of pre-BCK-algebras and some of its subvarieties, and to apply this theory to the study of some varieties arising naturally in universal algebra and algebraic logic.

Remark 1.1.2. Two fundamental restrictions are imposed on the scope of the work presented in this thesis. First, our study of pre-BCK-algebras does *not* extend to a study of quasivarieties of pre-BCK-algebras (with the obvious exception of the class of BCK-algebras), despite the fact that quasivarieties are the natural algebraic counterparts of algebraisable logics in the sense that the equivalent algebraic semantics of an algebraisable logic can always be taken to be a quasivariety. The rationale behind this (admittedly artificial) restriction is that the theory of subtractive varieties, which plays an important role in our investigation of the variety of pre-BCK-algebras and its subvarieties, does not extend well to quasivarieties: see for instance Blok and Raftery [40, p. 181, Example 7.2] or van Alten [229, pp. 71–72]. Second, our study of pre-BCK-algebras does *not* extend to a study of algebras naturally generalising pocrims in some sense but which fail to be relatively point-regular. While the role played by residuated ordered monoids in universal algebra and algebraic logic is an important motivation for the work undertaken in this thesis, it is not yet clear what form such a generalisation of these monoids should take, or even if such a generalisation of these monoids is of any intrinsic interest: *cf.* Higgs [109, p. 72] and the remarks of §4.2.1. ∎

1.1.3. Organisation. This dissertation is organised as follows. In an attempt to keep this thesis self-contained, in the remainder of this chapter we introduce some notation and terminology and review those parts of universal algebra and algebraic logic pertinent to the study of pre-BCK-algebras, including: the theory of bands; Leech's theory of non-commutative and skew lattices; the theory of varieties with equationally definable principal congruences due to Blok, Köhler and Pigozzi; the theory of BCK-algebras and BCK-lattices due to Iséki, Tanaka, Idziak and others; Agliano and Ursini's theory of ideals and subtractive varieties in universal algebra; and the theory of assertional and algebraisable logics due to Blok, Pigozzi and Raftery. Our survey is leisurely: we take the opportunity to present the occasional new result and example, and generally to tidy up some loose ends. Chapter 2 is devoted to the study proper of the theory of pre-BCK-algebras. Adopting the approach of Iséki and Tanaka's survey paper of BCK-algebras [126], we show the elementary theory of pre-BCK-algebras closely parallels that of BCK-algebras. In Chapter 3 we apply the theory of pre-BCK-algebras to three important classes of algebras arising naturally in algebraic logic and universal algebra, namely: subtractive varieties with equationally definable principal ideals; binary discriminator varieties; and pointed ternary discriminator varieties. In Chapter 4 we briefly review the work undertaken in this thesis and make some suggestions for future work.

1.2 Notation and Terminology

In this section we fix some of the fundamental notation used throughout this thesis and introduce some terminology of universal algebra and algebraic logic. For notation and terminology not explicitly introduced either here or in the sequel we generally follow Burris and Sankappanavar [55] or Blok and Pigozzi [36].

1.2.1. Ordered Sets. A *quasiorder* (also *preorder* in the literature) is a reflexive transitive relation. A *quasiordered set* is a pair $\langle A;\ \preceq\rangle$ where A is a set and $\preceq$ is a quasiorder. Let $\langle A;\ \preceq\rangle$ be a quasiordered set. For $B \subseteq A$ and $a \in A$ we define:

$$\begin{aligned} (B] &:= \{b \in A : b \preceq a \text{ for some } a \in B\}, \qquad & (a] &:= (\{a\}] \\ [B) &:= \{b \in A : a \preceq b \text{ for some } a \in B\}, \qquad & [a) &:= [\{a\}). \end{aligned}$$

The set $(a]$ is called the *principal* (*order*) *ideal generated by* a *in* $\langle A;\ \preceq\rangle$. A subset B of A is said to be *hereditary* in $\langle A;\ \preceq\rangle$ if $B = (B]$. An element $m \in A$ is *minimal* [*maximal*] if $m \preceq a$ [$a \preceq m$] for any $a \in A$. Observe that minimal and maximal elements of a quasiordered set need not be unique in general. A minimal element [maximal element] of a quasiordered set $\langle A;\ \preceq\rangle$ is called a *least element* [*greatest element*] if it is unique. Let $B \subseteq A$. An element $c \in A$ is a *lower bound* [*upper bound*] of B if $c \preceq b$ [$b \preceq c$] for all $b \in B$. An element $d \in A$ is a *greatest lower bound* [*least upper bound*] of B if d is a lower bound [upper bound] of B and $c \preceq d$ [$d \preceq c$] for any lower bound [upper bound] c of B. Observe that greatest lower and least upper bounds of B need not be unique in general. The set of greatest lower [least upper] bounds of B is denoted glb B [lub B]. For $B := \{a_1, \ldots, a_n\}$ the set glb B [lub B] is alternatively denoted glb $\{a_1, \ldots, a_n\}$ [lub $\{a_1, \ldots, a_n\}$]. Given elements a, b of a quasiordered set $\langle A;\ \preceq\rangle$ with $a \preceq b$, the *interval* $[a, b]$ is the set $\{c \in A : a \preceq c \preceq b\}$.

A *partial order* is a reflexive, symmetric and transitive relation. A partially ordered set is a pair $\langle A;\ \leq\rangle$, where A is a set and $\leq$ is a partial order. We abbreviate the term 'partially ordered set' by *poset*. The following lemma is folklore.

Lemma 1.2.2. *[196, Theorem I§5.2] Let $\langle A;\ \preceq\rangle$ be a quasiordered set and let Ξ be the binary relation defined by:*

$$a \,\Xi\, b \quad \textit{iff} \quad a \preceq b \textit{ and } b \preceq a$$

for any $a, b \in A$. Then Ξ is an equivalence relation on A. Moreover, the binary relation $\leq$ defined on A/Ξ by:

$$[a]_\Xi \leq [b]_\Xi \quad \textit{iff} \quad a \preceq b$$

for any $[a]_\Xi, [b]_\Xi \in A/\Xi$ with $a, b \in A$ is a partial ordering on A/Ξ.

Lemma 1.2.3. *Let $\langle A;'\preceq\rangle$ be a quasiordered set and let Ξ be the equivalence relation on A induced by $\preceq$ in the sense of Lemma 1.2.2. The following statements hold for any $a, b \in A$:*

1. *If $d_1 \in A$ is a greatest lower bound of $\{a, b\}$ and $d_2 \,\Xi\, d_1$ for $d_2 \in A$, then d_2 is also a greatest lower bound of $\{a, b\}$;*

2. *If $d_1, d_2 \in A$ are both greatest lower bounds of $\{a, b\}$, then $d_1 \,\Xi\, d_2$.*

Proof. For (1), let $d_1 \in A$ be a greatest lower bound of $\{a, b\}$ and suppose $d_2 \equiv d_1$ for $d_2 \in A$. Since $d_2 \preceq d_1$, d_2 is a lower bound of $\{a, b\}$. If $c \in A$ is a lower bound of $\{a, b\}$, then $c \preceq d_1 \preceq d_2$ since d_1 is a greatest lower bound of $\{a, b\}$. Thus d_2 is also a greatest lower bound of $\{a, b\}$. For (2), let $d_1, d_2 \in A$ be greatest lower bounds of $\{a, b\}$. Since d_1 is a lower bound of $\{a, b\}$ and d_2 is a greatest lower bound of $\{a, b\}$ we have $d_1 \preceq d_2$; likewise we have $d_2 \preceq d_1$. Thus $d_1 \equiv d_2$ as required. ∎

Let $\langle A; \preceq\rangle$ be a quasiordered set. The *proper part* of $\preceq$, denoted $\prec$, is defined as $\preceq$ but not $\equiv$. A subset B of A is called a *clique* of $\equiv$ if $B^2 \subseteq \equiv$. Let $\langle A; \leq\rangle$ be a poset. A quasiorder $\preceq$ on A is said to be *admissible* if, for all $a, b \in A$, $a \leq b$ implies $a \preceq b$. Further details concerning quasiorderings and partial orderings may be found in Wechler [236, pp. 31–35] and Cleave [61, Chapter 5§5–6].

1.2.4. Languages.[1] We fix a countably infinite set $\mathbf{X}$ of *variables* for use throughout this thesis. In an algebraic [logical] context, we usually write $x, y, z, \ldots$ [$p, q, r, \ldots$] for metavariables ranging over $\mathbf{X}$. In the sequel we confine our attention to *algebraic* languages unless otherwise stated. Thus a *language* $\boldsymbol{\mathcal{L}}$ consists of a set $\mathcal{L}$ of *function symbols* together with an *arity* function *ar* that assigns a natural number to each function symbol in $\mathcal{L}$. By abuse of notation we often identify a language $\boldsymbol{\mathcal{L}}$ with its set of function symbols $\mathcal{L}$, while by abuse of language we sometimes describe an (algebraic) language as a *type*. Given a language $\boldsymbol{\mathcal{L}}$, in an algebraic context the members of $\mathbf{X}$ and $\mathcal{L}$ are called *individual variables* and *operation symbols* respectively, while in a logical context the members of $\mathbf{X}$ and $\mathcal{L}$ are respectively referred to as *propositional variables* and *logical connectives.* The *term* or *formula algebra of type* $\boldsymbol{\mathcal{L}}$ *over* $\mathbf{X}$, denoted $\mathbf{T}_{\mathcal{L}}(\mathbf{X})$ or $\mathbf{Fm}_{\mathcal{L}}$, is the absolutely free algebra of type $\boldsymbol{\mathcal{L}}$ over $\mathbf{X}$. In an algebraic context elements $s, t, u, v, \ldots$ of $\mathrm{T}_{\mathcal{L}}(\mathrm{X})$ are called $\boldsymbol{\mathcal{L}}$*-terms*, while in a logical context elements $\varphi, \psi, \chi, \ldots$ of $\mathrm{Fm}_{\mathcal{L}}$ are called $\boldsymbol{\mathcal{L}}$*-formulas.* An $\boldsymbol{\mathcal{L}}$*-substitution* is an endomorphism of the formula algebra over $\boldsymbol{\mathcal{L}}$; notice that an $\boldsymbol{\mathcal{L}}$-substitution σ may be identified with its restriction to $\mathbf{X}$ by the universal mapping property [55, Lemma II§10.6]. We often

[1] The material in this subsection is adapted from van Alten [229, Chapter 0].

drop the prefix $\mathcal{L}$ from the phrases '$\mathcal{L}$-substitution', '$\mathcal{L}$-term' and '$\mathcal{L}$-formula' when $\mathcal{L}$ is understood.

1.2.5. Universal Algebra. Algebras are denoted $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ and their respective universes are denoted $A, B, C, \ldots$. An algebra $\mathbf{A}$ is *trivial* if $|A| = 1$ and *non-trivial* otherwise. Classes of algebras are denoted $\mathsf{K}, \mathsf{V}, \ldots$. Throughout this thesis we make standard use of the algebraic class operators $\mathbf{I}, \mathbf{H}, \mathbf{S}, \mathbf{P}$, $\mathbf{P_s}$ (for subdirect products) and $\mathbf{P_u}$ (for ultraproducts). For a class K of similar algebras, we write $\mathbf{V}(\mathsf{K})$ for the variety $\mathbf{HSP}(\mathsf{K})$ generated by K [55, Theorem II§9.5] and $\mathbf{Q}(\mathsf{K})$ for the quasivariety $\mathbf{ISPP_u}(\mathsf{K})$ generated by K [102]; recall $\mathbf{Q}(\mathsf{K}) = \mathbf{ISP}(\mathsf{K})$ when K is finite. We also write $\mathsf{K}_{\mathsf{FIN}}$ for the subclass of finite algebras of K and K_{SI} for the subclass of subdirectly irreducible members of K.

Let A be a set. The set of equivalence relations on A is denoted $\mathrm{Eq}(A)$ and the lattice of equivalence relations on A is denoted $\mathbf{Eq}(A)$. The set of all partitions of A is denoted $\Pi(A)$ and the lattice of all partitions of A is denoted $\mathbf{\Pi}(A)$. For $\pi \in \Pi(A)$ let $\theta(\pi) := \{\langle a, b\rangle \in A^2 : \{a, b\} \subseteq B \text{ for some } B \in \pi\}$. By [55, Theorem I§4.11] $\mathbf{\Pi}(A)$ is isomorphic to $\mathbf{Eq}(A)$ under the mapping $\pi \mapsto \theta(\pi)$ $(\pi \in \Pi(A))$. Let $\theta \in \mathrm{Eq}(A)$. We write variously $a\theta b$, $a \equiv b\,(\mathrm{mod}\,\theta)$ and $a \equiv_\theta b$ for $\langle a, b\rangle \in \theta$. For every $a \in A$, we denote the equivalence class of a modulo θ by $[a]_\theta$. We denote the quotient set by A/θ. The identity relation on A is denoted ω_A and the universal relation $A \times A$ is denoted ι_A.

The set of congruences on an algebra $\mathbf{A}$ is denoted $\mathrm{Con}\,\mathbf{A}$ and the lattice of congruences on $\mathbf{A}$ is denoted $\mathbf{Con}\,\mathbf{A}$. For $B \subseteq A^2$, the congruence on $\mathbf{A}$ generated by B is denoted $\Theta^{\mathbf{A}}(B)$. For $a, b \in A$ we abbreviate $\Theta^{\mathbf{A}}(\{\langle a, b\rangle\})$ by $\Theta^{\mathbf{A}}(a, b)$. The identity congruence on $\mathbf{A}$ is denoted $\omega_{\mathbf{A}}$ and the universal congruence $A \times A$ is denoted $\iota_{\mathbf{A}}$.

Let K be a class of similar algebras and let $\mathbf{A} \in \mathsf{K}$. A *constant term* of $\mathbf{A}$ or K is any nullary or constant unary term function, or, less precisely, the element of $\mathbf{A}$ (or of each member of K) that constitutes the range of such a function [34, p. 551]. A *constant* is a nullary fundamental operation. We say $\mathbf{A}$ is *with* $\mathsf{0}$ if $\mathsf{0}$ is a constant term of $\mathbf{A}$. We say K is *with* $\mathsf{0}$ if $\mathsf{0}$ is a constant term of K.

We say K is *pointed* if it is with $\mathbf{0}$ for some constant term $\mathbf{0}$; observe that a pointed class may have more than one constant term.

Let $\mathbf{A}$ be an algebra. Elements $a, b \in A$ are said to be *residually distinct* if they have distinct images in every non-trivial homomorphic image of $\mathbf{A}$; in symbols, $\Theta^{\mathbf{A}}(a, b) = \iota_{\mathbf{A}}$ [34, p. 551]. We say $\mathbf{A}$ is *with* $\{\mathbf{0}, \mathbf{1}\}$ if $\mathbf{A}$ has constant terms $\mathbf{0}$ and $\mathbf{1}$ such that $\mathbf{0}^{\mathbf{A}}$ and $\mathbf{1}^{\mathbf{A}}$ are residually distinct. We say a class K of similar algebras is *with* $\{\mathbf{0}, \mathbf{1}\}$ if K has constant terms $\mathbf{0}$ and $\mathbf{1}$ such that $\mathbf{0}^{\mathbf{A}}$ and $\mathbf{1}^{\mathbf{A}}$ are residually distinct in any non-trivial member $\mathbf{A}$ of K. We say K is *double-pointed* if it is with $\{\mathbf{0}, \mathbf{1}\}$ for some constant terms $\mathbf{0}$ and $\mathbf{1}$; notice that a double-pointed class may have more than two constant terms. Observe that any variety V with $\mathbf{0}$ may be associated with a class V^{+} with $\{\mathbf{0}, \mathbf{1}\}$ upon adjoining a new nullary operation symbol $\mathbf{1}$ to the language of V and defining:

$$\mathsf{V}^{+} := \left\{ \langle \mathbf{A};\ 1 \rangle : \mathbf{A} \in \mathsf{V} \text{ and } \mathbf{0}^{\langle \mathbf{A};\, 1 \rangle}, \mathbf{1}^{\langle \mathbf{A};\, 1 \rangle} \in A \text{ are residually distinct} \right\}$$

where $\langle \mathbf{A};\ 1 \rangle$ is the algebra obtained from $\mathbf{A}$ by enriching the signature of $\mathbf{A}$ with the nullary operation symbol $\mathbf{1}$ whose interpretation is a fixed element $1 \in A$. V^{+} is called the *generic double-pointed expansion of* V. In certain circumstances V^{+} is always guaranteed to be a variety: see Blok and Pigozzi [34, p. 551] for details.

Let $\mathbf{A}$ be an algebra with $\mathbf{0}$. We say $\mathbf{A}$ is $\mathbf{0}$-*regular* if $[0]_\theta = [0]_\phi$ implies $\theta = \phi$ for all $\theta, \phi \in \operatorname{Con} \mathbf{A}$. A variety V with $\mathbf{0}$ is $\mathbf{0}$-*regular* if every $\mathbf{A} \in \mathsf{V}$ is $\mathbf{0}$-regular. V is said to be *point regular* if it is $\mathbf{0}$-regular for some constant term $\mathbf{0}$. V is *strongly* $\mathbf{0}$-*regular* if V is $\mathbf{0}$-regular and, for any $\mathbf{A} \in \mathsf{V}$, every compact congruence of $\mathbf{A}$ is principal [181, p. 483]. V is *strongly point regular* if it is strongly $\mathbf{0}$-regular for some constant term $\mathbf{0}$.

Proposition 1.2.6. *[17, Lemma 1.4.10]; [104, Corollary 1.7] Let V be a variety with $\mathbf{0}$. V is $\mathbf{0}$-regular iff there exist binary terms $d_1, \ldots, d_n$ of V such that the identities:*

$$d_i(x, x) \approx \mathbf{0}, \qquad\qquad 1 \le i \le n$$

and the quasi-identity:

$$d_1(x,y) \approx \mathbf{0} \ \& \ \ldots \ \& \ d_n(x,y) \approx \mathbf{0} \supset x \approx y$$

hold in V.

Let K be a quasivariety and let $\mathbf{A} \in \mathsf{K}$. A congruence θ on $\mathbf{A}$ is called a K-*congruence* if $\mathbf{A}/\theta \in \mathsf{K}$. The set of all K-congruences on $\mathbf{A}$ is denoted $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$; notice $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A} = \mathrm{Con}\,\mathbf{A}$ if K is a variety. When ordered by inclusion $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ gives rise to an algebraic lattice $\mathbf{Con}_{\mathsf{K}}\,\mathbf{A}$ [38, p. 633]. We say $\mathbf{A}$ is K-*subdirectly irreducible* if $\mathbf{A}$ has a minimal non-trivial K-congruence. By a result of Mal'cev [156], every algebra $\mathbf{B} \in \mathsf{K}$ is isomorphic to a subdirect product of K-subdirectly irreducible members of K (that are homomorphic images of $\mathbf{B}$). We say $\mathbf{A}$ is K-*congruence distributive* if $\mathbf{Con}_{\mathsf{K}}\,\mathbf{A}$ is a distributive lattice; K is called K-*congruence distributive* if every member of K is K-congruence distributive. We say $\mathbf{A}$ has the K-*congruence extension property* if for any $\mathbf{B} \in \mathbf{S}(\mathbf{A})$ and any K-congruence θ of $\mathbf{B}$, there is a K-congruence ϕ of $\mathbf{A}$ such that $\phi \cap (B \times B) = \theta$. We say K has the K-*congruence extension property* if every member of K has the K-congruence extension property. Suppose $\mathbf{A}$ is with $\mathbf{0}$. $\mathbf{A}$ is called K-$\mathbf{0}$-*regular* if $[0]_\theta = [0]_\phi$ implies $\theta = \phi$ for all $\theta, \phi \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$; K is said to be K-$\mathbf{0}$-*regular* if every member of K is K-$\mathbf{0}$-regular.

Let V be a variety, let $\mathbf{A} \in \mathsf{V}$ and let $\mathsf{K} \subsetneq \mathsf{V}$ be a fixed subquasivariety of V. A congruence θ on $\mathbf{A}$ is called a V/K-*congruence* if $\mathbf{A}/\theta \in \mathsf{K}$. The set of all V/K-congruences on $\mathbf{A}$ is denoted $\mathrm{Con}_{\mathsf{V}/\mathsf{K}}\,\mathbf{A}$ and yields an algebraic lattice $\mathbf{Con}_{\mathsf{V}/\mathsf{K}}\,\mathbf{A}$ under inclusion [88, Section 4.2.1, p. 80]. We say $\mathbf{A}$ is V/K-*congruence distributive* if $\mathbf{Con}_{\mathsf{V}/\mathsf{K}}\,\mathbf{A}$ is a distributive lattice; V is V/K-*congruence distributive* if every member of V is V/K-congruence distributive. We say $\mathbf{A}$ has the V/K-*congruence extension property* if for any $\mathbf{B} \in \mathbf{S}(\mathbf{A})$ and any V/K-congruence θ of $\mathbf{B}$, there is a V/K-congruence ϕ of $\mathbf{A}$ such that $\phi \cap (B \times B) = \theta$; V has the V/K-*congruence extension property* if every member of V has the V/K-congruence extension property. Suppose $\mathbf{A}$ is with $\mathbf{0}$. $\mathbf{A}$ is called V/K-$\mathbf{0}$-*regular* if $[0]_\theta = [0]_\phi$ implies $\theta = \phi$ for all $\theta, \phi \in \mathrm{Con}_{\mathsf{V}/\mathsf{K}}\,\mathbf{A}$; V is said to be V/K-$\mathbf{0}$-*regular* if every member of V is V/K-$\mathbf{0}$-regular.

Remark 1.2.7. Let K be a class of similar algebras and let $\mathbf{A} := \langle A; \cdots \rangle$ be an algebra of the same type. In some recent literature in algebraic logic a 'relative congruence' is a congruence $\theta \in \operatorname{Con}\mathbf{A}$ such that $\mathbf{A}/\theta \in \mathsf{K}$: see for instance Blok and Raftery [38] or Ferreirim [88]. Although this notion of 'relative congruence' clearly subsumes both the notion of K-congruence and the notion of V/K-congruence as defined above, we explicitly distinguish between K- and V/K-congruences for the sake of clarity in the sequel. ∎

Theorem 1.2.8 (Principle of the Maximal V/K-Homomorphic Image). *(cf. [62, Proposition 1.7]) Let V be a variety, let $\mathbf{A} \in \mathsf{V}$ and let $\mathsf{K} \subseteq \mathsf{V}$ be a fixed subquasivariety of V. Then the intersection ρ of all V/K-congruences on $\mathbf{A}$ exists and is a V/K-congruence. Thus $\mathbf{A}/\rho$ is the* maximal V/K-homomorphic image of $\mathbf{A}$ *in the sense that $\mathbf{A}/\rho \in \mathsf{K}$, and every other homomorphic image $\mathbf{B}$ of $\mathbf{A}$ such that $\mathbf{B} \in \mathsf{K}$ is a homomorphic image of $\mathbf{A}/\rho$.*

Proof. Let V, $\mathbf{A}$ and K be as in the statement of the theorem. Since the V/K-congruences of $\mathbf{A}$ are closed under arbitrary intersection the intersection ρ of all V/K-congruences of $\mathbf{A}$ exists and is itself a V/K-congruence. Let $\mathbf{B}$ be any homomorphic image of $\mathbf{A}$ such that $\mathbf{B} \in \mathsf{K}$. By the homomorphism theorem [55, Theorem II§6.12], $\mathbf{B} \cong \mathbf{A}/\theta$ for some $\theta \in \operatorname{Con}\mathbf{A}$. By hypothesis, $\mathbf{A}/\theta \in \mathsf{K}$. Thus θ is a V/K-congruence, and so $\rho \subseteq \theta$ by definition of ρ. But then $\mathbf{A}/\rho$ is a homomorphic image of $\mathbf{A}/\theta$ (by [55, Theorem II§6.8; Exercise II§6.6]), so $\mathbf{A}/\rho$ is a homomorphic image of $\mathbf{B}$. ∎

Let V be a variety, let $\mathbf{A} \in \mathsf{V}$ and let $\mathsf{K} \subseteq \mathsf{V}$ be a fixed subquasivariety of V. Remark 1.2.7 notwithstanding, we invariably describe the maximal V/K-homomorphic image of $\mathbf{A}$ as the *maximal K-homomorphic image of* $\mathbf{A}$, in keeping with the spirit of the existing literature: see for instance Clifford and Preston [62, p. 18].

1.2.9. Algebraic Logic.[2] Let $\mathcal{L}$ be a language. A pair $\langle \Gamma, \varphi \rangle$, where Γ is a finite set of $\mathcal{L}$-formulas and φ is an $\mathcal{L}$-formula, is called an (*inference*) *rule* (*over* $\mathcal{L}$). An *axiom* is an inference rule of the form $\langle \varnothing, \varphi \rangle$; we invariably identify an axiom $\langle \varnothing, \varphi \rangle$ with the $\mathcal{L}$-formula φ. Let $Ax \cup Ir$ be a set of axioms

[2]The material in this subsection is adapted from van Alten [229, Chapter 0].

and inference rules over $\mathcal{L}$ and let $\Delta \cup \{\psi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$. A *derivation of* ψ *from* Δ (*with respect to* $Ax \cup Ir$) is a non-empty finite sequence $\varphi_1, \ldots, \varphi_n$ of $\mathcal{L}$-formulas such that $\varphi_n = \psi$ and for $i = 1, \ldots, n$, one of the following conditions holds: (i) $\varphi_i \in \Delta$; (ii) φ_i is a substitution instance of an axiom; or (iii) there exists an inference rule $\langle \Gamma, \chi \rangle$ of $Ax \cup Ir$ and a substitution σ such that $\varphi_i = \sigma(\chi)$ and $\sigma(v) \in \{\varphi_1, \ldots, \varphi_{i-1}\}$ for each $v \in \Gamma$. A *deductive* or *Hilbert system* $\mathbb{S}$ (*over* $\mathcal{L}$) is a pair $\langle \mathcal{L}, \vdash_{\mathbb{S}} \rangle$ where the binary relation $\vdash_{\mathbb{S}}$: $\mathbb{P}(\mathrm{Fm}_{\mathcal{L}}) \to \mathrm{Fm}_{\mathcal{L}}$ is defined by $\Gamma \vdash_{\mathbb{S}} \varphi$ iff φ is derivable from Γ with respect to $Ax \cup Ir$. The relation $\vdash_{\mathbb{S}}$ is called the *consequence relation* of $\mathbb{S}$, and a deductive system is sometimes identified with its consequence relation. Typical examples of deductive systems include $\mathbb{CPC}$, the classical propositional calculus, and $\mathbb{IPC}$, the intuitionistic propositional calculus.

Let $\mathbb{S}$ be a deductive system over a language $\mathcal{L}$ determined by a set $Ax \cup Ir$ of axioms and inference rules. The set $Ax \cup Ir$ is called an *axiomatisation* of $\mathbb{S}$ and the axioms and inference rules in $Ax \cup Ir$ are called the *axioms* and *inference rules* of $\mathbb{S}$, respectively. Clearly a Hilbert system may have more than one axiomatisation. A Hilbert system for which there exists a finite axiomatisation is said to be *finitely axiomatisable*. For an inference rule $\langle \Gamma, \varphi \rangle$ of $\mathbb{S}$, we usually write $\Gamma \vdash_{\mathbb{S}} \varphi$; we also write $\vdash_{\mathbb{S}} \varphi$ for $\varnothing \vdash_{\mathbb{S}} \varphi$. An $\mathcal{L}$-formula φ for which $\vdash_{\mathbb{S}} \varphi$ is called a *theorem* of $\mathbb{S}$. We adopt the following conventions concerning sets of $\mathcal{L}$-formulas Γ, Δ and $\mathcal{L}$-formulas $\varphi_1, \ldots, \varphi_n, \psi$:

$$
\begin{array}{rll}
\varphi_1, \ldots, \varphi_n \vdash_{\mathbb{S}} \psi & \text{abbreviates} & \{\varphi_1, \ldots, \varphi_n\} \vdash_{\mathbb{S}} \psi; \\
\Gamma, \varphi \vdash_{\mathbb{S}} \psi & \text{abbreviates} & \Gamma \cup \{\varphi\} \vdash_{\mathbb{S}} \psi; \\
\Gamma \vdash_{\mathbb{S}} \Delta & \text{abbreviates} & \Gamma \vdash_{\mathbb{S}} \varphi \quad \text{for all } \varphi \in \Delta; \\
\Gamma \dashv\vdash_{\mathbb{S}} \Delta & \text{abbreviates} & \Gamma \vdash_{\mathbb{S}} \Delta \text{ and } \Delta \vdash_{\mathbb{S}} \Gamma.
\end{array}
$$

For sets of $\mathcal{L}$-formulas $\Gamma_1, \ldots, \Gamma_n, \Delta$ and $\mathcal{L}$-formulas $\varphi_1, \ldots, \varphi_n, \psi$, we also write:

$$
\frac{\Gamma_1 \vdash_{\mathbb{S}} \varphi_1 \quad \cdots \quad \Gamma_n \vdash_{\mathbb{S}} \varphi_n}{\Delta \vdash_{\mathbb{S}} \psi}
$$

as shorthand notation for '$\Gamma_1 \vdash_{\mathbb{S}} \varphi_1$ and ... and $\Gamma_n \vdash_{\mathbb{S}} \varphi_n$ implies $\Delta \vdash_{\mathbb{S}} \psi$'; by abuse of language we call such a metalogical implication a *rule* of $\mathbb{S}$.

Let $\mathbb{S}$ be a deductive system over a language $\mathcal{L}$. For any $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$ and $\Gamma, \Delta \subseteq \mathrm{Fm}_{\mathcal{L}}$, the consequence relation $\vdash_{\mathbb{S}}$ is easily seen to satisfy the following three conditions:

1. $\varphi \in \Gamma$ implies $\Gamma \vdash_{\mathbb{S}} \varphi$;
2. $\Gamma \vdash_{\mathbb{S}} \varphi$ and $\Gamma \subseteq \Delta$ implies $\Delta \vdash_{\mathbb{S}} \varphi$;
3. $\Gamma \vdash_{\mathbb{S}} \varphi$ and $\Delta \vdash_{\mathbb{S}} \psi$ for each $\psi \in \Gamma$ implies $\Delta \vdash_{\mathbb{S}} \varphi$.

Moreover, $\vdash_{\mathbb{S}}$ is *finitary* in the sense that:

4. $\Gamma \vdash_{\mathbb{S}} \varphi$ implies $\Delta \vdash_{\mathbb{S}} \varphi$ for some finite $\Delta \subseteq \Gamma$;

and $\vdash_{\mathbb{S}}$ is also *structural* in the sense that:

5. $\Gamma \vdash_{\mathbb{S}} \varphi$ implies $\sigma[\Gamma] \vdash_{\mathbb{S}} \sigma(\varphi)$ for every substitution $\sigma \in \Sigma$.

Conversely, the Łoś-Suszko theorem [153] (see also [238, Chapter 3§2]) asserts that every relation between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas satisfying conditions (1)–(5) above is the consequence relation of some Hilbert system $\mathbb{S}$ over $\mathcal{L}$. Without loss of generality, therefore, a deductive system $\langle \mathcal{L}, \vdash_{\mathbb{S}} \rangle$ may be defined as a relation $\vdash_{\mathbb{S}}: \mathbb{P}(\mathrm{Fm}_{\mathcal{L}}) \to \mathrm{Fm}_{\mathcal{L}}$ satisfying (1)–(5) above; defining axioms and inference rules need not be assumed. The above remarks notwithstanding, for binary $\mathcal{L}$-formulas $\wedge, \vee, \to, \Delta$, a unary $\mathcal{L}$-formula $\neg$ and an n-ary logical connective ϖ of $\mathcal{L}$, we identify and earmark the following rules for use in the sequel [238, Section 2.3.1]:

(AD)	$p, q \vdash_{\mathbb{S}} p \wedge q$	(Adjunction)
(SP)	$\left.\begin{array}{r} p \wedge q \vdash_{\mathbb{S}} p \\ p \wedge q \vdash_{\mathbb{S}} q \end{array}\right\}$	(Simplification)
(AT)	$\left.\begin{array}{r} p \vdash_{\mathbb{S}} p \vee q \\ p \vdash_{\mathbb{S}} q \vee p \end{array}\right\}$	(Addition)

(SM)	$\dfrac{\Gamma, \varphi \vdash_{\mathbb{S}} \chi \quad \Gamma, \psi \vdash_{\mathbb{S}} \chi}{\Gamma, \varphi \vee \psi \vdash_{\mathbb{S}} \chi}$	(Summation)
(MP)	$p, p \to q \vdash_{\mathbb{S}} q$	(Modus Ponens)
(DT)	$\dfrac{\Gamma, \varphi \vdash_{\mathbb{S}} \psi}{\Gamma \vdash_{\mathbb{S}} \varphi \to \psi}$	(Deduction Rule)
(CN)	$p, \neg p \vdash_{\mathbb{S}} q$	(Contradiction)
(RA$_J$)	$\dfrac{\Gamma, \varphi \vdash_{\mathbb{S}} \neg \varphi}{\Gamma \vdash_{\mathbb{S}} \neg \varphi}$	(Weak Reductio ad Absurdum)
(R)	$\vdash_{\mathbb{S}} \varphi \Delta \varphi$	(Reflexivity)
(S)	$\varphi \Delta \psi \vdash_{\mathbb{S}} \psi \Delta \varphi$	(Symmetry)
(T)	$\varphi \Delta \psi, \psi \Delta \chi \vdash_{\mathbb{S}} \varphi \Delta \chi$	(Transitivity)
(Δ-MP)	$\varphi, \varphi \Delta \psi \vdash_{\mathbb{S}} \psi$	(Δ-Detachment)
(CP-ϖ)	$\varphi_1 \Delta \psi_1, \ldots, \varphi_n \Delta \psi_n \vdash_{\mathbb{S}} \varpi(\varphi_1, \ldots, \varphi_n) \Delta \varpi(\psi_1, \ldots, \psi_n)$	(ϖ-Compatibility)

A binary $\mathcal{L}$-formula $\wedge$ [$\vee$; $\to$] is a *conjunction* [*disjunction*; *conditional*] for $\mathbb{S}$ if the entailments (AD) and (SP) [(AT) and (SM); (MP) and (DT)] are satisfied by $\mathbb{S}$. A binary $\mathcal{L}$-formula Δ is called a *G-identity* for $\mathbb{S}$ if the the entailments (R), (S), (T), (Δ-MP) and (CP-ϖ) (for every n-ary logical connective $\varpi \in \mathcal{L}$) are satisfied by $\mathbb{S}$. A unary $\mathcal{L}$-formula $\neg$ is a *weak negation* for $\mathbb{S}$ if the entailments (CN) and (RA$_J$) are satisfied by $\mathbb{S}$. Given a conjunction $\wedge$ and a conditional $\to$ for $\mathbb{S}$, a *biconditional* for $\mathbb{S}$ is the derived connective:

$$p \leftrightarrow q := (p \to q) \wedge (q \to p).$$

Let $\mathbb{S}$ be a deductive system over a language $\mathcal{L}$. An *extension* of $\mathbb{S}$ is any system $\mathbb{S}' := \langle \mathcal{L}, \vdash_{\mathbb{S}'} \rangle$ over the same language such that $\Gamma \vdash_{\mathbb{S}} \varphi$ implies $\Gamma \vdash_{\mathbb{S}'} \varphi$

for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$. $\mathbb{S}'$ is said to be *axiomatic* if it obtained by adjoining new axioms to $\mathbb{S}$ only (that is, if the inference rules of $\mathbb{S}$ are left fixed). For a sublanguage $\mathcal{L}' \subseteq \mathcal{L}$, let $\vdash_{\mathbb{S}'}$ denote the restriction of $\vdash_{\mathbb{S}}$ to $\mathcal{L}'$ in the sense that $\Gamma \vdash_{\mathbb{S}'} \varphi$ iff $\Gamma \vdash_{\mathbb{S}} \varphi$ and $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}'}$. The resulting deductive system $\mathbb{S}' := \langle \mathcal{L}', \vdash_{\mathbb{S}'} \rangle$ is called the *$\mathcal{L}'$-fragment* of $\mathbb{S}$.

1.3 Bands, Monoids and Semilattices

Various classes of bands, monoids and semilattices play an important role in this thesis. Here we briefly review and summarise some of the theory of these classes that will be needed in the sequel.

1.3.1. Semigroups. A *groupoid* is an algebra $\langle A;\ \cdot \rangle$ of type $\langle 2 \rangle$. A groupoid whose operation is associative is a *semigroup*. The binary operation $\cdot$ of a semigroup $\langle A;\ \cdot \rangle$ is called *multiplication*; given $a, b \in A$, the multiplication $a \cdot b$ is (informally) written ab if the context is clear. Given a semigroup $\langle A;\ \cdot \rangle$ and $a_1, \ldots, a_n \in A$, the product $a_1 a_2 \cdots a_n$ is defined inductively by:

$$a_1 a_2 \cdots a_n := \begin{cases} a_1 & \text{if } n = 1 \\ (a_1 a_2 \cdots a_{n-1}) a_n & \text{otherwise.} \end{cases}$$

An easy proof by induction [103, Proposition I§2.1] shows this definition of product has an unambiguous meaning.

Example 1.3.2. [87, Section 2] On any non-empty set A two semigroups can always be constructed, *viz.*:

1. The *left zero* semigroup $\mathbf{A}_L$ on A, with multiplication $a \cdot^{\mathbf{A}_L} b := a$ for any $a, b \in A$; and

2. The *right zero* semigroup $\mathbf{A}_R$ on A, with multiplication $a \cdot^{\mathbf{A}_R} b := b$ for any $a, b \in A$.

■

A *semigroup with zero* $\langle A;\ \cdot \rangle$ is a semigroup with an element 0 such that $a0 = 0 = 0a$ for all $a \in A$; the element 0 is called the *zero* of $\langle A;\ \cdot \rangle$. A *semigroup with identity* is defined dually. By abuse of language and notation we will often confuse a semigroup with zero $\mathbf{A}$ with the algebra $\langle A;\ \cdot, 0\rangle$ obtained from $\mathbf{A}$ by enriching the type of $\mathbf{A}$ with a new nullary operation symbol $\mathbf{0}$ whose canonical interpretation on $\langle A;\ \cdot, 0\rangle$ is $0 \in A$, where 0 is the zero of $\mathbf{A}$. Like remarks apply concerning semigroups with identity. It is always possible to adjoin a zero to a semigroup $\langle A;\ \cdot\rangle$. Let $0 \notin A$ and define:

$$a \cdot^0 b := \begin{cases} a \cdot b & \text{if } a, b \in A \\ 0 & \text{otherwise.} \end{cases}$$

The resulting algebra $\langle A \cup \{0\};\ \cdot^0\rangle$ is a semigroup with zero and is called a *semigroup with a zero adjoined.*

1.3.3. Polrims and Pocrims. A *monoid* is a semigroup with identity. Let $\langle A;\ \oplus, 0\rangle$ be a monoid whose identity element 0 is the least element of a partial order $\leq$ on A, compatible with the binary operation $\oplus$ in the sense that $a \leq b$ implies both $a \oplus c \leq b \oplus c$ and $c \oplus a \leq c \oplus b$ for any $a, b, c \in A$. If for every $a, b \in A$ there is a least element c (denoted $a \dot{-} b$ and called the (*left*) *residual* of a and b) of A such that $a \leq c \oplus b$, then the algebra $\mathbf{A} := \langle A;\ \oplus, \dot{-}, 0\rangle$ is a *partially ordered left residuated integral monoid* (briefly, *polrim*) [38, pp. 81–82], [229, p. 16]. Polrims arise naturally in algebraic logic and have been considered by Raftery and van Alten [192] and van Alten [229] among others. A *partially ordered commutative residuated integral monoid*, or *pocrim* for short, is a polrim $\langle A;\ \oplus, \dot{-}, 0\rangle$ whose monoid operation is commutative [229, Example 1.5]. Pocrims also arise naturally in algebraic logic and have been studied by several authors, including Blok and Raftery [39], Fleischer [90], Higgs [109] and Iséki [123]. See also Bosbach [45], Ono and Komori [176], Raftery and van Alten [192] and van Alten [229].

1.3.4. Dually Relatively Pseudocomplemented Semilattices. A *join* or *upper semilattice* is a poset $\langle A;\ \leq\rangle$ for which $\text{lub}\{a, b\}$ exists for all $a, b \in A$. Let $\langle A;\ \vee, 0\rangle$ be an arbitrary join semilattice with least element and let $a, b \in A$. Recall that the *dual relative pseudocomplement* $a * b$ *of* b *with respect to* a is (if it exists) the unique element of A satisfying $a \leq b \vee c$

iff $a * b \leq c$ for all $c \in A$ [38, Example II]. If $a * b$ exists for all $a, b \in A$, then $\langle A;\ \vee, 0\rangle$ is said to be *dually relatively pseudocomplemented.* In the case where the operation of dual relative pseudocomplementation is distinguished, the resulting algebra $\langle A;\ \vee, *, 0\rangle$ is called a *dual Brouwerian semilattice* [129, Definition III§4.1]. Dually relatively pseudocomplemented semilattices were introduced (in dually isomorphic form) by Birkhoff in [23, pp. 147–149], while dual Brouwerian semilattices have been studied by Köhler in [135] and by Nemitz [175] in dually isomorphic form under the name *implicative semilattices.* A result of Blok and Pigozzi shows that the class of dual Brouwerian semilattices is precisely the class of all pocrims for which the monoid operation is idempotent [34, Corollary 1.23] (see also Cornish [67] and Blok and Raftery [39, p. 294]). It is folklore that the lattice of varieties of dual Brouwerian semilattices has a unique atom [39, p. 295], namely the class of all dual Brouwerian semilattices $\mathbf{A}$ such that $\mathbf{A} \models x * (y * x) \approx x$: in this case the semilattice ordering on $\mathbf{A}$ is a lattice ordering and $a * b$ is the complement of b in the interval $[0, a \vee b]$ for any $a, b \in A$. Thus the unique atom in the lattice of varieties of dual Brouwerian semilattices is termwise definitionally equivalent to the variety GBA of *generalised Boolean algebras*, namely the class of all relatively complemented distributive lattices with zero in which the operations of zero and relative complementation are distinguished. For notational purposes, from this point forth we will always denote the operation of relative complementation in a generalised Boolean algebra by a/b.

An algebra $\langle A;\ \wedge, \vee, 0\rangle$ of type $\langle 2, 2, 0\rangle$ is called a *dually relatively pseudocomplemented lattice* if: (i) the reduct $\langle A;\ \wedge, \vee\rangle$ is a lattice; and (ii) the reduct $\langle A;\ \vee, 0\rangle$ is dually relatively pseudocomplemented. A *dual Brouwerian lattice* is an algebra $\langle A;\ \wedge, \vee, *, 0\rangle$ of type $\langle 2, 2, 2, 0\rangle$ such that the reduct $\langle A;\ \wedge, \vee, 0\rangle$ is a lattice with zero and the reduct $\langle A;\ \vee, *, 0\rangle$ is a dual Brouwerian semilattice. Such a lattice is always distributive: see Curry [75, Theorem 4§C.5]. Dual Brouwerian lattices have been studied extensively in the literature: see for instance McKinsey and Tarski [161, 162] where they are studied in dually isomorphic form under the name *Brouwerian lattices.*

1.3.5. Pseudocomplemented Semilattices. A *meet* or *lower semilattice* is a poset $\langle A;\ \leq\rangle$ for which $\mathrm{glb}\{a,b\}$ exists for all $a,b \in A$. Let $\langle A;\ \wedge, 0\rangle$ be a meet semilattice with zero. An element $a^* \in A$ is called a *pseudocomplement* of $a \in A$ if $a \wedge a^* = 0$, and $a \wedge b = 0$ implies $b \leq a^*$ [101, p. 58]. An element can have at most one pseudocomplement. An algebra $\langle A;\ \wedge, {}^*, 0\rangle$ of type $\langle 2,1,0\rangle$ is called a *pseudocomplemented semilattice* if the reduct $\langle A;\ \wedge, 0\rangle$ is a meet semilattice with zero and for any $a \in A$, the pseudocomplement of a exists and is a^*; observe this definition implies any pseudocomplemented semilattice has a greatest element $1 := 0^*$ with respect to the underlying semilattice order. Pseudocomplemented semilattices have been studied by several authors, including Frink [96], Jones [128] and Ribenboim [200], to whom the following theorem is due.

Theorem 1.3.6. *[200] An algebra $\langle A;\ \wedge, {}^*, 0\rangle$ of type $\langle 2,1,0\rangle$ is a pseudocomplemented semilattice iff the reduct $\langle A;\ \wedge, 0\rangle$ is a meet semilattice with zero and the following identities are satisfied:*

$$x \wedge x^* \approx 0 \tag{1.8}$$

$$x \wedge (x \wedge y)^* \approx x \wedge y^* \tag{1.9}$$

$$x \wedge \mathbf{0}^* \approx x \tag{1.10}$$

$$\mathbf{0}^{**} \approx \mathbf{0}. \tag{1.11}$$

Thus the class PCSL *of pseudocomplemented semilattices is a variety.*

Let $\mathbf{B} := \langle B;\ \wedge, \vee, {}', 0, 1\rangle$ be a non-trivial Boolean algebra with least element 0 and unit element 1. Let $\check{B} := B \cup \{m\}$ where $a < m$ for all $a \in B$. Then $\langle \check{B};\ \leq\rangle$ is a meet semilattice with zero. Moreover, $\check{B}$ is pseudocomplemented: for any $b \in \check{B}$,

$$b^* = \begin{cases} m & \text{if } b = 0 \\ b' & \text{if } b \in B,\ b \neq 0 \\ 0 & \text{if } b = m. \end{cases}$$

We denote the resulting pseudocomplemented semilattice by $\check{\mathbf{B}}$.

Theorem 1.3.7. *[128, Theorem 1.2] A non-trivial pseudocomplemented semi-*

lattice is subdirectly irreducible iff it is isomorphic to $\breve{\mathbf{B}}$ *for some Boolean algebra* $\mathbf{B}$.

Theorem 1.3.8. *[128, Theorem 11.1] The class of pseudocomplemented semilattices is generated (as a variety) by the 3-element chain* $\mathbf{3}$ *(considered as a pseudocomplemented semilattice).*

Corollary 1.3.9. *[128, Theorem 11.1] The lattice of varieties of pseudocomplemented semilattices is a 3-element chain. The only non-trivial subvariety of the variety of pseudocomplemented semilattices is the class* $\{\mathbf{A} \in \mathsf{PCSL} : \mathbf{A} \models x^{**} \approx x\}$*, and this class is termwise definitionally equivalent to the variety of Boolean algebras.*

Let $\mathbf{A}$ be a pseudocomplemented semilattice. The *skeleton* $\mathrm{S}(\mathbf{A})$ of $\mathbf{A}$ is the set $\{a^* : a \in A\}$ and the *dense set* $\mathrm{D}(\mathbf{A})$ of $\mathbf{A}$ is the set $\{a \in A : a^* = 0\}$. The following properties of $\mathrm{S}(\mathbf{A})$ and $\mathrm{D}(\mathbf{A})$ are essentially well known [14, p. 153], [101, p. 59]: (i) $\{0,1\} \subseteq \mathrm{S}(\mathbf{A})$; (ii) $1 \in \mathrm{D}(\mathbf{A})$; (iii) $a \in \mathrm{S}(\mathbf{A})$ iff $a = a^{**}$ for any $a \in A$; and (iv) $a, b \in \mathrm{S}(\mathbf{A})$ implies $a \wedge b \in \mathrm{S}(\mathbf{A})$.

Theorem 1.3.10 (Glivenko-Frink Theorem). *[101, Theorem I§6.4] Let* $\mathbf{A}$ *be a pseudocomplemented semilattice with skeleton* $\mathrm{S}(\mathbf{A})$*. Then the underlying partial ordering of* $\mathbf{A}$ *partially orders* $\mathrm{S}(\mathbf{A})$ *and makes* $\mathrm{S}(\mathbf{A})$ *into a Boolean lattice. For* $a, b \in \mathrm{S}(\mathbf{A})$ *we have* $a \wedge b \in \mathrm{S}(\mathbf{A})$*, and the join in* $\mathrm{S}(\mathbf{A})$ *is described by:*

$$a \vee b = (a^* \wedge b^*)^*.$$

An algebra $\langle A; \wedge, \vee, {}^*, 0\rangle$ of type $\langle 2,2,1,0\rangle$ is called a *distributive lattice with pseudocomplementation* if: (i) the reduct $\langle A; \wedge, \vee\rangle$ is a distributive lattice; and (ii) the reduct $\langle A; \wedge, {}^*, 0\rangle$ is a pseudocomplemented semilattice. Clearly the class DLPC of all distributive lattices with pseudocomplementation is a variety. Distributive lattices with pseudocomplementation have been extensively studied by many authors; standard references include Grätzer [101, Chapter 3] and Balbes and Dwinger [14, Chapter VIII].

1.3.11. Bands. An element e of a semigroup $\langle A; \cdot\rangle$ is an *idempotent* if $e^2 = e$; $\langle A; \cdot\rangle$ is an *idempotent semigroup*, or a *band*, if all its elements are idempotent. The study of bands dates back to the 1950s and the papers of Kimura [133, 134], McLean [163], and Yamada and Kimura [244]. A detailed development of the theory of bands may be found in Petrich [180, Chapter II]; see also Howie [111, Sections 4.4–4.6].

Example 1.3.12. (*cf.* [111, Theorem 1.1.3]) A semigroup $\langle A;\ \cdot\rangle$ satisfying $aba = a$ for any $a, b \in A$ is called a *rectangular band.* Rectangular bands are precisely the bands satisfying the identity $x \cdot y \cdot z \approx x \cdot z$. Any such semigroup is isomorphic to a semigroup of the form $\langle B \times C;\ \cdot\rangle$, where B and C are non-empty sets and multiplication on $B \times C$ is defined as:

$$(b_1, c_1)(b_2, c_2) := (b_1, c_2)$$

for any $b_1, b_2 \in B$ and $c_1, c_2 \in C$. The name 'rectangular' stems from this last property: if (b_1, c_1) and (b_2, c_2) are construed as vertices of a rectangle in the Cartesian plane, the products $(b_1, c_1)(b_2, c_2) = (b_1, c_2)$ and $(b_2, c_2)(b_1, c_1) = (b_2, c_1)$ comprise the remaining two vertices of the rectangle. ∎

Let $\langle A;\ \cdot\rangle$ be a band. The *Green's quasiorders* on $\langle A;\ \cdot\rangle$ are the relations $\preceq_{\mathcal{L}}$, $\preceq_{\mathcal{R}}$ and $\preceq_{\mathcal{D}}$ defined respectively by [208, Section 0]:

$$a \preceq_{\mathcal{L}} b \text{ iff } ab = a, \quad a \preceq_{\mathcal{R}} b \text{ iff } ba = a, \quad a \preceq_{\mathcal{D}} b \text{ iff } aba = a$$

for any $a, b \in A$; these relations are all proper quasiorderings. Each of the relations $\preceq_{\mathcal{R}}$ and $\preceq_{\mathcal{L}}$ is contained in $\preceq_{\mathcal{D}}$, and $\preceq_{\mathcal{D}}$ is called the *natural quasiordering* [208, Proposition 1]. In the sequel we write simply $\preceq$ for $\preceq_{\mathcal{D}}$ if there is no danger of confusion. The relation $\leq_{\mathcal{H}}$ defined on $\langle A;\ \cdot\rangle$ by:

$$a \leq_{\mathcal{H}} b \quad \text{iff} \quad a \preceq_{\mathcal{L}} \text{ and } \preceq_{\mathcal{R}} b \quad \text{iff} \quad ab = a = ba$$

for any $a, b \in A$ is the *natural partial order* [103, Proposition II§1.1]; it is properly a partial ordering and is denoted by $\leq$ hereafter if the context is clear. The following lemma is folklore.

Lemma 1.3.13. *For a band* $\mathbf{A} := \langle A;\ \cdot\rangle$ *and a fixed* $a \in A$ *the following assertions hold:*

1. $(a] = \{c : c \leq a\} = aAa$ (*for* $aAa := \{ada : d \in A\}$);

2. $(a]$ *is a subuniverse of* A;

3. *The* principal subalgebra $(\mathbf{a}] := \langle (a];\ \cdot^{\mathbf{A}|_{(a]}} \rangle$ generated by a *is a monoid.*

Proof. For (1), let $b \in aAa$. Then $b = ada$ for some $d \in A$, whence $a(ada) = ada = (ada)a$. Thus $ada \leq a$, and $b = ada \in \{c : c \leq a\}$. Conversely, let $b \in \{c : c \leq a\}$. Then $b \leq a$, whence $b = bb = abba = aba \in aAa$. Thus $aAa = \{c : c \leq a\}$. Items (2) and (3) now follow trivially. ∎

The *Green's relations* $\mathcal{L}, \mathcal{R}$ and $\mathcal{D}$ on a band $\langle A;\ \cdot \rangle$ are the symmetric parts of the Green's quasiorders, namely [208, Section 0]:

$$\begin{aligned} \mathcal{L} &:= \{(a,b) : a \preceq_{\mathcal{L}} b \text{ and } b \preceq_{\mathcal{L}} a\}, \\ \mathcal{R} &:= \{(a,b) : a \preceq_{\mathcal{R}} b \text{ and } b \preceq_{\mathcal{R}} a\}, \\ \mathcal{D} &:= \{(a,b) : a \preceq b \text{ and } b \preceq a\}. \end{aligned}$$

Each of $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{D}$ is an equivalence relation. Note that since $\leq$ is a partial ordering, no non-trivial equivalence is induced by $\leq$ on a band.

Theorem 1.3.14 (Clifford-McLean Theorem). *[111, Theorem 4.4.1] Each $\mathcal{D}$-class of a band* $\mathbf{A}$ *is a maximal rectangular subalgebra of* $\mathbf{A}$. *Moreover, the equivalence $\mathcal{D}$ is a congruence on* $\mathbf{A}$ *which induces the maximal semilattice homomorphic image* $\mathbf{A}/\mathcal{D}$ *of* $\mathbf{A}$.

1.3.15. Left [Right] Normal Bands. A band is called *regular* if it satisfies the identity $x \cdot y \cdot x \cdot z \cdot x \approx x \cdot y \cdot z \cdot x$. For a band $\mathbf{A}$, the following are equivalent [180, Proposition II.3.6], [133, Theorem 4, Corollary 4]: (i) $\mathbf{A}$ is regular; (ii) the relations $\mathcal{L}$ and $\mathcal{R}$ are congruences on $\mathbf{A}$; (iii) $\mathbf{A}$ decomposes uniquely to within isomorphism as the fibre product $\mathbf{A}/\mathcal{L} \times_{\mathbf{A}/\mathcal{D}} \mathbf{A}/\mathcal{R}$. A band is *normal* if the identity $x \cdot y \cdot z \cdot x \approx x \cdot z \cdot y \cdot x$ holds, and a normal band is regular [244, Lemma 1]. The following lemma is folklore.

Lemma 1.3.16. *For any band* $\mathbf{A}$ *and* $a, b, c, \in A$, *the following are equivalent:*

1. $\mathbf{A}$ *is normal;*

2. *For any $a \in A$, the principal subalgebra $(\mathbf{a}]$ generated by a is a semilattice;*

3. *If $a \leq b$ then $ac \leq bc$ and $ca \leq cb$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in A$ and let $b, c \in (a]$. Since the principal subalgebra $(\mathbf{a}]$ is a monoid with identity a (by Lemma 1.3.13(3)), we deduce $bc = abca = acba = cb$ by normality.

(2) $\Rightarrow$ (3) Let $a \leq b$ and let $c \in A$. Since $bab \leq b$ and $bcb \leq b$ we deduce $bcac = bc(bab)c = (bcb)(bab)c = (bab)(bcb)c = a(bcb)c = (ab)cbc = acbc$ by (2). But $acbc = (ab)cbc = a(bc)bc = a(bc) = (ab)c = ac$ and so $ac \leq bc$. The inclusion $ca \leq cb$ is handled similarly.

(3) $\Rightarrow$ (1) Assume (3) holds and notice this is equivalent to the implication:

$$a \leq c \quad \text{and} \quad b \leq d \quad \text{implies} \quad ab \leq cd \tag{1.12}$$

for any $a, b, c, d \in A$. Let $a, b, c \in A$. From (1.12), $acba \leq a$ and $abca \leq abca$ we have $acba = acba(abca)acba \leq a(abca)a = abca$, just because $acba\mathcal{D}abca$. Similarly, $abca \leq acba$ so $acba = abca$ follows. ∎

A band with zero $\mathbf{A}$ is *locally Boolean* if for every $a \in A$ the principal subalgebra $(\mathbf{a}]$ is a Boolean lattice with respect to the natural partial ordering; notice this implies $\mathbf{A}$ is normal by Lemma 1.3.16. The following useful technical result is due to the author and the author's Ph.D. supervisor.

Lemma 1.3.17. *Let $\mathbf{A}$ be a locally Boolean band. Then $\mathbf{A}/\mathcal{D}$ is the maximal semilattice with zero homomorphic image of $\mathbf{A}$. Moreover, every principal order ideal of $\mathbf{A}/\mathcal{D}$ is a Boolean sublattice of $\mathbf{A}/\mathcal{D}$ under the semilattice partial ordering, so $\mathbf{A}/\mathcal{D}$ is locally Boolean.*

Proof. By the Clifford-McLean theorem, $\mathbf{A}/\mathcal{D}$ is the maximal semilattice with zero homomorphic image of $\mathbf{A}$. Denote by ν the canonical epimorphism mapping $\mathbf{A}$ onto $\mathbf{A}/\mathcal{D}$. If $\mathbf{B}$ is a subsemilattice of $\mathbf{A}$ then it is immediate from the definition of $\mathcal{D}$ that the restriction of ν to $\mathbf{B}$ is one-to-one and is thus an isomorphism from $\mathbf{B}$ onto $\nu[\mathbf{B}]$, the image of $\mathbf{B}$ under ν.

Let now I be a principal order ideal of $\mathbf{A}/\mathcal{D}$. Then I is generated by some element $[b]_{\mathcal{D}} \in A/\mathcal{D}$ where b is some element of A. For any such element b let $(\boldsymbol{b}]$ be the principal subalgebra of $\mathbf{A}$ that it generates. Since $(\boldsymbol{b}]$ is a subsemilattice of $\mathbf{A}$, $(\boldsymbol{b}]$ is isomorphic to $\nu[(\boldsymbol{b}]]$, and hence $\nu[(\boldsymbol{b}]]$ is a Boolean lattice. But $[c]_{\mathcal{D}} \in \nu[(b]]$ iff $[c]_{\mathcal{D}} \in (\nu(b)]_{\mathbf{A}/\mathcal{D}}$ iff $[c]_{\mathcal{D}} \leq^{\mathbf{A}/\mathcal{D}} [b]_{\mathcal{D}}$, so $\nu[(b]] = I$, which completes the proof. ∎

A band is *left regular* [*right regular*] if the identity $x\cdot y\cdot x \approx x\cdot y$ [$x\cdot y\cdot x \approx y\cdot x$] is satisfied. For a band $\mathbf{A}$, the following are equivalent [180, Proposition II.3.12]: (i) $\mathbf{A}$ is left regular [right regular]; (ii) $\mathcal{L} = \mathcal{D}$ [$\mathcal{R} = \mathcal{D}$]. Trivially a left regular [right regular] band is regular. A band is *left normal* [*right normal*] if the identity $x \cdot y \cdot z \approx x \cdot z \cdot y$ [$x \cdot y \cdot z \approx y \cdot x \cdot z$] is satisfied. Clearly a left normal [right normal] band is both normal and left regular [right regular] [244, Lemma 1]. The variety lnB of left normal bands has been studied by several authors in the literature, including Vagner [227] and Schein [203], to whom the following theorem is due.

Theorem 1.3.18. *[203] Up to isomorphism, the only subdirectly irreducible left normal bands are* $\mathbf{2}$, $\mathbf{2}_L$ *and* $\mathbf{3}_L$, *where* $\mathbf{2}$ *is the one element semilattice with a zero adjoined,* $\mathbf{2}_L$ *is the left normal band on* $\{a, b\}$ *and* $\mathbf{3}_L$ *is the band* $\mathbf{2}_L$ *with a zero adjoined. In symbols,* $\mathsf{lnB}_{\mathrm{SI}} = \{\mathbf{2}, \mathbf{2}_L, \mathbf{3}_L\}$.

In the statement of the following corollary and in the sequel lnB_0 denotes the variety of left normal bands with zero.

Corollary 1.3.19. *Up to isomorphism, the only subdirectly irreducible left normal bands with zero are* $\mathbf{2}$ *and* $\mathbf{3}_L$. *In symbols,* $\mathsf{lnB}_{0\mathrm{SI}} = \{\mathbf{2}, \mathbf{3}_L\}$.

Proof. Let $\langle A;\ \cdot, 0\rangle$ be a subdirectly irreducible left normal band with zero. Its band reduct $\langle A;\ \cdot\rangle$ must also be subdirectly irreducible, since $\langle A;\ \cdot, 0\rangle$ and $\langle A;\ \cdot\rangle$ have the same congruences. Hence $\langle A;\ \cdot\rangle$ must be either $\mathbf{2}$ or $\mathbf{3}_L$. ∎

Example 1.3.20. Let $\mathbf{L}_n := \langle L;\ \cdot, 0\rangle$ be an algebra of cardinality $n+1$ equipped with a distinguished element 0 and a binary operation $\cdot$ defined by:

$$ab := \begin{cases} 0 & \text{if } b = 0 \\ a & \text{otherwise} \end{cases}$$

for any $a, b \in L$. Then $\mathbf{L}_n$ is a left normal band with zero whose only $\mathcal{D}$-equivalence classes are $\{0\}$ and $L - \{0\}$. For every $0 \neq a \in L$, therefore, $(a]$ is a two element Boolean lattice. Hence $\mathbf{L}_n$ is locally Boolean and (by Lemma 1.3.17) $\mathbf{L}_n/\mathcal{D}$ is isomorphic to $\mathbf{2}$, the one element semilattice with a zero adjoined. ∎

1.4 Skew Boolean Algebras and Discriminator Varieties

Discriminator varieties have been called 'the most successful generalisation of Boolean algebras to date, successful because we obtain Boolean product representations' [55, Chapter IV§9, p. 186]. Skew Boolean $\cap$-algebras are a class of structurally enriched non-commutative lattices that enjoy deep connections with (pointed) discriminator varieties. We briefly review some of these connections in this section.

1.4.1. Non-commutative Lattices. Recall from [145] that a *double band* is an algebra $\langle A;\ \wedge, \vee\rangle$ of type $\langle 2, 2\rangle$ such that the reducts $\langle A;\ \wedge\rangle$ and $\langle A;\ \vee\rangle$ are bands; given the associativity of the operations $\wedge$ and $\vee$ we omit parentheses in the sequel where no ambiguity can arise. In view of the remarks of §1.3.11, on any double band $\langle A;\ \wedge, \vee\rangle$ there arise eight Green's quasiorderings: the four quasiorderings $\preceq_{\mathcal{L}}^{\langle A;\,\wedge\rangle}$, $\preceq_{\mathcal{R}}^{\langle A;\,\wedge\rangle}$, $\preceq_{\mathcal{D}}^{\langle A;\,\wedge\rangle}$ and $\leq_{\mathcal{H}}^{\langle A;\,\wedge\rangle}$ on $\langle A;\ \wedge\rangle$; and the four quasiorderings $\preceq_{\mathcal{L}}^{\langle A;\,\vee\rangle}$, $\preceq_{\mathcal{R}}^{\langle A;\,\vee\rangle}$, $\preceq_{\mathcal{D}}^{\langle A;\,\vee\rangle}$ and $\leq_{\mathcal{H}}^{\langle A;\,\vee\rangle}$ on $\langle A;\ \vee\rangle$. A *non-commutative lattice* is a double band $\langle A;\ \wedge, \vee\rangle$ for which at least one of the quasiorders induced by the operation $\vee$ is dual to one of the quasiorders induced by the operation $\wedge$ in the sense that $a \preceq_{\mathcal{G}_1}^{\langle A;\,\wedge\rangle} b$ iff $b \preceq_{\mathcal{G}_2}^{\langle A;\,\vee\rangle} a$ for $\mathcal{G}_1, \mathcal{G}_2 \in \{\mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{H}\}$; this definition is implicit in Leech [145, Section 1].

Remark 1.4.2. Traditionally, a non-commutative lattice has been understood as an algebra $\langle A;\ \wedge, \vee\rangle$ of type $\langle 2, 2\rangle$ whose idempotent and associative operations $\wedge$ and $\vee$ satisfy certain absorption identities. Non-commutative lattices in this sense have a long history and have been studied by many authors; see the survey paper of Leech [150, Section 0] for details. Our motivation for departing from this tradition stems from lattice theory: if $\langle A;\ \wedge, \vee\rangle$ is a lattice, then each of the operations induces a partial ordering on A that dualises the other [145, Section 1]. For a further discussion and justification of the study of non-commutative lattices in the sense of this thesis, see [150, Section 0] and [145, Section 1]. ∎

In this thesis we shall be exclusively concerned with three particular classes of non-commutative lattices: quasilattices, paralattices and (above all) skew lattices.

1.4.3. Quasilattices. A *quasilattice* is a double band $\langle A;\ \wedge, \vee\rangle$ such that the quasiorder $\preceq_{\mathcal{D}}^{\langle A;\ \wedge\rangle}$ and the quasiorder $\preceq_{\mathcal{D}}^{\langle A,\ \vee\rangle}$ dualise in the sense that $a \preceq_{\mathcal{D}}^{\langle A;\ \wedge\rangle} b$ iff $b \preceq_{\mathcal{D}}^{\langle A;\ \vee\rangle} a$ for any $a, b \in A$ [145, Section 1]. In view of this duality we work solely with the quasiorder $\preceq_{\mathcal{D}}^{\langle A;\ \wedge\rangle}$ in the sequel; to simplify notation we write simply $a \preceq b$ for $a \preceq_{\mathcal{D}}^{\langle A;\ \wedge\rangle} b$ for any $a, b \in A$ when no confusion can exist. The relation $\preceq$ is called the *natural quasilattice quasiorder* (*cf.* [149, Section 3.8]). Leech has shown that the class of quasilattices is a variety [145, Section 1], axiomatised relative to the variety of double bands by the identities $x \wedge (y \vee x \vee y) \wedge x \approx x$ and $y \vee (x \wedge y \wedge x) \vee y \approx y$. Leech has also shown that quasilattices satisfy a modified form of the Clifford-McLean theorem: every quasilattice is a lattice of its maximal rectangular subalgebras [145, Corollary 3]. Further information about quasilattices may be found in Leech [145, Sections 1, 2, 3, 4 and 6].

1.4.4. Paralattices. A double band $\langle A;\ \wedge, \vee\rangle$ for which the partial orders $\leq_{\mathcal{H}}^{\langle A;\ \wedge\rangle}$ and $\leq_{\mathcal{H}}^{\langle A;\ \vee\rangle}$ dualise in the sense that $a \leq_{\mathcal{H}}^{\langle A;\ \wedge\rangle} b$ iff $b \leq_{\mathcal{H}}^{\langle A;\ \vee\rangle} a$ for any $a, b \in A$ is called a *paralattice* [145, Section 1]. In view of this duality we work solely with the partial order $\leq_{\mathcal{H}}^{\langle A;\ \wedge\rangle}$ in the sequel, simply writing $a \leq b$ for $a \leq_{\mathcal{H}}^{\langle A;\ \wedge\rangle} b$ when no confusion can arise. The relation $\leq$ is called the *natural paralattice partial order*; see [146, Section 1.1]. Leech has observed that the class of paralattices is a variety [145, Section 1], axiomatised relative to the variety of double bands by the identities $x \wedge (x \vee y \vee x) \approx x \approx (x \vee y \vee x) \wedge x$ and $x \vee (x \wedge y \wedge x) \approx x \approx (x \wedge y \wedge x) \vee x$. A further study of paralattices may be found in [145, Sections 1, 5 and 6].

1.4.5. Skew Lattices. A *skew lattice* $\langle A;\ \wedge, \vee\rangle$ is a double band such that

$\preceq_{\mathcal{L}}^{\langle A;\,\wedge\rangle}$ dualises $\preceq_{\mathcal{R}}^{\langle A;\,\vee\rangle}$ and $\preceq_{\mathcal{R}}^{\langle A;\,\wedge\rangle}$ dualises $\preceq_{\mathcal{L}}^{\langle A;\,\vee\rangle}$ in the sense that $a \preceq_{\mathcal{L}}^{\langle A;\,\wedge\rangle} b$ iff $b \preceq_{\mathcal{R}}^{\langle A;\,\vee\rangle} a$ and $a \preceq_{\mathcal{R}}^{\langle A;\,\wedge\rangle} b$ iff $b \preceq_{\mathcal{L}}^{\langle A;\,\vee\rangle} a$ for any $a, b \in A$ [145, Section 1]. For any skew lattice these *absorption dualities* are equivalent to the following *absorption identities* [146, Section 1.1]:

$$x \wedge (x \vee y) \approx x \tag{1.13}$$

$$x \vee (x \wedge y) \approx x \tag{1.14}$$

$$(y \vee x) \wedge x \approx x \tag{1.15}$$

$$(y \wedge x) \vee x \approx x. \tag{1.16}$$

It follows that the class of skew lattices is a variety [146, Theorem 1.2]. From the obvious equational axiomatisation for the variety of skew lattices it is clear that any skew lattice $\mathbf{A} := \langle A;\ \wedge, \vee\rangle$ is *self-dual* in the sense that it is closed under three distinct dualisations, *viz.*:

1. The *horizontal dual*: $\mathbf{A}^h := \langle A;\ \wedge^h, \vee^h\rangle$, where $a \wedge^h b := b \wedge a$ and $a \vee^h b := b \vee a$, for any $a, b \in A$;

2. The *vertical dual*: $\mathbf{A}^v := \langle A;\ \wedge^v, \vee^v\rangle$, where $a \wedge^v b := a \vee b$ and $a \vee^v b := a \wedge b$, for any $a, b \in A$;

3. The *double dual*: $\mathbf{A}^d := \mathbf{A}^{hv}$.

For convenience, when we refer in the sequel to *skew lattice duality*, we will mean any of the three distinct dualisations cited above.

Example 1.4.6. [146, Section 1.1] Lattices provide an immediate example of a class of skew lattices, since every lattice is clearly a skew lattice satisfying the additional identities $x \vee y \approx y \vee x$ and $x \wedge y \approx y \wedge x$. ∎

Example 1.4.7. ([146, Section 1.4]; *cf.* [87, Section 2]) A class of skew lattices distinct from lattices is the class of *rectangular skew lattices*. Let L and R be non-empty sets and let $D := L \times R$ be their Cartesian product. Define the operations $\wedge$ and $\vee$ for any $a, b \in L$ and $a', b' \in R$ as follows:

$$\langle a, b\rangle \wedge \langle a', b'\rangle := \langle a, b'\rangle$$

$$\langle a, b\rangle \vee \langle a', b'\rangle := \langle a', b\rangle$$

A sequence of easy checks confirms that the algebra $\langle D;\ \wedge, \vee\rangle$ is a skew lattice. ∎

Lemma 1.4.8. *(cf. [145, Section 1]) Any skew lattice $\langle A;\ \wedge, \vee\rangle$ is a paralattice. That is, the partial orders $\leq_{\mathcal{H}}^{\langle A;\,\wedge\rangle}$ and $\leq_{\mathcal{H}}^{\langle A;\,\vee\rangle}$ dualise in the sense that $a \leq_{\mathcal{H}}^{\langle A;\,\wedge\rangle} b$ iff $b \leq_{\mathcal{H}}^{\langle A;\,\vee\rangle} a$ for any $a, b \in A$.*

Proof. Let $\mathbf{A}$ be a skew lattice and $a, b \in A$. Suppose $a \leq_{\mathcal{H}}^{\langle A;\,\wedge\rangle} b$. Then $b \vee a = b \vee (b \wedge a) = b$ by (1.14). Moreover, $a \vee b = (a \wedge b) \vee b = b$ by (1.16). Thus $b \leq_{\mathcal{H}}^{\langle A;\,\vee\rangle} a$. Similarly $b \leq_{\mathcal{H}}^{\langle A;\,\vee\rangle} a$ implies $a \leq_{\mathcal{H}}^{\langle A;\,\wedge\rangle} b$ using (1.13) and (1.15). ∎

Lemma 1.4.9. *(cf. [145, Section 2]) Any skew lattice $\langle A;\ \wedge, \vee\rangle$ is a quasilattice. That is, the quasiorders $\preceq_{\mathcal{D}}^{\langle A;\,\wedge\rangle}$ and $\preceq_{\mathcal{D}}^{\langle A;\,\vee\rangle}$ dualise in the sense that $a \preceq_{\mathcal{D}}^{\langle A;\,\wedge\rangle} b$ iff $b \preceq_{\mathcal{D}}^{\langle A;\,\vee\rangle} a$ for any $a, b \in A$.*

Proof. Let $\mathbf{A}$ be a skew lattice and $a, b \in A$. Suppose $a \preceq_{\mathcal{D}}^{\langle A;\,\wedge\rangle} b$. Observe this implies:

$$\begin{aligned} b \wedge a &= \big(a \wedge (b \wedge a)\big) \vee (b \wedge a) && \text{by (1.16)} \\ &= a \vee (b \wedge a) && \text{as } a \preceq_{\mathcal{D}}^{\langle A;\,\wedge\rangle} b \end{aligned} \tag{1.17}$$

and hence that:

$$\begin{aligned} b \vee a &= (b \vee a) \wedge \big((b \vee a) \vee (b \wedge a)\big) && \text{by (1.13)} \\ &= (b \vee a) \wedge \big(b \vee (a \vee (b \wedge a))\big) && \\ &= (b \vee a) \wedge \big(b \vee (b \wedge a)\big) && \text{by (1.17)} \\ &= (b \vee a) \wedge b && \text{by (1.14).} \end{aligned} \tag{1.18}$$

Therefore we conclude:

$$\begin{aligned} b \vee a \vee b &= (b \vee a) \vee b && \\ &= \big((b \vee a) \wedge b\big) \vee b && \text{by (1.18)} \\ &= b && \text{by (1.16)} \end{aligned}$$

and so $b \preceq_{\mathcal{D}}^{\langle A;\, \vee\rangle} a$. Similarly $b \preceq_{\mathcal{D}}^{\langle A;\, \vee\rangle} a$ implies $a \preceq_{\mathcal{D}}^{\langle A;\, \wedge\rangle} b$ by skew lattice duality. ∎

The Green's relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{D}$ on a skew lattice $\langle A;\ \wedge, \vee\rangle$ are the symmetric parts of the Green's quasiorders, namely:

$$\begin{aligned}
\mathcal{L} &:= \mathcal{L}^{\langle A;\, \wedge\rangle} = \mathcal{R}^{\langle A;\, \vee\rangle} := \big\{(a,b) : a \preceq_{\mathcal{L}} b \text{ and } b \preceq_{\mathcal{L}} a\big\} \\
\mathcal{R} &:= \mathcal{R}^{\langle A;\, \wedge\rangle} = \mathcal{L}^{\langle A;\, \vee\rangle} := \big\{(a,b) : a \preceq_{\mathcal{R}} b \text{ and } b \preceq_{\mathcal{R}} a\big\} \\
\mathcal{D} &:= \mathcal{D}^{\langle A;\, \wedge\rangle} = \mathcal{D}^{\langle A;\, \vee\rangle} := \big\{(a,b) : a \preceq b \text{ and } b \preceq a\big\}
\end{aligned}$$

for any $a, b \in A$, where $\mathcal{L}^{\langle A;\, \wedge\rangle}, \mathcal{R}^{\langle A;\, \wedge\rangle}$ and $\mathcal{D}^{\langle A;\, \wedge\rangle}$ $[\mathcal{L}^{\langle A;\, \vee\rangle}, \mathcal{R}^{\langle A;\, \vee\rangle}$ and $\mathcal{D}^{\langle A;\, \vee\rangle}]$ denote the various Green's relations on the reduct $\langle A;\ \wedge\rangle$ $[\langle A;\ \vee\rangle]$ [150, Section 1.3]. In the sequel the Green's relation $\mathcal{D}$ on a skew lattice is called *equivalence* [146, Section 1.6].

Theorem 1.4.10 (Clifford-McLean Theorem for Skew Lattices). *[146, Theorem 1.7] Let* **A** *be a skew lattice. Then equivalence as defined above is a congruence relation. The $\mathcal{D}$-equivalence classes are the maximal rectangular subalgebras of* **A**, *while the quotient algebra* $\mathbf{A}/\mathcal{D}$ *is the maximal lattice homomorphic image of* **A**. *For all $a, b \in A$, the following conditions are equivalent:*

1. *$a\mathcal{D}b$;*
2. *$a \wedge b \wedge a = a$ and $b \wedge a \wedge b = b$;*
3. *$a \wedge b = b \vee a$.*

A skew lattice $\langle A;\ \wedge, \vee\rangle$ is said to be *left handed* if the reduct $\langle A;\ \wedge\rangle$ is left regular; right handed skew lattices are defined dually.

Theorem 1.4.11 (Second Decomposition Theorem for Skew Lattices). *[146, Theorem 1.15] Let* **A** *be a skew lattice. Then* **A** *is* regular. *That is,* **A** *satisfies the identities:*

$$\begin{aligned}
x \wedge y \wedge x \wedge z \wedge x &\approx x \wedge y \wedge z \wedge x \\
x \vee y \vee x \vee z \vee x &\approx x \vee y \vee z \vee x.
\end{aligned}$$

Thus $\mathcal{L}$ is a congruence on $\mathbf{A}$, *and* $\mathbf{A}_R := \mathbf{A}/\mathcal{L}$ *is the maximal right handed image of* $\mathbf{A}$. *Dually,* $\mathcal{R}$ *is a congruence and* $\mathbf{A}_L := \mathbf{A}/\mathcal{R}$ *is the maximal left handed image of* $\mathbf{A}$. *Finally, the following commuting diagram forms a pullback of skew lattices:*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\mathcal{L}"] \arrow[d, "\mathcal{R}"'] & \mathbf{A}_R \arrow[d, "\mathcal{R}"] \\ \mathbf{A}_L \arrow[r, "\mathcal{L}"'] & \mathbf{A}/\mathcal{D} \end{tikzcd}$$

Skew lattices were introduced by Leech in a 1989 study [146] of bands of idempotents in rings. Let $\mathbf{A} := \langle A;\ +, \cdot \rangle$ be a ring. For any $a, b \in A$, define the derived operations $\wedge$ and $\vee$ by $a \wedge b := a \cdot b$ and $a \vee b := a + b - (a \cdot b)$. Let $\mathrm{E}(\mathbf{A})$ denote the set of idempotents of $\mathbf{A}$. If $B \subseteq \mathrm{E}(\mathbf{A})$ is closed under both $\wedge$ and $\vee$ as defined above, then $\langle B;\ \wedge, \vee \rangle$ is a skew lattice [146, Theorem 2.6]. Even more is true: every multiplicative band in $\mathrm{E}(\mathbf{A})$ that is maximal with respect to being right regular is a skew lattice [146, Theorem 3.2]. Since their introduction skew lattices have been studied by a number of authors, including Bignall and Leech [19], Leech [146, 147, 148, 150, 145], and Spinks [210, 212].

1.4.12. Skew Lattices with Zero. Let $\mathbf{A}$ be a skew lattice. A *maximal* element of $\mathbf{A}$ is an element m such that $a \wedge m \wedge a = a$, or, equivalently, $a \preceq m$ for all $a \in A$. When they exist maximal elements form an equivalence class under $\mathcal{D}$ called the *maximal class* [147, Section 1.4]. *Minimal elements* of $\mathbf{A}$ and the *minimal class* are defined dually. A *skew lattice with zero* is a skew lattice $\langle A;\ \wedge, \vee \rangle$ for which there exists $0 \in A$ (the *zero* of $\langle A;\ \wedge, \vee \rangle$) such that 0 is the least element under the natural skew lattice partial order; a *skew lattice with identity* is defined dually. For any skew lattice $\mathbf{A}$ the following are equivalent: (i) $\mathbf{A}$ is a skew lattice with zero; (ii) for all $a \in A$, $a \wedge 0 = 0 = 0 \wedge A$; (iii) for all $a \in A$, $a \vee 0 = a = 0 \vee a$; (iv) $\mathbf{A}$ has a unique minimal element. By abuse of language and notation we will often identify a skew lattice with zero $\mathbf{A} := \langle A;\ \wedge, \vee \rangle$ with the algebra $\langle A;\ \wedge, \vee, 0 \rangle$ obtained from $\mathbf{A}$ by enriching the language of $\mathbf{A}$ with a new nullary operation symbol $\mathbf{0}$ whose canonical interpretation on $\langle A;\ \wedge, \vee, 0 \rangle$ is $0 \in A$, where 0 is the zero of

A; like remarks apply for skew lattices with identity. The following lemma is immediate.

Lemma 1.4.13. *Let $\langle A;\ \wedge, 0\rangle$ [$\langle A;\ \wedge, \vee, 0\rangle$] be a band with zero [skew lattice with zero]. Then $0 \preceq a$ for all $a \in A$, and $a\,\mathcal{D}\,0$ iff $a = 0$. Thus for any elements $a, b, c \in A$, if $a \wedge c = 0$ then $a \wedge b \wedge c = 0$.*

1.4.14. Left Handed [Right Handed] Skew Lattices. Let $\mathbf{A}$ be a skew lattice and $a, b \in A$. By [146, Section 1.13], [150, Section 1.5] and [210, Section 3.3] the following are equivalent: (i) $\mathbf{A}$ is left handed; (ii) $\mathcal{D} = \mathcal{L}$; (iii) $\mathbf{A} \models x \wedge y \wedge x \approx x \wedge y$ and $\mathbf{A} \models x \vee y \vee x \approx y \vee x$; (iv) if $a\mathcal{D}b$ then $a \wedge b = a$ and $a \vee b = b$; (v) $\mathbf{A} \models x \wedge (y \vee x) \approx x$ and $\mathbf{A} \models (x \wedge y) \vee x \approx x$. Clearly left handed [right handed] skew lattices form a variety. To within isomorphism every skew lattice uniquely decomposes as the fibred product of a right handed skew lattice with a left handed skew lattice over a common underlying maximal lattice homomorphic image [146, Corollary 1.16]: this is an immediate consequence of Theorem 1.4.11. See also [150, Section 1.5, Theorem 1.6].

1.4.15. Symmetric Skew Lattices. A skew lattice $\mathbf{A}$ is called *meet symmetric* if $a \vee b = b \vee a$ implies $a \wedge b = b \wedge a$ for any $a, b \in A$. For a skew lattice $\mathbf{A}$, the following are equivalent [210, Section 3.5]: (i) $\mathbf{A}$ is meet symmetric; (ii) $\mathbf{A} \models x \wedge y \wedge (x \vee y \vee x) \approx (x \vee y \vee x) \wedge y \wedge x$; (iii) $\mathbf{A} \models x \wedge y \wedge (x \vee y) \approx (y \vee x) \wedge (y \wedge x)$. A skew lattice $\mathbf{A}$ is called *join symmetric* if $a \wedge b = b \wedge a$ implies $a \vee b = b \vee a$ for any $a, b \in A$. For a skew lattice $\mathbf{A}$, the following are equivalent [210, Section 3.5]: (i) $\mathbf{A}$ is join symmetric; (ii) $\mathbf{A} \models x \vee y \vee (x \wedge y \wedge x) \approx (x \wedge y \wedge x) \vee y \vee x$; (iii) $\mathbf{A} \models x \vee y \vee (x \wedge y) \approx (y \wedge x) \vee y \vee x$. Meet symmetry for an arbitrary skew lattice does not imply join symmetry, nor conversely; see [210, Section 3.5] for details. A skew lattice is *symmetric* if it is both meet symmetric and join symmetric. For a skew lattice $\mathbf{A}$ the following are equivalent [146, Proposition 2.3, Theorem 2.4]: (i) $\mathbf{A}$ is symmetric; (ii) $\mathbf{A}$ is *biconditionally commutative* (that is, $a \vee b = b \vee a$ iff $a \wedge b = b \wedge a$) [146, Section 2.3]; (iii) the subalgebra $\mathbf{B}$ generated from any non-empty, element-wise $\wedge$-commuting subset B of $\mathbf{A}$ is a sublattice. The motivation for the study of symmetric skew lattices comes from skew lattices in rings, which are symmetric—see [146, Section 2] and [150, Section 2] for details.

1.4.16. Local Skew Lattices. A skew lattice $\langle A;\ \wedge, \vee\rangle$ is said to be *local* when its reduct $\langle A;\ \wedge\rangle$ is normal. For a skew lattice $\mathbf{A}$, the following are equivalent: (i) $\mathbf{A}$ is local; (ii) $\mathbf{A} \models x \wedge y \wedge z \wedge x \approx x \wedge z \wedge y \wedge x$; (iii) the natural skew lattice partial order is preserved under meets; (iv) for all $a \in A$, the principal subalgebra $(\mathbf{a}]$ generated by a is a sublattice. Local skew lattices have been studied under the name *normal skew lattices* by Leech [148] in conformance with standard semigroup terminology; see also [147, Section 2]. A version of the following lemma is asserted without proof in [148].

Lemma 1.4.17. *(cf. [148, Section 2.1]) A local skew lattice is meet symmetric. Thus a local skew lattice is symmetric iff it is join symmetric.*

Proof. Let $\mathbf{A}$ be a local skew lattice and let $a, b \in A$. For the first statement it is sufficient in view of the remarks of §1.4.15 to show $a \vee b = b \vee a$ implies $a \wedge b = b \wedge a$. So suppose $a \vee b = b \vee a$. We have:

$$
\begin{aligned}
a \wedge b &= \big((b \vee a) \wedge a\big) \wedge \big(b \wedge (b \vee a)\big) && \text{by (1.15) and (1.13)}\\
&= \big((a \vee b) \wedge a\big) \wedge \big(b \wedge (a \vee b)\big) && \text{since } a \vee b = b \vee a\\
&= \big((a \vee b) \wedge b\big) \wedge \big(a \wedge (a \vee b)\big) && \text{by normality}\\
&= b \wedge a && \text{by (1.15) and (1.13).}
\end{aligned}
$$

Thus $\mathbf{A}$ is meet symmetric. The second statement now follows trivially. ∎

As with symmetric skew lattices, the study of normal skew lattices is motivated by the study of skew lattices in rings: every maximal normal band of idempotents in a ring forms a normal skew lattice which is the full set of idempotents in the subring it generates [148, p. 1], [147, Theorem 2.2].

1.4.18. Distributive Skew Lattices. There are several different notions of distributivity for skew lattices in the literature. In this thesis a skew lattice is *distributive* if it satisfies the following *middle distributive* identities [150, Section 2.5]:

$$x \wedge (y \vee z) \wedge x \approx (x \wedge y \wedge x) \vee (x \wedge z \wedge x) \tag{1.19}$$

$$x \vee (y \wedge z) \vee x \approx (x \vee y \vee x) \wedge (x \vee z \vee x). \tag{1.20}$$

By Theorem 2.8 of Leech [146] skew lattices arising in rings satisfy the identities (1.19)–(1.20): this provides the motivation for the study of distributive skew lattices. Since their introduction distributive skew lattices have been studied in a number of contexts; see for instance [150, Section 0.5] and [152, pp. 13 ff]. In [148, Section 2] Leech gave a range of conditions under which the middle distributivity identities are equivalent for skew lattices, and in [150, Section 2.5] posed the following problem: *Are the identities* (1.19) *and* (1.20) *equivalent for skew lattices, as they are for lattices?* The following nine element counterexample, found using the model generating program SEM [246], answers this question in the negative.

Example 1.4.19. The clauses:

$$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$$
$$x \vee (y \vee z) \approx (x \vee y) \vee z$$
$$x \wedge (x \vee y) \approx x$$
$$x \vee (x \wedge y) \approx x$$
$$(y \vee x) \wedge x \approx x$$
$$(y \wedge x) \vee x \approx x$$
$$x \wedge (y \vee z) \wedge x \approx (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$$
$$A \vee (B \wedge C) \vee A \not\approx (A \vee B \vee A) \wedge (A \vee C \vee A)$$

have the following model:

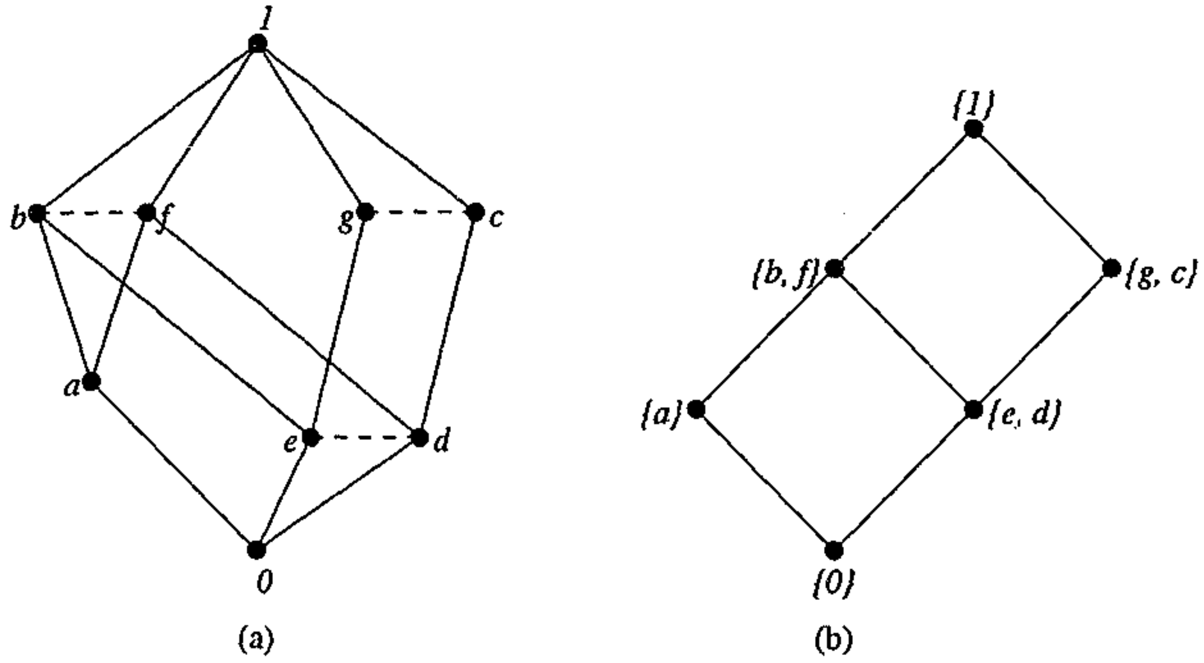

Figure 1.1. (a): The skew lattice of Example 1.4.19; (b) Its maximal lattice homomorphic image.

Model (found by SEM 1.7).

$\wedge$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	a	a	0	0	0	a	0	a
b	0	a	b	d	d	e	f	e	b
c	0	0	e	c	d	e	d	g	c
d	0	0	e	d	d	e	d	e	d
e	0	0	e	d	d	e	d	e	e
f	0	a	b	d	d	e	f	e	f
g	0	0	e	c	d	e	d	g	g
1	0	a	b	c	d	e	f	g	1

$\vee$	0	a	b	c	d	e	f	g	1
0	0	a	b	c	d	e	f	g	1
a	a	a	b	1	f	b	f	1	1
b	b	b	b	1	b	b	b	1	1
c	c	1	1	c	c	c	1	c	1
d	d	f	f	c	d	d	f	c	1
e	e	b	b	g	e	e	b	g	1
f	f	f	f	1	f	f	f	1	1
g	g	1	1	g	g	g	1	g	1
1	1	1	1	1	1	1	1	1	1

$A := a$; $B := b$; $C := c$. ∎

The skew lattice of Example 1.4.19 is depicted in Figure 1.1(a); its maximal lattice homomorphic image is illustrated in Figure 1.1(b). In the Hasse diagram of Figure 1.1(a) the unbroken lines depict the natural skew lattice partial ordering, while the broken horizontal lines connect elements lying in the same non-trivial clique (that is, elements lying in the same non-trivial $\mathcal{D}$-class); we remark that we adopt these diagramming conventions without further comment in the sequel in relation to the depiction of bands, (structurally enriched) skew lattices, and more generally any algebra with an underlying partial order and an underlying admissible quasiorder. The $\mathcal{D}$-classes form the cliques in Figure 1.1(b); from Figure 1.1(b) it is evident that there are three non-trivial $\mathcal{D}$-classes $\{b, f\}$, $\{g, c\}$ and $\{e, d\}$, and three trivial classes $\{1\}$, $\{a\}$ and $\{0\}$. Since $\mathcal{D}$-equivalent elements i and j satisfy $i \wedge j = j$ and $i \vee j = i$, the example is right handed. It follows that the entire skew lattice is determined from $\mathcal{D}$-equivalence and the natural partial ordering. In particular, 0 is the zero element since $i \wedge 0 = 0 = 0 \wedge i$. Similarly 1 is the identity element since $1 \vee i = 1 = i \vee 1$ holds. In general, calculations of meets and joins in a skew lattice is trivial if either: (i) the elements involved are comparable; or (ii) the involved meet or join classes are trivial. A fully general account of how non-trivial meets and joins are determined from the geometric structure of a skew lattice is given in [150]. With respect to the non-trivial cases involving $\{b, f\}$, $\{g, c\}$ and the meet class $\{e, d\}$, the situation is as follows. Since $e \leq b, g$ and $d \leq f, c$, the meets $i \wedge g = e$ and $i \wedge c = d$ obtain for $i \in \{b, f\}$, and $i \wedge b = e$ and $i \wedge f = d$ for $i \in \{g, c\}$. Thus it is apparent that (1.19) holds, but $A \vee (B \wedge C) \vee A = f$ iff $(A \vee B \vee A) \wedge (A \vee C \vee A) = b$ for $\langle A, B, C\rangle \in \{\langle a, b, c\rangle, \langle a, e, c\rangle\}$ and $A \vee (B \wedge C) \vee A = b$ iff $(A \vee B \vee A) \wedge (A \vee C \vee A) = f$ for $\langle A, B, C\rangle \in \{\langle a, d, g\rangle, \langle a, f, g\rangle\}$.

In addition to the counterexample given above, SEM was subsequently able to exhibit a further three non-isomorphic nine element counterexamples satisfying either (1.19) or (1.20) but not both. These remaining counterexamples arise by skew lattice duality in the following manner. Let $\mathbf{T}$ denote the skew lattice of Example 1.4.19. Then $\mathbf{T}^h$ is a left handed skew lattice with the same diagram as that of Figure 1.1(a). Likewise $\mathbf{T}^v$ and $\mathbf{T}^d$ both share a distinct diagram, which is the result of 'flipping' the illustration of Figure 1.1(a) across the *bfgc* axis. This diagram is used to determine a unique left handed structure $\mathbf{T}^v$ and a unique right handed structure $\mathbf{T}^d$. All four skew lattices have the same maximal lattice image as that given in Figure 1.1(b).

Theorem 1.4.20. *The four skew lattices* $\mathbf{T}$, $\mathbf{T}^h$, $\mathbf{T}^v$ *and* $\mathbf{T}^d$ *form a complete set of counterexamples of minimal order showing that for the skew lattice identities, neither (1.19) nor (1.20) implies the other. This set of counterexamples has the property that any one member is sufficient to generate the remaining three.*

Proof. The first statement of the theorem obtains because the search performed by SEM is exhaustive, no models of order 8 or less were found during its search, and no other models of order 9 distinct from $\mathbf{T}$, $\mathbf{T}^h$, $\mathbf{T}^v$ and $\mathbf{T}^d$ exist. The second statement follows upon observing that if $\mathbf{T}'$ is any dual of $\mathbf{T}$, then the set of duals of $\mathbf{T}$ and the set of duals of $\mathbf{T}'$ coincide. ∎

Neither the skew lattice of Example 1.4.19 nor any of its duals are symmetric. This observation has lead Leech to ask [151]: *Are the identities* (1.19) *and* (1.20) *equivalent for symmetric skew lattices?* The following theorem of the author answers this question. The long proof obtained using the automated theorem prover OTTER [158] is omitted, but may be found in [210].

Theorem 1.4.21. *[210, Section 3.7]; [212, Theorem 2.3] For a symmetric skew lattice* $\mathbf{A}$, *the following are equivalent:*

1. $\mathbf{A} \models x \vee (y \wedge z) \vee x \approx (x \vee y \vee x) \wedge (x \vee z \vee x)$;
2. $\mathbf{A} \models x \wedge (y \vee z) \wedge x \approx (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$.

For symmetrical local skew lattices the situation in relation to distributivity is even more pleasing, as the following theorem of Leech [150] shows.

Proposition 1.4.22. *[150, Theorem 3.2] For a local skew lattice* $\mathbf{A}$, *the following are equivalent:*

1. *For each* $a \in A$, *the sublattice* $(a]$ *is distributive;*
2. *The maximal lattice homomorphic image* $\mathbf{A}/\mathcal{D}$ *is distributive;*
3. $\mathbf{A} \models x \vee (y \wedge z) \vee x \approx (x \vee y \vee x) \wedge (x \vee z \vee x)$;
4. $\mathbf{A} \models x \wedge (y \vee z) \wedge x \approx (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$.

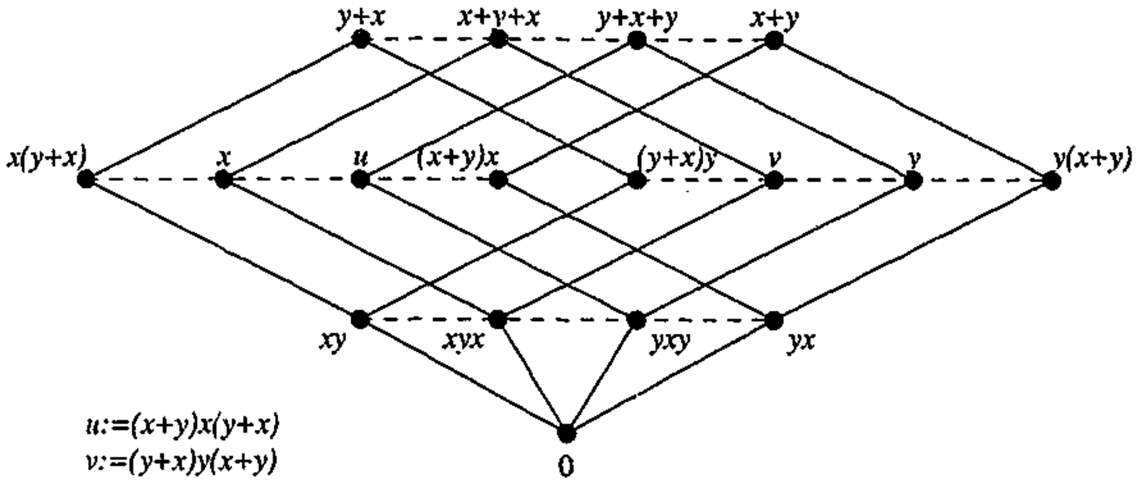

Figure 1.2. The free symmetric local skew lattice with zero on two free generators $\overline{x}, \overline{y}$.

Moreover, the skew lattice subvariety of local skew lattices that are both symmetric and distributive is characterised by the identities:

$$x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z) \tag{1.21}$$

$$(x \vee y) \wedge z \approx (x \wedge z) \vee (y \wedge z). \tag{1.22}$$

Example 1.4.23. Let $\mathbf{2}$ be the one-element semilattice with a zero adjoined, let $\mathbf{R}$ denote the right handed rectangular skew lattice on $\{a, b\}$ and let $\mathbf{L}$ denote its left handed dual. Let $\mathbf{S} := \mathbf{L} \times \mathbf{R} \times \mathbf{2} \times \mathbf{2}$. Upon adjoining a zero to $\mathbf{S}$ we obtain (by [149, Theorem 4.10]) a skew lattice isomorphic to $\mathbf{F}(\overline{x}, \overline{y})$, the free symmetric local skew lattice with zero on two free generators $\overline{x}, \overline{y}$: see Figure 1.2. (For notational reasons, the free generators $\overline{x}, \overline{y}$ are denoted simply by x, y in the diagram (and like remarks apply to products of $\overline{x}, \overline{y}$); also sum (+) and product (juxtaposition) notation is used in the figure instead of the familiar symbols $\vee$ and $\wedge$ for join and meet respectively.) From Proposition 1.4.22 we may conclude that $\mathbf{F}(\overline{x}, \overline{y})$ is distributive, simply because $\mathbf{F}(\overline{x}, \overline{y})/\mathcal{D}$ is the free distributive lattice with zero on two free generators. It follows from these remarks that $\mathbf{F}(\overline{x}, \overline{y})$ coincides with the free distributive symmetric local skew lattice on two free generators. In contrast, the free symmetric skew lattice on two free generators is infinite: see Leech [149, Theorem 4.12]. ■

Throughout the remainder of this section, all skew lattices of interest are distributive symmetric local skew lattices with zero.

1.4.24. Skew Boolean Algebras. A skew lattice $\mathbf{A}$ is called *locally Boolean* if: (i) it is symmetric; (ii) it has a zero; and (iii) each principal subalgebra $(a]$ generated by $a \in A$ is a Boolean lattice [150, Section 3.1]. From the remarks of §1.4.16 and Proposition 1.4.22 it is clear that any locally Boolean skew lattice is distributive symmetric local. The following result, which is folklore, properly characterises the locally Boolean skew lattices among the distributive symmetric local skew lattices; the proof is due to the author and the author's Ph.D. supervisor.

Proposition 1.4.25. *(cf. [65, Proposition 2.3]) Let* $\mathbf{A}$ *be a distributive symmetric local skew lattice with zero. The following are equivalent:*

1. $\mathbf{A}$ *is locally Boolean;*
2. $\mathbf{A}/\mathcal{D}$ *is a relatively complemented distributive lattice with zero;*
3. *For all* $a, b \in A$ *there is a unique* $c \in A$ *such that* $(a \wedge b \wedge a) \vee c = a$ *and* $c \wedge (a \wedge b \wedge a) = 0$*;*
4. $\mathbf{A}$ *has* relative complements*: if* $b \leq a$*, then* b *has a unique complement in* $(a]$.

Proof. (1) $\Rightarrow$ (2) Suppose $\mathbf{A}$ is locally Boolean. By previous remarks and Lemma 1.3.17, $\mathbf{A}/\mathcal{D}$ is a distributive lattice with zero in which every principal order ideal is a Boolean lattice under the semilattice partial ordering. This is sufficient to guarantee $\mathbf{A}/\mathcal{D}$ is relatively complemented: see Cornish and Hickman [72] or Cornish [64].

(2) $\Rightarrow$ (3) Suppose $\mathbf{A}/\mathcal{D}$ is a relatively complemented distributive lattice with zero. To simplify notation we write $\overline{a}$ for the equivalence class $[a]_{\mathcal{D}}$ in $A/\mathcal{D}$ containing $a \in A$. In $\mathbf{A}/\mathcal{D}$ we have that $\overline{a} \wedge \overline{b}$ is complemented in the interval from $\overline{0}$ to $\overline{a}$, so there is a $\overline{w} \leq \overline{a}$ such that $\overline{w \wedge b} = \overline{0}$ and $\overline{a \wedge b} \vee \overline{w} = \overline{a}$. In view of these remarks we have:

$$\overline{a} = \overline{a} \wedge \overline{a} \wedge \overline{a}$$

$$
\begin{aligned}
&= \overline{a} \wedge \big((\overline{a \wedge b}) \vee \overline{w}\big) \wedge \overline{a} \\
&= (\overline{a} \wedge \overline{a \wedge b} \wedge \overline{a}) \vee (\overline{a} \wedge \overline{w} \wedge \overline{a}) && \text{by middle distributivity} \\
&= \overline{a \wedge b \wedge a} \vee \overline{a \wedge w \wedge a}.
\end{aligned}
$$

Thus $a \equiv (a \wedge b \wedge a) \vee c\,(\mathrm{mod}\ \mathcal{D})$, where $c = a \wedge w \wedge a$. By the Clifford Mc-Lean theorem,

$$
\begin{aligned}
a &= a \wedge \big((a \wedge b \wedge a) \vee c\big) \wedge a \\
&= \big(a \wedge (a \wedge b \wedge a) \wedge a\big) \vee \big(a \wedge c \wedge a\big) && \text{by middle distributivity} \\
&= (a \wedge b \wedge a) \vee \big(a \wedge (a \wedge w \wedge a) \wedge a\big) \\
&= (a \wedge b \wedge a) \vee (a \wedge w \wedge a) \\
&= (a \wedge b \wedge a) \vee c.
\end{aligned}
$$

On the other hand, from $\overline{w \wedge b} = \overline{0}$ we have:

$$
\begin{aligned}
0 &= a \wedge 0 \wedge a \\
&= a \wedge w \wedge b \wedge a && \text{by Lemma 1.4.13} \\
&= (a \wedge w \wedge a) \wedge (a \wedge b \wedge a) && \text{by regularity} \\
&= c \wedge (a \wedge b \wedge a).
\end{aligned}
$$

To see c is unique, suppose $(a \wedge b \wedge a) \vee c_1 = a = (a \wedge b \wedge a) \vee c_2$ while $a \wedge b \wedge a \wedge c_1 = 0$ and $a \wedge b \wedge a \wedge c_2 = 0$. We have:

$$
\begin{aligned}
a \wedge c_1 &= \big((a \wedge b \wedge a) \vee c_1\big) \wedge c_1 \\
&= (a \wedge b \wedge a \wedge c_1) \vee (c_1 \wedge c_1) && \text{by (1.22)} \\
&= 0 \vee c_1 \\
&= c_1.
\end{aligned}
$$

Also,

$$
a \wedge c_1 = \big((a \wedge b \wedge a) \vee c_2\big) \wedge c_1
$$

$$
\begin{aligned}
&= (a \wedge b \wedge a \wedge c_1) \vee (c_2 \wedge c_1) && \text{by (1.22)} \\
&= 0 \vee (c_2 \wedge c_1) \\
&= c_2 \wedge c_1.
\end{aligned}
$$

Similarly, $c_2 \wedge a = c_2$ and $c_2 \wedge a = c_2 \wedge c_1$. Thus $c_1 = a \wedge c_1 = c_2 \wedge c_1 = c_2 \wedge a = c_2$, completing the proof.

(3) $\Rightarrow$ (4) Suppose $b \leq a$. Then $b \wedge a = b = a \wedge b$, whence $a \wedge b \wedge a = b$. Since a is clearly the maximal element of the sublattice $(\boldsymbol{a}]$, by (3) we have that c is the unique complement of b in $(\boldsymbol{a}]$.

(4) $\Rightarrow$ (1) Let $a \in A$ and let $b \in (a]$. Then $b \leq a$, and by (4) there is a unique complement of b in $(\boldsymbol{a}]$. ■

Let $\mathbf{A}$ be a locally Boolean skew lattice. The *standard difference* of $a, b \in A$ is $a \backslash b$, the complement of $a \wedge b \wedge a$ in $(\boldsymbol{a}]$ [19, Definition 3.1]. A *skew Boolean algebra* is an algebra $\langle A; \vee, \wedge, \backslash, 0\rangle$ of type $\langle 2, 2, 2, 0\rangle$ that satisfies all the identities determining distributive symmetric local skew lattices with zero, together with the identities [19, Definition 3.1]:

$$(x \backslash y) \vee (x \wedge y \wedge x) \approx x \tag{1.23}$$
$$(x \wedge y \wedge x) \vee (x \backslash y) \approx x \tag{1.24}$$

$$(x \backslash y) \wedge (x \wedge y \wedge x) \approx \mathbf{0} \tag{1.25}$$
$$(x \wedge y \wedge x) \wedge (x \backslash y) \approx \mathbf{0}. \tag{1.26}$$

Clearly the class SBA of skew Boolean algebras is a variety [147, Theorem 1.8]; for axiomatisations see Spinks [210, Section 4], Cornish [65, Section 2] and Leech [150, Section 3.3]. An important consequence of the identities (1.23)–(1.26) is Proposition 1.4.27 below, which is part of the folklore of non-commutative lattice theory.

Lemma 1.4.26. *The variety of skew Boolean algebras satisfies the following* *identities:*

$$x \wedge (x \backslash y) \approx x \backslash y \tag{1.27}$$

$$(x \backslash y) \wedge x \approx x \backslash y. \tag{1.28}$$

Proof. Let $\mathbf{A}$ be a skew Boolean algebra and let $a, b \in A$. We have $a \backslash b = \big((a \wedge b \wedge a) \vee (a \backslash b)\big) \wedge (a \backslash b) = a \wedge (a \backslash b)$ by (1.15) and (1.24), which establishes (1.27). Also $a \backslash b = (a \backslash b) \wedge \big((a \backslash b) \vee (a \wedge b \wedge a)\big) = (a \backslash b) \wedge a$ by (1.13) and (1.23), which establishes (1.28). ∎

Proposition 1.4.27. *(Bignall) Let* $\mathbf{A}$ *be a skew Boolean algebra. If* $\theta \in \operatorname{Con}\langle A;\ \wedge, \vee\rangle$*, then* $\theta \in \operatorname{Con}\mathbf{A}$*. Thus* $\operatorname{Con}\langle A;\ \wedge, \vee\rangle = \operatorname{Con}\mathbf{A}$.

Proof. It is sufficient to show that $\theta \in \operatorname{Con}\langle A;\ \wedge, \vee\rangle$ has the substitution property for the $\backslash$ operation. Let $a \equiv a_1 \,(\operatorname{mod}\theta)$ and $b \equiv b_1 \,(\operatorname{mod}\theta)$ and notice $a \wedge b \wedge a \equiv_\theta a_1 \wedge b_1 \wedge a_1$. We have:

$$\begin{aligned}
a \backslash b &= a \wedge (a \backslash b) && \text{by (1.27)}\\
&\equiv_\theta a_1 \wedge (a \backslash b)\\
&= \big((a_1 \wedge b_1 \wedge a_1) \vee (a_1 \backslash b_1)\big) \wedge (a \backslash b) && \text{by (1.24)}\\
&= \big((a_1 \wedge b_1 \wedge a_1) \wedge (a \backslash b)\big) \vee \big((a_1 \backslash b_1) \wedge (a \backslash b)\big) && \text{by (1.22)}\\
&\equiv_\theta \big((a \wedge b \wedge a) \wedge (a \backslash b)\big) \vee \big((a_1 \backslash b_1) \wedge (a \backslash b)\big)\\
&= 0 \vee \big((a_1 \backslash b_1) \wedge (a \backslash b)\big) && \text{by (1.26)}\\
&= (a_1 \backslash b_1) \wedge (a \backslash b).
\end{aligned}$$

Also,

$$\begin{aligned}
a_1 \backslash b_1 &= (a_1 \backslash b_1) \wedge a_1 && \text{by (1.28)}\\
&\equiv_\theta (a_1 \backslash b_1) \wedge a\\
&= (a_1 \backslash b_1) \wedge \big((a \wedge b \wedge a) \vee (a \backslash b)\big) && \text{by (1.24)}\\
&= \big((a_1 \backslash b_1) \wedge (a \wedge b \wedge a)\big) \vee \big((a_1 \backslash b_1) \wedge (a \backslash b)\big) && \text{by (1.21)}\\
&\equiv_\theta \big((a_1 \backslash b_1) \wedge (a_1 \wedge b_1 \wedge a_1)\big) \vee \big((a_1 \backslash b_1) \wedge (a \backslash b)\big)\\
&= 0 \vee \big((a_1 \backslash b_1) \wedge (a \backslash b)\big) && \text{by (1.25)}
\end{aligned}$$

$$= (a_1\backslash b_1) \wedge (a\backslash b).$$

We conclude $a\backslash b \equiv_\theta a_1\backslash b_1$, and the proposition is proved. ∎

Example 1.4.28. ([147, Example 1.7(b)]; *cf.* [65, p. 287]) Let $\mathbf{A} := \langle A;\ \wedge, \vee, 0\rangle$ be a rectangular skew lattice with a zero adjoined. Then $\mathbf{A}$ is relatively complemented: upon distinguishing the operation of standard difference we obtain a *primitive* skew Boolean algebra consisting of 0 together with a single non-zero equivalence class A. ∎

Primitive skew Boolean algebras play an important role in the theory of skew Boolean algebras. In the statement of the following theorem, $\mathbf{2}^p$ denotes the two element primitive skew Boolean algebra, while $\mathbf{3}^p_L$ and $\mathbf{3}^p_R$ denote the primitive left and right handed three-element skew Boolean algebras respectively.

Theorem 1.4.29. *[147, Theorem 1.13] A skew Boolean algebra is directly indecomposable iff it is primitive. Up to isomorphism, the only subdirectly irreducible skew Boolean algebras are the algebras $\mathbf{2}^p$, $\mathbf{3}^p_L$ and $\mathbf{3}^p_R$.*

Let $\mathbf{A}$ be a skew Boolean algebra. $\mathbf{A}$ is called *left handed* [right handed] if its skew lattice reduct $\langle A;\ \wedge, \vee\rangle$ is left handed [right handed]. The subvariety of left handed skew Boolean algebras [right handed skew Boolean algebras] is denoted lhSBA [rhSBA]. Left handed skew Boolean algebras were introduced by Cornish in [65] under the name *Boolean skew algebras.* Skew Boolean algebras were introduced in full generality under the name *skew quasi-Boolean algebras* by Leech in [147]. It is not immediately apparent that Cornish's class of Boolean skew algebras coincides with the class of left handed skew Boolean algebras; see Spinks [210, Section 4.1] for a discussion and proof. The motivation for the study of skew Boolean algebras comes in the first instance from ring theory: every maximal normal band of idempotents in a ring forms a skew Boolean algebra [146, Theorem 2.2].

Let $\mathbf{A}$ be a skew Boolean algebra with maximal class M. An algebra $\langle A;\ \wedge, \vee, \backslash, 0, 1\rangle$ of type $\langle 2, 2, 2, 0, 0\rangle$ obtained from $\mathbf{A}$ by adjoining to the language of $\mathbf{A}$ a new nullary operation symbol $\mathbf{1}$ whose canonical interpretation on $\langle A;\ \wedge, \vee, \backslash, 0, 1\rangle$ is a fixed element $1 \in M$ is a *quasi-bounded skew Boolean* *algebra.* Somewhat confusingly, quasi-bounded skew Boolean algebras were introduced by Leech in [147] under the name *skew Boolean algebras.*

Example 1.4.30. Let $\mathbf{A} := \langle A;\ \wedge, \vee, {}^*, 0\rangle$ be a distributive lattice with pseudocomplementation. For any $a, b \in A$, define the operations:

$$
\begin{aligned}
a \mathbin{\overline{\wedge}} b &:= a \wedge b^{**} \\
a \mathbin{\overline{\vee}} b &:= (a \wedge b^*) \vee b \\
a \backslash b &:= a \wedge b^*;
\end{aligned}
$$

also, recall $1 := 0^*$. An easy but tedious verification shows the induced algebra $\langle A;\ \overline{\wedge}, \overline{\vee}, \backslash, 0, 1\rangle$ is a left handed quasi-bounded skew Boolean algebra iff $\mathbf{A}$ is a Stone algebra, namely a distributive lattice with pseudocomplementation satisfying $x^* \vee x^{**} \approx \mathbf{1}$. Further, $\langle A;\ \overline{\wedge}, \overline{\vee}, \backslash, 0, 1\rangle$ is term equivalent to a Boolean algebra iff $\langle A;\ \wedge, \vee, {}^*, 0, 1\rangle$ is a Boolean algebra. These remarks yield a new solution to Birkhoff [23, Problem 70]. ∎

Example 1.4.31. Let $\mathbf{A} := \langle A;\ \wedge, \vee, \rightarrow, 0, 1\rangle$ be a Heyting algebra, namely a Brouwerian lattice with distinguished least element. It is well known (see for instance [14, p. 174]) that upon defining $a^* := a \rightarrow 0$ for any $a \in A$ the induced algebra $\mathbf{A}^* := \langle A;\ \wedge, \vee, {}^*, 0\rangle$ is a distributive lattice with pseudocomplementation with greatest element $1 = 0^*$. For any $a, b \in A$, define the operations $\overline{\wedge}, \overline{\vee}$ and $\backslash$ as in the preceding example. An easy verification using the identities and quasi-identities of [75, Chapter 4, Section C.2] shows that if $\mathbf{A}$ is a linearly ordered Heyting algebra (that is, if $\mathbf{A} \models (x \rightarrow y) \vee (y \rightarrow x) \approx \mathbf{1}$) then $\mathbf{A} \models (x \rightarrow y) \vee \big((x \rightarrow y) \rightarrow y\big) \approx \mathbf{1}$. It follows that the polynomial reduct $\mathbf{A}^*$ is a Stone algebra, and thus that the induced algebra $\langle A;\ \overline{\wedge}, \overline{\vee}, \backslash, 0, 1\rangle$ is a left handed quasi-bounded skew Boolean algebra. ∎

1.4.32. Skew Boolean $\cap$-Algebras. Let B be a subset of the universe of a skew lattice $\mathbf{A}$. The infimum of B with respect to the underlying natural partial ordering of $\mathbf{A}$, if it exists, is called the *intersection* of B in A, and is denoted $\bigcap B$ [19, Definition 2.5]. A skew lattice is said to have [*finite*] *intersections* if every non-empty [finite] subset has an intersection. A *skew*

Boolean algebra with intersections is a skew Boolean algebra for which the skew lattice reduct has finite intersections [19, Section 1.4].

Example 1.4.33. Let $\mathbf{A} := \langle A;\ \wedge, \vee, \setminus, 0\rangle$ be a finite skew Boolean algebra. Then for every $a \in A$, the principal subalgebra $(a]$ of $\mathbf{A}$ is finite. Since the reduct $\langle A;\ \wedge, \vee, 0\rangle$ is a local skew lattice with zero, $\mathbf{A}$ has intersections by [19, Proposition 2.10]. ∎

In the sequel we view skew lattices having finite intersections as as algebras $\langle A;\ \wedge, \vee, \cap\rangle$ of type $\langle 2, 2, 2\rangle$. That the class of such algebras is a variety is an immediate consequence of the following proposition.

Proposition 1.4.34. *[19, Proposition 2.6] A skew lattice having finite intersections is an algebra $\langle A;\ \wedge, \vee, \cap\rangle$ of type $\langle 2, 2, 2\rangle$ such that $\langle A;\ \cap\rangle$ is a meet semilattice, $\langle A;\ \wedge, \vee\rangle$ is a skew lattice, and for which the following identities hold:*

$$x \cap (x \wedge y \wedge x) \approx x \wedge y \wedge x$$
$$x \wedge (x \cap y) \approx x \cap y \approx (x \cap y) \wedge x.$$

A *skew Boolean intersection algebra*, or *skew Boolean $\cap$-algebra* for short, is an algebra $\langle A;\ \wedge, \vee, \setminus, \cap, 0\rangle$ of type $\langle 2, 2, 2, 2, 0\rangle$ such that: (i) the reduct $\langle A;\ \wedge, \vee, \setminus, 0\rangle$ is a skew Boolean algebra; and (ii) the reduct $\langle A;\ \wedge, \vee, \cap\rangle$ is a skew lattice with intersections. Clearly the class SBIA of skew Boolean $\cap$-algebras is a variety.

Example 1.4.35. [19, Example 3.2b] Let $\langle A;\ \vee, \wedge, \setminus, 0\rangle$ be a primitive skew Boolean algebra, as per Example 1.4.28. Primitive skew Boolean algebras possess arbitrary intersections, with $\bigcap a_i$ being the common value when all the a_i are equal, and 0 otherwise; this property characterises the primitive skew Boolean algebras among all the non-trivial skew Boolean algebras with intersections. A *primitive* skew Boolean $\cap$-algebra is a primitive skew Boolean algebra with intersections in which the intersection operation is distinguished: see Leech [150, Section 4.4]. ∎

Theorem 1.4.36. *[19, Theorem 3.5] The following assertions hold in the variety of skew Boolean $\cap$-algebras:*

1. *The primitive skew Boolean* $\cap$*-algebras are the non-trivial simple algebras;*

2. *The primitive skew Boolean* $\cap$*-algebras are the subdirectly irreducible algebras;*

3. *Every non-trivial skew Boolean* $\cap$*-algebra is a subdirect product of primitive algebras.*

A skew Boolean $\cap$-algebra is said to be *left handed* [*right handed*] if its skew lattice reduct is left handed [right handed]. The subvariety of left handed skew Boolean $\cap$-algebras [right handed skew Boolean $\cap$-algebras] is denoted lhSBIA [rhSBIA]. Skew Boolean $\cap$-algebras were introduced in their left handed form under the name *quasi-Boolean skew lattices* by Bignall in [17], and in full generality by Bignall and Leech in [19]. See also [150, Section 4].

1.4.37. Discriminator Varieties and Dual Discriminator Varieties. The *ternary discriminator* and *dual ternary discriminator* on a set A are the functions $t : A^3 \to A$ and $d : A^3 \to A$ defined respectively by [183, 95]:

$$t(a, b, c) := \begin{cases} c & \text{if } a = b \\ a & \text{otherwise} \end{cases} \quad \text{and} \quad d(a, b, c) := \begin{cases} a & \text{if } a = b \\ c & \text{otherwise.} \end{cases}$$

A ternary term $t(x, y, z)$ [$d(x, y, z)$] that realises the ternary discriminator [dual ternary discriminator] on an algebra $\mathbf{A}$ is called a *ternary discriminator term* [*dual ternary discriminator term*] for $\mathbf{A}$. An algebra $\mathbf{A}$ is said to be a *ternary discriminator algebra* [dual ternary discriminator algebra] if it has a ternary discriminator term [dual ternary discriminator term]. If K is a class of algebras of the same similarity type with a common ternary discriminator term [a common dual ternary discriminator term], then the variety $\mathbf{V}(\mathsf{K})$ generated by K is called a *ternary discriminator variety* [*dual ternary discriminator variety*]. Observe that any ternary discriminator variety is a dual ternary discriminator variety, since $d(x, y, z) \approx t(x, t(x, y, z), z)$ [95, Section 1]; conversely, a dual ternary discriminator variety is a discriminator variety iff it is congruence permutable [95]. We will always drop the qualifier 'ternary' in the sequel if the context is clear. In the statement of the following theorem and in the sequel, $x\Delta y$ abbreviates (for skew Boolean $\cap$-algebras) the term $(x\backslash(x\cap y)) \vee (y\backslash(y\cap x))$.

Theorem 1.4.38. *[19, Theorem 4.4, Corollary 4.9] The class of skew Boolean $\cap$-algebras is a discriminator variety, with discriminator term given by:*

$$t(x,y,z) := \big(z\backslash(x\Delta y)\big) \vee \big(x\backslash(x\cap y)\big).$$

The *pure pointed discriminator variety* is the pointed discriminator variety of type $\langle 3, 0\rangle$ generated by the class of all pointed discriminator algebras $\langle A;\ t, 0\rangle$ where t is the discriminator function on A and 0 is a nullary operation [19, Definition 4.6]. In the statement of the following theorem and in the sequel the pure pointed discriminator variety is denoted by $\mathsf{PD_0}$.

Theorem 1.4.39. *[19, Theorem 4.7] The variety $\mathsf{PD_0}$ is termwise definitionally equivalent to the variety of left handed skew Boolean $\cap$-algebras. In particular, given $\langle A;\ t, 0\rangle \in \mathsf{PD_0}$, left handed skew Boolean $\cap$-operations $\vee$, $\wedge$ and $\backslash$ and $\cap$ are defined on A by:*

$$\begin{aligned} a \wedge b &:= t\big(b, t(b,0,a), a\big)\\ a \vee b &:= t(b,0,a)\\ a\backslash b &:= t(0,b,a)\\ a \cap b &:= t\big(a, t(a,b,0), 0\big)\end{aligned}$$

for any $a, b \in A$. Conversely, given a left handed skew Boolean $\cap$-algebra $\langle A;\ \wedge,\vee,\backslash,\cap,0\rangle$ and $a,b,c \in A$, the operation $t(a,b,c) := (c\backslash b) \vee (c \wedge a) \vee \big(a\backslash(a\cap b)\big)$ yields an algebra $\langle A;\ t, 0\rangle$ in $\mathsf{PD_0}$.

Bignall and Leech have noted [19, p. 396] that $\mathsf{PD_0}$ is also termwise definitionally equivalent to the variety of right handed skew Boolean $\cap$-algebras.

Corollary 1.4.40. *[19, Corollary 4.8] Any algebra $\mathbf{A}$ in a pointed discriminator variety has a left handed skew Boolean $\cap$-algebra polynomial reduct whose congruences coincide with those of $\mathbf{A}$.*

Discriminator varieties were introduced by Pixley in [183], while dual discriminator varieties were introduced by Fried and Pixley in [95]. Discriminator varieties have been extensively studied in the literature: see [55, Chapter IV§9] for an introductory discussion, [237] for a comprehensive study, and Jònsson's survey of congruence distributive varieties [129, Chapter IV] for a more recent discourse.

1.5 Varieties with EDPC

Let $\mathbb{S}$ be a strongly algebraisable deductive system with equivalent variety semantics V. It was early understood in the study of algebraic logic that $\mathbb{S}$ satisfies some reasonable form of the deduction-detachment theorem iff V has equationally definable principal congruences. In conjunction with the realisation that varieties with equationally definable principal congruences are congruence distributive, this fact has lead to an intensive study of such varieties by Blok, Köhler, Pigozzi and others. We summarise some results of their investigations in this section.

1.5.1. Equationally Definable Principal Congruences. Let K be a class of similar algebras. A first-order formula $\varphi(x, y, u, v)$ in the language of K is said to *define principal congruences* in K if, for all $\mathbf{A} \in \mathsf{K}$ and $a, b \in A$,

$$\Theta^{\mathbf{A}}(a, b) = \{\langle c, d\rangle \in A \times A : \mathbf{A} \models \varphi[a, b, c, d]\}.$$

K is said to have *definable principal congruences* if there exists a first-order formula in the language of K that defines principal congruences in K [129, Definition III§2.1].

The study of varieties with definable principal congruences was initiated by Baldwin and Berman in [15], and in [94] Fried, Grätzer and Quackenbush introduced the notion of *equationally* definable principal congruences. A variety V has *Equationally Definable Principal Congruences* (*EDPC* for short) if there exist finitely many pairs $\langle p_1, q_1\rangle, \ldots, \langle p_n, q_n\rangle$, $i = 1, \ldots, n$, of 4-ary terms of V

such that for all $\mathbf{A} \in \mathsf{V}$ and all $a, b, c, d \in A$,

$$c \equiv d \left(\operatorname{mod} \Theta^{\mathbf{A}}(a,b)\right) \quad \text{iff} \quad p_i^{\mathbf{A}}(a,b,c,d) = q_i^{\mathbf{A}}(a,b,c,d)$$

for each $i = 1, \dots, n$. The following result is due variously to Blok, Köhler and Pigozzi [136] and van Alten [229]. In the statement of the theorem and in the sequel Cp $\mathbf{A}$ [**Cp A**] denotes the set [join semilattice] of compact congruences on an algebra $\mathbf{A}$.

Theorem 1.5.2. *([136]; [229, Proposition 5.19(i)]) For any variety* V*, the following are equivalent:*

1. V *has EDPC;*

2. *The join semilattice* $\langle \mathrm{Cp}\,\mathbf{A};\ \vee, \omega_{\mathbf{A}} \rangle$ *of compact congruences of* $\mathbf{A}$ *is dually relatively pseudocomplemented for any* $\mathbf{A} \in \mathsf{V}$.

Moreover, if V *has EDPC, the following statements hold:*

3. V *is congruence distributive and has the congruence extension property;*

4. V *is semisimple iff* V *is generated (as a variety) by a class of simple algebras.*

Since their introduction varieties with EDPC arising from algebraic logic have been systematically studied by Blok and Pigozzi in a series of papers [29, 30, 34, 35]; for a survey of much of this work beyond that presented below see Jònsson [129, Chapter III].

1.5.3. WBSO Varieties and QD Terms. Recall from [29, Lemma 2.7] that a *variety of weak Brouwerian semilattices with filter preserving operations* (briefly, a *WBSO variety*) is a variety V with $\mathbf{1}$ such that: (i) the join semilattice $\langle \mathrm{Cp}\,\mathbf{A};\ \vee, \omega_{\mathbf{A}} \rangle$ of compact congruences is dually relatively pseudocomplemented; and (ii) there exist binary terms $\rightarrow, \cdot$ and Δ in the language of V such that for any $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$,

$$\begin{aligned} \Theta^{\mathbf{A}}(a \rightarrow b, 1) &= \Theta^{\mathbf{A}}(b,1) * \Theta^{\mathbf{A}}(a,1) \\ \Theta^{\mathbf{A}}(a \cdot b, 1) &= \Theta^{\mathbf{A}}(a,1) \vee \Theta^{\mathbf{A}}(b,1) \end{aligned}$$

$$\Theta^{\mathbf{A}}(a\Delta b, 1) = \Theta^{\mathbf{A}}(a, b)$$

where $*$ denotes dual relative pseudocomplementation in $\langle \mathrm{Cp}\,\mathbf{A};\ \vee, \omega_{\mathbf{A}}\rangle$. An algebra $\mathbf{A}$ is called a *weak Brouwerian semilattice with filter preserving operations* if it is a member of a WBSO variety, or, equivalently, $\mathsf{V} \doteq \mathbf{V}(\mathbf{A})$ for some WBSO variety V. The terms $\rightarrow, \cdot$ and Δ are called *weak relative pseudocomplementation*, *weak meet* and *Gödel equivalence* terms respectively. In general, none of these terms need be unique; see [29, p. 357]. A *weak join* for a WBSO variety V is a binary term $x + y$ of V with the property that $\Theta^{\mathbf{A}}(a + b, 1) = \Theta^{\mathbf{A}}(a, 1) \cap \Theta^{\mathbf{A}}(b, 1)$ for all $\mathbf{A} \in \mathsf{V}$ [29, p. 370]. In general, a WBSO variety need not have a weak join; nor need a weak join, if it exists, be unique.

Let $\mathbf{A}$ be a weak Brouwerian semilattice with filter preserving operations. The binary relation $\preceq$ defined on A by the condition $a \preceq b$ iff $a \rightarrow b = 1$ for any $a, b \in A$ is a quasiordering. The equivalence $\sim$ induced by $\preceq$ is a congruence on $\langle A;\ \cdot, \rightarrow, 1\rangle$ and $\langle A;\ \cdot, \rightarrow, 1\rangle/{\sim}$ is a Brouwerian semilattice that is dually isomorphic to the dual Brouwerian semilattice $\langle \mathrm{Cp}\,\mathbf{A};\ \vee, *, \omega_{\mathbf{A}}\rangle$ under the map $a \mapsto \Theta^{\mathbf{A}}(a, 1)$ [29, p. 352], [35, p. 7]. $\mathbf{A}$ is a *weak Boolean algebra with filter preserving operations* if the dual Brouwerian semilattice of compact congruences $\langle \mathrm{Cp}\,\mathbf{A};\ \vee, *, \omega_{\mathbf{A}}\rangle$ is (termwise definitionally equivalent to) a generalised Boolean algebra. A subset $F \subseteq A$ is a *weak filter* of $\mathbf{A}$ if $1 \in F$, $a \cdot b \in F$ whenever $a, b \in F$, and $b \in F$ whenever $a \in F$ and $a \rightarrow b = 1$ [29, p. 351]; the set of weak filters of $\mathbf{A}$ is denoted $\mathrm{Wf}\,\mathbf{A}$. Blok, Köhler and Pigozzi have shown that the weak filters of $\mathbf{A}$ are exactly the subsets of A of the form $\bigcup G$, where G is a filter of $\langle A;\ \cdot, \rightarrow, 1\rangle/{\sim}$ [29, p. 354; Theorem 2.6]. Call a subset F of the universe of $\mathbf{A}$ an *implicative filter* if $1 \in F$, and $a, a \rightarrow b \in F$ implies $b \in F$. A version of the following result is mentioned without proof in [35, p. 7]; see also [29, p. 352].

Lemma 1.5.4. *(cf. [35, p. 7]; cf. [29, p. 352]) Let V be a WBSO variety. Let $\mathbf{A} \in \mathsf{V}$ and $\{1\} \subseteq F \subseteq A$. The following are equivalent:*

1. *$F = [1]_\theta$ for some $\theta \in \mathrm{Con}\,\mathbf{A}$;*

2. *F is a weak filter of $\mathbf{A}$;*

3. F is an implicative filter of **A**.

Proof. The equivalence of (1) and (2) is proved in [29, p. 352]. To complete the proof of the lemma it is sufficient in view of preceding remarks to show the equivalence of (2) and (3) in the context of Brouwerian semilattices; for this see either Meng *et al* [164, Theorem 10] or Rasiowa [195, Theorem IV§2.1]. ∎

Proposition 1.5.5. *[29, Lemma 2.2] Let* **A** *be a weak Brouwerian semilattice with filter preserving operations. The maps* $\theta \mapsto [1]_\theta$ *(*$\theta \in \mathrm{Con}\,\mathbf{A}$*) and* $F \mapsto \{\langle a, b\rangle : a\Delta b \in F\}$ *(*$F \in \mathrm{Wf}\,\mathbf{A}$*) are mutually inverse isomorphisms between the congruence and weak filter lattices of* **A**.

Proposition 1.5.6. *[29, Lemma 2.4] Let* V *be a WBSO variety. For any* $\mathbf{A} \in \mathsf{V}$ *the following assertions hold:*

1. *Every compact congruence of* **A** *is principal. In particular, for any* $a, b \in A$*,*

$$\Theta^{\mathbf{A}}\big(\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\big) = \\ \Theta^{\mathbf{A}}\big((\cdots(a_1\Delta^{\mathbf{A}} b_1) \cdot^{\mathbf{A}} \cdots) \cdot^{\mathbf{A}} (a_n\Delta^{\mathbf{A}} b_n), 1\big);$$

2. *For any* $a, b, c, d \in A$*,*

$$c \equiv d \left(\mathrm{mod}\, \Theta^{\mathbf{A}}(a, b)\right) \quad \textit{iff} \quad (a\Delta^{\mathbf{A}} b) \to^{\mathbf{A}} (c\Delta^{\mathbf{A}} d) = 1.$$

Thus V *has EDPC.*

Let **A** be an algebra. A *Quaternary Deductive* (*QD*) *term* on **A** is a term $q(x, y, z, w)$ such that for any $a, b, c, d \in A$ [29, p. 359],

$$q^{\mathbf{A}}(a, b, c, d) := \begin{cases} c & \text{if } a = b \\ d & \text{if } c \equiv d \left(\mathrm{mod}\, \Theta^{\mathbf{A}}(a, b)\right). \end{cases}$$

A QD term on a variety V is a term $q(x, y, z, w)$ such that $q(x, y, z, w)$ is a QD term on every member of V. The QD term may be regarded as a natural generalisation of the normal transform to non-semisimple varieties; see Blok, Köhler and Pigozzi [29, Section 3] and Blok and Pigozzi [34, p. 547] for details.

Theorem 1.5.7. *[34, Theorem 1.25(i)] A variety has a QD term iff it is congruence permutable and has EDPC.*

Theorem 1.5.8. *[34, Theorem 1.26] For a variety V with $\mathbf{1}$ the following are equivalent:*

1. *V is congruence permutable, $\mathbf{1}$-regular, and has EDPC;*
2. *V has a QD term and a Gödel equivalence term;*
3. *V is a congruence permutable WBSO variety.*

1.5.9. TD Terms and Fixedpoint Discriminator Varieties. A ternary term $e(x, y, z)$ is a *ternary deductive term* for a class K of similar algebras if $\mathsf{K} \models e(x, x, z) \approx z$ and, moreover, for all $\mathbf{A} \in \mathsf{K}$ and $a, b, c, d \in A$, $e^{\mathbf{A}}(a, b, c) = e^{\mathbf{A}}(a, b, d)$ if $c \equiv d\big(\mathrm{mod}\,\Theta^{\mathbf{A}}(a, b)\big)$. A version of the following lemma is proved in Blok and Pigozzi [34]. See also Bignall [18, Lemma 2.1] and McKenzie [159, Theorem 1.3].

Lemma 1.5.10. *(cf. [34, Theorem 2.3]) Let V be a variety with a TD term $e(x, y, z)$. For any algebra $\mathbf{A} \in \mathsf{V}$, $\mathrm{Con}\,\mathbf{A} = \mathrm{Con}\,\langle A; e^{\mathbf{A}}\rangle$.*

Proof. Let V be a variety with a TD term $e(x, y, z)$ and let $\mathbf{A} \in \mathsf{V}$. By [34, Theorem 2.3(iii)], $\mathbf{A}$ satisfies the identity:

$$e\big(x, y, f(z_1, \ldots, z_n)\big) \approx e\big(x, y, f(e(x, y, z_1), \ldots, e(x, y, z_n))\big)$$

for every n-ary operation symbol f in the type of $\mathbf{A}$. This implies the relation $\Theta(a, b)$ defined on $A \times A$ by:

$$\Theta(a, b) := \big\{\langle c, d\rangle \in A \times A : e^{\mathbf{A}}(a, b, c) = e^{\mathbf{A}}(a, b, d)\big\}$$

is a congruence relation on $\mathbf{A}$, and therefore on $\langle A;\ e^{\mathbf{A}}\rangle$ as well. Now if $c \equiv d\,\big(\mathrm{mod}\,\Theta(a, b)\big)$ and ϕ is any congruence on $\langle A;\ e^{\mathbf{A}}\rangle$ such that $c \equiv d\,(\mathrm{mod}\,\phi)$, then:

$$c = e^{\mathbf{A}}(a,a,c) \equiv_{\phi} e^{\mathbf{A}}(a,b,c) = e^{\mathbf{A}}(a,b,d) \equiv_{\phi} e^{\mathbf{A}}(a,a,b) = b,$$

which implies that $\Theta(a,b)$ is the principal congruence generated by $\langle a,b\rangle$ on both $\langle A;\ e^{\mathbf{A}}\rangle$ and $\mathbf{A}$. This means that $\langle A;\ e^{\mathbf{A}}\rangle$ and $\mathbf{A}$ have the same principal congruences. But by [55, Theorem II§5.7(d)] any congruence θ on $\langle A;\ e^{\mathbf{A}}\rangle$ is given by:

$$\theta = \bigvee\{\Theta^{\langle A;\, e^{\mathbf{A}}\rangle}(a,b) : a \equiv b\,(\operatorname{mod}\theta)\},$$

where the join is taken in the lattice of equivalence relations on A. By previous remarks, such a join must also yield a congruence relation on $\mathbf{A}$, whence $\operatorname{Con}\langle A;\ e^{\mathbf{A}}\rangle \subseteq \operatorname{Con}\mathbf{A}$. Since the converse is clear, the result follows. ■

Let V be a variety with a TD term $e(x,y,z)$. By the proof of the preceding lemma, $e^{\mathbf{A}}(a,b,c) = e^{\mathbf{A}}(a,b,d)$ iff $c \equiv d\big(\operatorname{mod}\Theta^{\mathbf{A}}(a,b)\big)$, whence V has EDPC [34, Corollary 2.5]. Moreover, V is congruence 3-permutable by [34, Theorem 2.9]; in general, V need not be congruence permutable. The following proposition is implicit in [29].

Proposition 1.5.11. *(cf. [29, p. 361]) For a variety V, the following assertions hold:*

1. *Suppose V has a TD term $e(x,y,z)$. If $p(x,y,z)$ is a Mal'cev term for V then the term:*

   $$q(x,y,z,w) := p\big(e(x,y,z), e(x,y,w), w\big)$$

   is a QD term for V;

2. *Suppose V has a QD term $q(x,y,z,w)$. If V is $\mathbf{1}$-regular (for some constant term $\mathbf{1}$) and $d_1(x,y),\ldots,d_n(x,y)$ are binary terms witnessing the $\mathbf{1}$-regularity of V in the sense of Proposition 1.2.6 then V is a WBSO variety with weak meet, weak relative pseudocomplementation and Gödel*

equivalence terms defined respectively by:

$$\begin{aligned}
x \to y &:= q(x, \mathbf{1}, y, \mathbf{1}) \\
x \cdot y &:= q(x, \mathbf{1}, y, x) \\
x \Delta y &:= (\cdots (d_1(x, y) \cdot d_2(x, y)) \cdot \cdots) \cdot d_n(x, y).
\end{aligned}$$

Moreover, $\mathsf{V} \models \mathbf{1} \to x \approx x$.

Proof. Let V be a variety. For (1), assume V has a TD term $e(x, y, z)$. Suppose first that $p(x, y, z)$ is a Mal'cev term for V. Let $\mathbf{A} \in \mathsf{V}$ and let $a, b, c, d \in A$. Suppose $c \equiv d\big(\text{mod}\, \Theta^{\mathbf{A}}(a, b)\big)$. Then $e^{\mathbf{A}}(a, b, c) = g = e^{\mathbf{A}}(a, b, d)$ for some $g \in A$ by previous remarks and thus:

$$\begin{aligned}
q^{\mathbf{A}}(a, b, c, d) &= p^{\mathbf{A}}\big(e^{\mathbf{A}}(a, b, c), e^{\mathbf{A}}(a, b, d), d\big) \\
&= p^{\mathbf{A}}(g, g, d) \\
&= d && \text{since } p(x, x, y) \approx y.
\end{aligned}$$

On the other hand, suppose $a = b$. Then:

$$\begin{aligned}
q^{\mathbf{A}}(a, b, c, d) &= p^{\mathbf{A}}\big(e^{\mathbf{A}}(a, a, c), e^{\mathbf{A}}(a, a, d), d\big) \\
&= p^{\mathbf{A}}(c, d, d) && \text{since } e(x, x, y) \approx y \\
&= c && \text{since } p(x, y, y) \approx x.
\end{aligned}$$

Thus $q(x, y, z, w)$ is a QD term for V as claimed.

For (2), assume $q(x, y, z, w)$ is a QD term for V. Assume also that V is $\mathbf{1}$-regular (for some constant term $\mathbf{1}$). By Theorem 1.5.7 V is congruence permutable with EDPC, so V is a congruence permutable WBSO variety by Theorem 1.5.8. Let $x \cdot y$, $x \to y$ and $x \Delta y$ be as in the assertion of the proposition. By [29, Theorem 3.5(i),Theorem 3.7(iii)] we have that the term $x \to y$ is a weak relative pseudocomplement for V such that $\mathsf{V} \models \mathbf{1} \to x \approx x$, while by [29, Theorem 3.5(ii)] we have that the term $x \cdot y$ is a weak meet. Let $d_1(x, y), \ldots, d_n(x, y)$ be terms witnessing the $\mathbf{1}$-regularity of V in the sense of

Proposition 1.2.6. Since $x \cdot y$ is a weak meet for V,

$$\begin{aligned}\Theta^{\mathbf{A}}(a\Delta^{\mathbf{A}}b, 1) &= \Theta^{\mathbf{A}}\big((\cdots(d_1^{\mathbf{A}}(a,b) \cdot^{\mathbf{A}} d_2^{\mathbf{A}}(a,b)) \cdot^{\mathbf{A}} \cdots) \cdot^{\mathbf{A}} d_n^{\mathbf{A}}(a,b), 1\big) \\ &= \Theta^{\mathbf{A}}\big(d_1^{\mathbf{A}}(a,b),1\big) \vee \Theta^{\mathbf{A}}\big(d_2^{\mathbf{A}}(a,b),1\big) \vee \cdots \vee \Theta^{\mathbf{A}}\big(d_n^{\mathbf{A}}(a,b),1\big) \\ &= \Theta^{\mathbf{A}}(a,b)\end{aligned}$$

by [29, Theorem 0.7]. Thus $x\Delta y$ is a Gödel equivalence term for V as claimed.
∎

A TD term $e(x,y,z)$ on an algebra $\mathbf{A}$ is *commutative* if $e^{\mathbf{A}}\big(a,b,e^{\mathbf{A}}(a',b',c)\big) = e^{\mathbf{A}}\big(a',b',e^{\mathbf{A}}(a,b,c)\big)$ for all $a,b,a',b',c \in A$; a TD term $e(x,y,z)$ on a variety V is *commutative* if it is commutative on every member of V. Let A be a set. A ternary operation $f : A^3 \to A$ is called a *fixedpoint discriminator* if there exists an element $d \in A$ such that [34, Definition 3.3]:

$$f(a,b,c) := \begin{cases} c & \text{if } a = b \\ d & \text{otherwise,} \end{cases}$$

in which case d is called the *discriminating element* of f. Note that in general the discriminating element associated with a fixedpoint discriminator in a fixedpoint discriminator variety need not be a constant term [34, p. 580]. An algebra $\mathbf{A}$ is called a *fixedpoint discriminator algebra* if there is a ternary term f of $\mathbf{A}$ that realises the fixedpoint discriminator on $\mathbf{A}$. A variety V is a *fixedpoint discriminator variety* if there is a ternary term f of V and a subclass K of V such that $f^{\mathbf{A}}$ is a fixedpoint discriminator on each $\mathbf{A} \in \mathsf{K}$ and $\mathsf{V} = \mathbf{V}(\mathsf{K})$. In this case f is called a *fixedpoint discriminator term* for V.

Theorem 1.5.12. *[34, Theorem 3.5] For any variety V the following are equivalent:*

1. *V is a fixedpoint discriminator variety;*

2. *V is semisimple and has a commutative TD term.*

Moreover, if the equivalent conditions (1)–(2) are met, then a ternary term $e(x,y,z)$ is a commutative TD term for V iff it is a fixedpoint discriminator term for V.

Let V be a variety with $\mathbf{0}$. By remarks due to Blok and Pigozzi [34, p. 582], if V is a ternary discriminator variety then it is a fixedpoint discriminator variety. Indeed, if $t(x, y, z)$ is a discriminator term for V then $f(x, y, z) := t\big(t(x, y, z), t(x, y, \mathbf{0}), \mathbf{0}\big)$ is a fixedpoint discriminator term for V. More generally, we have:

Theorem 1.5.13. *[34, Theorem 3.8] For any pointed variety* V *the following are equivalent:*

1. V *is a ternary discriminator variety;*
2. V *is a congruence permutable fixedpoint discriminator variety;*
3. V *is congruence permutable, semisimple and has a commutative TD term;*
4. V *is congruence permutable, semisimple and has EDPC;*
5. V *is a congruence permutable variety of weak Boolean algebras with filter preserving operations.*

If these conditions hold then any constant term can be taken to be the discriminating element of a fixedpoint discriminator term of V.

Let $\mathbf{A}$ be an algebra with $\mathbf{1}$. A TD term $e(x, y, z)$ on $\mathbf{A}$ such that $a \equiv b \,\big(\mathrm{mod}\,\Theta^{\mathbf{A}}(e^{\mathbf{A}}(a, b, 1), 1)\big)$ for all $a, b \in A$ is said to be *regular (for* $\mathbf{A}$*) with respect to* $\mathbf{1}$; note $e(x, y, \mathbf{1})$ witnesses $\mathbf{1}$-regularity in the sense of Proposition 1.2.6 by the remarks of [34, p. 585] and hence that $\mathbf{1}^{\mathbf{A}}$ is a regular element of $\mathbf{A}$ in the usual sense. Let V be a variety with $\mathbf{1}$ and let $e(x, y, z)$ be a TD term for V. Call $e(x, y, z)$ *regular (for* V*) with respect to* $\mathbf{1}$ if it is regular with respect to $\mathbf{1}$ for every member of V.

Theorem 1.5.14. *Let* V *be a variety with* $\{\mathbf{0}, \mathbf{1}\}$. *Suppose moreover that* V *is a fixedpoint discriminator variety generated by a class* $\mathsf{K} \subseteq \mathsf{V}$ *of fixedpoint discriminator algebras, that* $f(x, y, z)$ *is a fixedpoint discriminator term for* V *and that* $\mathbf{0}^{\mathbf{A}}$ *is the discriminating element on any* $\mathbf{A} \in \mathsf{K}$. *Then the following statements hold:*

1. $f(x, y, z)$ *is a commutative TD term for* V *that is regular with respect to* $\mathbf{1}$*;*

2. V *is a variety of weak Boolean algebras with filter preserving operations whose weak meet, weak relative pseudocomplement and Gödel equivalence terms are defined respectively by:*

$$\begin{aligned} x \cdot y &:= f(x, \mathbf{1}, y) \\ x \to y &:= f\big(f(x, \mathbf{1}, y), f(x, \mathbf{1}, \mathbf{1}), \mathbf{1}\big) \\ x \Delta y &:= f(x, y, \mathbf{1}). \end{aligned}$$

Proof. The first assertion is just [34, Corollary 4.8]. In view of the first assertion, Theorem 1.5.12 and [34, Theorem 4.4], V is a semisimple WBSO variety with the stated weak meet, weak relative pseudocomplementation and Gödel equivalence terms. Since any semisimple WBSO variety is a variety of weak Boolean algebras with filter preserving operations (by [30, Corollary 4.3]), the second assertion follows. ■

The fixedpoint discriminator was introduced by Blok and Pigozzi in their study of varieties with EDPC [34] as a generalisation of the ternary discriminator to varieties for which congruence permutability fails, while the TD term was introduced in the same paper as a generalisation of the fixedpoint discriminator. See [34, Definition 3.3; pp. 580–583; pp. 588-590] and [35] for more details.

1.6 BCK-Algebras

BCK-algebras play a central role in this thesis. Here we briefly survey the elementary theory of BCK-algebras and some related classes.

1.6.1. BCK-Algebras. For the sake of convenience we repeat here the definition given in §1.1.1. An algebra $\langle A;\ \dot{-}, 0\rangle$ of type $\langle 2, 0\rangle$ is called a *BCK-algebra* iff it satisfies the following identities and quasi-identity [126, Definition 1]:

$$\big((x \dot{-} y) \dot{-} (x \dot{-} z)\big) \dot{-} (z \dot{-} y) \approx \mathbf{0} \tag{1.1}$$

$$\big(x \dot{-} (x \dot{-} y)\big) \dot{-} y \approx \mathbf{0} \tag{1.2}$$

$$x \dot{-} x \approx 0 \tag{1.3}$$

$$0 \dot{-} x \approx 0 \tag{1.4}$$

$$x \dot{-} y \approx 0 \;\&\; y \dot{-} x \approx 0 \supset x \approx y. \tag{1.5}$$

It is immediate that the class BCK of BCK-algebras is a quasivariety; results due to Wroński [240] and Higgs [109] show BCK is not a variety. Because of (1.3) the class of BCK-algebras may be construed as a quasivariety of groupoids; consequently we (informally) denote BCK difference $\dot{-}$ in the sequel by juxtaposition when no confusion can arise.

BCK-algebras were introduced by Imai and Iséki in a 1966 paper [119]. Historically, the motivation behind the introduction of BCK-algebras was twofold. First, Imai and Iséki wished to give an abstract characterisation of set difference and its properties; and second, Imai and Iséki were interested in investigating systems of implicational calculi related to combinatory logic, particularly the BCK system of Meredith [186, p. 316]. The connection between the two motivations arises from the close relationship observed between set difference in set theory and implication in propositional calculi. Since their introduction BCK-algebras have been the subject of a vast amount of critical exegesis (see for instance the survey articles [126] and [70] and the more recent paper [38] of Blok and Raftery), and in particular connections with the original motivations of Imai and Iséki have been clarified. Indeed, results due to Pałasiński [178], Ono and Komori [176], Fleischer [90] and Wroński [242] show that an algebra $\langle A; \ \dot{-}, 0\rangle$ of type $\langle 2, 0\rangle$ is a BCK-algebra iff it is the $\langle \dot{-}, 0\rangle$-subreduct of a pocrim; see Blok and Raftery [39] and van Alten [229, Chapter 1] for details. On the other hand, Blok and Pigozzi [31, Section 5.2.3] have shown BCK is termwise definitionally equivalent to the equivalent algebraic semantics (in the sense of [31]) of BCK logic, while Bunder [51, Theorem 1] has proved that an algebra $\langle A; \ \dot{-}, 0\rangle$ of type $\langle 2, 0\rangle$ is a BCK-algebra iff it satisfies the quasi-identity (1.5) and the algebraic analogues of the **B**-combinator $(p \to q) \to \big((q \to r) \to (p \to r)\big)$, the **C**-combinator $\big(p \to (q \to r)\big) \to \big(q \to (p \to r)\big)$ and the **K**-combinator $p \to (q \to p)$, respectively the identity (1.1) and the identities:

$$(x \dot{-} y) \dot{-} z \approx (x \dot{-} z) \dot{-} y \tag{1.29}$$

$$(x \dot{-} y) \dot{-} x \approx \mathbf{0}. \tag{1.30}$$

Another important identity known to be satisfied by the quasivariety of BCK-algebras is [126, Theorem 2]:

$$x \dot{-} \mathbf{0} \approx x \tag{1.7}$$

and in fact Higgs [109, p. 70] and Blok and Raftery [38, Section 1] have shown independently that an algebra $\langle A;\ \dot{-}, 0\rangle$ of type $\langle 2, 2, 0\rangle$ is a BCK-algebra iff it satisfies the identities (1.1), (1.4), (1.7) and the quasi-identity (1.5).

Lemma 1.6.2. *[126, p. 4] Let $\langle A;\ \dot{-}, 0\rangle$ be a BCK-algebra and let $\leq_0$ be the relation defined by $a \leq_0 b$ iff $ab = 0$ for any $a, b \in A$. Then $\langle A;\ \leq_0\rangle$ is a partially ordered set with 0 as its least element. Moreover, right [left] multiplication by a fixed element of A is isotone [antitone].*

On an arbitrary BCK-algebra $\mathbf{A}$ there exists a second partial order $\leq_1$, coarser than $\leq_0$, and defined by $a \leq_1 b$ iff $a \in bA$ for any $a, b \in A$ (where $bA := \{bc : c \in A\}$); this observation is due to Guzmán [105, Proposition 3.2(a)]. Proposition 1.6.4 below, which sharpens Guzmán's result, will be needed in the sequel; we remark that this proposition does not seem to have been reported in the literature previously.

Lemma 1.6.3. *(cf. [105, Lemma 1.2(i), Proposition 2.2(c)]) The quasivariety of BCK-algebras satisfies the following identities:*

$$x \dot{-} \big(x \dot{-} (x \dot{-} y)\big) \approx x \dot{-} y \tag{1.31}$$

$$(x \dot{-} y) \dot{-} z \approx x \dot{-} \big(x \dot{-} ((x \dot{-} y) \dot{-} z)\big) \tag{1.32}$$

$$(x \dot{-} y) \dot{-} \big(x \dot{-} (z \dot{-} (z \dot{-} y))\big) \approx \mathbf{0} \tag{1.33}$$

$$(x \dot{-} y) \dot{-} \big((x \dot{-} y) \dot{-} z\big) \approx \mathbf{0}. \tag{1.34}$$

Proof. Let $\langle A;\ \dot{-}, 0\rangle$ be a BCK-algebra and let $a, b, c \in A$. To see (1.31) holds, observe that $a(ab) \leq_0 b$ by (1.2), whence $ab \leq_0 a\big(a(ab)\big)$ by Lemma 1.6.2. On the other hand, note that $a\big(a(ab)\big) \leq_0 ab$ by (1.2). Thus $a\big(a(ab)\big) = ab$ by (1.5). For (1.32), we have $0 = \big(a(a((ab)c))\big)\big((ab)c\big)$ by (1.2), whence $a\big(a((ab)c)\big) \leq_0 (ab)c$. Conversely, put $\alpha := a$ and $\beta := (ab)c$. We have:

$$
\begin{aligned}
0 &= \big((ab)c\big)\big((ab)c\big) && \text{by (1.3)}\\
&= \big((ab)((ab)c)\big)c && \text{by (1.29)}\\
&= \big((a((ab)c))b\big)c && \text{by (1.29)}\\
&= \big((\alpha\beta)b\big)c &&\\
&= \big((\alpha(\alpha(\alpha\beta)))b\big)c && \text{by (1.31)}\\
&= \big((a(a(a((ab)c))))b\big)c &&\\
&= \big((ab)(a(a((ab)c)))\big)c && \text{by (1.29)}\\
&= \big((ab)c\big)\big(a(a((ab)c))\big) && \text{by (1.29).}
\end{aligned}
$$

Thus $(ab)c \leq_0 a\big(a((ab)c)\big)$, which establishes (1.32). For (1.33), we have $0 = \big((ab)(a(c(cb)))\big)\big((c(cb))b\big) = (ab)\big(a(c(cb))\big)$ by (1.1), (1.2) and (1.7). For (1.34), we have $0 = \big((ab)(a(bc))\big)\big((bc)b\big) = (ab)\big(a(bc)\big)$ by (1.1), (1.30) and (1.7). ∎

Proposition 1.6.4. *Let* **A** *be a BCK-algebra. The following statements hold:*

1. *The binary relation $\leq_1$ defined for any $a, b \in A$ by:*

$$a \leq_1 b \quad \textit{iff} \quad a \in bA \quad \textit{iff} \quad a \in b \cap A \quad \textit{iff} \quad b \cap a = a$$

where $b \cap a := b(ba)$ and $b \cap A := \{b \cap c : c \in A\}$, is a partial order on A. Moreover for any $a, b \in A$, the relation $\leq_1$ enjoys the following properties:

(i) *If $a \leq_1 b$ then $a \leq_0 b$;*

(ii) *If $a \leq_1 b$ then $ac \leq_1 bc$;*

(iii) $0 \leq_1 a$.

2. *The binary relation $\leq_2$ defined for any $a, b \in A$ by:*

$$a \leq_2 b \quad \textit{iff} \quad \big(b(ba)\big)(ba) = a \quad \textit{iff} \quad b \cap a = a \textit{ and } a(ba) = a$$

is a partial order on A. Moreover for any $a, b \in A$, the relation $\leq_2$ enjoys the following properties:

(i) *If $a \leq_2 b$ then $a \leq_1 b$;*

(ii) *If $a \leq_2 b$ then $ac \leq_2 bc$;*

(iii) $0 \leq_2 a$.

Proof. To prove $\leq_1$ is a partial ordering under the stated conditions it is sufficient to verify the equivalences $a \in bA$ iff $a \in b \cap A$ iff $b \cap a = a$, just because the relation $\leq_1$ defined by $a \leq_1 b$ iff $a \in bA$ is a partial order on A by [105, Proposition 3.2(a)]. So suppose that $a \in bA$. Then $a = bc$ for some $c \in A$, whence $a = bc = b\big(b(bc)\big) \in b \cap A$ by (1.31). Suppose $a \in b \cap A$. Then $a = b(bc)$ for some $c \in A$. By (1.31) we have $a = b(bc) = b\big(b(b(bc))\big) = b \cap a$. Suppose $b \cap a = a$. Then $a = b(ba) \in bA$, and so $a \in bA$ iff $a \in b \cap A$ iff $b \cap a = a$ as required. To see Item (1)(i) holds, observe $a \leq_1 b$ implies $0 = (b(ba))b = ab$ by (1.30) and hence that $a \leq_0 b$. For (1)(ii), suppose $a \leq_1 b$. We have:

$$\begin{aligned}
(bc)\big((bc)(ac)\big) &= (bc)\big((bc)((b(ba))c)\big) && \text{since } b \cap a = a \\
&= (bc)\big((bc)((bc)(ba))\big) && \text{by (1.29)} \\
&= (bc)(ba) && \text{by (1.)} \\
&= \big(b(ba)\big)c && \text{by (1.2)} \\
&= ac && \text{since } b \cap a = a.
\end{aligned}$$

Thus $ac \leq_1 bc$. Item (1)(iii) is clear.

To prove $\leq_2$ is a partial ordering under the stated conditions we first show $\big(b(ba)\big)(ba) = a$ iff $b \cap a = a$ and $a(ba) = a$ for any $a, b \in A$. Suppose $\big(b(ba)\big)(ba) = a$. We have:

$$b \cap a = \big(b(ba)\big)0 \qquad \text{by (1.7)}$$

$$
\begin{aligned}
&= \big(b(ba)\big)\big((b(ba))a\big) && \text{by (1.2)} \\
&= \big(b(ba)\big)\big((b(ba))((b(ba))(ba))\big) && \text{since } \big(b(ba)\big)(ba) = a \\
&= \big(b(ba)\big)(ba) && \text{by (1.31)} \\
&= a && \text{since } \big(b(ba)\big)(ba) = a.
\end{aligned}
$$

It follows that $a = \big(b(ba)\big)(ba) = a(ba)$. Conversely, if $b \cap a = a$ and $a(ba) = a$ then $a = a(ba) = \big(b(ba)\big)(ba)$. Thus $\big(b(ba)\big)(ba) = a$ iff $b \cap a = a$ and $a(ba) = a$ as claimed. To see $\leq_2$ is a partial order we verify the properties of reflexivity, anti-symmetry and transitivity directly. For reflexivity just note $a \leq_2 a$ from $\big(a(aa)\big)(aa) = (a0)0 = a$. For anti-symmetry, observe $a \leq_2 b$ implies $a \leq_1 b$ and likewise $b \leq_2 a$ implies $b \leq_1 a$, whence $a = b$. For transitivity, suppose $a \leq_2 b$ and $b \leq_2 c$. It is sufficient to show $c \cap a = a$ and $a(ca) = a$. To see $c \cap a = a$, put $\alpha := c$, $\beta := cb$, $\gamma := ba$ and observe:

$$
\begin{aligned}
a &= b(ba) && \text{since } b \cap a = a \\
&= \big(c(cb)\big)(ba) && \text{since } c \cap b = b \\
&= (\alpha\beta)\gamma && \\
&= \alpha\big(\alpha((\alpha\beta)\gamma)\big) && \text{by (1.32)} \\
&= c\big(c((c(cb))(ba))\big) && \\
&= c\big(c(b(ba))\big) && \text{since } c \cap b = b \\
&= c(ca) && \text{since } b \cap a = a.
\end{aligned}
$$

To see $a = a(ca)$, observe:

$$
\begin{aligned}
a(ca) &= \big(b(ba)\big)(ca) && \text{since } b \cap a = a \\
&= \big(b(ca)\big)(ba) && \text{by (1.29)} \\
&= \big((c(cb))(ca)\big)(ba) && \text{since } c \cap b = b \\
&= \big((c(ca))(cb)\big)(ba) && \text{by (1.29)} \\
&= \big(a(cb)\big)(ba) && \text{since } c \cap a = a \\
&= \big((b(ba))(cb)\big)(ba) && \text{since } b \cap a = a \\
&= \big((b(cb))(ba)\big)(ba) && \text{by (1.29)}
\end{aligned}
$$

$$
\begin{aligned}
&= \big(b(ba)\big)(ba) && \text{since } b(cb) = b \\
&= a(ba) && \text{since } b \cap a = a \\
&= a && \text{since } a(ba) = a.
\end{aligned}
$$

Thus $\leq_2$ is transitive and hence a partial order. Item (2)(i) is clear from preceding remarks. For (2)(ii), it is sufficient to show $a \leq_2 b$ implies $ac \leq_1 bc$ and $(ac)\big((bc)(ac)\big) = ac$. So suppose $a \leq_2 b$. Then $a \leq_1 b$ and so $ac \leq_1 bc$. Moreover, $\big((ac)((bc)(ac))\big)(ac) = \big((ac)(ac)\big)\big((bc)(ac)\big) = 0\big((bc)(ac)\big) = 0$ by (1.29) and (1.7) and so $(ac)\big((bc)(ac)\big) \leq_0 ac$. On the other hand, put $\alpha := ac$, $\beta := ba$ and $\gamma := bc$. We have:

$$
\begin{aligned}
(ac)\big((ac)((bc)(ac))\big) &= (ac)\big((ac)((bc)((b(ba))c))\big) && \text{since } b \cap a = a \\
&= (ac)\big((ac)((bc)((bc)(ba)))\big) && \text{by (1.29)} \\
&= \big((a(ba))c\big)\big((ac)((bc)((bc)(ba)))\big) && \text{since } a(ba) = a \\
&= \big((ac)(ba)\big)\big((ac)((bc)((bc)(ba)))\big) && \text{by (1.29)} \\
&= (\alpha\beta)\big(\alpha(\gamma(\gamma\beta))\big) \\
&= 0 && \text{by (1.33)}.
\end{aligned}
$$

Thus $ac \leq_0 (ac)\big((bc)(ac)\big)$. We conclude $(ac)\big((bc)(ac)\big) = ac$ and thus $a \leq_2 b$ implies $ac \leq_2 bc$ as claimed. Item (2)(iii) is clear. ∎

In the sequel we work primarily with the partial order $\leq_0$; we write simply $\leq$ for $\leq_0$ when there is no danger of confusion.

Example 1.6.5. The partial orders $\leq_0$, $\leq_1$ and $\leq_2$ are distinct on an arbitrary BCK-algebra. To see this, consider the BCK-algebra $\mathbf{A} := \langle A; \dot{-}, 0\rangle$ where $A := \{0, a, b, c, 1\}$ and BCK difference is defined by the following operation

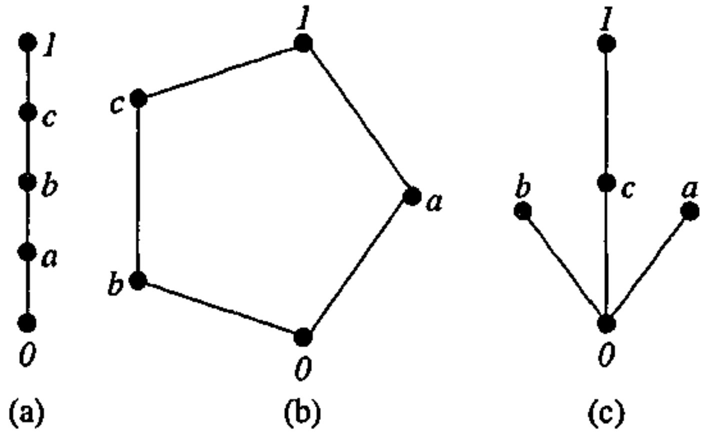

Figure 1.3. The BCK-algebra of Example 1.6.5: (a) Under the partial order $\leq_0$; (b) Under the partial order $\leq_1$; (c) Under the partial order $\leq_2$.

table:

$\dot{-}^{\mathbf{A}}$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	c	b	0	0
1	1	c	b	a	0

The Hasse diagrams corresponding to the underlying partially ordered sets $\langle A;\ \leq_0\rangle$, $\langle A;\ \leq_1\rangle$ and $\langle A;\ \leq_2\rangle$ are depicted in Figure 1.3. From these diagrams it is clear that the partial orders $\leq_0$, $\leq_1$ and $\leq_2$ are distinct. ■

Remark 1.6.6. In contrast to $\leq_0$, $\leq_1$ and $\leq_2$ are not in general antitone in each of their positions. To see this, consider the BCK-algebra of Example 1.6.5. One easily checks that $b \leq_1 c$, but $a = 1c \not\leq_1 1b = b$. Likewise $0 \leq_2 c$ but $a = 1c \not\leq_2 10 = 1$. ■

Remark 1.6.7. Define the $\langle \dot{-}, 0\rangle$-terms $x \dot{-} y^n$, $n \in \omega$ inductively by:

$$x \dot{-} y^0 := x$$

$$x \dot{-} y^{k+1} := (x \dot{-} y^k) \dot{-} y \qquad \text{for } k \geq 0.$$

Owing to (1.30) these terms form a descending chain in an arbitrary BCK-algebra: $xy^k \leq \ldots \leq xy^2 \leq xy \leq x$. Let **A** be a BCK-algebra and $a, b \in A$. In the preceding parlance, the BCK partial orders $\leq_0$, $\leq_1$ and $\leq_2$ can be described by:

$$\begin{aligned} a \leq_0 b \quad &\text{iff} \quad ab = 0 \\ a \leq_1 b \quad &\text{iff} \quad b(ba)^1 = a \\ a \leq_2 b \quad &\text{iff} \quad b(ba)^2 = a. \end{aligned}$$

From the description above of $\leq_0$, $\leq_1$ and $\leq_2$ it is natural to anticipate that the family of relations $\{\leq_n : n \geq 1\}$ defined by $a \leq_n b$ iff $b(ba)^n = a$ is an infinitely descending chain of partial orders on A. This is not the case: using (1.31) and (1.32) and the $\langle \dot{-}, 0\rangle$-terms xy^n it is possible to show directly that, for any $n > 2$, $a \leq_{n+1} b$ iff $a \leq_n b$. ∎

Problem 1.6.8. Investigate the behaviour of the partial orders $\leq_i$, $i = 0, 1, 2$, on a BCK-algebra. Does there exist a natural family of partial orders on an arbitrary BCK-algebra generalising the orders $\leq_i$, $i = 0, 1, 2$? ∎

Let **A** be a BCK-algebra. An *ideal* of **A** is a subset I of A such that $0 \in I$ and $a, ba \in I$ implies $b \in I$. It is folklore that the set $\mathrm{I}(\mathbf{A})$ of all ideals of **A** forms an algebraic lattice $\mathbf{I}(\mathbf{A})$ under set inclusion. Let $\varnothing \neq B \subseteq A$. The *ideal* $\langle B\rangle_{\mathbf{A}}$ *of* **A** *generated by* B is $\bigcap\{J \in \mathrm{I}(\mathbf{A}) : B \subseteq J\}$, the intersection of all ideals of **A** containing B. By definition, $\langle\varnothing\rangle_{\mathbf{A}} := \{0\}$. The following characterisation of $\langle B\rangle_{\mathbf{A}}$ for non-empty $B \subseteq A$ is due to Iséki [120, Theorem 3]:

$$\langle B\rangle_{\mathbf{A}} := \big\{a \in A : (\exists\, n \in \omega)(\exists\, b_1, \ldots, b_n \in B)(ab_1 \ldots b_n = 0)\big\}.$$

The following technical result, which will be needed in the sequel, is due to Pałasinski [177] and Cornish [70].

Proposition 1.6.9. *For any BCK-algebra* **A**, *the following assertions hold:*

1. *[177, Theorem 1] The ideal lattice of* **A** *is distributive;*

2. *[70, Theorem 4.1]* **A** *enjoys the* ideal extension property*: whenever* $\mathbf{B} \in \mathbb{S}(\mathbf{A})$ *and* $I \in \mathrm{I}(\mathbf{B})$ *there exists* $J \in \mathrm{I}(\mathbf{A})$ *such that* $J \cap B = I$*. In*

particular, $\langle I\rangle_{\mathbf{A}}$, the ideal of $\mathbf{A}$ *generated by I, can always be taken as a suitable J.*

In general, an ideal I of $\mathbf{A}$ is the 0-class of at least one, and possibly many congruences on $\mathbf{A}$, of which the largest $\phi_I := \{(a,b) \in A \times A : ab, ba \in I\}$ is actually a BCK-congruence. Conversely, for any $\theta \in \operatorname{Con}\mathbf{A}$, the 0-class $\{a \in A : (a,0) \in \theta\}$ is an ideal of $\mathbf{A}$.

Theorem 1.6.10. *[38, Proposition 1] For a BCK-algebra* $\mathbf{A}$ *the following assertions hold:*

1. *The maps $I \overset{\xi}{\mapsto} \phi_I$ $(I \in \mathrm{I}(\mathbf{A}))$ and $\theta \overset{\xi^{-1}}{\mapsto} [0]_\theta$ $(\theta \in \mathrm{Con}_{\mathsf{BCK}}\,\mathbf{A})$ are mutually inverse lattice isomorphisms between the* BCK*-congruence lattice of* $\mathbf{A}$ *and the ideal lattice of* $\mathbf{A}$*;*

2. $\mathbf{A}$ *is* BCK-0*-regular,* BCK*-congruence distributive and enjoys the* BCK*-congruence extension property;*

3. $\mathsf{H}(\mathbf{A}) \subseteq \mathsf{BCK}$ *iff* $\mathbf{A}$ *is* 0*-regular, in which case* $\mathbf{A}$ *is also congruence distributive. If* $\mathsf{HS}(\mathbf{A}) \subseteq \mathsf{BCK}$*, then* $\mathbf{A}$ *has the congruence extension property.*

It is known that the congruences (in the absolute sense) of BCK-algebras are not well-behaved in general. In particular, Wroński [241, Theorem 5, Theorem 6] and Nagayama [173, Theorem 1.3] (see also Example 2.3.10 in the sequel) have shown that the congruence lattices of BCK-algebras need satisfy no lattice identities beyond those satisfied by all lattices. An example due to Blok and Raftery [37] shows also that BCK-algebras do not in general enjoy the congruence extension property. In contrast, Proposition 1.6.9 and Theorem 1.6.10 show the situation to be quite different for *BCK-varieties*, namely those varieties of algebras of type $\langle 2, 0\rangle$ whose members are BCK-algebras. In the remainder of this section we describe some BCK-varieties of relevance to the sequel, together with several equational classes of BCK-algebras augmented with additional operations.

1.6.11. Commutative BCK-Algebras. A BCK-algebra $\langle A;\ \dot{-}, 0\rangle$ for which the underlying partially ordered set $\langle A;\ \leq\rangle$ is a meet semilattice is called a *commutative BCK-algebra*; in this case $a \cap b := a(ab)$ (the *BCK meet*) is the greatest lower bound of $\{a, b\}$ for any $a, b \in A$. For any BCK-algebra **A**, the following are equivalent [126, Theorem 3]: (i) **A** is a commutative BCK-algebra; (ii) $\mathbf{A} \models x \cap y \approx y \cap x$. From (ii) it follows easily that the class cBCK of all commutative BCK-algebras is a variety [245]. Problem 1.6.8 notwithstanding, we also have the following characterisation of commutative BCK-algebras, which will be needed in the sequel. See also Cornish [71, Proposition 1.8].

Lemma 1.6.12. *A BCK-algebra is commutative iff its underlying partial orders $\leq_0$ and $\leq_1$ coincide.*

Proof. Let **A** be a BCK-algebra and let $a, b \in A$.

($\Rightarrow$) Suppose **A** is commutative. If $a \leq_1 b$ then $a \leq_0 b$ by Proposition 1.6.4(1)(i). For the converse, if $a \leq_0 b$ then $a \cap b = a$, and hence also $b \cap a = a$ since **A** is commutative. But this means that $a \leq_1 b$ by Proposition 1.6.4(1)(i).

($\Leftarrow$) Suppose $a \leq_0 b$ iff $a \leq_1 b$. Because $a \cap b \leq_0 b$, by hypothesis we have that $a \cap b \leq_1 b$, whence $b \cap (a \cap b) = a \cap b$ by Proposition 1.6.4(1)(i). Put $\alpha := b$, $\beta := a$ and $\gamma := ab$. We have:

$$\begin{aligned}
(a \cap b)(b \cap a) &= \big(b \cap (a \cap b)\big)(b \cap a) \\
&= \big(b(b(a(ab)))\big)\big(b(ba)\big) \\
&= \big(b(b(ba))\big)\big(b(a(ab))\big) && \text{by (1.29)} \\
&= (ba)\big(b(a(ab))\big) && \text{by (1.31)} \\
&= (\alpha\beta)\big(\alpha(\beta\gamma)\big) \\
&= 0 && \text{by (1.34).}
\end{aligned}$$

Thus $a \cap b \leq_0 b \cap a$, and by symmetry $b \cap a \leq_0 a \cap b$. Therefore $a \cap b = b \cap a$, and **A** is commutative. ∎

Commutative BCK-algebras were introduced by Tanaka [214] and have been studied subsequently by several authors, including Traczyck [216], Cornish [64, Section 3], Romanowska and Traczyck [202, 201], Cornish, Sturm and Traczyk [73] and Yutani [245]. In particular, results due to Traczyk [216, Lemma 2.1, Theorem 2.4] show that the underlying meet semilattice $\langle A;\ \cap\rangle$ of a commutative BCK-algebra $\mathbf{A}$ is in fact a distributive nearlattice; recall from [70, p. 112] and [64, p. 489] that a *[distributive] nearlattice* is a lower semilattice $\langle A;\ \wedge\rangle$ in which each principal order ideal $(m] := \{a \in A : a \leq m\}$ is a [distributive] lattice. For $a \in (m]$, the map $N_m : (m] \to (m]$ defined by $N_m(a) := ma$ is an involution (dual order isomorphism that is its own inverse), whence the supremum of $a, b \in (m]$ is $a \cup b := N_m\big(N_m(a) \cap N_m(b)\big)$. See Cornish [64, Lemma 3.1] and Cornish and Hickman [72].

1.6.13. Positive Implicative BCK-Algebras. Let $\mathbf{A}$ be a BCK-algebra with underlying poset $\langle A;\ \leq\rangle$. For any $a, b, c \in A$, $(ab)c \leq (ac)(bc)$ [126, p. 12]; in general, the opposite inclusion does not hold. A *positive implicative BCK-algebra* is a BCK-algebra for which the inequality $(ac)(bc) \leq (ab)c$ is identically satisfied. For a BCK-algebra $\mathbf{A}$, the following are equivalent [38, Proposition 13]: (i) $\mathbf{A}$ is positive implicative; (ii) $\mathbf{A} \models (x \dot{-} y) \dot{-} y \approx x \dot{-} y$; (iii) $\mathbf{A} \models \big(x \dot{-} (x \dot{-} y)\big) \dot{-} (y \dot{-} x) \approx \big(y \dot{-} (y \dot{-} x)\big) \dot{-} (x \dot{-} y)$. Henkin [108] appears to have been among the first to consider positive implicative BCK-algebras: they are precisely his class of *implicative models*. Since Henkin's 1950 paper positive implicative BCK-algebras have been independently investigated by a number of authors, including Diego [79] (in dually isomorphic form under the name *Hilbert algebras*—see Kondo [140]), Rasiowa [195, Section II§2] (in dually isomorphic form under the name *positive implication algebras*), Iseki and Tanaka [126] and more recently Blok and Raftery [39]. Results due to Diego [79] show the class pBCK of positive implicative BCK-algebras is a locally finite variety, while results due to Blok and Pigozzi [34, Corollary 1.23], Cornish [67] and Blok and Raftery [39, p. 294] show pBCK is precisely the class of all $\langle \dot{-}, 0\rangle$-subreducts of dual Brouwerian semilattices. The following lemma is a variant on this last.

Lemma 1.6.14. *[7, Lemma 3.2] If each two elements a, b from a non-empty subset B of a join semilattice $\mathbf{A}$ have a dual relative pseudocomplement $a * b$ that belongs to B, then $\langle B;\ *\rangle$ is a positive implicative BCK-algebra.*

Recall from Remark 1.6.7 that the $\langle \dot{-}, 0\rangle$-terms $x \dot{-} y^n$, $n \in \omega$, are defined inductively by:

$$x \dot{-} y^0 := x$$

$$x \dot{-} y^{k+1} := (x \dot{-} y^k) \dot{-} y \qquad \text{for } k \geq 0.$$

In [68] Cornish studied BCK-algebras satisfying:

$$x \dot{-} y^{n+1} \approx x \dot{-} y^n \tag{E$_n$}$$

where $1 \leq n \in \omega$ as a natural generalisation of positive implicative BCK-algebras; the variety of positive implicative BCK-algebras is just the class of all BCK-algebras satisfying (E_1). For each $n \in \omega$, the class $\mathsf{e_n BCK}$ of BCK-algebras satisfying (E_n) is a variety [68, Theorem 1.4], the members of which are known as *n-potent* BCK-algebras. Cornish has shown these varieties form a strictly increasing chain [70, Section 3.6]. Since their introduction the varieties $\mathsf{e_n BCK}$, $n \in \omega$, have been studied by a number of authors, including Cornish [70, Section 4], Blok and Raftery [38, 39] and Pałasiński [179], to whom the following theorem is collectively due.

Theorem 1.6.15. *[70, Corollary 4.2]; [179]; [39, Theorem 4.2] For a variety* V *of BCK-algebras the following assertions hold:*

1. V *has a commutative TD term iff* V *is a subvariety of* $\mathsf{e_n BCK}$ *for some* $n \in \omega$. *If* V *is a subvariety of* $\mathsf{e_n BCK}$, $n \in \omega$, *then a commutative (but not regular) TD term for* V *is:*

$$e(x, y, z) := \big(z \dot{-} (x \dot{-} y)^n\big) \dot{-} (y \dot{-} x)^n.$$

2. V *has EDPC iff* V *is a subvariety of* $\mathsf{e_n BCK}$ *for some* $n \in \omega$. *For any algebra* $\mathbf{A} \in \mathsf{e_n BCK}$, $n \in \omega$, *and* $a, b, c, d \in A$,

$$\begin{aligned} c \equiv d \left(\operatorname{mod} \Theta^{\mathbf{A}}(a, b)\right) \quad &\textit{iff} \quad \big(c(ab)^n\big)(ba)^n = \big(d(ab)^n\big)(ba)^n \\ &\textit{iff} \quad \big((cd)(ab)^n\big)(ba^n) = \big((dc)(ab)^n\big)(ba)^n. \end{aligned}$$

1.6.16. Implicative BCK-Algebras. Let $\mathbf{A} := \langle A;\ /, 0\rangle$ be a BCK-algebra. If $\mathbf{A} \models x/(y/x) \approx x$ then $\mathbf{A}$ is called an *implicative* BCK-algebra. A well known result of Iséki [126, Theorems 9,10] asserts that a BCK-algebra is implicative iff it is both commutative and positive implicative, whence the class of implicative BCK-algebras is a variety. The following equational characterisation is due independently to Kalman [131] and Abbot [2].

Theorem 1.6.17. *([131, p. 402]; [2, Section 1]) An algebra $\langle A;\ /, 0\rangle$ of type $\langle 2, 0\rangle$ is an implicative BCK-algebra iff it satisfies the following identities:*

$$x/(y/x) \approx x \tag{1.35}$$

$$x/(x/y) \approx y/(y/x) \tag{1.36}$$

$$(x/y)/z \approx (x/z)/y \tag{1.37}$$

$$x/x \approx \mathbf{0}. \tag{1.38}$$

Thus the class iBCK *of implicative BCK-algebras is a variety.*

Example 1.6.18. Let $\mathbf{C}_n := \langle C;\ /, 0\rangle$ be an algebra of type $\langle 2, 0\rangle$ with cardinality $n+1$ and operation $/$ defined by:

$$ab := \begin{cases} a & \text{if } a \neq b \\ 0 & \text{otherwise} \end{cases}$$

for any $a, b \in C$. Then $\mathbf{C}_n$ is an implicative BCK-algebra; we call $\mathbf{C}_n$ a *flat* implicative BCK-algebra on $n+1$ elements. ∎

Because of (1.3)–(1.4) and (1.7) any two-element implicative BCK-algebra is flat and so may be identified with $\mathbf{C}_1$. It is easy to see that $\mathbf{C}_1$ embeds into any non-trivial BCK-algebra, and hence that $\mathbf{Q}(\mathbf{C}_1)$ is the smallest non-trivial subquasivariety of BCK-algebras. In fact $\mathbf{Q}(\mathbf{C}_1)$ is the unique atom in the lattice of *varieties* of BCK-algebras; this is a consequence of the following result, due to Kalman, which will be needed in the sequel.

Theorem 1.6.19. *[131, Lemma 2] Up to isomorphism,* $\mathsf{iBCK}_{\mathrm{SI}} = \{\mathbf{C}_1\}$.

Combining Theorem 1.6.19 with Birkhoff's subdirect representation theorem yields the following result, which is also due to Kalman [131] (for a proof of

Theorem 1.6.20 that does not assume the axiom of choice, see Cornish [66, Corollary 1.5]). We remark that it is this result that justifies the change in notation for implicative BCK difference from $\dot{-}$ to $/$ encountered in this section and in the sequel.

Theorem 1.6.20. *[131] The class* iBCK *is precisely the class of all* $\{/, 0\}$*-subreducts of Boolean algebras* $\langle A;\ \wedge, \vee, ', 0, 1\rangle$*, where* $a/b := a \wedge b'$ *for any* $a, b \in A$. ∎

Implicative BCK-algebras were first introduced by Monteiro [168] (in dually isomorphic form under the name *Tarski algebras*) and independently by Kalman in [131] (under the name *flocks*). Implicative BCK-algebras have been studied subsequently by a range of authors, including Abbott [2, 1] (in dually isomorphic form under the name *implication algebras*), Rasiowa [195, Sections II§5–7] (likewise), and Iséki and Tanaka [126]. The following representation theorem is due to Abbott [2, 1]. In the statement of the theorem, a *semi-Boolean algebra* is a locally Boolean meet semilattice: that is, a meet semilattice $\langle A;\ \wedge, 0\rangle$ in which for each $a \in A$ the principal subalgebra $(\mathbf{a}]$ is a Boolean lattice under the semilattice partial ordering.

Theorem 1.6.21. *[2, Theorem 6, Theorem 7] Every implicative BCK-algebra* $\langle A;\ /, 0\rangle$ *induces a semi-Boolean algebra* $\langle A;\ \sqcap, 0\rangle$ *upon defining* $a \sqcap b := a/(a/b)$ *for any* $a, b \in A$*. Conversely, every semi-Boolean algebra* $\langle A;\ \sqcap, 0\rangle$ *determines an implicative BCK-algebra* $\langle A;\ /, 0\rangle$ *under the operation* $a/b := (a \sqcap b)'_{(\mathbf{a}]}$ *for any* $a, b \in A$*, where* $(a \sqcap b)'_{(\mathbf{a}]}$ *denotes the complement of* $a \sqcap b$ *in the principal subalgebra* $(\mathbf{a}]$ *generated by* a*. Moreover, these correspondences are inverse to each other.*

A BCK-algebra $\mathbf{A}$ is *bounded* if there exists $1 \in A$ such that $a \leq 1$ for any $a \in A$. As usual, by abuse of language and notation we will often confuse a bounded BCK-algebra $\mathbf{A}$ with its expansion to $\langle A;\ /, 0, 1\rangle$, where $\mathbf{1}$ is a new nullary operation symbol adjoined to the language of $\mathbf{A}$ whose canonical interpretation on $\langle A;\ /, 0, 1\rangle$ is $1 \in A$.

Corollary 1.6.22. *The underlying poset $\langle A;\ \leq\rangle$ of a bounded implicative BCK-algebra $\langle A;\ /, 0, 1\rangle$ is a Boolean lattice. For any $a, b \in A$,*

$$a \wedge b = a \cap b$$
$$a \vee b = 1/\big((1/a) \cap (1/b)\big).$$

1.6.23. BCK-[Semi]Lattices. An algebra $\langle A;\ \wedge, \dot{-}, 0\rangle$ of type $\langle 2, 2, 0\rangle$ is called a *lower BCK-semilattice* if: (i) the reduct $\langle A;\ \dot{-}, 0\rangle$ is a BCK-algebra; and (ii) the following conditions are satisfied with respect to the BCK partial order $\leq$ for any $a, b, c \in A$ [116, p. 840]:

$$a \wedge b \leq a \tag{1.39}$$
$$a \wedge b \leq b \tag{1.40}$$
$$c \leq a \text{ and } c \leq b \text{ imply } c \leq a \wedge b. \tag{1.41}$$

An *upper BCK-semilattice* is defined analogously as an algebra $\langle A;\ \vee, \dot{-}, 0\rangle$ of type $\langle 2, 2, 0\rangle$ such that: (i) the reduct $\langle A;\ \dot{-}, 0\rangle$ is a BCK-algebra; and (ii) for any $a, b \in A$, $a \vee b$ is the least upper bound of the doubleton $\{a, b\}$ with respect to the BCK partial order. A *BCK-lattice* is an algebra $\langle A;\ \wedge, \vee, \dot{-}, 0\rangle$ of type $\langle 2, 2, 2, 0\rangle$ such that: (i) the reduct $\langle A;\ \wedge, \dot{-}, 0\rangle$ is a lower BCK-semilattice; and (ii) the reduct $\langle A;\ \vee, \dot{-}, 0\rangle$ is an upper BCK-semilattice. The following characterisation of BCK-[semi]lattices is implicit in the proof of [116, Theorem 1].

Lemma 1.6.24. *(cf. [116, Theorem 1]) An algebra $\langle A;\ \wedge, \dot{-}, 0\rangle$ [$\langle A;\ \vee, \dot{-}, 0\rangle$] of type $\langle 2, 2, 0\rangle$ is a lower BCK-semilattice [upper BCK-semilattice] iff the following conditions hold:*

1. *The reduct $\langle A;\ \wedge\rangle$ [$\langle A;\ \vee\rangle$] is a meet semilattice [join semilattice];*
2. *The reduct $\langle A;\ \dot{-}, 0\rangle$ is a BCK-algebra;*
3. *The BCK partial order coincides with [dualises] the semilattice partial order.*

An algebra $\langle A;\ \wedge, \vee, \dot{-}, 0\rangle$ of type $\langle 2,2,2,0\rangle$ is a BCK-lattice iff the following conditions hold:

1'. The reduct $\langle A;\ \wedge, \vee\rangle$ is a lattice;

2'. The reduct $\langle A;\ \dot{-}, 0\rangle$ is a BCK-algebra;

3'. The BCK partial order coincides with the lattice partial order.

Proof. To prove the lemma it is sufficient to prove the first statement in the context of lower BCK-semilattices. Let $\mathbf{A}$ be a lower BCK-semilattice and let $a, b, c, d \in A$. Clearly conditions (1) and (2) are satisfied. From (1.39) we have $\big(a \dot{-} (a \dot{-} b)\big) \wedge b \leq^{\langle A;\ \dot{-}, 0\rangle} a \dot{-} (a \dot{-} b)$, while from $a \dot{-} (a \dot{-} b) \leq^{\langle A;\ \dot{-}, 0\rangle} a \dot{-} (a \dot{-} b)$, $a \dot{-} (a \dot{-} b) \leq^{\langle A;\ \dot{-}, 0\rangle} b$ and (1.41) we have $a \dot{-} (a \dot{-} b) \leq^{\langle A;\ \dot{-}, 0\rangle} \big(a \dot{-} (a \dot{-} b)\big) \wedge b$. Thus $\big(a \dot{-} (a \dot{-} b)\big) \wedge b = a \dot{-} (a \dot{-} b)$. If $c \leq^{\langle A;\ \dot{-}, 0\rangle} d$ then $c = c \dot{-} 0 = c \dot{-} (c \dot{-} d) = \big(c \dot{-} (c \dot{-} d)\big) \wedge d = (c \dot{-} 0) \wedge d = c \wedge d$, so $c \leq^{\langle A;\ \wedge\rangle} d$. On the other hand, from (1.39) we have $a \wedge b \leq^{\langle A;\ \dot{-}, 0\rangle} b$, and so $(a \wedge b) \dot{-} b = 0$. If $c \leq^{\langle A;\ \wedge\rangle} d$ then $0 = (c \wedge d) \dot{-} d = c \dot{-} d$ and so $c \leq^{\langle A;\ \dot{-}, 0\rangle} d$. Thus (3) holds, and $\mathbf{A}$ satisfies (1)–(3). Since the converse is clear, the lemma is proved. ∎

Theorem 1.6.25. *[116, Theorem 1] The class* lBS *[*uBS*] of lower [upper] BCK-semilattices is a variety. Therefore the class* BL *of BCK-lattices is also a variety.*

BCK-[semi]lattices were introduced by Idziak in [116]. They have since been studied by Idziak [115, 117], Raftery and Sturm [190], Kondo [141] and Ono and Komori [176] among others. Examples of BCK-[semi]lattices abound in the literature, and include dual Brouwerian semilattices (see [164] and [60] for details) and generalised Boolean algebras (see Corollary 3.3.56 in the sequel).

1.7 Ideals and Subtractive Varieties

The theory of ideals in universal algebra and the theory of subtractive varieties, as developed by Agliano, Ursini and others, are the major tools we employ in our study of pre-BCK-algebras. We summarise here the parts of these theories that we exploit in the sequel. Unless otherwise stated, throughout this section all classes of algebras considered are pointed; typically we always assume the existence of a constant term $\mathbf{0}$.

1.7.1. Ideal Terms. Let K be a class of algebras of the same similarity type. A term $p(\vec{x},\vec{y})$ in the language of K is a K-*ideal term in* $\vec{y}$ (in symbols $p(\vec{x},\vec{y}) \in \mathrm{IT}_{\mathsf{K}}(\vec{y})$) if the identity $p(\vec{x},\mathbf{0},\ldots,\mathbf{0}) \approx \mathbf{0}$ holds in K. A non-empty subset I of $\mathbf{A} \in \mathsf{K}$ is a K-*ideal* of $\mathbf{A}$ if for any $p(\vec{x},\vec{y}) \in \mathrm{IT}_{\mathsf{K}}(\vec{y})$ we have $p^{\mathbf{A}}(\vec{a},\vec{b}) \in I$ for $\vec{a} \in A$ and $\vec{b} \in I$ [104, p. 46]. The intersection of K-ideals is itself a K-ideal and one easily sees that the set $\mathrm{I}_{\mathsf{K}}(\mathbf{A})$ of all K-ideals of $\mathbf{A}$ forms an algebraic lattice $\mathbf{I}_{\mathsf{K}}(\mathbf{A})$ under inclusion [104, Lemma 1.2]. For any $B \subseteq A$, the ideal $\langle B\rangle^{\mathsf{K}}_{\mathbf{A}}$ generated by B is the set $\{p^{\mathbf{A}}(\vec{a},\vec{b}) : p(\vec{x},\vec{y}) \in \mathrm{IT}_{\mathsf{K}}(\vec{y});\ \vec{a} \in A, \vec{b} \in B\}$. A K-ideal is *compact* when it is finitely generated; for $B := \{a_1,\ldots, a_n\}$, the compact ideal $\langle B\rangle^{\mathsf{K}}_{\mathbf{A}}$ is denoted $\langle a_1,\ldots, a_n\rangle^{\mathsf{K}}_{\mathbf{A}}$. By [11, p. 360] the set $\mathrm{CI}_{\mathsf{K}}(\mathbf{A})$ of compact K-ideals of $\mathbf{A}$ forms a join subsemilattice $\mathbf{CI}_{\mathsf{K}}(\mathbf{A})$ of $\mathbf{I}_{\mathsf{K}}(\mathbf{A})$ under inclusion. A K-ideal is said to be *principal* when it is generated by a single element; the principal ideal $\langle B\rangle^{\mathsf{K}}_{\mathbf{A}}$ generated by $B := \{a\}$ is denoted $\langle a\rangle^{\mathsf{K}}_{\mathbf{A}}$. Clearly $\langle 0\rangle^{\mathsf{K}}_{\mathbf{A}} = \{0\}$. When K is $\{\mathbf{A}\}$ (or, equivalently, the variety $\mathbf{V}(\mathbf{A})$ generated by $\mathbf{A}$) then a K-ideal is simply called an *ideal* and all affixes and suffixes in sight are dropped, provided the context is clear. The set $\big\{[0]_\theta : \theta \in \mathrm{Con}\,\mathbf{A}\big\}$ is denoted by $\mathrm{N}(\mathbf{A})$, and any element of $\mathrm{N}(\mathbf{A})$ is called a *normal set*; trivially $\mathrm{N}(\mathbf{A}) \subseteq \mathrm{I}(\mathbf{A}) \subseteq \mathrm{I}_{\mathsf{K}}(\mathbf{A})$. Clearly $\mathrm{N}(\mathbf{A})$ inherits in a natural way the lattice structure of $\mathrm{Con}\,\mathbf{A}$: see [222, p. 205]. We say K [an algebra $\mathbf{A}$] has *normal ideals* if $\mathrm{I}_{\mathsf{K}}(\mathbf{A}) = \mathrm{N}(\mathbf{A})$ for all $\mathbf{A} \in \mathsf{K}$ [if $\mathrm{I}(\mathbf{A}) = \mathrm{N}(\mathbf{A})$]. If K has normal ideals, the set of all ideals [lattice of ideals] of any $\mathbf{A} \in \mathsf{K}$ may be denoted simply by $\mathrm{I}(\mathbf{A})$ [$\mathbf{I}(\mathbf{A})$] without any reference to K; we adopt this convention in the sequel. We say an algebra $\mathbf{A}$ [a class K] is *ideal simple* if the only ideals of $\mathbf{A}$ [of all $\mathbf{A} \in \mathsf{K}$] are $\{0\}$ and A. We say $\mathbf{A}$ is *hereditarily ideal simple* if every subalgebra of $\mathbf{A}$ is ideal simple; K is *hereditarily ideal simple* if every member of K is hereditarily ideal simple. Finally, an ideal I of $\mathbf{A}$ is said to be *proper* if $I \neq A$, and *improper* otherwise.

The study of K-ideals was initiated by Gumm and Ursini in [104], wherein they proposed the syntactic notion of ideal described above as an abstraction of the familiar closure properties of 'ideals' arising in commonly occurring classes of algebras. Since their introduction, K-ideals in universal algebra have been studied by Agliano and Ursini in several contexts; see [8, 218, 219] for details.

Remark 1.7.2. Blok and Raftery [40] have recently proposed a still more general notion of ideal than that of Gumm and Ursini, applicable to general (not necessarily pointed) quasivarieties. Inspired by [31], Blok and Raftery view the closure systems of (K-) congruences of algebras in a quasivariety K as models of a certain deductive system $\mathbb{S}(\mathsf{K}, \tau)$ 'extracted' from the quasi-equational theory of K by means of a suitable translation τ (for details, see [40, Section 5]). For $\mathbf{A} \in \mathsf{K}$ a *strong ideal* of $\mathbf{A}$ is simply an $\mathbb{S}(\mathsf{K}, \tau)$-filter of $\mathbf{A}$, namely a subset of A closed under the axioms and inference rules of $\mathbb{S}(\mathsf{K}, \tau)$. In [40, Section 7] Blok and Raftery compare their notion of ideal with that proposed by Gumm and Ursini, and show in [40, Theorem 7.4] that for subtractive varieties the two notions coincide. Because we restrict ourselves to the investigation of subtractive varieties in this thesis, we are free to work just with the syntactic notion of ideal proposed by Gumm and Ursini; this we (mostly) do in the sequel. ∎

Application of Gumm and Ursini's theory of ideals is primarily directed in the literature towards ideal determined varieties. Recall from [104] that a variety V is *ideal determined* if for all $\mathbf{A} \in \mathsf{V}$, any V-ideal is the 0-class of exactly one congruence, or equivalently, if the map $\theta \mapsto [0]_\theta$ ($\theta \in \mathrm{Con}\,\mathbf{A}$) is a lattice isomorphism from $\mathbf{Con\,A}$ onto $\mathbf{I}(\mathbf{A})$.

Proposition 1.7.3. *[104, Corollary 1.9] A variety* V *with* $\mathbf{0}$ *is ideal determined iff it is* $\mathbf{0}$*-regular and there exists a binary term* $s(x,y)$ *in the language of* V *such that:*

$$s(x,x) \approx \mathbf{0} \tag{1.42}$$

$$s(x,\mathbf{0}) \approx x. \tag{1.43}$$

1.7.4. Subtractive Varieties. A variety V is *subtractive* if there exists a binary term $s(x,y)$ of V such that the identities (1.42) and (1.43) are satisfied in V; an algebra $\mathbf{A}$ is *subtractive* if $\mathbf{V}(\mathbf{A})$ is subtractive. A variety V is *congruence* $\mathbf{0}$*-permutable* if for any $\mathbf{A} \in \mathsf{V}$ and $\theta, \phi \in \mathrm{Con}\,\mathbf{A}$, $[0]_{\theta\circ\phi} = [0]_{\phi\circ\theta}$.

For a variety V, the following are equivalent [222, Proposition 1.2]: (i) V is subtractive; (ii) V is congruence $\mathbf{0}$-permutable; and (iii) for each $\mathbf{A} \in \mathsf{V}$, the map $\theta \mapsto [0]_\theta$ ($\theta \in \mathbf{Con}\,\mathbf{A}$) is a lattice epimorphism from $\mathbf{Con}\,\mathbf{A}$ into $\mathrm{I}_{\mathsf{V}}(\mathbf{A})$.

Proposition 1.7.5. *[222, Proposition 1.4] Let* V *be a subtractive variety and let* $\mathbf{A} \in \mathsf{V}$*. Then every* $I \in \mathrm{I}_{\mathsf{V}}(\mathbf{A})$ *is a congruence class. That is,* $\mathrm{I}_{\mathsf{V}}(\mathbf{A}) = \mathrm{N}(\mathbf{A})$*, and so* V *has normal ideals.*

Subtractive varieties were introduced by Ursini in [222] and have been systematically investigated by Agliano and Ursini [9, 10, 11, 222, 225] in a program strongly influenced by recent developments in universal algebra, including commutator theory and the theory of varieties with equationally definable principal congruences. Particularly important among subtractive varieties are those satisfying the identity:

$$s(0, x) \approx 0. \tag{1.44}$$

Indeed, recall from [81, Definition 2] that a variety V is *congruence* $\mathbf{0}$*-distributive* if for any $\mathbf{A} \in \mathsf{V}$ and $\theta, \phi, \psi \in \mathrm{Con}\,\mathbf{A}$, $[0]_{(\theta\vee\phi)\wedge\psi} = [0]_{(\theta\wedge\psi)\vee(\phi\wedge\psi)}$. A variety that is both congruence $\mathbf{0}$-permutable and congruence $\mathbf{0}$-distributive is said to be *arithmetical at* $\mathbf{0}$. A variety V is *ideal distributive* if $\mathbf{I}_{\mathsf{V}}(\mathbf{A})$ is a distributive lattice for any $\mathbf{A} \in \mathsf{V}$. For a variety V, the following are equivalent [81, Theorem 4], [9, Proposition 4.1]: (i) there exists a binary term $s(x, y)$ of V such that V satisfies (1.42)–(1.44); (ii) V is arithmetical at 0; and (iii) V is subtractive and ideal distributive.

1.7.6. Congruences of Subtractive Varieties. In general, a subtractive variety need not be ideal determined. Let $\mathbf{A}$ be a (subtractive) algebra and $I \in \mathrm{I}(\mathbf{A})$. Let $\mathrm{CON}(I) := \{\theta \in \mathrm{Con}\,\mathbf{A} : [0]_\theta = I\}$ denote the set of congruences of $\mathbf{A}$ whose 0-classes coincide with I. Let $I^\delta := \bigwedge \mathrm{CON}(I)$ $[I^\epsilon := \bigvee \mathrm{CON}(I)]$ denote the least [greatest] congruence of $\mathbf{A}$ whose 0-class is I. If $\mathbf{A}$ is an algebra with normal ideals, $\mathrm{CON}(I) = [I^\delta, I^\epsilon]$ is an interval in $\mathbf{Con}\,\mathbf{A}$ [10, Proposition 1.3]. Let $(\ )^\delta : \mathbf{I}(\mathbf{A}) \to \mathbf{Con}\,\mathbf{A}$ and $(\ )^\epsilon : \mathbf{I}(\mathbf{A}) \to \mathbf{Con}\,\mathbf{A}$ be the maps defined by $I \mapsto I^\delta$ and $I \mapsto I^\epsilon$ respectively. For $\theta \in \mathrm{Con}\,\mathbf{A}$ let θ_0 $[\theta_1]$ denote the least [greatest] congruence $([0]_\theta)^\delta$ $[([0]_\theta)^\epsilon]$ on $\mathbf{A}$ whose 0-class is

$[0]_\theta$. Let $\hat\theta := \mathrm{CON}([0]_\theta)$; notice $\hat\theta = [\theta_0, \theta_1]$. Let $(\ )_0 : \mathrm{Con}\,\mathbf{A} \to \mathrm{Con}\,\mathbf{A}$ and $(\ )_1 : \mathrm{Con}\,\mathbf{A} \to \mathrm{Con}\,\mathbf{A}$ be the maps defined respectively by $\theta \mapsto \theta_0$ and $\theta \mapsto \theta_1$.

A *system of* $\mathbf{0}$*-terms* (*without parameters*) for a class K of similar algebras is a set $D := \{d_j(x,y) : j \in J\}$ of terms in the language of K such that:

1. $\mathsf{K} \models d_j(x,x) \approx \mathbf{0}$ for all $j \in J$;

2. For all $\mathbf{A} \in \mathsf{K}$ and $a \in A$, if $d_j^{\mathbf{A}}(0,a) = 0$ for all $j \in J$, then $a = 0$.

Let $\mathbf{A}$ be an algebra and $I \in \mathrm{I}(\mathbf{A})$. Define $I^D \subseteq A \times A$ by $(a,b) \in I^D$ iff for all $j \in J$, $d_j^{\mathbf{A}}(a,b) \in I$. For $\theta \in \mathrm{Con}\,\mathbf{A}$ let $\theta^D := [0]_\theta^D$. For a class K of similar algebras, a set $D := \{d_j(x,y) : j \in J\}$ of terms of K is a *system of ideal congruence terms* (*without parameters*) *for* K (in short, an *IC-system* (*without parameters*) *for* K) if $\theta^D \in \mathrm{Con}\,\mathbf{A}$ and $[0]_{\theta^D} = [0]_\theta$ for all $\mathbf{A} \in \mathsf{K}$. Observe that if K has normal ideals, D is an IC-system for K iff for all $\mathbf{A} \in \mathsf{K}$ and $I \in \mathrm{I}(\mathbf{A})$, $I^D \in \mathrm{CON}(I)$, or, equivalently, $I^D \in \mathrm{Con}\,\mathbf{A}$ and $[0]_{I^D} = I$.

Proposition 1.7.7. *[10, Proposition 3.8, Proposition 3.9] Let* V *be a variety with normal ideals and let* D *be a system of* $\mathbf{0}$*-terms for* V*. The following are equivalent:*

1. *For* $\mathbf{A} \in \mathsf{V}$, $\langle 0\rangle_{\mathbf{A}}^D \in \mathrm{Con}\,\mathbf{A}$*;*

2. D *is an IC-system for* V*;*

3. *For* $\mathbf{A} \in \mathsf{V}$ *and* $\theta \in \mathrm{Con}\,\mathbf{A}$, $\theta^D = \theta_1$*;*

4. *For* $\mathbf{A} \in \mathsf{V}$ *and* $I \in \mathrm{I}(\mathbf{A})$, $I^D = I^\epsilon$.

A variety is said to be *finitely congruential* if it has a finite IC-system without parameters; by [10, Theorem 3.10] a finitely congruential variety with normal ideals is subtractive. For subtractive varieties, finite congruentiality generalises point regularity: see Agliano and Ursini [10, Remark (4), p. 322].

1.7.8. Equationally Definable Principal Ideals. A class K of algebras of a given similarity type has *Equationally Definable Principal* K*-Ideals* (briefly, *EDPI*) if there are terms $p_i(x, y), q_i(x, y)$, $i = 1, \ldots, n$ in the language of K such that for any $\mathbf{A} \in \mathsf{K}$ and $a, b \in A$:

$$a \in \langle b \rangle_{\mathbf{A}} \text{ iff } p_i^{\mathbf{A}}(a, b) = q_i^{\mathbf{A}}(a, b) \qquad i = 1, \ldots, n.$$

Equationally definable principal ideals were introduced by Agliano and Ursini in [9] as the ideal-theoretic analogue of equationally definable principal congruences. Subtractive varieties with EDPI have been investigated at length by Agliano and Ursini [9, 11], to whom the following result is due.

Theorem 1.7.9. *[6, Theorem 4.1] For a subtractive variety* V*, the following are equivalent:*

1. V *has EDPI;*

2. *There is a binary term* $p(x, y)$ *of* V *such that:*

   $$a \in \langle b \rangle_{\mathbf{A}} \quad \textit{iff} \quad p^{\mathbf{A}}(a, b) = 0$$

   for any $\mathbf{A} \in \mathsf{V}$*;*

3. *The join semilattice* $\langle \mathrm{CI}(\mathbf{A}); \vee, \langle 0 \rangle_{\mathbf{A}} \rangle$ *of compact ideals is dually relatively pseudocomplemented for any* $\mathbf{A} \in \mathsf{V}$.

Moreover, if V *has EDPI then there exists a binary term* $x \dot{-} y$ *of* V *witnessing* both *subtractivity and EDPI in the sense of* (2) *above. That is, there exists a binary term* $x \dot{-} y$ *of* V *such that for any* $\mathbf{A} \in \mathsf{V}$ *and* $a, b \in A$*,*

$$a \dot{-}^{\mathbf{A}} a = 0$$

$$a \dot{-}^{\mathbf{A}} 0 = a$$

$$a \in \langle b \rangle_{\mathbf{A}} \quad \textit{iff} \quad a \dot{-}^{\mathbf{A}} b = 0.$$

Proposition 1.7.10. *([11, Theorem 3.1]; [4, Corollary 2]) Let* V *be a subtractive variety. If* V *has EDPC then* V *has EDPI. Conversely, if* V *is ideal determined and has EDPI, then* V *has EDPC and the map* $\theta \mapsto [0]_\theta$ $(\theta \in \mathrm{Con}\,\mathbf{A})$ *is a dual Brouwerian semilattice isomorphism from* $\mathbf{Cp}\ \mathbf{A}$ *onto* $\mathbf{CI}(\mathbf{A})$ *for any* $\mathbf{A} \in \mathsf{V}$.

Example 1.7.11. [11, Example 3.7] An algebra $\langle A; \to, 1\rangle$ of type $\langle 2, 0\rangle$ is called a *MINI-algebra* if the following identities are satisfied:

$$x \to \mathbf{1} \approx \mathbf{1} \tag{1.45}$$

$$\mathbf{1} \to x \approx x \tag{1.46}$$

$$\big(x \to (y \to z)\big) \to \big((x \to y) \to (x \to z)\big) \approx \mathbf{1} \tag{1.47}$$

$$x \to (y \to x) \approx \mathbf{1}. \tag{1.48}$$

Hilbert algebras are precisely those MINI-algebras for which the quasi-identity:

$$x \to y \approx \mathbf{1} \;\&\; y \to x \approx \mathbf{1} \supset x \approx y \tag{1.49}$$

holds. Hence the variety of MINI-algebras is a natural generalisation of the variety of Hilbert algebras to the subtractive but not **1**-regular case. In consequence, the class of MINI-algebras provides a natural example of a subtractive variety with EDPI (that does not have EDPC); indeed, the term $y \to x$ witnesses *both* subtractivity and EDPI in the variety of MINI-algebras in the sense of Theorem 1.7.9 above. ∎

Proposition 1.7.12. *[11, Corollary 3.6] For a variety* V *with language* $\langle \dot{-}, \mathbf{0}\rangle$ *of type* $\langle 2, 0\rangle$ *the following are equivalent:*

1. *For all* $\mathbf{A} \in \mathsf{V}$ *and* $a, b \in A$,

$$a \in \langle b\rangle_{\mathbf{A}} \quad \textit{iff} \quad a \dot{-} b = 0$$

$$a \dot{-} b = 0 \textit{ and } b \dot{-} a = 0 \textit{ implies } a = b;$$

2. V *is a variety of positive implicative BCK-algebras.*

Let V be a variety. A *meet generator term* for V is a binary term $\sqcap$ in the language of V such that for any $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$ [11, p. 378],

$$\langle a\rangle_{\mathbf{A}} \wedge^{\mathbf{I}(\mathbf{A})} \langle b\rangle_{\mathbf{A}} = \langle a \sqcap^{\mathbf{A}} b\rangle_{\mathbf{A}}.$$

A *join generator term* for V is a binary term $\sqcup$ of V such that for any $\mathbf{A} \in \mathsf{V}$

and $a, b \in A$ [11, p. 379],

$$\langle a\rangle_{\mathbf{A}} \vee^{\mathbf{I}(\mathbf{A})} \langle b\rangle_{\mathbf{A}} = \langle a \sqcup^{\mathbf{A}} b\rangle_{\mathbf{A}}.$$

Subtractive varieties with a meet [join] generator term have been studied by Agliano and Ursini in [11, Section 4, Section 5]. For a subtractive variety V, the following are equivalent: (i) V is ideal distributive and the meet of two principal ideals is principal; (ii) V has a meet generator term [11, p. 378; Theorem 4.2]. Concerning join generator terms, the following proposition obtains for arbitrary (that is, not necessarily subtractive) varieties.

Proposition 1.7.13. *[11, Proposition 4.3] For a variety V, the following are equivalent:*

1. *For any $\mathbf{A} \in \mathsf{V}$, the join of two principal ideals is principal;*
2. *For any $\mathbf{A} \in \mathsf{V}$, every compact ideal is principal;*
3. *There is a binary term $\sqcup$ and two ternary terms $r(x, y, z)$ and $t(x, y, z)$ of V such that the identities:*

$$\begin{aligned}
&\mathbf{0} \sqcup \mathbf{0} \approx \mathbf{0}\\
&r(x, y, \mathbf{0}) \approx \mathbf{0}\\
&t(x, y, \mathbf{0}) \approx \mathbf{0}\\
&r(x, y, x \sqcup y) \approx x\\
&t(x, y, x \sqcup y) \approx y
\end{aligned}$$

   hold in V;

4. *V has a join generator term.*

If any of (1)–(4) hold, the binary term $\sqcup$ of (3) is a join generator term for V.

Let K be a class of similar algebras. For any $\mathbf{A} \in \mathsf{K}$ let $\approx_{\mathbf{A}}$ be the relation defined on A by [6, p. 2]:

$$a \approx_{\mathbf{A}} b \text{ iff } \Theta^{\mathbf{A}}(0, a) = \Theta^{\mathbf{A}}(0, b)$$

for any $a, b \in A$. If K has normal ideals, then $a \approx_{\mathbf{A}} b$ iff $\mathrm{N}_{\mathbf{A}}(a) = \mathrm{N}_{\mathbf{A}}(b)$ (by Agliano [7, p. 5]), where $\mathrm{N}_{\mathbf{A}}(a), \mathrm{N}_{\mathbf{A}}(b)$ denote the normal sets generated by $\{a\}, \{b\}$ respectively. Therefore if K is a subtractive variety then $a \approx_{\mathbf{A}} b$ iff $\langle a\rangle_{\mathbf{A}} = \langle b\rangle_{\mathbf{A}}$.

Lemma 1.7.14. *Let V be a subtractive variety with EDPI and let $x \dot{-} y$ be a binary term witnessing* both *subtractivity and EDPI for V in the sense of Theorem 1.7.9. For any $\mathbf{A} \in \mathsf{V}$,*

$$\approx_{\mathbf{A}} = \left\{(a, b) : a \dot{-}^{\mathbf{A}} b = 0 = b \dot{-}^{\mathbf{A}} a\right\}.$$

Proof. Let V and $\dot{-}$ be as in the statement of the lemma. Let $\mathbf{A} \in \mathsf{V}$. For any $a, b \in A$, we have that $a \approx_{\mathbf{A}} b$ iff $\langle a\rangle_{\mathbf{A}} = \langle b\rangle_{\mathbf{A}}$ iff $a \in \langle b\rangle_{\mathbf{A}}, b \in \langle a\rangle_{\mathbf{A}}$ iff $a \dot{-}^{\mathbf{A}} b = 0 = b \dot{-}^{\mathbf{A}} a$ (by EDPI). ∎

Because of Lemma 1.7.14, the following theorem may be inferred from results due to Agliano and Ursini [11].

Theorem 1.7.15. *Let V be a subtractive variety with EDPI. Let $x \dot{-} y$ witness* both *subtractivity and EDPI for V in the sense of Theorem 1.7.9 and let $\mathbf{A} \in \mathsf{V}$. The following assertions hold:*

1. *[11, Theorem 3.4(2)] $\left\langle \mathrm{PI}(\mathbf{A});\ *, \langle 0\rangle_{\mathbf{A}}\right\rangle$ is a positive implicative BCK-algebra isomorphic with $\langle A;\ \dot{-}, 0\rangle/{\approx_{\mathbf{A}}}$;*

2. *[11, p. 383] If V has a meet generator term $x \sqcap y$, the compact ideals of $\mathbf{A}$ are closed under intersection. Thus $\left\langle \mathrm{CI}(\mathbf{A});\ \cap, \vee, *, \langle 0\rangle_{\mathbf{A}}\right\rangle$ is a dually Brouwerian lattice and $\left\langle \mathrm{PI}(\mathbf{A});\ \cap, *, \langle 0\rangle_{\mathbf{A}}\right\rangle$ is a $\langle \cap, *\rangle$-subreduct of $\left\langle \mathrm{CI}(\mathbf{A});\ \cap, \vee, *, \langle 0\rangle_{\mathbf{A}}\right\rangle$ isomorphic with $\langle A;\ \sqcap, \dot{-}, 0\rangle/{\approx_{\mathbf{A}}}$;*

3. *[11, Theorem 5.1(2)] If V has a join generator term $x \sqcup y$, then $\left\langle \mathrm{PI}(\mathbf{A});\ \vee, *, \langle 0\rangle_{\mathbf{A}}\right\rangle$ is a dual Brouwerian semilattice isomorphic with $\langle A;\ \sqcup, \dot{-}, 0\rangle/{\approx_{\mathbf{A}}}$;*

4. *[11, Theorem 5.6(2)] If V has both a meet and join generator term then $\left\langle \mathrm{PI}(\mathbf{A});\ \cap, \vee, *, \langle 0\rangle_{\mathbf{A}}\right\rangle$ is a dually Brouwerian lattice isomorphic with $\langle A;\ \sqcap, \sqcup, \dot{-}, 0\rangle/{\approx_{\mathbf{A}}}$.*

Let K be a class of similar algebras. For any $\mathbf{A} \in \mathsf{K}$, the relation $\approx_{\mathbf{A}}$ is an equivalence relation. If $\approx_{\mathbf{A}} = \omega_{\mathbf{A}}$ for any $\mathbf{A} \in \mathsf{K}$, K is said to be *congruence orderable*. A point-regular congruence orderable variety is *Fregean*. Congruence orderable varieties were introduced by Büchi and Owens in [49] and have been studied by Idziak, Somczyńska and Wroński [118], Pigozzi [181] and Agliano [7], to whom the following proposition is due.

Proposition 1.7.16. *[7, Theorem 2.1] Let V be a subtractive variety and let $s(x, y)$ witness subtractivity for V. If V is congruence orderable then V is 0-regular and the terms $d_1(x, y) := s(x, y)$ and $d_2(x, y) := s(y, x)$ witness 0-regularity for V in the sense of Proposition 1.2.6. Thus a congruence orderable subtractive variety is Fregean.*

Let K be a class of similar algebras. If $\approx_{\mathbf{A}} \in \operatorname{Con} \mathbf{A}$ for any $\mathbf{A} \in \mathsf{K}$ then K is called *weakly congruence orderable* or *congruence quasi-orderable*. Weak congruence orderability was introduced by Agliano in [6] as a weakening of the concept of congruence orderability. For subtractive varieties with EDPI, weak congruence orderability has been studied by Agliano and Ursini [11] and Agliano [6], to whom the following result is due.

Lemma 1.7.17. *[6, Corollary 2.5] Let $\mathbf{A}$ be a subtractive algebra. Then $\mathbf{A}$ is weakly congruence orderable iff for any binary term $s(x, y)$ witnessing subtractivity for $\mathbf{A}$ one has:*

$$\approx_{\mathbf{A}} = \{(a, b) : s^{\mathbf{A}}(a, b) = 0 = s^{\mathbf{A}}(b, a)\} = \langle 0 \rangle^{\epsilon}_{\mathbf{A}}.$$

Proposition 1.7.18. *Let V be a subtractive variety and let $s(x, y)$ be a term witnessing the subtractivity of V. Then V is congruence orderable iff V is weakly congruence orderable and the binary terms $d_1(x, y) := s(x, y)$, $d_2(x, y) := s(y, x)$ witness the 0-regularity of V in the sense of Proposition 1.2.6.*

Proof. Let V be a subtractive variety and let $s(x, y)$ be a term witnessing the subtractivity of V. Suppose V is congruence orderable. Trivially V is weakly congruence orderable. Moreover, by Proposition 1.7.16 the terms $d_1(x, y) := s(x, y)$, $d_2(x, y) := s(y, x)$ witness the 0-regularity of V in the sense of Proposition 1.2.6. Conversely, suppose V is weakly congruence orderable and moreover that the binary terms $d_1(x,y) := s(x,y)$, $d_2(x,y) := s(y,x)$ witness the $\mathbf{0}$-regularity of V. Let $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$. By Lemma 1.7.17, $a \approx_{\mathbf{A}} b$ implies $s^{\mathbf{A}}(a,b) = 0 = s^{\mathbf{A}}(b,a)$, whence $d_1^{\mathbf{A}}(a,b) = 0 = d_2^{\mathbf{A}}(a,b)$, whence $a = b$ by Proposition 1.2.6. Thus V is congruence orderable. ∎

Proposition 1.7.19. *[6, Theorem 4.2] Let V be a weakly congruence orderable subtractive variety with EDPI. Then V_ϵ is a congruence orderable subtractive variety with EDPC.*

1.7.20. Binary and Dual Binary Discriminator Varieties. Let A be a set. For a fixed but arbitrary $0 \in A$ the *binary discriminator* and *dual binary discriminator* on A are the functions $b : A^2 \to A$ and $h : A^2 \to A$ defined respectively by [58, Section 2]:

$$b(a,c) := \begin{cases} a & \text{if } c = 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h(a,c) := \begin{cases} 0 & \text{if } c = 0 \\ a & \text{otherwise} \end{cases}$$

for any $a, c \in A$; the element $0 \in A$ is called the *discriminating element*. An algebra $\mathbf{A}$ with $\mathbf{0}$ is called a *binary discriminator algebra* [*dual binary discriminator algebra*] if there is a binary term b [a binary term h] of $\mathbf{A}$ whose canonical interpretation on $\mathbf{A}$ is the binary discriminator [dual binary discriminator] with discriminating element $\mathbf{0}^{\mathbf{A}}$. A variety V with $\mathbf{0}$ is said to be a *binary discriminator variety* [*dual binary discriminator variety*] if there is a binary term b of V [h of V] and a subclass K of V such that $b^{\mathbf{A}}$ is the binary discriminator [$h^{\mathbf{A}}$ is the dual binary discriminator] with discriminating element $\mathbf{0}^{\mathbf{A}}$ on each $\mathbf{A} \in \mathsf{K}$ and $\mathsf{V} = \mathbf{V}(\mathsf{K})$; b [h] is called a *binary discriminator term* [*dual binary discriminator term*] for V. Note that any binary discriminator variety is a dual binary discriminator variety, since $h(x,y) \approx b\big(x, b(x,y)\big)$; conversely, a dual binary discriminator variety is a binary discriminator variety iff it is subtractive [58, Theorem 2.1(1)].

Theorem 1.7.21. *[58, Theorem 5.1] For a variety V the following are equivalent:*

1. *V is a binary discriminator variety;*

2. *The following assertions hold for V and a binary term $b(x,y)$ of V:*

   (a) *V satisfies the identities:*

$$
\begin{aligned}
&b(x,0) \approx x,\\
&b(0,x) \approx x,\\
&b(x,x) \approx 0,\\
&b\big(x, b(y,x)\big) \approx x;
\end{aligned}
$$

   (b) *V satisfies the identity:*

$$b\big(f(x_1,\dots,x_n),y\big) \approx b\big(f(b(x_1,y),\dots,b(x_n,y)),y\big)$$

   for every n-ary operation symbol f in the type of V;

   (c) *V is generated by a class $\mathsf{K} \subseteq \mathsf{V}$ whose members are ideal simple;*

3. *There exists a binary term $b(x,y)$ of V satisfying (2)(a)–(2)(b) above and V is generated by a class $\mathsf{K} \subseteq \mathsf{V}$ whose algebras have no proper congruence kernels.*

The binary discriminator and dual binary discriminator were introduced by Chajda, Halaš and Rosenberg in a 1999 paper [58] in an attempt to generalise the ternary discriminator and dual ternary discriminator to varieties exhibiting congruence permutability and congruence distributivity only locally at 0 respectively. For a brief discussion contrasting the binary discriminator with the ternary discriminator, see [58, p. 242, pp. 247–248, p. 249].

1.8 Algebraisable and Assertional Logics

In [31] Blok and Pigozzi introduced an abstract notion of algebraisability based on a generalisation of the classical Lindenbaum-Tarski process in an attempt to formalise the precise connection between EDPC and the deduction theorem [182, p. 125]. Since the publication of the seminal monograph [31] algebraic logic has been intensively developed by Blok, Czelakowski, Herrmann, Pigozzi and others. In this section we briefly review Blok and Pigozzi's theory of algebraisable logics, and also consider some recent developments in algebraic logic concerning the assertional logics of pointed quasivarieties.

1.8.1. Algebraisable Deductive Systems. Let K be a class of algebras of type $\mathcal{L}$ and let $\mathbb{S}$ be a deductive system of the same type. K is called an *algebraic semantics* for $\mathbb{S}$ if $\vdash_{\mathbb{S}}$ can be interpreted in $\models_{\mathsf{K}}$ in the following sense: there exist finite families $\{\delta_1, \ldots, \delta_r\}$ and $\{\varepsilon_1, \ldots, \varepsilon_r\}$ of equations in one variable such that for all $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ and $l = 1, \ldots, r$,

$$\Gamma \vdash_{\mathbb{S}} \varphi \quad \text{iff} \quad \left\{\delta_t(\chi) \approx \varepsilon_t(\chi) : \chi \in \Gamma;\ t = 1, \ldots, r\right\} \models_{\mathsf{K}} \delta_l(\varphi) \approx \varepsilon_l(\varphi). \tag{1.50}$$

K is said to be *equivalent* to $\mathbb{S}$ if $\models_{\mathsf{K}}$ can be interpreted in $\mathbb{S}$ in the following sense: there exists a finite system $\{\Delta_1, \ldots, \Delta_m\}$ of formulas in two variables such that for all $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,

$$\varphi \approx \psi \dashv\models_{\mathsf{K}} \left\{\delta_t(\varphi \Delta_i \psi) \approx \varepsilon_t(\varphi \Delta_i \psi) : i = 1, \ldots, m;\ t = 1, \ldots, r.\right\} \tag{1.51}$$

The equations $\delta_t \approx \varepsilon_t$, $t = 1, \ldots, r$, are called the *defining equations* for $\mathbb{S}$ and K while the family $\{\Delta_1, \ldots, \Delta_m\}$ of composite binary connectives is called a system of *equivalence formulas* for $\mathbb{S}$ and K. A deductive system is said to be *algebraisable* if it has an equivalent algebraic semantics. Suppose $\mathbb{S}$ is algebraisable with equivalent algebraic semantics K. Because of [31, Corollary 2.11], K may be identified with the quasivariety $\mathbf{Q}(\mathsf{K})$ it generates; that is, K is an *equivalent quasivariety semantics.* If K is a variety then K is an *equivalent variety semantics*, and in this case $\mathbb{S}$ is said to be *strongly algebraisable.* The following intrinsic characterisation of algebraisable deductive systems is due to Blok and Pigozzi [31].

Theorem 1.8.2. *[31, Theorem 4.7] A deductive system $\mathbb{S}$ is algebraisable iff there exists a finite family $\{\Delta_1, \ldots, \Delta_m\}$ of formulas in two variables and finite families $\{\delta_1, \ldots, \delta_r\}$ and $\{\varepsilon_1, \ldots, \varepsilon_r\}$ of equations in a single variable such that* *for any connective* ϖ *(of rank n) and formulas* $\varphi_1, \ldots, \varphi_n$, $\psi_1, \ldots, \psi_n$, φ, ψ, χ *the following five conditions hold for* $j = 1, \ldots, m$:

1. $\vdash_{\mathbb{S}} \varphi\Delta_j\varphi$;

2. $\{\varphi\Delta_i\psi : i = 1, \ldots, m\} \vdash_{\mathbb{S}} \psi\Delta_j\varphi$;

3. $\{\varphi\Delta_i\psi : i = 1, \ldots, m\} \cup \{\psi\Delta_i\chi : i = 1, \ldots, m\} \vdash_{\mathbb{S}} \Delta_j(\varphi, \chi)$;

4. $\{\varphi_k\Delta_i\psi_k : i = 1, \ldots, m;\ k = 1, \ldots, n\} \vdash_{\mathbb{S}} \varpi(\varphi_1, \ldots, \varphi_n)\Delta_j\varpi(\psi_1, \ldots, \psi_n)$;

5. $\chi \dashv\vdash_{\mathbb{S}} \{\delta_t(\chi)\Delta_i\varepsilon_t(\chi) : i = 1, \ldots, m;\ t = 1, \ldots, r\}$.

In this event $\{\Delta_1, \ldots, \Delta_m\}$ *and* $\delta_t \approx \varepsilon_t$, $t = 1, \ldots, r$ *are systems of equivalence formulas and defining equations for* $\mathbb{S}$.

It is possible for distinct algebraisable deductive systems to have the same equivalent algebraic semantics: see Blok and Pigozzi [31, Chapter 5.2.4]. On the other hand, the equivalent algebraic semantics associated with a given algebraisable deductive system is unique [31, Theorem 2.15] and is determined by the algorithm of the following theorem.

Theorem 1.8.3. *[31, Theorem 2.17] Let* $\mathbb{S}$ *be a deductive system given by a set of axioms Ax and a set of inference rules Ir. Assume* $\mathbb{S}$ *is algebraisable with equivalence formulas* $\{\Delta_1, \ldots, \Delta_m\}$ *and defining equations* $\delta_t \approx \varepsilon_t$, $t = 1, \ldots, r$. *Then the unique equivalent quasivariety semantics for* $\mathbb{S}$ *is axiomatised by the identities:*

1. $\delta_t(\varphi) \approx \varepsilon_t(\varphi), \quad t = 1, \ldots, r,$
   for each $\varphi \in Ax$;

2. $\delta_t(x\Delta_i x) \approx \varepsilon(x\Delta_i x), \quad i = 1, \ldots, m;\ t = 1, \ldots, r$;

together with the following quasi-identities:

3. $\&_{u=1}^{n}\&_{t=1}^{r}\delta_t(\chi_u) \approx \varepsilon_t(\chi_u) \supset \delta_l(\varphi) \approx \varepsilon_l(\varphi), \quad l = 1, \ldots, r$
   for each $\langle\{\chi_1, \ldots, \chi_n\}, \varphi\rangle \in Ir$;

4. $\&_{i=1}^{m}\&_{t=1}^{r}\delta_t(x\Delta_i y) \approx \varepsilon_t(x\Delta_i y) \supset x \approx y$.

Example 1.8.4. Let $\mathcal{L} := \{\to\}$ be a language of type $\langle 2 \rangle$. *BCK logic* is the deductive system $\mathbb{BCK} := \langle \mathcal{L}, \vdash_{\mathbb{BCK}} \rangle$ defined by the following axioms and inference rule [28, Lecture 6, Section 2.2]:

$$(p \to q) \to \big((q \to r) \to (p \to r)\big) \tag{B}$$
$$\big(p \to (q \to r)\big) \to \big(q \to (p \to r)\big) \tag{C}$$
$$p \to (q \to p) \tag{K}$$

$$p, p \to q \vdash_{\mathbb{BCK}} q. \tag{BCK-MP}$$

BCK logic has been extensively studied in the literature: see [80, Section 4] for a survey and references. Results due to Blok and Pigozzi [31, Theorem 5.10, Theorem 5.11] show $\mathbb{BCK}$ is algebraisable; its equivalent algebraic semantics is termwise definitionally equivalent to the quasivariety BCK^D of *dual BCK-algebras* introduced by Blok and Pigozzi in [32, Example 7.3] and axiomatised by the following set of identities and quasi-identities:

$$(x \to y) \to \big((y \to z) \to (x \to z)\big) \approx \mathbf{1}$$
$$x \to \big((x \to y) \to y\big) \approx \mathbf{1}$$
$$x \to x \approx \mathbf{1}$$
$$x \to \mathbf{1} \approx \mathbf{1}$$

$$x \to y \approx \mathbf{1} \;\&\; y \to x \approx \mathbf{1} \supset x \approx y.$$

Clearly BCK^D is itself termwise definitionally equivalent to (in fact, is dually isomorphic to) the quasivariety of BCK-algebras—for details, see [32, Example 7.3]. ∎

Following the publication of the seminal monograph [31] algebraisable logics have been the object of intense investigation; for a (partial) survey of the literature, see the tutorials of Blok and Pigozzi [36] and Blok and Jònsson [28].

1.8.5. Matrix Semantics. An $\mathcal{L}$*-matrix* (or simply a *matrix* when $\mathcal{L}$ is understood) is a tuple $\langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is an algebra of type $\mathcal{L}$ and F is a subset of A. For any $\mathcal{L}$-matrix $\langle \mathbf{A};\ F\rangle$ and $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, let $\models_{\langle \mathbf{A};\ F\rangle}$: $\mathbb{P}(\mathrm{Fm}_{\mathcal{L}}) \to \mathrm{Fm}_{\mathcal{L}}$ be the relation defined by $\Gamma \models_{\langle \mathbf{A};\ F\rangle} \varphi$ if:

$$h(\psi) \in F \text{ for every } \psi \in \Gamma \text{ implies } h(\varphi) \in F, \quad \text{for every } h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}.$$

If M is a class of $\mathcal{L}$-matrices, then $\Gamma \models_{\mathsf{M}} \varphi$ if $\Gamma \models_{\langle \mathbf{A};\ F\rangle} \varphi$ for every $\langle \mathbf{A};\ F\rangle \in \mathsf{M}$.

Let $\mathbb{S}$ be a deductive system of type $\mathcal{L}$, $\mathbf{A}$ be an algebra of the same type and let $F \subseteq A$. The subset F of A is called an $\mathbb{S}$-*filter* of $\mathbf{A}$, or simply a *filter* when $\mathbb{S}$ is understood, if $\Gamma \vdash_{\mathbb{S}} \varphi$ implies $\Gamma \models_{\langle \mathbf{A};\ F\rangle} \varphi$ for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$; the set of all $\mathbb{S}$-filters of $\mathbf{A}$ is denoted $\mathrm{Fi}_{\mathbb{S}}\,\mathbf{A}$. If F is an $\mathbb{S}$-filter, then the $\mathcal{L}$-matrix $\langle \mathbf{A};\ F\rangle$ is called a *matrix model* of $\mathbb{S}$; the class of all matrix models of $\mathbb{S}$ is denoted $\mathsf{Mat}\,\mathbb{S}$. A congruence θ on $\mathbf{A}$ is said to be *compatible* with F if $a \in F$ and $(a, b) \in \theta$ implies $b \in F$ [31, Section 1.4]; the largest congruence on $\mathbf{A}$ compatible with F is called the *Leibniz congruence* on $\mathbf{A}$ over F and is denoted $\Omega_{\mathbf{A}}(F)$ [31, Theorem 1.5]. The natural map $\Omega_{\mathbf{A}} : \mathrm{Fi}_{\mathbb{S}}\,\mathbf{A} \to \mathrm{Con}\,\mathbf{A}$ defined by $F \mapsto \Omega_{\mathbf{A}}(F)$ is called the *Leibniz operator* on $\mathbf{A}$. An $\mathbb{S}$-matrix is said to be *reduced* if $\Omega_{\mathbf{A}}(F) = \omega_{\mathbf{A}}$, and the class of all reduced $\mathbb{S}$-matrices is denoted $\mathsf{Mat}^*\,\mathbb{S}$.

Theorem 1.8.6 (Reduced Matrix Completeness Theorem). *[36, Theorem 3.5] Let $\mathbb{S}$ be a deductive system and let $\mathsf{Mat}^*\,\mathbb{S}$ be the class of all reduced $\mathbb{S}$-matrices. For all $\Gamma \cup \{\varphi\}$,*

$$\Gamma \vdash_{\mathbb{S}} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathsf{Mat}^*\mathbb{S}} \varphi.$$

For a deductive system $\mathbb{S}$ over a language $\mathcal{L}$, a class M of $\mathcal{L}$-matrices is said to be a *matrix semantics* of $\mathbb{S}$ if, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathbb{S}} \varphi$ iff $\Gamma \models_{\mathsf{M}} \varphi$. For a deductive system $\mathbb{S}$, a quasivariety K and a system $\delta_t \approx \varepsilon_t$, $t = 1, \ldots, r$ of equations in one variable the following are equivalent [31, Theorem 2.4]: (i) the class of matrices $\{\langle \mathbf{A}, \{F_{\mathbf{A}}^{\delta\approx\varepsilon}\}\rangle : \mathbf{A} \in \mathsf{K}\}$, where $F_{\mathbf{A}}^{\delta\approx\varepsilon} = \{a \in A : \delta_t^{\mathbf{A}}(a) = \varepsilon_t^{\mathbf{A}}(a),\ t = 1, \ldots, r\}$, is a matrix semantics for $\mathbb{S}$; and (ii) K is an algebraic semantics for $\mathbb{S}$ with defining equations $\delta_t \approx \varepsilon_t$, $t = 1, \ldots, r$.

Theorem 1.8.7. *[31, Corollary 5.3] Let $\mathbb{S}$ be an algebraisable deductive system. Let K be the equivalent quasivariety semantics of $\mathbb{S}$ and let $\mathsf{Mat}^*\,\mathbb{S}$ be* *the class of all reduced* $\mathbb{S}$*-matrices. Then* K *is the class of all algebra reducts of* $\mathsf{Mat}^*\,\mathbb{S}$*, viz.:*

$$\mathsf{K} = \{\mathbf{A} : \langle \mathbf{A}, F\rangle \in \mathsf{Mat}^*\,\mathbb{S} \textit{ for some } \mathbb{S}\textit{-filter } F \textit{ of } \mathbf{A}\}.$$

Lemma 1.8.8. *Let* $\mathbb{S}_1$ *and* $\mathbb{S}_2$ *be deductive systems over the same language* $\mathcal{L}$*. If* $\mathbb{S}_1$ *and* $\mathbb{S}_2$ *are algebraisable with the same defining equations and the same equivalent algebraic semantics then they coincide.*

Proof. Let $\mathbb{S}_1$ and $\mathbb{S}_2$ both be algebraisable with equivalent quasivariety semantics K and defining equations $\delta_t \approx \varepsilon_t$, $t = 1, \ldots, r$. By previous remarks and Theorem 1.8.7, $\mathsf{Mat}^*\,\mathbb{S}_1 = \{\langle \mathbf{A}, \{a \in A : \delta_t^{\mathbf{A}}(a) = \varepsilon_t^{\mathbf{A}}(a),\ \ t = 1, \ldots, r\}\rangle : \mathbf{A} \in \mathsf{K}\} = \mathsf{Mat}^*\,\mathbb{S}_2$. Hence $\mathbb{S}_1$ and $\mathbb{S}_2$ coincide by Theorem 1.8.6. ■

1.8.9. Assertional Logics of Pointed Quasivarieties. For a quasivariety K of algebras with $\mathbf{1}$ over a language $\mathcal{L}$, the *assertional logic of* K, in symbols $\mathbb{S}(\mathsf{K}, \mathbf{1})$, is the closure operator $\vdash_{\mathbb{S}(\mathsf{K},\mathbf{1})}$ defined by the class of matrices $\mathsf{M}(\mathsf{K}, \mathbf{1}) := \{\langle \mathbf{A}, \{\mathbf{1}^{\mathbf{A}}\}\rangle : \mathbf{A} \in \mathsf{K}\}$ in the sense that:

$$\Gamma \vdash_{\mathbb{S}(\mathsf{K},\mathbf{1})} \varphi \quad \text{iff} \quad \Gamma \models_{\mathsf{M}(\mathsf{K},\mathbf{1})} \varphi$$

for any $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$. Because K is a quasivariety, $\vdash_{\mathbb{S}(\mathsf{K},\mathbf{1})}$ is finitary and structural, and hence is a deductive system in the sense of this thesis. Since $\mathsf{M}(\mathsf{K}, \mathbf{1})$ is a matrix semantics for $\mathbb{S}(\mathsf{K}, \mathbf{1})$ by definition, the assertional logic $\mathbb{S}(\mathsf{K}, \mathbf{1})$ may be defined equivalently by specifying that, for any $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,

$$\Gamma \vdash_{\mathbb{S}(\mathsf{K},\mathbf{1})} \varphi \quad \text{iff} \quad \{\psi \approx \mathbf{1} : \psi \in \Gamma\} \models_{\mathsf{K}} \varphi \approx \mathbf{1}.$$

The entailment $\{\psi \approx \mathbf{1} : \psi \in \Gamma\} \models_{\mathsf{K}} \varphi \approx \mathbf{1}$ is itself equivalent to the existence of some finite $\Delta \subseteq \Gamma$ such that:

$$\mathsf{K} \models \mathop{\&}_{\psi \in \Delta} \psi(\vec{p}) \approx \mathbf{1} \supset \varphi(\vec{p}) \approx \mathbf{1}$$

whence it is a harmless notational convenience to assume, for any entailment $\Delta \vdash_{\mathbb{S}(\mathsf{K},\mathbf{1})} \varphi$, that $\Delta \cup \{\varphi\}$ is a *finite* set of terms, all of the form $\chi(\vec{p})$, where $\vec{p}$ is understood as a *finite* sequence of variables, including all that occur in $\Delta \cup \{\varphi\}$. See Raftery and van Alten [193, Section 4.1].

Example 1.8.10. Let $\mathsf{M}(\mathsf{BCK}^D, \mathbf{1})$ denote the class of matrices of the form:

$$\langle \mathbf{A}, \{1^{\mathbf{A}}\} : \mathbf{A} \in \mathsf{BCK}^D \rangle$$

where $\mathsf{BCK}^{\mathsf{D}}$ is the class of dual BCK-algebras (recall Example 1.8.4). By an argument analogous to that of [36, p. 24] $\mathsf{M}(\mathsf{BCK}^D, \mathbf{1})$ is precisely the class of reduced $\mathbb{BCK}$-matrices of $\mathbb{BCK}$. By the reduced matrix completeness theorem it follows that the deductive system $\mathbb{S}(\mathsf{BCK}^D, \mathbf{1})$ coincides with $\mathbb{BCK}$. ∎

Assertional logics of pointed classes were introduced by Pigozzi in [181, Section 2] (but see also Curry [75, pp. 64 ff.]). For recent work concerning assertional logics of pointed classes, see Blok and Raftery [41], [40, Section 6], Czelakowski and Jansana [77, Section 6], Czelakowski and Pigozzi [78, Section 2.1], Raftery and Barbour [16, Section 2.3] and Raftery and van Alten [193, Section 4.1]. (See also Ursini [224, Section 2] for related work.) For varieties with normal ideals, the associated assertional logics have been investigated by Agliano and Ursini in [10, pp. 314 ff.], wherein the following lemma is stated without proof. The proof we give below is implicit in Blok and Raftery [40, Lemma 5.1, Section 7]; see also Blok and Raftery [40, Theorem 7.4, Corollary 7.5].

Lemma 1.8.11. *Let* V *be a variety with normal ideals and let* $\mathbb{S}(\mathsf{V}, \mathbf{1})$ *be the assertional logic of* V. *For any* $\mathbf{A} \in \mathsf{V}$, *the* V*-ideals of* $\mathbf{A}$ *coincide with the* $\mathbb{S}(\mathsf{V}, \mathbf{1})$*-filters of* $\mathbb{S}(\mathsf{V}, \mathbf{1})$.

Proof. Let V, $\mathbb{S}(\mathsf{V}, \mathbf{1})$ and $\mathbf{A}$ be as in the statement of the lemma. By [40, Lemma 5.1] the 1-class of any congruence of $\mathbf{A}$ is a $\mathbb{S}(\mathsf{V}, \mathbf{1})$-filter of $\mathbf{A}$. Conversely, any $\mathbb{S}(\mathsf{V}, \mathbf{1})$-filter is a V-ideal by the remarks of [40, Section 7, p. 180]. ∎

The following lemma is also stated without proof by Agliano and Ursini in [10, p. 314].

Lemma 1.8.12. *Let* V *be a variety with normal ideals. For any* $\mathbf{A} \in \mathsf{V}$ *and* $I \in \mathrm{I}(\mathbf{A})$, $\Omega_{\mathbf{A}}(I) = I^{\epsilon}$.

Proof. Let V, $\mathbf{A}$ and I be as stated. In view of the preceding lemma, by [40, Proposition 6.1] we have that $I = [1]_{\Omega_{\mathbf{A}}(I)}$ for any $I \in \mathrm{I}(\mathbf{A})$, which implies $\Omega_{\mathbf{A}}(I) = I^{\epsilon}$ as required. ∎

For a variety V with normal ideals, let $\mathsf{V}_{\epsilon} := \{\mathbf{A} : \mathbf{A} \cong \mathbf{B}/\langle 1\rangle^{\epsilon}_{\mathbf{B}}$ for some $\mathbf{B} \in \mathsf{V}\}$ be the class of *reduced algebras* of V.

Remark 1.8.13. In [10, p. 296, p. 315] Agliano and Ursini define $\mathsf{V}_{\epsilon} := \{\mathbf{A}/\langle 0\rangle^{\epsilon}_{\mathbf{A}} : \mathbf{A} \in \mathsf{V}\}$ for any variety V with normal ideals. This definition is in error [226]: V_{ϵ} must be closed under isomorphic copies by [10, Proposition 3.3]. The definition of V_{ϵ} used throughout this thesis reflects this correction. ∎

Theorem 1.8.14. *Let* V *be a variety with normal ideals. The following are equivalent:*

1. V *is finitely congruential;*
2. $\mathbb{S}(\mathsf{V}, \mathbf{1})$ *is algebraisable;*
3. V_{ϵ} *is a quasivariety.*

Moreover, if $\mathbb{S}(\mathsf{V}, \mathbf{1})$ *is algebraisable then* V_{ϵ} *is the equivalent algebraic semantics of* $\mathbb{S}(\mathsf{V}, \mathbf{1})$.

Proof. The first assertion is [10, Theorem 3.13]. By [77, Proposition 6.9] and Lemma 1.8.12, V_{ϵ} is exactly the class of algebra reducts of the reduced matrices of $\mathbb{S}(\mathsf{V}, \mathbf{1})$, which implies V_{ϵ} is the equivalent algebraic semantics of $\mathbb{S}(\mathsf{V}, \mathbf{1})$ by Theorem 1.8.7. ∎

Theorem 1.8.15. *[10, Corollary 3.17] Let* V *be a variety with normal ideals. The following are equivalent:*

1. V *is ideal determined;*
2. $\mathsf{V} = \mathsf{V}_{\epsilon}$*;*

3. $\mathbb{S}(\mathsf{V}, \mathbf{1})$ *is strongly algebraisable and its equivalent algebraic semantics is exactly* V.

Generalisations of the previous theorem to the effect that a quasivariety K with $\mathbf{1}$ is K-$\mathbf{1}$-regular iff K is the equivalent algebraic semantics of its algebraisable assertional logic $\mathbb{S}(\mathsf{K}, \mathbf{1})$ have very recently been obtained by several authors: see Blok and Raftery [40] and Barbour and Raftery [16, Corollary 50].

1.8.16. The Deduction Theorem in Algebraic Logic. Let $\mathbb{S}$ be a deductive system. $\mathbb{S}$ is said to have a *Deduction-Detachment Theorem* (DDT) if there exists a finite set $\Sigma := \Sigma(p,q) := \{\zeta_i(p,q) : i = 1, \ldots, n\}$ of formulas of $\mathbb{S}$ such that for any set $\Gamma \cup \{\varphi, \psi\}$ of formulas of $\mathbb{S}$ [36, Section 4],

$$\Gamma, \varphi \vdash_{\mathbb{S}} \psi \quad \text{iff} \quad \varphi \vdash_{\mathbb{S}} \Sigma(\varphi, \psi).$$

In this case, Σ is called a *deduction-detachment set* for $\mathbb{S}$. Observe that the existence of a DDT for $\mathbb{S}$ does not imply the existence of a conditional for $\mathbb{S}$; of course, the converse does obtain.

Theorem 1.8.17. *[36, Theorem 7.3] Let $\mathbb{S}$ be a strongly algebraisable deductive system and let V be its equivalent variety semantics. Then $\mathbb{S}$ has a DDT iff V has EDPC.*

The deduction theorem in algebraic logic has been extensively investigated by Blok, Pigozzi, Czelakowski and others: see for instance [33, 36, 76]. For a history of the deduction theorem in logic see Porte [185].

Chapter 2

The Theory of Pre-BCK-Algebras

In this chapter we investigate the theory of pre-BCK-algebras. Our study is based on and guided by Iséki and Tanaka's standard survey paper *An introduction to the theory of BCK-algebras* [126]. Thus the scope of our study is largely limited to the variety of pre-BCK-algebras *simpliciter* and to some natural pre-BCK-algebraic counterparts of the varieties of commutative, positive implicative and implicative BCK-algebras. Our study of pre-BCK-algebras does *not* extend to pre-BCK-algebraic analogues of the varieties of n-potent BCK-algebras described in §1.6.13, or to pre-BCK-algebraic analogues of other varieties of BCK-algebras that have been considered in the literature, such as the residuation subreducts of hoops.

Our main goal in this chapter is to show that pre-BCK-algebras admit a coherent elementary theory. In particular, our aim is to demonstrate that much of the first-order theory of BCK-algebras, suitably generalised, extends to pre-BCK-algebras. (By fiat, we highlight differences between the theory of pre-BCK-algebras and the theory of BCK-algebras where these occur.) Nonetheless, our ultimate motivation in studying pre-BCK-algebras is not simply an interest in generalisation for its own sake; rather, the driving force behind our study lies in the development of the applications of the sequel. This focus is most clearly reflected in this chapter in the emphasis given herein to the development of the order theory and the ideal theory of pre-BCK-algebras.

2.1 Pre-BCK-Algebras

In this section we study pre-BCK-algebras as a generalisation of pre-BCK-algebras to the subtractive but not point-regular case.

In §2.1.1 pre-BCK-algebras proper are introduced. It is shown that the variety PBCK of pre-BCK-algebras coincides with a certain variety of algebras generalising BCK and considered by Blok and Raftery in [38] and independently (in dually isomorphic form) by Agliano and Ursini in [10]. We also prove that the variety of pre-BCK-algebras is a subvariety of the varietal closure of the quasivariety of left residuation algebras. In one of the two main results of the section, a 'Clifford-McLean'-type theorem for pre-BCK-algebras, we show that for a pre-BCK-algebra $\mathbf{A}$, the equivalence Ξ induced by the natural pre-BCK quasiordering $\preceq$ (in the sense of Lemma 1.2.2) is a congruence on $\mathbf{A}$ such that the quotient algebra $\mathbf{A}/\Xi$ is the maximal BCK-algebra homomorphic image of A.

For a suitable notion of pre-ideal, the pre-ideal theory of pre-BCK-algebras is studied in §2.1.20. We provide a simple characterisation of pre-ideal generation in pre-BCK-algebras. For a pre-BCK-algebra $\mathbf{A}$, it is shown that a pre-ideal of $\mathbf{A}$ is just the inverse image of an ideal of the maximal BCK-algebra homomorphic image $\mathbf{A}/\Xi$. We also establish some other properties of pre-ideals of pre-BCK-algebras.

In §2.1.25 the relationship between pre-ideals and congruences in pre-BCK-algebras is investigated. It is shown that every pre-ideal of a pre-BCK-algebra $\mathbf{A}$ is the 0-class of a PBCK/BCK-congruence on $\mathbf{A}$. Hence we deduce that for pre-BCK-algebras, pre-ideals coincide with the (PBCK-) ideals of Gumm and Ursini described in §1.7.1. We prove that the lattice of all ideals of a pre-BCK-algebra is isomorphic to its lattice of PBCK/BCK-congruences. Further, we establish the existence of a commutative square of isomorphisms between the ideal and PBCK/BCK-congruence lattices of a pre-BCK-algebra and the ideal and BCK-congruence lattices of its maximal BCK-algebra homomorphic image.

The assertional logic of the variety of pre-BCK-algebras is studied in §2.1.33. We prove the variety of pre-BCK-algebras is finitely congruential, and hence show the assertional logic of the variety of pre-BCK-algebras is algebraisable with equivalent algebraic semantics (termwise definitionally equivalent to) BCK. In consequence we infer the other main result of the section: a quasi-identity of the form $\mathop{\&}_{i=1}^{n} s_i(\vec{x}) \approx 0 \supset t(\vec{x}) \approx 0$ is satisfied by PBCK iff it is satisfied by BCK.

2.1.1. Pre-BCK-Algebras. An algebra $\langle A; \dot{-}, 0\rangle$ of type $\langle 2, 0\rangle$ is called a *pre-BCK-algebra (in the sense of Cornish)* if the following identities are satisfied [71, Section 1]:

$$\big((x \dot{-} y) \dot{-} (x \dot{-} z)\big) \dot{-} (z \dot{-} y) \approx 0 \tag{2.1}$$

$$\big(x \dot{-} (x \dot{-} y)\big) \dot{-} y \approx 0 \tag{2.2}$$

$$x \dot{-} x \approx 0 \tag{2.3}$$

$$0 \dot{-} x \approx 0. \tag{2.4}$$

Because of (2.3), the class of pre-BCK-algebras may be understood as a variety of pointed groupoids; consequently we (informally) denote pre-BCK difference by juxtaposition in the sequel when no confusion can arise. Clearly a pre-BCK-algebra is a BCK-algebra iff it satisfies the quasi-identity:

$$x \dot{-} y \approx 0 \;\&\; y \dot{-} x \approx 0 \supset x \approx y. \tag{2.5}$$

Recall from [121, 123] that a BCK-algebra $\mathbf{A} := \langle A; \dot{-}, 0\rangle$ satisfies *Iséki's condition (S)* iff there exists a largest element $a + b$ of the subset $\{c \in A : ca \leq b\}$ for any $a, b \in A$, or, equivalently, $\mathbf{A}$ possesses another binary operation $+$ such that $a(b + c) = (ab)c$ is identically satisfied. Pre-BCK-algebras were introduced by Cornish in [71] as a means of constructing BCK-algebras from BCK-algebras with condition (S). Cornish's construction, reproduced in Example 2.1.2 below, generalises Wroński's example [240] showing that the class of BCK-algebras is not a variety; see [71, Theorem 2.2] and the remarks thereafter.

Example 2.1.2. [71, Section 2] Let $\langle A;\ \dot{-}, +, 0\rangle$ be a BCK-algebra with condition (S) and let t be any element of A. Let $A' := \{a' : a \in A\}$ and $A'' := \{a'' : a \in A\}$ be two sets equipotent with A and let $W_t(A)$ be the (disjoint) union of A, A' and A''. For any $a, b \in W_t(A)$ let the product $\dot{-}^{\mathbf{W}_t(A)}$ be defined as follows:

$$\begin{aligned}
&ab := a \dot{-}^{\mathbf{A}} b,\\
&a'b' = a''b'' := b \dot{-}^{\mathbf{A}} a,\\
&ab' = ab'' := 0,\\
&a'b := (a+b)',\\
&a''b := (a+b)'',\\
&a'b'' = a''b' := (b+t)a.
\end{aligned}$$

By [71, Lemma 2.1] $\mathbf{W}_t(A) := \left\langle W_t(A);\ \dot{-}^{\mathbf{W}_t(A)}, 0\right\rangle$ is a pre-BCK-algebra. Wroński's example [240] showing that the class of BCK-algebras is not a variety is $\mathbf{W}_1(\omega)$. ∎

Let $\langle A;\ \dot{-}, 0\rangle$ be a pointed groupoid with operation $\dot{-}$ defined by $ab = 0$ for any $a, b \in A$. It is clear from (2.1)–(2.4) that $\langle A;\ \dot{-}, 0\rangle$ is a pre-BCK-algebra. This observation is indicative of the fact that interesting classes of pre-BCK-algebras are those possessing one or more additional properties. Let $\langle A;\ \dot{-}, +, 0\rangle$ be a BCK-algebra with condition (S) and let t be any element of A. Because $a + 0 = a$ for any $a \in A$, $\mathbf{W}_t(A)$ is a pre-BCK-algebra satisfying the identity $x \dot{-} \mathbf{0} \approx x$. Thus it is consistent with Cornish's original construction to mean by 'pre-BCK-algebra' a pre-BCK-algebra (in the sense of Cornish) satisfying $x \dot{-} \mathbf{0} \approx x$. In this thesis, therefore, a *pre-BCK-algebra* is a pre-BCK-algebra (in the sense of Cornish) satisfying:

$$x \dot{-} \mathbf{0} \approx x. \tag{2.6}$$

The variety PBCK of pre-BCK-algebras is thus the class of algebras axiomatised by (2.1)–(2.4) and (2.6). Given these definitions, the following result is immediate.

Theorem 2.1.3. *The variety of pre-BCK-algebras is subtractive witness $x \dot{-} y$.*

Example 2.1.4. The identities (2.3), (2.4) and (2.6) assert that up to isomorphism there is a single two-element pre-BCK-algebra, namely the two-element implicative BCK-algebra $\mathbf{C}_1$. It is easy to see $\mathbf{C}_1$ embeds into every non-trivial pre-BCK-algebra and hence that $\mathbf{Q}(\mathbf{C}_1)$ is the smallest non-trivial subquasivariety of PBCK; *cf.* [231, p. 6]. Even more is true: because $\mathbf{C}_1$ generates the class of implicative BCK-algebras as a variety (recall Theorem 1.6.19), iBCK is the unique atom in the lattice of *varieties* of pre-BCK-algebras. ∎

Example 2.1.5. Denote by $\mathbf{B}_2$ the algebra $\langle\{0,1,2\};\ \dot{-},0\rangle$ of type $\langle 2,0\rangle$ with operation $\dot{-}$ defined by:

$\dot{-}^{\mathbf{B}_2}$	0	1	2
0	0	0	0
1	1	0	0
2	2	0	0

The 3-element pointed groupoid $\mathbf{B}_2$ is the simplest example of a pre-BCK-algebra that is distinct from a BCK-algebra. In their Siena paper [54] Burris and Berman have recently catalogued *all* 3-element groupoids; the Siena catalogue number of $\mathbf{B}_2$ is 216. From this remark and [54, p. 390] the following facts are known about the variety $\mathbf{V}(\mathbf{B}_2)$ generated by $\mathbf{B}_2$:

- The set of types (in the sense of Hobby and McKenzie's tame congruence theory [110]) realised in $\mathbf{V}(\mathbf{B}_2)$ is $\{\mathbf{1},\mathbf{3}\}$;
- $\mathbf{V}(\mathbf{B}_2)$ is not congruence distributive, congruence modular or congruence permutable;
- $\mathbf{V}(\mathbf{B}_2)$ does not have a decidable first-order theory;
- $|F(0)| = 1$, $|F(1)| = 2$ and $|F(2)| = 7$, where $|F(n)|$ denotes the cardinality of the $\mathbf{V}(\mathbf{B}_2)$-free algebra $\mathbf{F}(n)$ on n free generators.

∎

Classes of algebras generalising BCK but which fail in any sense to be point-regular have been considered by several authors in the literature: for instance, see Bunder [52], Blok and Raftery [38], Agliano and Ursini [10] and Humberstone [113]. However, such classes have invariably been introduced in the context of a wider field of study and have not been considered extensively in their own right. Apropos this remark, let V denote the variety of algebras with language $\langle \dot{-}, 0\rangle$ of type $\langle 2, 0\rangle$ axiomatised by the identities (2.1), (2.4), (2.6) and introduced by Blok and Raftery in [38, Section 4] in the context of their investigation into the quasivariety of BCK-algebras and its subvarieties.

Lemma 2.1.6. *An algebra $\langle A; \dot{-}, 0\rangle$ of type $\langle 2, 0\rangle$ is a pre-BCK-algebra iff it satisfies* (2.1), (2.4) *and* (2.6). *Thus the variety of pre-BCK-algebras coincides with Blok and Raftery's variety* V.

Proof. Let $\langle A; \dot{-}, 0\rangle$ be an algebra of type $\langle 2, 0\rangle$ satisfying (2.1), (2.4) and (2.6). Let $a, b \in A$. Notice that $0 = \big((a \dot{-} 0) \dot{-} (a \dot{-} b)\big) \dot{-} (b \dot{-} 0) = \big(a \dot{-} (a \dot{-} b)\big) \dot{-} b$ by (2.1) and (2.6). Thus (2.2) holds. Moreover, by (2.6) we obtain $0 = \big((a \dot{-} (a \dot{-} 0)\big) \dot{-} 0 = a \dot{-} a$ and so (2.3) holds also. The converse is clear. ∎

Let X be the variety of algebras with language $\langle \dot{-}, 0\rangle$ of type $\langle 2, 0\rangle$ axiomatised by the identities (2.1), (2.3), (2.4), (2.6) and the identity:

$$\big((x \dot{-} y) \dot{-} (z \dot{-} y)\big) \dot{-} (x \dot{-} z) \approx 0. \tag{2.7}$$

The variety X was introduced by Agliano and Ursini (in dually isomorphic form) in a case study [10, Example 4.5, pp. 330–332] concerning the relationship between ideals and congruences in subtractive varieties.

Lemma 2.1.7. *The variety of pre-BCK-algebras satisfies the following identities:*

$$(x \dot{-} y) \dot{-} \big(x \dot{-} (z \dot{-} (z \dot{-} y))\big) \approx 0 \tag{2.8}$$

$$\big(x \dot{-} (y \dot{-} z)\big) \dot{-} \big(x \dot{-} ((u \dot{-} z) \dot{-} (u \dot{-} y))\big) \approx 0. \tag{2.9}$$

Proof. Let **A** be a pre-BCK-algebra and let $a, b, c, d \in A$. For (2.8), put $\alpha := a, \beta := b$ and $\gamma := c(cb)$. We have:

$$\begin{aligned}
0 &= \big((\alpha\beta)(\alpha\gamma)\big)(\gamma\beta) && \text{by (2.1)}\\
&= \big((\alpha\beta)(\alpha\gamma)\big)\big((c(cb))b\big) &&\\
&= \big((\alpha\beta)(\alpha\gamma)\big)0 && \text{by (2.2)}\\
&= (\alpha\beta)(\alpha\gamma) && \text{by (2.6)}\\
&= (ab)\big(a(c(cb))\big). &&
\end{aligned}$$

For (2.9), put $\alpha := a, \beta := bc$ and $\gamma := (dc)(db)$. We have:

$$\begin{aligned}
0 &= \big((\alpha\beta)(\alpha\gamma)\big)(\gamma\beta) && \text{by (2.1)}\\
&= \big((\alpha\beta)(\alpha\gamma)\big)\big(((dc)(db))(bc)\big) &&\\
&= \big((\alpha\beta)(\alpha\gamma)\big)0 && \text{by (2.1)}\\
&= (\alpha\beta)(\alpha\gamma) && \text{by (2.6)}\\
&= \big(a(bc)\big)\big(a((dc)(db))\big). &&
\end{aligned}$$

■

Proposition 2.1.8. *The variety of pre-BCK-algebras satisfies the following identities:*

$$\big((x \mathbin{\dot{-}} y) \mathbin{\dot{-}} (z \mathbin{\dot{-}} y)\big) \mathbin{\dot{-}} (x \mathbin{\dot{-}} z) \approx \mathbf{0}\,, \tag{2.7}$$

$$\big((x \mathbin{\dot{-}} y) \mathbin{\dot{-}} z\big) \mathbin{\dot{-}} \big((x \mathbin{\dot{-}} z) \mathbin{\dot{-}} y\big) \approx \mathbf{0}. \tag{2.10}$$

Proof. Let **A** be a pre-BCK-algebra and let $a, b, c \in A$. We first derive (2.10). For (2.10), put $\alpha := (ab)c, \beta := (ac), \gamma := b$ and $\delta := a$. We have:

$$\begin{aligned}
0 &= \big(\alpha(\beta\gamma)\big)\big(\alpha((\delta\gamma)(\delta\beta))\big) && \text{by (2.9)}\\
&= \big(((ab)c)((ac)b)\big)\big(((ab)c)((ab)(a(ac)))\big). &&
\end{aligned}$$

Put $\alpha := ab$, $\beta := c$ and $\gamma := a$. We have:

$$\begin{aligned}
&\big(((ab)c)((ac)b)\big)\big(((ab)c)((ab)(a(ac)))\big) \\
&\quad = \big(((ab)c)((ac)b)\big)\big((\alpha\beta)(\alpha(\gamma(\gamma\beta)))\big) \\
&\quad = \big(((ab)c)((ac)b)\big)0 && \text{by (2.8)} \\
&\quad = \big((ab)c\big)\big((ac)b\big) && \text{by (2.6).}
\end{aligned}$$

For (2.7), put $\alpha := ab, \beta := cb$ and $\gamma := ac$. We have:

$$\begin{aligned}
0 &= \big((\alpha\beta)\gamma\big)\big((\alpha\gamma)\beta\big) && \text{by (2.10)} \\
&= \big((\alpha\beta)\gamma\big)\big(((ab)(ac))(cb)\big) \\
&= \big((\alpha\beta)\gamma\big)0 && \text{by (2.1)} \\
&= (\alpha\beta)\gamma && \text{by (2.6)} \\
&= \big((ab)(cb)\big)(ac).
\end{aligned}$$

■

Lemma 2.1.9. *An algebra $\langle A;\ \dot{-}, 0\rangle$ of type $\langle 2, 0\rangle$ is a pre-BCK-algebra iff it satisfies* (2.1), (2.3), (2.4), (2.6), *and* (2.7). *Thus the variety of pre-BCK-algebras coincides with Agliano and Ursini's variety* X.

Proof. Let $\mathbf{A}$ be a pre-BCK-algebra. Then $\mathbf{A} \models (2.1), (2.3), (2.4), (2.6)$ by definition. Also $\mathbf{A} \models (2.7)$ by Proposition 2.1.8, so $\mathbf{A} \in \mathsf{X}$. Conversely, if $\langle A;\ \dot{-}, 0\rangle$ is an algebra of type $\langle 2, 0\rangle$ satisfying (2.1), (2.3), (2.4), (2.6) and (2.7) then $\langle A;\ \dot{-}, 0\rangle$ is a pre-BCK-algebra by Lemma 2.1.6. ■

Several K-0-regular quasivarieties K generalising BCK have also been considered in the literature, of which the most important is the class of left residuation algebras. A *left residuation algebra* is a $\langle \dot{-}, 0\rangle$-subreduct of a polrim [229, Chapter 1, p. 17]; recall the definition of a polrim from §1.3.3. By van Alten [229, Proposition 1.4(i)] the class LR of all left residuation algebras is a quasivariety, axiomatised by the identities (2.4), (2.6), (2.7) and the quasi-identity (2.5); LR is not a variety [139, Theorem 9]. Because of (2.5) and [78, Theorem 2.3], LR is LR-0-regular, while by van Alten and Raftery [231, Lemma 3.1] BCK is exactly the subquasivariety of left residuation algebras axiomatised by either of the identities:

$$(x \dot{-} y) \dot{-} z \approx (x \dot{-} z) \dot{-} y \quad \text{or} \quad \big(x \dot{-} (x \dot{-} y)\big) \dot{-} y \approx \mathbf{0}.$$

Left residuation algebras were introduced under the (misleading—see van Alten [229, Proposition 2.10; p. 40]) name *BCC-algebras* by Komori in [139]. They have since been studied by several authors, including Komori [138, 139], Dudek [82, 83, 84], Ono and Komori [176] and Wroński [242]. A recent major study of left residuation algebras (and their associated assertional logics) is van Alten [229]; see also Raftery and van Alten [192] and van Alten and Raftery [230, 231]. Van Alten [229, Chapter 3] and Komori [138] have also studied the varietal closure $\mathsf{H}(\mathsf{LR})$ of the variety of left residuation algebras; the following theorem is due to Komori [138].

Theorem 2.1.10. *[138, Theorem 6] The variety* $\mathsf{H}(\mathsf{LR})$ *generated by the class* LR *of all left residuation algebras is finitely based and axiomatised by the identities* (2.4), (2.6) *and* (2.7).

Proposition 2.1.11. *(cf. [231, Lemma 3.1]) An algebra* $\mathbf{A} := \langle A; \dot{-}, 0\rangle$ *of type* $\langle 2, 0\rangle$ *is a pre-BCK-algebra iff* $\mathbf{A} \in \mathsf{H}(\mathsf{LR})$ *and moreover* $\mathbf{A}$ *satisfies:*

$$\big(x \dot{-} (x \dot{-} y)\big) \dot{-} y \approx \mathbf{0}. \tag{2.2}$$

Thus the variety of pre-BCK-algebras is an equational subclass of $\mathsf{H}(\mathsf{LR})$*, the variety generated by the class of all left residuation algebras.*

Proof. ($\Rightarrow$) Let $\mathbf{A}$ be a pre-BCK-algebra. By definition $\mathbf{A} \models (2.2)$. Moreover $\mathbf{A} \models (2.7)$ by Proposition 2.1.8; since $\mathbf{A} \models (2.4), (2.6)$ by definition, we have that $\mathbf{A} \in \mathsf{H}(\mathsf{LR})$ by Theorem 2.1.10.

($\Leftarrow$) Let $\mathbf{A} \in \mathsf{H}(\mathsf{LR})$ be such that $\mathbf{A} \models (2.2)$ and let $a, b, c \in A$. Throughout the proof we denote $\dot{-}^{\mathbf{A}}$ by juxtaposition for ease of notation. Put $\alpha := a, \beta := b$ and $\gamma := c(cb)$. We have:

$$0 = \big((\alpha\beta)(\gamma\beta)\big)(\alpha\gamma) \qquad \text{by (2.7)}$$

$$
\begin{aligned}
&= \big((\alpha\beta)((c(cb))b)\big)(\alpha\gamma) && \\
&= \big((\alpha\beta)0\big)(\alpha\gamma) && \text{by (2.2)} \\
&= (\alpha\beta)(\alpha\gamma) && \text{by (2.6)} \\
&= (ab)\big(a(c(cb))\big). && (2.11)
\end{aligned}
$$

Put $\alpha := a, \beta := bc$ and $\gamma := (bd)(cd)$. We also have:

$$
\begin{aligned}
0 &= \big((\alpha\beta)(\gamma\beta)\big)(\alpha\gamma) && \text{by (2.7)} \\
&= \big((\alpha\beta)(((bd)(cd))(bc))\big)(\alpha\gamma) && \\
&= \big((\alpha\beta)0\big)(\alpha\gamma) && \text{by (2.7)} \\
&= (\alpha\beta)(\alpha\gamma) && \text{by (2.6)} \\
&= \big(a(bc)\big)\big(a((bd)(cd))\big). && (2.12)
\end{aligned}
$$

For (2.1), put $\alpha := (ab)(ac), \beta := cb$ and $\gamma := ab$. We have:

$$
\begin{aligned}
0 &= (\alpha\beta)\big(\alpha(\gamma(\gamma\beta))\big) && \text{by (2.11)} \\
&= \big(((ab)(ac))(cb)\big)\big(((ab)(ac))((ab)((ab)(cb)))\big). &&
\end{aligned}
$$

Put $\alpha := ab$, $\beta := a$, $\gamma := c$ and $\delta := b$. We have:

$$
\begin{aligned}
&\big(((ab)(ac))(cb)\big)\big(((ab)(ac))((ab)((ab)(cb)))\big) && \\
&\quad = \big(((ab)(ac))(cb)\big)\big((\alpha(\beta\gamma))(\alpha((\beta\delta)(\gamma\delta)))\big) && \\
&\quad = \big(((ab)(ac))(cb)\big)0 && \text{by (2.12)} \\
&\quad = \big((ab)(ac)\big)(cb) && \text{by (2.6).}
\end{aligned}
$$

Thus $\mathbf{A} \models (2.1)$. By Theorem 2.1.10 we have that $\mathbf{A} \models (2.4), (2.6)$, which implies $\mathbf{A}$ is a pre-BCK-algebra by Lemma 2.1.6. ∎

Because of Proposition 2.1.11, results obtained by van Alten [229], Komori [138, 139] and others about $\mathbf{H}(\mathsf{LR})$ are applicable to PBCK. In particular, the following useful technical lemma may be regarded as a specialisation to pre-BCK-algebras of a result due to van Alten [229].

Lemma 2.1.12. *(cf. [229, Lemma 1.2]) Let* $\mathbf{A}$ *be a pre-BCK-algebra and let* $\preceq$ *be the binary relation defined on* A *by* $a \preceq b$ *iff* $ab = 0$. *Then* $\langle A; \preceq\rangle$ *is a quasiordered set with least element* 0. *Moreover, the relation* $\preceq$ *satisfies the following conditions for any* $a, b, c \in A$*:*

1. *If* $a \preceq b$ *then* $cb \preceq ca$;
2. *If* $a \preceq b$ *then* $ac \preceq bc$;
3. $ab \preceq a$.

Proof. Let $\mathbf{A}$ be a pre-BCK-algebra and let $a, b, c \in A$. By (2.3), $a \preceq a$. Suppose $a \preceq b$ and $b \preceq c$. Then $ab = 0$ and $bc = 0$, so $ac = \big((ac)0\big)0 = \big((ac)(ab)\big)(bc) = 0$ by (2.6) and (2.1). Thus $a \preceq c$ and $\preceq$ is a quasiorder on A. Also, $0 \preceq a$ for any $a \in A$ by (2.4), so 0 is a minimal element under $\preceq$. Suppose $0 \neq m \in A$ is another element minimal under $\preceq$. Then for all $a \in A$ we have $m \preceq a$, and in particular $m \preceq 0$. Thus $m0 = 0$. But $m0 = m$ by (2.6), so $0 = m$, a contradiction. Therefore 0 is unique and so is the least element of A under $\preceq$. To complete the proof of the lemma it remains to show (1)–(3). For (1), suppose $a \preceq b$. Then $ab = 0$ and $(cb)(ca) = \big((cb)(ca)\big)0 = \big((cb)(ca)\big)(ab) = 0$ by (2.6) and (2.1). Thus $cb \preceq ca$. For (2), suppose $a \preceq b$. Then $ab = 0$ and $(ac)(bc) = \big((ac)(bc)\big)0 = \big((ac)(bc)\big)(ab) = 0$ by (2.6) and (2.7). Thus $ac \preceq bc$. For (3), just note $(ab)a = \big((ab)(a0)\big)(0b) = 0$ by (2.6), (2.4) and (2.1), whence $ab \preceq a$. ∎

Remark 2.1.13. (*cf.* [70, Example 3.1]) Let $\langle A; \preceq\rangle$ be a quasiordered set with least element $0 \in A$. For any $a, b \in A$, let:

$$a \dot{-} b := \begin{cases} 0 & \text{if } a \preceq b \\ a & \text{otherwise.} \end{cases}$$

Then the induced algebra $\langle A; \dot{-}, 0\rangle$ is a pre-BCK-algebra, whose underlying quasiordering is consistent with the original quasiordering on A. Hence the underlying quasiordering on a pre-BCK-algebra has no interesting properties in general. ∎

The preceding remark notwithstanding, the equivalence Ξ induced by the pre-order $\preceq$ (in the sense of Lemma 1.2.2) plays a special role in the theory of pre-BCK-algebras. This is shown in the following result, which may be understood as a kind of 'Clifford-McLean theorem' for pre-BCK-algebras.

Theorem 2.1.14. *For any pre-BCK-algebra* $\mathbf{A}$ *the following assertions hold:*

1. *The equivalence* Ξ *induced by* $\preceq$ *in the sense of Lemma 1.2.2 is a congruence on* $\mathbf{A}$, *and* $\mathbf{A}/\Xi$ *is a BCK-algebra;*
2. Ξ *is the smallest congruence on* $\mathbf{A}$ *whose quotient algebra is a BCK-algebra, and so* $\mathbf{A}/\Xi$ *is the maximal BCK-algebra homomorphic image of* $\mathbf{A}$*;*
3. $[0]_\Xi = \{0\}$*;*
4. $\Xi = \langle 0 \rangle^{\mathrm{c}}_{\mathbf{A}}$.

Proof. Let $\mathbf{A}$ be a pre-BCK-algebra.

For (1), suppose $a_1 \Xi b_1$, $a_2 \Xi b_2$ for $a_1, b_1, a_2, b_2 \in A$. From $a_1 \preceq b_1$ we have $a_1 a_2 \preceq b_1 a_2$ by Lemma 2.1.12(2) and from $b_2 \preceq a_2$ we have $b_1 a_2 \preceq b_1 b_2$ by Lemma 2.1.12(1). By transitivity, $a_1 a_2 \preceq b_1 b_2$. Also, from $a_2 \preceq b_2$ we have $b_1 b_2 \preceq b_1 a_2$ by Lemma 2.1.12(1) and from $b_1 \preceq a_1$ we have $b_1 a_2 \preceq a_1 a_2$ by Lemma 2.1.12(2). By transitivity, $b_1 b_2 \preceq a_1 a_2$. Thus $a_1 a_2 \Xi b_1 b_2$ and Ξ is a congruence on $\mathbf{A}$. To see $\mathbf{A}/\Xi$ is a BCK-algebra it is sufficient to show $\mathbf{A}/\Xi \models (2.5)$. Suppose $ab \Xi 0$ and $ba \Xi 0$. From $ab \preceq 0$, $ba \preceq 0$ we have $ab = 0 = ba$ by Lemma 2.1.12, so $a \Xi b$. Thus $\mathbf{A}/\Xi \models (2.5)$ and $\mathbf{A}/\Xi$ is a BCK-algebra.

For (2), let $\theta \in \operatorname{Con} \mathbf{A}$ be such that $\mathbf{A}/\theta$ is a BCK-algebra. Suppose $a \Xi b$ for $a, b \in A$. Then $ab = 0$ and $ba = 0$, and thus $ab \equiv_\theta 0$ and $ba \equiv_\theta 0$. Since $\mathbf{A}/\theta \models (2.5)$ we have that $a \equiv_\theta b$. Thus Ξ is the smallest congruence on $\mathbf{A}$ whose quotient algebra is a BCK-algebra, and $\mathbf{A}/\Xi$ is the maximal BCK-algebra homomorphic image of $\mathbf{A}$.

For (3), let $a \in A$ be such that $a \Xi 0$. Then $a \preceq 0$, so $a = 0$ by Lemma 2.1.12.

For (4), it is sufficient by definition of $\langle 0\rangle^{\mathfrak{c}}_{\mathbf{A}}$ to show Ξ is the largest congruence $\theta \in \operatorname{Con}\mathbf{A}$ such that $[0]_\theta = \{0\}$. Now Ξ is a congruence on $\mathbf{A}$ such that $[0]_\Xi = \{0\}$ by (1) and (3). Let $\theta \in \operatorname{Con}\mathbf{A}$ be such that $[0]_\theta = \{0\}$ and suppose $a \equiv_\theta b$ for $a, b \in A$. Then $ab \equiv_\theta aa = 0$ and $ba \equiv_\theta bb = 0$, so $ab, ba \in [0]_\theta$. But $[0]_\theta = \{0\}$, so $ab = 0$ and $ba = 0$, which implies $a \Xi b$. ∎

Corollary 2.1.15. *For any pre-BCK-algebra* $\mathbf{A}$ *(with underlying quasiorder* $\preceq$*) the partial orders* $\preceq/\Xi$ *and* $\leq^{\mathbf{A}/\Xi}$ *coincide, where* $\preceq/\Xi$ *denotes the partial order on* A/Ξ *induced by* $\preceq$ *in the sense of Lemma 1.2.2 and* $\leq^{\mathbf{A}/\Xi}$ *denotes the underlying partial order of the maximal BCK-algebra homomorphic image* $\mathbf{A}/\Xi$ *of* $\mathbf{A}$.

Proof. Let $\mathbf{A}$, $\mathbf{A}/\Xi$, $\preceq/\Xi$ and $\leq^{\mathbf{A}/\Xi}$ be as in the statement of the corollary. Throughout the proof to simplify notation we write $\overline{a}$ for the equivalence class $[a]_\Xi$ in A/Ξ containing $a \in A$. Let $\overline{a}, \overline{b} \in A/\Xi$ with $a, b \in A$. We have:

$$
\begin{aligned}
\overline{a} \preceq/\Xi\, \overline{b} \quad &\text{iff} \quad a \preceq b && \text{by Lemma 1.2.2}\\
&\text{iff} \quad a \mathbin{\dot{-}}^{\mathbf{A}} b = 0^{\mathbf{A}} && \text{by Theorem 2.1.14(3)}\\
&\text{iff} \quad \overline{a} \mathbin{\dot{-}}^{\mathbf{A}/\Xi} \overline{b} = 0^{\mathbf{A}/\Xi} &&\\
&\text{iff} \quad \overline{a} \leq^{\mathbf{A}/\Xi} \overline{b}. &&
\end{aligned}
$$

∎

Corollary 2.1.16. *Let* $\mathbf{A}$ *be a pre-BCK-algebra. An identity of the form* $t(\vec{x}) \approx 0$ *is satisfied by* $\mathbf{A}$ *iff it is satisfied by its maximal BCK-algebra homomorphic image* $\mathbf{A}/\Xi$. *In symbols,*

$$\mathbf{A} \models t(\vec{x}) \approx 0 \quad \textit{iff} \quad \mathbf{A}/\Xi \models t(\vec{x}) \approx 0.$$

Proof. Let $\mathbf{A}$ be a pre-BCK-algebra. Clearly $\mathbf{A} \models t(\vec{x}) \approx 0$ implies $\mathbf{A}/\Xi \models t(\vec{x}) \approx 0$. For the converse, suppose $\mathbf{A}/\Xi \models t(\vec{x}) \approx 0$. Let $\vec{a} \in A$ and let $\nu : \mathbf{A} \to \mathbf{A}/\Xi$ denote the natural map. By assumption, $\nu\big(t^{\mathbf{A}}(\vec{a})\big) = \nu(0^{\mathbf{A}})$; since $\nu(0^{\mathbf{A}}) = [0^{\mathbf{A}}]_\Xi = \{0^{\mathbf{A}}\}$ (by Theorem 2.1.14(3)), we infer $t^{\mathbf{A}}(\vec{a}) = 0^{\mathbf{A}}$. Thus $\mathbf{A} \models t(\vec{x}) \approx 0$ as desired. ∎

Corollary 2.1.17. *The class* PBCK_ϵ *of reduced algebras of* PBCK *is exactly the class* BCK *of BCK-algebras.*

Proof. Just observe $\mathbf{A} \in \mathsf{PBCK}_\epsilon$ iff $\langle 0\rangle^\epsilon_{\mathbf{A}} = \omega_{\mathbf{A}}$ iff $\Xi = \omega_{\mathbf{A}}$ (by Theorem 2.1.14(4)) iff $\mathbf{A} \in \mathsf{BCK}$. ∎

Corollary 2.1.18. *For any* $\mathbf{A} \in \mathsf{PBCK}$, $\mathrm{Con}_{\mathsf{PBCK/BCK}}\,\mathbf{A} \subsetneq [\Xi, \iota_{\mathbf{A}}]$.

Proof. Let $\theta \in \mathrm{Con}_{\mathsf{PBCK/BCK}}\,\mathbf{A}$. By Theorem 2.1.14(2) we have $\Xi \leq \theta \leq \iota_{\mathbf{A}}$, and so $\theta \in [\Xi, \iota_{\mathbf{A}}]$. Thus $\mathrm{Con}_{\mathsf{PBCK/BCK}}\,\mathbf{A} \subsetneq [\Xi, \iota_{\mathbf{A}}]$. ∎

Remark 2.1.19. The converse of the preceding corollary does not hold in general. Consider Wroński's example $\mathbf{W_1}(\omega)$ (Example 2.1.2) that shows the class of BCK-algebras is not a variety. Since $\mathbf{W_1}(\omega)$ is a BCK-algebra, $\Xi = \omega_{\mathbf{W_1}(\omega)}$. But there exists $\theta \geq \Xi$ such that $\mathbf{B}_2 \cong \mathbf{W_1}(\omega)/\theta$ (see Wroński [240]), whence $\mathrm{Con}_{\mathsf{PBCK/BCK}}\,\mathbf{W_1}(\omega) \not\subseteq [\Xi, \iota_{\mathbf{W_1}(\omega)}]$. ∎

2.1.20. Pre-Ideal Theory of Pre-BCK-Algebras. Let I be a non-empty subset of (the universe of) a pre-BCK-algebra $\mathbf{A}$. I is called a *pre-ideal* if the following conditions are satisfied:

$$0 \in I, \tag{2.13}$$

$$a \in I, ba \in I \text{ implies } b \in I. \tag{2.14}$$

The set of all pre-ideals of a pre-BCK-algebra $\mathbf{A}$ is denoted by $\mathrm{Pre}(\mathbf{A})$. The following easy lemma collects together some useful facts about pre-ideals.

Lemma 2.1.21. *(cf. [229, Lemma 4.16]) For a pre-BCK-algebra* $\mathbf{A}$ *the following assertions hold:*

1. *A pre-ideal of* $\mathbf{A}$ *is a hereditary subset of* $\langle A; \preceq\rangle$*;*

2. *If* I *is a pre-ideal of* $\mathbf{A}$*, and there exist* $a \in A$ *and* $b_1, \ldots, b_n \in I$ *such that* $\big(\cdots((ab_1)b_2)\cdots\big)b_n = 0$*, then* $a \in I$*;*

3. *For any congruence* θ *on* $\mathbf{A}$*, the* 0*-class* $[0]_\theta = \big\{a \in A : (a, 0) \in \theta\big\}$ *of* θ *is a pre-ideal of* $\mathbf{A}$*;*

4. A pre-ideal of **A** *is a subuniverse of* **A**.

Proof. Let **A** be a pre-BCK-algebra. For (1), let I be a pre-ideal. From $a \in I$ and $b \preceq a$ we have $ba = 0 \in I$, whence $b \in I$ by the definition of I as a pre-ideal. Since $0 \in I$, (2) now follows by repeated application of (1). For (3), let $\theta \in \operatorname{Con} \mathbf{A}$ and let $a, ba \in [0]_\theta$. Since $b = b0 \equiv_\theta ba \equiv_\theta 0$ we have $b \equiv_\theta 0$; that is to say $b \in [0]_\theta$. Since $0 \in [0]_\theta$, we have that $[0]_\theta$ is a pre-ideal. For (4), let I be a pre-ideal of **A** and let $a, b \in I$. By Lemma 2.1.12(3) we have $ab \preceq a$; thus $ab \in I$ by (1) and I is a subuniverse of **A**. ∎

The conditions (2.13)–(2.14) defining pre-ideals, in conjunction with the results of Lemma 2.1.21, suggest intuitively that every pre-ideal of a pre-BCK-algebra is the inverse image of an ideal of its maximal BCK-algebra homomorphic image. This intuition is made precise in the following theorem, the proof of which is due *mutatis mutandis* to the author's Ph.D. supervisor. See also Corollary 2.1.29 in the sequel.

Theorem 2.1.22. *[17, Lemma 1.1.9] The set* $\operatorname{Pre}(\mathbf{A})$ *of all pre-ideals of a pre-BCK-algebra* **A** *forms an algebraic lattice* $\mathbf{Pre}(\mathbf{A})$ *when ordered by inclusion, which is isomorphic to the lattice of ideals of* $\mathbf{A}/\Xi$ *under the map* $I \overset{\psi}{\mapsto} I/\Xi$ *for any* $I \in \operatorname{Pre}(\mathbf{A})$.

Proof. It is clear that if I is a pre-ideal of **A**, then $\psi(I)$ is an ideal of $\mathbf{A}/\Xi$. Conversely, if J is an ideal of $\mathbf{A}/\Xi$, then $I = \{a \in A : a \in [b]_\Xi \text{ for some } [b]_\Xi \in J\}$ is a pre-ideal of **A** with the property that $\psi(I) = J$. Thus ψ is a bijection between the pre-ideals of **A** and the ideals of $\mathbf{A}/\Xi$. Suppose now that I_1 and I_2 are pre-ideals of **A** such that $I_1 \subseteq I_2$. Then $[a]_\Xi \in \psi(I_1)$ implies $\{b \in A : b \Xi a\} \subseteq I_1 \subseteq I_2$, which implies $[a]_\Xi \in \psi(I_2)$, and so $\psi(I_1) \subseteq \psi(I_2)$. Conversely if J_1 and J_2 are ideals of $\mathbf{A}/\Xi$ such that $J_1 \subseteq J_2$, then $\psi^{-1}(J_1) = \{a \in A : a \in [b]_\Xi \text{ for some } [b]_\Xi \in J_1\} \subseteq \{a \in A : a \in [b]_\Xi \text{ for some } [b]_\Xi \in J_2\} = \psi^{-1}(J_2)$. Thus ψ is an order isomorphism, and hence a lattice isomorphism between the algebraic lattices of pre-ideals and ideals of **A** and $\mathbf{A}/\Xi$. ∎

In light of the preceding theorem, it is natural to anticipate that results concerning ideals of BCK-algebras cited in §1.6.1 extend to pre-ideals of pre-BCK-algebras. Certainly this is the case in relation to the following proposition, which characterises the pre-ideal $\langle B\rangle_{\mathbf{A}} := \bigcap\{J \in \mathrm{Pre}(\mathbf{A}) : B \subseteq J\}$ generated by non-empty $B \subseteq A$ of any pre-BCK-algebra $\mathbf{A}$ (of course, $\langle B\rangle_{\mathbf{A}}$ exists because pre-ideals of $\mathbf{A}$ are closed under arbitrary intersections). The result is due to van Alten [229, Proposition 4.16(iv)], who first stated and proved the proposition in the context of pre-ideals of left residuation algebras. Because of Proposition 2.1.11, van Alten's proof generalises to pre-BCK-algebras without modification.

Proposition 2.1.23. *(cf. [125, Theorem 3]) Let* $\mathbf{A}$ *be a pre-BCK-algebra. For any* $\varnothing \neq B \subseteq A$,

$$\langle B\rangle_{\mathbf{A}} = \big\{a \in A : (\exists\, n \in \omega)(\exists\, c_1, \ldots, c_n \in B) \textit{ such that } \big(\cdots(ac_1)\cdots\big)c_n = 0\big\}.$$

In particular, for any $b \in A$,

$$\langle b\rangle_{\mathbf{A}} = \big\{a \in A : (\exists\, n \in \omega) \textit{ such that } ab^n = 0\big\}.$$

Proof. Let $\mathbf{A}$ be a pre-BCK-algebra and let $\varnothing \neq B \subseteq A$. Let $D := \big\{a \in A : (\exists\, n \in \omega)(\exists\, c_1, \ldots, c_n \in B)$ such that $\big(\cdots(ac_1)\cdots\big)c_n = 0\big\}$. By Lemma 2.1.21(2) we have $D \subseteq \langle B\rangle_{\mathbf{A}}$. Also, $B \subseteq D$, so it remains only to establish that D is a pre-ideal of $\mathbf{A}$. Let $a \in A$ and $b, ab \in D$, say:

$$\big(\cdots((ab)c_1)\cdots\big)c_n = 0 = \big(\cdots(bd_1)\cdots\big)d_m,$$

where the c_i, d_j are in B. Then:

$$\begin{aligned}
&\big(\cdots(((\cdots(ad_1)\cdots)d_m)c_1)\cdots\big)c_n \\
&\quad = \big(\cdots((((\cdots(ad_1)\cdots)d_m)0)c_1)\cdots\big)c_n \\
&\quad = \big(\cdots((((\cdots(ad_1)\cdots)d_m)((\cdots(bd_1)\cdots)d_m))c_1)\cdots\big)c_n \\
&\quad \preceq \big(\cdots((ab)c_1)\cdots c_n\big) \qquad \text{by (2.7) and Lemma 2.1.12(2)} \\
&\quad = 0
\end{aligned}$$

so $a \in D$, as required. ∎

Some further properties of pre-ideals of pre-BCK-algebras analogous to properties of ideals of BCK-algebras are established in the following proposition. Concerning the statement of the proposition, a pre-BCK-algebra $\mathbf{A}$ is said to enjoy the *pre-ideal extension property* if for any $\mathbf{B} \in \mathsf{S}(\mathbf{A})$ and any $I \in \mathrm{Pre}(\mathbf{B})$ there is a pre-ideal $J \in \mathrm{Pre}(\mathbf{A})$ such that $J \cap B = I$.

Proposition 2.1.24. *For any pre-BCK-algebra* $\mathbf{A}$*, the following assertions hold:*

1. $\mathrm{Pre}(\mathbf{A})$ *is a distributive lattice;*

2. $\mathbf{A}$ *enjoys the pre-ideal extension property.*

Proof. (1) follows immediately from Theorem 2.1.22 and Proposition 1.6.9(1). For (2), suppose $\mathbf{A} \in \mathsf{PBCK}$, $\mathbf{B} \in \mathsf{S}(\mathbf{A})$ and $I \in \mathrm{Pre}(\mathbf{B})$. Then for $\mathbf{A}/\Xi \in \mathsf{BCK}$, $\mathbf{B}/\Xi \in \mathsf{S}(\mathbf{A}/\Xi)$ and $\psi(I) \in \mathrm{Pre}(\mathbf{B}/\Xi)$ there exists an ideal $J \in \mathrm{Pre}(\mathbf{A}/\Xi)$ such that $J \cap B/\Xi = \psi(I)$, where $\psi(I)$ is the image of I under the map ψ of Theorem 2.1.22. Indeed, by Proposition 1.6.9(2) we can pick $J = \langle \psi(I) \rangle_{\mathbf{A}/\Xi}$. Now $\psi^{-1}(J)$ is a pre-ideal of $\mathbf{A}$ with the property that $\psi^{-1}(J) \cap B = I$. ∎

2.1.25. Pre-Ideals and Congruences of Pre-BCK-algebras. Recall from §1.6.1 that for a BCK-algebra $\mathbf{A}$, any ideal of $\mathbf{A}$ (in the sense of §1.6.1) is the 0-class of a BCK-congruence on $\mathbf{A}$, and conversely that the 0-class of any (BCK-) congruence on $\mathbf{A}$ is an ideal of $\mathbf{A}$. Inasmuch as pre-ideals and PBCK/BCK-congruences are (for pre-BCK-algebras) the pre-BCK-algebraic analogues of ideals and BCK-congruences (of BCK-algebras), respectively, the preceding remarks invite (for pre-BCK-algebras) a study of the relationship between pre-ideals and PBCK/BCK-congruences.

Theorem 2.1.26. *For any pre-BCK-algebra* $\mathbf{A}$ *and* $I \in \mathrm{Pre}(\mathbf{A})$*, the relation* ϕ_I *defined on* $A \times A$ *by:*

$$\phi_I := \{(a,b) \in A \times A : ab, ba \in I\}$$

is a congruence on $\mathbf{A}$*, and the quotient algebra* $\mathbf{A}/\phi_I$ *is a BCK-algebra.*

Proof. Let $\mathbf{A}$ be a pre-BCK-algebra, let $I \in \mathrm{Pre}(\mathbf{A})$ and let ϕ_I be as in the statement of the theorem. To see ϕ_I is an equivalence relation, we show (for any $a, b, c \in A$):

(i) $a \equiv_{\phi_I} a$;
(ii) $a \equiv_{\phi_I} b$ implies $b \equiv_{\phi_I} a$;
(iii) $a \equiv_{\phi_I} b$, $b \equiv_{\phi_I} c$ implies $a \equiv_{\phi_I} c$.

For (i), we have that $aa = 0 \in I$ by (2.3), so $a \equiv_{\phi_I} a$.

For (ii), we have that $a \equiv_{\phi_I} b$ implies $b \equiv_{\phi_I} a$ by definition of ϕ_I.

For (iii), suppose $a \equiv_{\phi_I} b$ and $b \equiv_{\phi_I} c$. By (2.1) applied twice, we have that $\big((ac)(ab)\big)(bc) = 0 \in I$ and $\big((ca)(cb)\big)(ba) = 0 \in I$. Since $ab, bc \in I$ and $cb, ba \in I$ by assumption, we have that $ac, ca \in I$ by Lemma 2.1.21(2) applied twice. Hence $a \equiv_{\phi_I} c$.

By (i), (ii) and (iii), ϕ_I is an equivalence relation. To see ϕ_I is a congruence on $\mathbf{A}$, it is sufficient to show (for any $a_1, a_2, b_1, b_2 \in A$):

(iv) $a_1 \equiv_{\phi_I} a_2$ and $b_1 \equiv_{\phi_I} b_2$ implies $a_1 b_1 \equiv_{\phi_I} a_2 b_2$.

To prove (iv), we show:

(iv)(a) $b_1 \equiv_{\phi_I} b_2$ implies $a_1 b_1 \equiv_{\phi_I} a_1 b_2$;
(iv)(b) $a_1 \equiv_{\phi_I} a_2$ implies $a_1 b_2 \equiv_{\phi_I} a_2 b_2$.

For (iv)(a), suppose $b_1 \equiv_{\phi_I} b_2$. By (2.1) applied twice, $\big((a_1 b_1)(a_1 b_2)\big)(b_2 b_1) = 0 \in I$ and $\big((a_1 b_2)(a_1 b_1)\big)(b_1 b_2) = 0 \in I$. Since $b_2 b_1 \in I, b_1 b_2 \in I$ by hypothesis, $(a_1 b_1)(a_1 b_2) \in I, (a_1 b_2)(a_1 b_1) \in I$ by Lemma 2.1.21(1) applied twice. Hence $a_1 b_1 \equiv_{\phi_I} a_1 b_2$.

For (iv)(b), suppose $a_1 \equiv_{\phi_I} a_2$. By (2.7) applied twice, $\big((a_1 b_2)(a_2 b_2)\big)(a_1 a_2) = 0 \in I$ and $\big((a_2 b_2)(a_1 b_2)\big)(a_2 a_1) = 0 \in I$. Since $a_1 a_2 \in I, a_2 a_1 \in I$ by hypothesis, $(a_1 b_2)(a_2 b_2) \in I, (a_2 b_2)(a_1 b_2) \in I$ by Lemma 2.1.21(1) applied twice. Hence $a_1 b_2 \equiv_{\phi_I} a_2 b_2$.

By (iv)(a), (iv)(b) and (iii), we infer that $a_1 \equiv_{\phi_I} a_2$ and $b_1 \equiv_{\phi_I} b_2$ implies $a_1 b_1 \equiv_{\phi_I} a_2 b_2$, which establishes (iv).

By (i), (ii), (iii) and (iv), ϕ_I is a congruence on $\mathbf{A}$. Hence $\mathbf{A}/\phi_I$ is a pre-BCK-algebra. To complete the proof, it is sufficient in view of the remarks of §2.1.1 to show $\mathbf{A}/\phi_I \models (2.5)$. So suppose $ab \equiv_{\phi_I} 0$ and $ba \equiv_{\phi_I} 0$ for $a, b \in A$. Then

$(ab)0 \in I$ and $(ba)0 \in I$, so $ab \in I$ and $ba \in I$ by (2.6); that is to say $a \equiv_{\phi_I} b$. Hence $\mathbf{A}/\phi_I \models$ (2.5). Therefore $\mathbf{A}/\phi_I$ is a BCK-algebra, and the proof is complete. ∎

By Proposition 1.7.5, any subtractive variety has normal ideals. This result does *not* extend to (even K-0-regular) quasivarieties K with a binary term $s(x, y)$ witnessing (1.42), (1.43): for a counterexample see van Alten [229, pp. 71–72]. Nonetheless, Proposition 4.15 of van Alten [229] implies that, for a BCK-algebra $\mathbf{A}$, a non-empty subset $I \subseteq A$ is a BCK-ideal of $\mathbf{A}$ iff $I \in \mathrm{N}(\mathbf{A})$ iff I is an ideal of $\mathbf{A}$ (in the sense of §1.6.1). For pre-BCK-algebras, Theorem 2.1.26 in conjunction with the following lemma yields the crucial connection between pre-ideals and PBCK-ideals presented in Proposition 2.1.28 below.

Lemma 2.1.27. *Let* $\mathbf{A}$ *be a pre-BCK-algebra, let* $I \in \mathrm{Pre}(\mathbf{A})$ *and let* ϕ_I *be the congruence induced by* I *in the sense of Theorem 2.1.26. Then* $I = [0]_{\phi_I}$*, and* ϕ_I *is the largest congruence on* $\mathbf{A}$ *with this property.*

Proof. Let $\mathbf{A}$ be a pre-BCK-algebra, let $I \in \mathrm{Pre}(\mathbf{A})$ and let ϕ_I be the congruence induced by I in the sense of Theorem 2.1.26. Let $a \in I$. By (2.6), $a0 = a \in I$, while $0a = 0 \in I$ by (2.4). Hence $a \equiv_{\phi_I} 0$. That is to say, $a \in [0]_{\phi_I}$, so $I \subseteq [0]_{\phi_I}$. Conversely, let $a \in [0]_{\phi_I}$. Then $a \equiv_{\phi_I} 0$, and so $a0 \in I$. Since $0 \in I$, we infer that $a \in I$ by the definition of I as a pre-ideal. Thus $[0]_{\phi_I} \subseteq I$ and $I = [0]_{\phi_I}$.

Suppose now that θ is a congruence on $\mathbf{A}$ such that $[0]_\theta = I$. Let $a \equiv_\theta b$ for $a, b \in A$. Then $ab \equiv_\theta aa = 0$ and $ba \equiv_\theta bb = 0$, so $ab, ba \in [0]_\theta$; that is to say $a \equiv_{\phi_{[0]_\theta}} b$. Since $[0]_\theta = I$, $a \equiv_{\phi_I} b$. Thus $\theta \subseteq \phi_I$, and the proof is complete. ∎

Proposition 2.1.28. *A non-empty subset* $I \subseteq A$ *of a pre-BCK-algebra* $\mathbf{A}$ *is a pre-ideal of* $\mathbf{A}$ *iff it is a* PBCK*-ideal of* $\mathbf{A}$*. Hence* $\mathrm{I}_{\mathsf{PBCK}}(\mathbf{A}) = \mathrm{Pre}(\mathbf{A})$.

Proof. Let $\mathbf{A}$ be a pre-BCK-algebra with $\{0\} \subseteq I \subseteq A$.

($\Rightarrow$) Suppose $I \in \mathrm{Pre}(\mathbf{A})$. By Lemma 2.1.27, $I = [0]_{\phi_I}$, where ϕ_I is the congruence on $\mathbf{A}$ of Theorem 2.1.26. Hence $I \in \mathrm{N}(\mathbf{A})$. Since $\mathbf{A}$ is subtractive, $I \in \mathrm{I}_{\mathsf{PBCK}}(\mathbf{A})$ by Proposition 1.7.5. Hence $\mathrm{Pre}(\mathbf{A}) \subseteq \mathrm{I}_{\mathsf{PBCK}}(\mathbf{A})$.

($\Leftarrow$) Suppose $I \in \mathrm{I}_{\mathsf{PBCK}}(\mathbf{A})$. Since $\mathbf{A}$ is subtractive, $I \in \mathrm{N}(\mathbf{A})$ by Proposition 1.7.5. That is, $I = [0]_\theta$ for some $\theta \in \mathrm{Con}\,\mathbf{A}$. Hence $I \in \mathrm{Pre}(\mathbf{A})$ by Lemma 2.1.21(3). Thus $\mathrm{I}_{\mathsf{PBCK}}(\mathbf{A}) \subseteq \mathrm{Pre}(\mathbf{A})$. ∎

In view of the preceding proposition, from this point forwards we will always employ the notation and terminology of §1.7.1 in connection with (pre-) ideals of pre-BCK-algebras.

Corollary 2.1.29. *The set* $\mathrm{I}(\mathbf{A})$ *of all ideals of a pre-BCK-algebra* $\mathbf{A}$ *yields a distributive algebraic lattice* $\mathbf{I}(\mathbf{A})$ *when ordered by inclusion. For any* $I, J \in \mathrm{I}(\mathbf{A})$,

$$\begin{aligned} I \wedge J &= I \cap J = \{a \sqcap b : a \in I, b \in J\} \\ I \vee J &= \{b : \textit{for some } a \in I,\ ba \in J\} \end{aligned}$$

where $a \sqcap b := a(ab)$ *for any* $a, b \in A$.

Proof. Let $\mathbf{A}$ be a pre-BCK-algebra. By the remarks of §1.7.1, the set $\mathrm{I}(\mathbf{A})$ of ideals of $\mathbf{A}$ yields an algebraic lattice $\mathbf{I}(\mathbf{A})$ under inclusion, which is distributive by Proposition 2.1.28 and Proposition 2.1.24(1). Let $I, J \in \mathrm{I}(\mathbf{A})$. It is clear that $I \wedge J = I \cap J$. To see $I \cap J = \{a \sqcap b : a \in I, b \in J\}$, let $c := a \sqcap b$ where $a \in I$ and $b \in J$. Then $c \preceq a \in I$ and $c \preceq b \in J$ by Lemma 2.1.12(3) and (2.2) respectively, so $c \in I$ and $c \in J$ by Lemma 2.1.21(1); that is to say $c \in I \cap J$. Hence $\{a \sqcap b : a \in I, b \in J\} \subseteq I \cap J$ and, since the opposite inclusion is trivial, equality holds. That $I \vee J = \{b : \text{for some } a \in I,\ ba \in J\}$ follows immediately from the remarks of §1.7.1 and Part (4) of [222, Proposition 1.3]. ∎

By Theorem 1.6.10, the maps $I \overset{\xi}{\mapsto} \phi_I$ $(I \in \mathrm{I}(\mathbf{A}))$ and $\theta \overset{\xi^{-1}}{\mapsto} [0]_\theta$ $(\theta \in \mathrm{Con}_{\mathsf{BCK}}\,\mathbf{A})$ are, for any BCK-algebra $\mathbf{A}$, mutually inverse lattice isomorphisms between the ideal lattice of $\mathbf{A}$ and BCK-congruence lattice of $\mathbf{A}$, which result suggests the following theorem.

Theorem 2.1.30. *For any pre-BCK-algebra* $\mathbf{A}$, *the maps* $I \overset{\varphi}{\mapsto} \phi_I$ $(I \in \mathrm{I}(\mathbf{A}))$ *and* $\theta \overset{\varphi^{-1}}{\mapsto} [0]_\theta$ $(\theta \in \mathrm{Con}_{\mathsf{PBCK/BCK}}\,\mathbf{A})$ *are mutually inverse lattice isomorphisms between the* PBCK/BCK-*congruence lattice of* $\mathbf{A}$ *and the ideal lattice of* $\mathbf{A}$.

Proof. Let $\mathbf{A}$ be a pre-BCK-algebra. To prove the theorem it suffices to show:

(i) The maps φ and φ^{-1} are mutually inverse bijections;

(ii) The maps φ and φ^{-1} are order preserving.

For (i), it is sufficient to show $\varphi^{-1}\circ\varphi = \omega_{\mathrm{I}(\mathbf{A})}$ and $\varphi\circ\varphi^{-1} = \omega_{\mathrm{Con}_{\mathsf{PBCK/BCK}}\mathbf{A}}$. In the former case $\varphi^{-1}\circ\varphi = \omega_{\mathrm{I}(\mathbf{A})}$ since $I = [0]_{\phi_I}$ for every $I \in \mathrm{I}(\mathbf{A})$ by Lemma 2.1.27 In the latter case $\theta \leq \phi_{[0]_\theta}$ (for $\theta \in \mathrm{Con}_{\mathsf{PBCK/BCK}}\,\mathbf{A}$) by Lemma 2.1.27, so it remains only to show $\phi_{[0]_\theta} \leq \theta$. So let $a \equiv b \,(\mathrm{mod}\, \phi_{[0]_\theta})$ for $a, b \in A$. Then $ab, ba \in [0]_\theta$ by definition of $\phi_{[0]_\theta}$; that is to say $ab, ba \equiv_\theta 0$. Since $\mathbf{A}/\theta \in \mathsf{BCK}$ by hypothesis, we infer that $a \equiv_\theta b$ by (2.5). Hence $\phi_{[0]_\theta} \subseteq \theta$. Thus $\phi_{[0]_\theta} = \theta$, and so $\varphi \circ \varphi^{-1} = \omega_{\mathrm{Con}_{\mathsf{PBCK/BCK}}\mathbf{A}}$.

For (ii), to see φ is order preserving let $I, J \in \mathrm{I}(\mathbf{A})$ with $I \subseteq J$. Suppose $a \equiv_{\phi_I} b$ for $a, b \in A$. Then $ab, ba \in I$, and since $I \subseteq J$ we have also that $ab, ba \in J$. Thus $a \equiv_{\phi_J} b$ and $\phi_I \subseteq \phi_J$. Hence φ is order preserving. To see φ^{-1} is order preserving, let $\theta, \psi \in \mathrm{Con}_{\mathsf{PBCK/BCK}}\,\mathbf{A}$ with $\theta \subseteq \psi$. Suppose $a \in [0]_\theta$. Then $a \equiv_\theta 0$, and since $\theta \subseteq \psi$ we have also that $a \equiv_\psi 0$. Thus $a \in [0]_\psi$ and $[0]_\theta \subseteq [0]_\psi$. Hence φ^{-1} is order preserving. ∎

Corollary 2.1.31. *For any pre-BCK-algebra* $\mathbf{A}$*, the following assertions hold:*

1. $\mathbf{A}$ *is* PBCK/BCK-0-*regular;*
2. $\mathbf{A}$ *is* PBCK/BCK-*congruence distributive;*
3. $\mathbf{A}$ *has the* PBCK/BCK-*congruence extension property.*

Proof. Let $\mathbf{A}$ be a pre-BCK-algebra. For (1), let θ and ψ be PBCK/BCK-congruences on $\mathbf{A}$ such that $[0]_\theta = [0]_\psi$. By Lemma 2.1.21(3), $[0]_\theta$ and $[0]_\psi$ are both ideals, which coincide by assumption of the equality of $[0]_\theta$ and $[0]_\psi$. Because of Theorem 2.1.30, we infer that $\theta = \psi$, so $\mathbf{A}$ is PBCK/BCK-0-regular and (1) holds. The remaining assertions of the corollary now follow trivially from PBCK/BCK-0-regularity and Proposition 2.1.24. ∎

From Theorem 1.6.10, Theorem 2.1.22 and Theorem 2.1.30, we may also infer:

Theorem 2.1.32. *For any pre-BCK-algebra* $\mathbf{A}$*, the map* $\theta \overset{\upsilon}{\mapsto} \theta/\Xi$ $(\theta \in \mathrm{Con}_{\mathsf{PBCK/BCK}}\,\mathbf{A})$ *is an isomorphism from the* PBCK/BCK*-congruence lattice of* $\mathbf{A}$ *onto the* BCK*-congruence lattice of* $\mathbf{A}/\Xi$*. Therefore if* $\mathsf{H}(\mathbf{A}/\Xi) \subseteq \mathsf{BCK}$ *then* $\mathbf{Con}_{\mathsf{PBCK/BCK}}\,\mathbf{A}$ *and* $\mathbf{Con}\,\mathbf{A}/\Xi$ *are isomorphic.*

Proof. For any pre-BCK-algebra $\mathbf{A}$, we have that $\mathbf{Con}_{\mathsf{PBCK/BCK}}\,\mathbf{A} \cong \mathbf{I}(\mathbf{A})$ by Theorem 2.1.30; that $\mathbf{I}(\mathbf{A}) \cong \mathbf{I}(\mathbf{A}/\Xi)$ by Theorem 2.1.22; and that $\mathbf{I}(\mathbf{A}/\Xi) \cong \mathbf{Con}_{\mathsf{BCK}}\,\mathbf{A}/\Xi$ by Theorem 1.6.10. By composition of maps, it follows that $\mathrm{Con}_{\mathsf{PBCK/BCK}}\,\mathbf{A} \cong \mathrm{Con}_{\mathsf{BCK}}\,\mathbf{A}/\Xi$ under the mapping $\theta \overset{\upsilon}{\mapsto} \phi_{[0]_{(\theta/\Xi)}}$, which simplifies to $\theta \overset{\upsilon}{\mapsto} \theta/\Xi$ by PBCK/BCK-0-regularity as claimed. The remaining assertion of the theorem now follows, because the condition $\mathsf{H}(\mathbf{A}/\Xi) \subseteq \mathsf{BCK}$ guarantees that every congruence on $\mathbf{A}/\Xi$ is a BCK-congruence. ∎

We conclude this subsection by noting that the proof of Theorem 2.1.32 implicitly establishes the existence of a commutative square of isomorphisms connecting the ideal and PBCK/BCK-congruence lattices of a pre-BCK-algebra to the ideal and BCK-congruence lattices of its maximal BCK-algebra homomorphic image. In more detail: given a pre-BCK-algebra $\mathbf{A}$ (with maximal BCK-algebra homomorphic image $\mathbf{A}/\Xi$) and the maps:

$$\begin{array}{ll} I \overset{\xi}{\mapsto} \phi_I, & I \in \mathrm{I}(\mathbf{A}/\Xi) \\ I \overset{\psi}{\mapsto} I/\Xi, & I \in \mathrm{I}(\mathbf{A}) \\ I \overset{\varphi}{\mapsto} \phi_I, & I \in \mathrm{I}(\mathbf{A}) \\ \theta \overset{\upsilon}{\mapsto} \theta/\Xi, & \theta \in \mathrm{Con}\,\mathbf{A} \end{array}$$

of Theorem 1.6.10, Theorem 2.1.22, Theorem 2.1.30 and Theorem 2.1.32 respectively, the diagram:

$$\begin{tikzcd} \mathbf{I}(\mathbf{A}) \arrow[r, "\psi"] \arrow[d, "\varphi"'] & \mathbf{I}(\mathbf{A}/\Xi) \arrow[d, "\xi"] \\ \mathbf{Con}_{\mathsf{PBCK/BCK}}\,\mathbf{A} \arrow[r, "\upsilon"'] & \mathbf{Con}_{\mathsf{BCK}}\,\mathbf{A}/\Xi. \end{tikzcd}$$

commutes in the category of lattices.

2.1.33. The Assertional Logic of the Variety PBCK. Let the class PBCK^D

of *dual pre-BCK-algebras* be the variety of algebras with language $\langle \to, 1\rangle$ of type $\langle 2, 0\rangle$ axiomatised by the following identities:

$$(x \to y) \to \big((y \to z) \to (x \to z)\big) \approx \mathbf{1} \tag{2.15}$$

$$x \to \big((x \to y) \to y\big) \approx \mathbf{1} \tag{2.16}$$

$$x \to x \approx \mathbf{1} \tag{2.17}$$

$$x \to \mathbf{1} \approx \mathbf{1} \tag{2.18}$$

$$\mathbf{1} \to x \approx x; \tag{2.19}$$

notice that the class BCK^D of dual BCK-algebras (recall Example 1.8.4) is exactly the subquasivariety of PBCK^D axiomatised by the quasi-identity:

$$x \to y \approx \mathbf{1} \mathbin{\&} y \to x \approx \mathbf{1} \supset x \approx y.$$

Let $\mathrm{T}_{\mathcal{L}}(\mathrm{X})$ denote the term algebra of type $\mathcal{L}$ over $\mathbf{X}$, where $\mathcal{L}$ is the language of pre-BCK-algebras. Also, let $\mathbf{T}_{\mathcal{L}^D}(\mathrm{X})$ denote the term algebra of type $\mathcal{L}^D$ over X, where $\mathcal{L}^D$ is the language of dual pre-BCK-algebras. Consider the maps $\eta : \mathrm{T}_{\mathcal{L}}(\mathrm{X}) \to \mathrm{T}_{\mathcal{L}^D}(\mathrm{X})$ and $\xi : \mathrm{T}_{\mathcal{L}^D}(\mathrm{X}) \to \mathrm{T}_{\mathcal{L}}(\mathrm{X})$ defined respectively by:

$$\begin{aligned} \eta(\mathbf{0}) &:= \mathbf{1} & & \\ \eta(x) &:= x & & x \in \mathbf{X} \\ \eta(p \mathbin{\dot{-}} q) &:= \eta(q) \to \eta(p) & & p, q \in \mathrm{T}_{\mathcal{L}}(\mathrm{X}) \end{aligned}$$

and:

$$\begin{aligned} \xi(\mathbf{1}) &:= \mathbf{0} & & \\ \xi(x) &:= x & & x \in \mathbf{X} \\ \xi(r \to s) &:= \xi(s) \mathbin{\dot{-}} \xi(r) & & r, s \in \mathrm{T}_{\mathcal{L}^D}(\mathrm{X}). \end{aligned}$$

Because of the axiomatisation of PBCK by (2.1)–(2.5) and the axiomatisation of PBCK^D by (2.15)–(2.19), the proof of the following lemma is trivial and hence is omitted.

Lemma 2.1.34. *For* $p, q \in \mathrm{T}_{\mathcal{L}}(\mathrm{X})$ *and* $r, s \in \mathrm{T}_{\mathcal{L}^D}(\mathrm{X})$ *the following assertions* *hold:*

1. *If* $\mathsf{PBCK} \models p \approx q$ *then* $\mathsf{PBCK}^D \models \eta(p) \approx \eta(q)$*;*

2. *If* $\mathsf{PBCK}^D \models r \approx s$ *then* $\mathsf{PBCK} \models \xi(r) \approx \xi(s)$*.*

Moreover, $\xi \circ \eta = \omega_{\mathbf{T}_{\mathcal{L}}(\mathbf{X})}$ *and* $\eta \circ \xi = \omega_{\mathbf{T}_{\mathcal{L}^D}(\mathbf{X})}$.

Recall from §1.8.9 that for any quasivariety K with $\mathbf{1}$, the inherent assertional logic $\mathbb{S}(\mathsf{K}, \mathbf{1})$ of K may be defined by specifying that, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathbb{S}(\mathsf{K},\mathbf{1})} \varphi$ iff $\{\psi \approx \mathbf{1} : \psi \in \Gamma\} \models_{\mathsf{K}} \varphi \approx \mathbf{1}$. Because the variety of pre-BCK-algebras is termwise definitionally equivalent to (in fact, is dually isomorphic to) the variety of dual pre-BCK-algebras (by Lemma 2.1.34), the preceding observation implies the assertional logics $\mathbb{S}(\mathsf{PBCK}, \mathbf{0})$ and $\mathbb{S}(\mathsf{PBCK}^D, \mathbf{1})$ are definitionally equivalent. In a sense, it is thus a harmless notational convenience to ascribe to the variety of pre-BCK-algebras the intrinsic assertional logic $\mathbb{S}(\mathsf{PBCK}^D, \mathbf{1})$ of the variety PBCK^D of dual pre-BCK-algebras; of course, like remarks apply concerning BCK and $\mathbb{S}(\mathsf{BCK}^D, \mathbf{1})$. In the sequel, therefore, we shall not hesitate to denote by $\mathbb{S}(\mathsf{PBCK}^D, \mathbf{1})$ [$\mathbb{S}(\mathsf{BCK}^D, \mathbf{1})$] the inherent assertional logic of the variety of pre-BCK-algebras [the quasivariety of BCK-algebras] when convenient.

Proposition 2.1.35. *[10, Proposition 4.5]* $D := \{x \dot{-} y, y \dot{-} x\}$ *is an IC-system (without parameters) for* PBCK*. Hence* $I^{\epsilon} = \phi_I$ *for any ideal* I *and* $\theta^1 = \phi_{[0]_\theta}$ *for any congruence* θ *of a pre-BCK-algebra.*

Proof. Let $D := \{x \dot{-} y, y \dot{-} x\}$ and observe D a system of $\mathbf{0}$-terms for PBCK. Let $\mathbf{A} \in \mathsf{PBCK}$ and recall $I^D := \{(a,b) \in A \times A : ab, ba \in I\}$ for any ideal $I \in \mathrm{I}(\mathbf{A})$. Since $\langle 0 \rangle_{\mathbf{A}} = \{0\}$, $\langle 0 \rangle^D_{\mathbf{A}} = \{0\}^D = \{(a,b) \in A \times A : ab, ba = 0\} = \Xi \in \mathrm{Con}\,\mathbf{A}$ by Theorem 2.1.14(1). By Proposition 1.7.7(2) we have that D is an IC-system without parameters for PBCK. The second statement now follows immediately from Proposition 1.7.7(4). ∎

In [130] Kabziński studied the lattice $\Lambda^Q(\mathsf{K})$ of all quasivarieties K over the language $\langle \to, \mathbf{1} \rangle$ of BCK^D algebras for which $\models_{\mathsf{M}(\mathsf{K},\mathbf{1})} = \models_{\mathsf{M}(\mathsf{BCK}^D,\mathbf{1})}$, where $\models_{\mathsf{M}(\mathsf{K},\mathbf{1})}$ is the matrix consequence determined by the class of all matrices

$\{\langle \mathbf{A}, \{1^{\mathbf{A}}\}\rangle : \mathbf{A} \in \mathsf{K}\}$ and $\models_{\mathsf{M}(\mathsf{BCK}^D,\mathbf{1})}$ is the matrix consequence of Example 1.8.10, or equivalently all quasivarieties K for which K is an algebraic semantics for $\mathbb{BCK}$. By [130, Fact 1] BCK^D is the smallest element of $\Lambda^Q(\mathsf{K})$. In [130, Fact 2] Kabziński asserts without proof that PBCK^D is a member of $\Lambda^Q(\mathsf{K})$. The following theorem, which shows PBCK and BCK have the same assertional logic (namely $\mathbb{BCK}$), verifies Kabziński's result. For some recent results related to and generalising the theorem, see Blok and Raftery [41, Example 6.2].

Theorem 2.1.36. *The assertional logic* $\mathbb{S}(\mathsf{PBCK}^D, \mathbf{1})$ *of the variety of pre-BCK-algebras is algebraisable with equivalent algebraic semantics* BCK^D. *Thus the variety of pre-BCK-algebras and the quasivariety of BCK-algebras have the same assertional logic, namely* $\mathbb{BCK}$.

Proof. For the first assertion of the theorem, we have that PBCK is finitely congruential by Proposition 2.1.35, so $\mathbb{S}(\mathsf{PBCK}, \mathbf{0})$ is algebraisable with equivalent algebraic semantics PBCK_ϵ by Theorem 1.8.14. Because PBCK_ϵ coincides with BCK by Corollary 2.1.17, from Lemma 2.1.34 it follows that $\mathbb{S}(\mathsf{PBCK}^D, \mathbf{1})$ is algebraisable with equivalent algebraic semantics BCK^D. For the second assertion of the theorem, we have that $\mathbb{S}(\mathsf{BCK}^D, \mathbf{1})$ coincides with $\mathbb{BCK}$ by Example 1.8.10, so $\mathbb{S}(\mathsf{BCK}^D, \mathbf{1})$ is algebraisable with equivalent algebraic semantics BCK^D by Example 1.8.4. Hence $\mathbb{S}(\mathsf{PBCK}^D, \mathbf{1})$ and $\mathbb{S}(\mathsf{BCK}^D, \mathbf{1})$ have the same equivalent algebraic semantics, namely BCK^D. Since $\mathbb{S}(\mathsf{PBCK}^D, \mathbf{1})$ and $\mathbb{S}(\mathsf{BCK}^D, \mathbf{1})$ also have the same defining equation (*viz.*, $p \approx \mathbf{1}$), they are identical by Lemma 1.8.8. Hence PBCK and BCK have the same assertional logic, namely $\mathbb{BCK}$. ∎

Remark 2.1.37. It should not be supposed that PBCK is the largest quasivariety K with language $\langle \to, \mathbf{1}\rangle$ of type $\langle 2, 0\rangle$ such that $\models_{\mathsf{M}(\mathsf{K},\mathbf{1})} = \models_{\mathsf{M}(\mathsf{BCK}^D,\mathbf{1})}$. Indeed, by Kabziński [130, Fact 3] the greatest element of $\Lambda^Q(\mathsf{K})$ is the quasivariety axiomatised by the following identities and quasi-identity:

$$(x \to y) \to ((z \to x) \to (z \to y)) \approx \mathbf{1}$$
$$(y \to (z \to x)) \to (z \to (y \to x)) \approx \mathbf{1}$$
$$x \to (y \to x) \approx \mathbf{1}$$

$$x \to \mathbf{1} \approx \mathbf{1} \mathbin{\&} x \to y \approx \mathbf{1} \supset y \approx \mathbf{1}.$$

We remark that Kabziński's result is essentially a specialisation to $\mathbb{S}(\mathsf{PBCK}^D, \mathbf{1})$ of an observation due to Blok and Pigozzi [31, p. 16], which characterise the largest algebraic semantics of any deductive system having an algebraic semantics. ■

Corollary 2.1.38. *A quasi-identity of the form:*

$$\mathop{\&}_{i=1}^{n} s_i(\vec{x}) \approx \mathbf{0} \supset t(\vec{x}) \approx \mathbf{0} \tag{2.20}$$

is satisfied by PBCK *iff it is satisfied by* BCK. *In particular, an identity of the form:*

$$t(\vec{x}) \approx \mathbf{0}$$

is satisfied by PBCK *iff it is satisfied by* BCK.

Proof. Let $\{s_i(\vec{x}) : i = 1, \ldots, n\} \cup \{t(\vec{x})\}$ be $\langle \dot{-}, \mathbf{0}\rangle$-terms in the variables $\vec{x}$. Identify the $\langle \to, \mathbf{1}\rangle$-terms $\{\eta(s_i(\vec{x})) : i = 1, \ldots, n\} \cup \{\eta(t(\vec{x}))\}$ (where $\eta : \mathrm{T}_{\mathcal{L}}(\mathrm{X}) \to \mathrm{T}_{\mathcal{L}^D}(\mathrm{X})$ is the map of Lemma 2.1.34) in the variables $\vec{x}$ with the $\langle \to, \mathbf{1}\rangle$-formulas $\{\varphi_i(\vec{p}) : i = 1, \ldots, n\} \cup \{\psi(\vec{p})\}$ in the variables $\vec{p}$. Given this notation, from Theorem 2.1.36 and repeated application of Lemma 2.1.34 we have the following string of equivalences:

$$\begin{aligned}
\mathsf{BCK} \models \mathop{\&}\nolimits_{i=1}^{n} s_i(\vec{x}) \approx \mathbf{0} \supset t(\vec{x}) \approx \mathbf{0} \quad & \text{iff} \\
\{s_i(\vec{x}) \approx \mathbf{0} : i = 1, \ldots, n\} \models_{\mathsf{BCK}} t(\vec{x}) \approx \mathbf{0} \quad & \text{iff} \\
\{\eta(s_i(\vec{x})) \approx \mathbf{1} : i = 1, \ldots, n\} \models_{\mathsf{BCK}^D} \eta(t(\vec{x})) \approx \mathbf{1} \quad & \text{iff} \\
\{\varphi_i(\vec{p}) : i = 1, \ldots, n\} \vdash_{\mathbb{S}(\mathsf{BCK}^D,\mathbf{1})} \psi(\vec{p}) \quad & \text{iff} \\
\{\varphi_i(\vec{p}) : i = 1, \ldots, n\} \vdash_{\mathbb{S}(\mathsf{PBCK}^D,\mathbf{1})} \psi(\vec{p}) \quad & \text{iff} \\
\{\eta(s_i(\vec{x})) \approx \mathbf{1} : i = 1, \ldots, n\} \models_{\mathsf{PBCK}^D} \eta(t(\vec{x})) \approx \mathbf{1} \quad & \text{iff} \\
\{s_i(\vec{x}) \approx \mathbf{0} : i = 1, \ldots, n\} \models_{\mathsf{PBCK}} t(\vec{x}) \approx \mathbf{0} \quad & \text{iff} \\
\mathsf{PBCK} \models \mathop{\&}\nolimits_{i=1}^{n} s_i(\vec{x}) \approx \mathbf{0} \supset t(\vec{x}) \approx \mathbf{0}. &
\end{aligned}$$

Therefore $\mathsf{BCK} \models (2.20)$ iff $\mathsf{PBCK} \models (2.20)$, which establishes the first assertion of the theorem. The second assertion now follows trivially by specialisation. ∎

The following corollary seems first to have been proved (in a slightly more general form) using algebraic methods by van Alten and Raftery [231]. See also Raftery and van Alten [193, Corollary 10].

Corollary 2.1.39. *(cf. [231, Corollary 3.4]) A quasi-identity of the form:*

$$\mathop{\&}_{i=1}^{n} s_i(\vec{x}) \approx \mathbf{0} \supset t(\vec{x}) \approx \mathbf{0} \tag{2.20}$$

is satisfied by the varietal closure $\mathbf{H}(\mathsf{BCK})$ *of the quasivariety of BCK-algebras iff it satisfied by* BCK. *In particular, an identity of the form:*

$$t(\vec{x}) \approx \mathbf{0}$$

is satisfied by $\mathbf{H}(\mathsf{BCK})$ *iff it is satisfied by* BCK.

Proof. Suppose $\mathbf{H}(\mathsf{BCK}) \models (2.20)$. Because $\mathsf{BCK} \subseteq \mathbf{H}(\mathsf{BCK})$ we deduce $\mathsf{BCK} \models (2.20)$. Conversely, suppose $\mathsf{BCK} \models (2.20)$. From Theorem 2.1.38 we infer $\mathsf{PBCK} \models (2.20)$; since $\mathbf{H}(\mathsf{BCK}) \subseteq \mathsf{PBCK}$ (by Komori [138, Theorem 7]), we deduce $\mathbf{H}(\mathsf{BCK}) \models (2.20)$ as required. The second assertion follows trivially by specialisation. ∎

Recall that, for a quasivariety K, a *relative subvariety* of K is a quasivariety K' such that $\mathsf{K}' = \mathsf{K} \cap \mathbf{V}(\mathsf{K}'')$ for some $\mathsf{K}'' \subseteq \mathsf{K}$. Let $\mathbb{S}$ be an algebraisable deductive system with equivalent algebraic semantics K. By Blok and Jónsson [28, Lecture 6, p. 4, Theorem 1.4], the axiomatic extensions of $\mathbb{S}$ are in one-to-one correspondence with the relative subvarieties of K. In more detail, let $\mathbb{S}'$ be an axiomatic extension of $\mathbb{S}$. By Blok and Pigozzi [31, Corollary 4.9], $\mathbb{S}'$ is algebraisable, say with equivalent quasivariety semantics K', and by Theorem 1.8.3(1)–(2), (3)–(4), it follows that K' is a relative subvariety of K. Conversely, it follows from (1.51) and Theorem 1.8.3(1)–(2), (3)–(4) that if K'' is any relative subvariety of K, then there exists an axiomatic extension $\mathbb{S}''$

of $\mathbb{S}$ whose equivalent quasivariety semantics is K''. See also van Alten [229, Section 4.4; p. 78].

From the preceding discussion and Theorem 2.1.36, it follows that the inherent assertional logic $\mathbb{S}(\mathsf{V}, \mathbf{0})$ of any subvariety V of the variety of pre-BCK-algebras coincides with an axiomatic extension of $\mathbb{BCK}$ (in fact, with the axiomatic extension of $\mathbb{BCK}$ arising as the intrinsic assertional logic of the quasivariety of BCK-algebras axiomatised relative to V by (2.5)—see Proposition 2.2.4 below). Hence the inherent assertional logic of any variety of pre-BCK-algebras has a familiar description (recall Example 1.8.4). Because of this observation, we shall dismiss from further consideration the assertional logics of those varieties of pre-BCK-algebras we encounter in the sequel.

2.1.40. Quasi-Bounded Pre-BCK-Algebras. A *maximal element* of a pre-BCK-algebra $\mathbf{A}$ is an element $m \in A$ such that $a \preceq m$ for all $a \in A$. When they exist maximal elements form an equivalence class under Ξ called the *maximal class*; *cf.* [147, Section 1.4]. Let $\mathbf{A}$ be a pre-BCK-algebra with maximal class M. The algebra $\mathbf{A}^1 := \langle A;\ \dot{-}, 0, 1\rangle$ obtained from $\mathbf{A}$ upon enriching the language of $\mathbf{A}$ with a new nullary operation symbol $\mathbf{1}$ whose canonical interpretation on $\mathbf{A}^1$ is a fixed $1 \in M$ is called a *quasi-bounded pre-BCK-algebra.* Clearly the class PBCK^1 of quasi-bounded pre-BCK-algebras is a variety, axiomatised relative to the variety of pre-BCK-algebras by the identity $x \dot{-} \mathbf{1} \approx \mathbf{0}$.

Remark 2.1.41. In passing from a given pre-BCK-algebra $\mathbf{A}$ with maximal class M to a quasi-bounded pre-BCK-algebra $\mathbf{A}^1$ there is in general no natural choice of maximal element $1 \in M$. Indeed, it seems plausible that distinct choices of maximal element give rise to non-isomorphic quasi-bounded pre-BCK-algebras, although we have no proof of this. ∎

For a pre-BCK-algebra $\mathbf{A}$, the derived operation $a \sqcap b := a(ab)$ is called the *pre-BCK meet* (briefly, *meet*) of $a, b \in A$. Given a quasi-bounded pre-BCK-algebra $\mathbf{A}^1$, the *pre-BCK complement* (briefly, *complement*) of $a \in A$ is $a^* := 1a$, while the *pre-BCK join* of $a, b \in A$ (briefly, *join*) is $a \sqcup b := (a^* \sqcap b^*)^*$. The three derived operations *, $\sqcap$ and $\sqcup$ play an important role in the sequel;

the two trivial lemmas below, whose proofs are omitted, summarise some elementary properties of these operations that will be needed subsequently.

Lemma 2.1.42. *The variety of pre-BCK-algebras satisfies the following identities:*

$$x \sqcap \mathbf{0} \approx \mathbf{0} \tag{2.21}$$

$$\mathbf{0} \sqcap x \approx \mathbf{0} \tag{2.22}$$

$$x \sqcap x \approx x. \tag{2.23}$$

Moreover, if $\mathbf{A}$ *is a pre-BCK-algebra the following statements hold for any* $a, b \in A$*:*

1. $ab = 0$ *iff* $a \sqcap b = a$;

2. $a \sqcap b$ *is a lower bound of* $\{a, b\}$.

Lemma 2.1.43. *(cf. [126, Proposition 2]; cf. [126, Corollary 1]) The variety of quasi-bounded pre-BCK-algebras satisfies the following identities:*

$$x \sqcap \mathbf{1} \approx x \tag{2.24}$$

$$\mathbf{1} \sqcap x \approx x^{**} \tag{2.25}$$

$$x \sqcup x \approx x^{**} \tag{2.26}$$

$$x \sqcup \mathbf{1} \approx \mathbf{1} \tag{2.27}$$

$$\mathbf{1} \sqcup x \approx \mathbf{1}. \tag{2.28}$$

Moreover, if $\mathbf{A}^{\mathbf{1}}$ *is a quasi-bounded pre-BCK-algebra the following statements hold for any* $a, b \in A$*:*

1. $a^{**} \preceq a$;

2. $a^{*}b^{*} \preceq ba$;

3. $a \preceq b$ *implies* $b^{*} \preceq a^{*}$;

4. $a^{*}b \equiv b^{*}a$;

5. $a^{***} \equiv a^{*}$.

2.2 Varieties of Pre-BCK-Algebras

In this section we study varieties of pre-BCK-algebras, with a focus on the natural pre-BCK-algebraic counterparts of the varieties of commutative, positive implicative and implicative BCK-algebras.

Arbitrary varieties of pre-BCK-algebras are briefly investigated in §2.2.1. For any variety V of BCK-algebras, the natural pre-BCK-algebraic counterpart of V is the class $\{\mathbf{A} \in \mathsf{PBCK} : \mathbf{A}/\Xi \cong \mathbf{B}$ for some $\mathbf{B} \in \mathsf{V}\}$. In the main result of the section, it is shown that the natural pre-BCK-algebraic counterpart of any variety of BCK-algebras is itself always a variety. We also show that any variety of pre-BCK-algebras containing the 3-element pre-BCK-algebra $\mathbf{B}_2$ of Example 2.1.5 fails to enjoy many of the properties typically associated with a 'tractable' class of algebras.

In §2.2.9 the variety of commutative pre-BCK-algebras is studied as the natural pre-BCK-algebraic counterpart of the variety of commutative BCK-algebras. In particular, it is shown that the commutative pre-BCK-algebras are characterised among the pre-BCK-algebras by means of a certain natural condition on the pre-BCK quasiordering.

In §2.2.16 we investigate the variety of positive implicative pre-BCK-algebras, the natural pre-BCK-algebraic counterpart of the variety of positive implicative BCK-algebras. It is proved that a variety of pre-BCK-algebras is a variety of positive implicative pre-BCK-algebras iff it is subtractive, weakly congruence orderable and has EDPI (witness $x \dot{-} y$). For any positive implicative pre-BCK-algebra, we also give an internal characterisation of dual relative pseudocomplementation in the join semilattice of compact ideals.

In §2.2.28 the variety of implicative pre-BCK-algebras is considered as the natural pre-BCK-algebraic counterpart of the variety of implicative BCK-algebras. The variety of implicative pre-BCK-algebras is characterised as precisely the intersection of the varieties of commutative and positive implicative pre-BCK-algebras. For a suitable notion of prime ideal, we also show that an ideal I of an implicative pre-BCK-algebra $\mathbf{A}$ is prime iff it is maximal iff it is irreducible iff $\mathbf{A}/\phi_I$ is isomorphic to the 2-element implicative BCK-algebra $\mathbf{C}_1$.

2.2.1. Varieties of Pre-BCK-Algebras. With any variety V of BCK-algebras, we may naturally associate a class $\mathsf{V}_\flat := \{\mathbf{A} \in \mathsf{PBCK} : \mathbf{A}/\Xi \cong \mathbf{B}$ for some $\mathbf{B} \in \mathsf{V}\}$ of pre-BCK-algebras. We call $\mathsf{V}_\flat$ the *natural pre-BCK-algebraic counterpart* of V. Because of the following theorem of Blok and Raftery [38, Section 4] (see also Raftery and Sturm [191, Corollary 2.8], Idziak [114, Theorem 1], van Alten [229, Proposition 4.4] and Blok and la Falce [25, Theorem 4.3]), which characterises syntactically those subclasses of PBCK that are varieties of BCK-algebras, the class $\mathsf{V}_\flat$ is always a variety. Henceforth, for any algebra with language $\langle \dot{-}, \mathbf{0}\rangle$ of type $\langle 2, 0\rangle$, we denote by $x \dot{-} \prod_{i=1}^{n} u_i(\vec{x})$ the term $\big(\cdots(x \dot{-} u_1(\vec{x}))\cdots\big) \dot{-} u_n(\vec{x})$, where $n \in \omega$ and $u_1, \ldots, u_n$ are $\langle \dot{-}, \mathbf{0}\rangle$-terms in the variables $\vec{x}$.

Theorem 2.2.2. *[38, Corollary 10]; [191, Corollary 2.8] Let K be a class of algebras with language $\langle \dot{-}, \mathbf{0}\rangle$ of type $\langle 2, 0\rangle$. Then $\mathbf{V}(\mathsf{K})$ is a BCK-variety iff $\mathsf{K} \subseteq \mathsf{PBCK}$ and K satisfies some identity:*

$$x \dot{-} \prod_{i=1}^{n} u_i(x, y) \approx y \dot{-} \prod_{j=1}^{m} v_j(x, y) \tag{2.29}$$

for fixed $n, m \in \omega$ and $\langle \dot{-}, \mathbf{0}\rangle$-terms $u_1, \ldots, u_n, v_1, \ldots, v_m$ such that BCK *satisfies:*

$$u_i(x, x) \approx \mathbf{0} \approx v_j(x, x) \qquad i = 1, \ldots, n, \quad j = 1, \ldots, m.$$

In this case, $\mathbf{V}(\mathsf{K})$ is congruence 3-permutable.

Theorem 2.2.3. *Let V be a variety of BCK-algebras, axiomatised relative to* PBCK *by some identity:*

$$x \dot{-} \prod_{i=1}^{n} u_i(x, y) \approx y \dot{-} \prod_{j=1}^{m} v_j(x, y) \tag{2.29}$$

for fixed $n, m \in \omega$ and $\langle \dot{-}, \mathbf{0}\rangle$-terms $u_1, \ldots, u_n, v_1, \ldots, v_m$ such that BCK *satisfies:*

$$u_i(x,x) \approx 0 \approx v_j(x,x) \qquad i = 1,\dots,n, \quad j = 1,\dots,m.$$

Then $\mathsf{V}_\ni$ *is a variety, axiomatised relative to* PBCK *by the pair of identities:*

$$\Big(x \dot{-} \prod_{i=1}^{n} u_i(x,y)\Big) \dot{-} \Big(y \dot{-} \prod_{j=1}^{m} v_j(x,y)\Big) \approx \mathbf{0} \tag{2.30}$$

and:

$$\Big(y \dot{-} \prod_{j=1}^{m} v_j(x,y)\Big) \dot{-} \Big(x \dot{-} \prod_{i=1}^{n} u_i(x,y)\Big) \approx \mathbf{0}. \tag{2.31}$$

Proof. Let V be a variety of BCK-algebras axiomatised relative to PBCK by the identity (2.29) for a given choice of $n, m \in \omega$ and $\langle \dot{-}, \mathbf{0}\rangle$-terms $u_1,\dots, u_n$, $v_1,\dots, v_m$ and let W be the variety of pre-BCK-algebras axiomatised relative to PBCK by the pair of identities (2.30)–(2.31) for the same choice of $n, m \in \omega$ and $\langle \dot{-}, \mathbf{0}\rangle$-terms $u_1,\dots, u_n$, $v_1,\dots, v_m$. To prove the theorem it suffices to show $\mathsf{V}_\ni$ coincides with W.

Let $\mathbf{A} \in \mathsf{V}_\ni$. Then $\mathbf{A}/\Xi \cong \mathbf{B}$ for some $\mathbf{B} \in \mathsf{V}$. Now $\mathbf{B} \models (2.29)$, so $\mathbf{B} \models (2.30), (2.31)$ by (1.3). Therefore $\mathbf{A}/\Xi \models (2.30), (2.31)$, because identities are preserved by isomorphic copies. But then $\mathbf{A} \models (2.30), (2.31)$ by Corollary 2.1.16, so $\mathbf{A} \in \mathsf{W}$. Hence $\mathsf{V}_\ni \subseteq \mathsf{W}$. For the converse, let $\mathbf{A} \in \mathsf{W}$. Then $\mathbf{A}/\Xi \models (2.30), (2.31)$, so $\mathbf{A}/\Xi \models (2.29)$ by (2.5). Thus $\mathbf{A} \in \mathsf{V}_\ni$ and $\mathsf{W} \subseteq \mathsf{V}_\ni$. Hence $\mathsf{V}_\ni = \mathsf{W}$, and the proof is complete. ∎

Recall from §1.8.9 that, for any variety V with normal ideals, the class V_ϵ of reduced algebras of V is $\{\mathbf{B} : \mathbf{B} \cong \mathbf{A}/\langle 0\rangle^\epsilon_{\mathbf{A}} \text{ for some } \mathbf{A} \in \mathsf{V}\}$. Because of Theorem 2.1.14(4), if V is a variety of pre-BCK-algebras then $\mathsf{V}_\epsilon = \{\mathbf{B} : \mathbf{B} \cong \mathbf{A}/\Xi_{\mathbf{A}} \text{ for some } \mathbf{A} \in \mathsf{V}\}$. Moreover, one easily sees in this case that $(\mathsf{V}_\epsilon)_\ni = \mathsf{V}$. That is, V_ϵ may be understood as the *natural BCK-algebraic counterpart* of V. Consequently, the following proposition (which is a specialisation of a fragment of a more general theorem for subtractive varieties due to Agliano and Ursini [10]) can be considered as a kind of a converse of Theorem 2.2.3.

Proposition 2.2.4. *(cf. [10, Theorem 3.13]) Let* V *be a variety of pre-BCK-algebras. Then* V_ϵ *is the quasivariety of BCK-algebras axiomatised relative to* V *by the quasi-identity:*

$$x \mathbin{\dot{-}} y \approx \mathbf{0} \,\&\, y \mathbin{\dot{-}} x \approx \mathbf{0} \supset x \approx y. \tag{2.5}$$

Proof. Let V be a variety of pre-BCK-algebras. By Proposition 2.1.35, V is finitely congruential witness $D := \{x \mathbin{\dot{-}} y, y \mathbin{\dot{-}} x\}$. Therefore $I^D = I^\epsilon$ for any $I \in \mathrm{I}(\mathbf{A})$ and $\mathbf{A} \in \mathsf{V}$. It follows that:

$$\begin{aligned} \mathbf{A} \in \mathsf{V}_\epsilon \quad &\text{iff} \quad \langle 0 \rangle_{\mathbf{A}}^\epsilon = \omega_{\mathbf{A}} \\ &\text{iff} \quad a \mathbin{\dot{-}}^{\mathbf{A}} b = 0 = b \mathbin{\dot{-}}^{\mathbf{A}} a \text{ implies } a = b \end{aligned}$$

for any $a, b \in A$. Hence V_ϵ is exactly the quasivariety of BCK-algebras axiomatised relative to V by the quasi-identity (2.5). ∎

Let V be a variety of BCK-algebras with natural pre-BCK-algebraic counterpart $\mathsf{V}_\backepsilon$. Because of Theorem 2.2.3 and the definition of $\mathsf{V}_\backepsilon$, Corollary 2.1.14 and Corollary 2.1.38 together indicate that the first-order theory of $\mathsf{V}_\backepsilon$ stands in relation to V as the first-order theory of PBCK stands in relation to BCK; support for this contention is provided by our study of the natural pre-BCK-algebraic counterparts of the varieties of commutative, positive implicative and implicative BCK-algebras in §2.2.9, §2.2.16 and §2.2.28 respectively in the sequel.

The preceding remarks and Corollary 2.1.31 notwithstanding, in general the second-order theory of $\mathsf{V}_\backepsilon$ bears little resemblance to the second order-theory of V. This is exemplified by the final result (Corollary 2.2.6 below) of this brief subsection, which shows that any variety of pre-BCK-algebras that contains the 3-element pre-BCK-algebra $\mathbf{B}_2$ of Example 2.1.5 is not $\mathbf{0}$-regular, congruence distributive or congruence n-permutable for any $n \geq 2$ (contrast this result with Theorem 1.6.10). The corollary obtains as an easy consequence of the following proposition of Blok and Raftery [38, Section 4], which shows BCK is the splitting quasivariety associated with the algebra $\mathbf{B}_2$ in PBCK. In the statement of the proposition and in the sequel, for any quasivariety K we denote by $\Lambda^Q(\mathsf{K})$ the 'lattice of subquasivarieties' of K, namely the dual of the lattice of corresponding implicational theories of K.

Proposition 2.2.5. *[38, Proposition 2] The pair $\langle \mathbf{Q}(\mathbf{B}_2), \mathsf{BCK} \rangle$ splits the lattice $\Lambda^Q(\mathsf{PBCK})$. That is, for every quasivariety $\mathsf{K} \subseteq \mathsf{PBCK}$, either $\mathsf{K} \subseteq \mathsf{BCK}$ or $\mathbf{Q}(\mathbf{B}_2) \subseteq \mathsf{K}$.*

Proof. Let $\mathsf{K} \in \Lambda^Q(\mathsf{PBCK})$ with $\mathsf{K} \not\subseteq \mathsf{BCK}$. By hypothesis, $\mathsf{K} \not\models (2.5)$, so there exists an algebra $\mathbf{A} \in \mathsf{K}$ with elements $a, b \in A$ such that $ab = 0 = ba$, but $a \neq b$. Then $\{0, a, b\}$ is the universe of a subalgebra of $\mathbf{A}$ isomorphic to $\mathbf{B}_2$. Hence $\mathbf{B}_2 \in \mathsf{K}$ and $\mathbf{Q}(\mathbf{B}_2) \subseteq \mathsf{K}$. ∎

Corollary 2.2.6. *(cf. [25, Corollary 3.5]) For any variety V of pre-BCK-algebras the following assertions hold:*

1. *V is $\mathbf{0}$-regular iff $\mathsf{V} \subseteq \mathsf{BCK}$;*
2. *V is congruence distributive iff $\mathsf{V} \subseteq \mathsf{BCK}$;*
3. *V is congruence n-permutable for some $n \geq 2$ iff $\mathsf{V} \subseteq \mathsf{BCK}$. In particular, if $\mathsf{V} \subseteq \mathsf{BCK}$, then V is congruence 3-permutable (and not congruence permutable).*

Proof. Suppose $\mathsf{V} \subseteq \mathsf{BCK}$. Then V is both $\mathbf{0}$-regular and congruence distributive by Theorem 1.6.10(2); also V is congruence n-permutable for $n = 3$ by Theorem 2.2.2. However, V is not congruence permutable, since the smallest non-trivial variety of BCK-algebras (namely, the variety of implicative BCK-algebras) is not congruence permutable (by results due to Mitschke [167, Theorem 2] and Blok and Pigozzi [34, pp. 583–584]). For the converse, suppose $\mathsf{V} \not\subseteq \mathsf{BCK}$. Then $\mathbf{Q}(\mathbf{B}_2) \subseteq \mathsf{V}$ by Proposition 2.2.5 and therefore $\mathbf{V}(\mathbf{B}_2) \subseteq \mathsf{V}$. But $\mathbf{V}(\mathbf{B}_2)$ is not $\mathbf{0}$-regular (since $\mathbf{B}_2$ itself is not $\mathbf{0}$-regular) and is neither congruence distributive nor congruence n-permutable for any $n \geq 2$ (by Blok and Raftery [38, Proposition 3]). Hence V is not $\mathbf{0}$-regular, congruence distributive or congruence n-permutable for any $n \geq 2$. ∎

Remark 2.2.7. In their paper on the lattice of subquasivarieties of BCK-algebras, Blok and Raftery report that [38, Corollary 4]:

1. BCK is the largest subquasivariety of PBCK that is BCK-0-regular; and
2. BCK is the largest subquasivariety of PBCK that is BCK-congruence distributive or that satisfies any non-trivial BCK-congruence identity.

Unfortunately, the proofs provided for both these claims appear flawed by an ambiguous usage of the term 'relative congruence' [24]. We have been unable to exhibit an alternative proof that establishes these claims. ∎

Problem 2.2.8. Is BCK the largest subquasivariety of PBCK that is BCK-0-regular, BCK-congruence distributive or that satisfies any non-trivial BCK-congruence identity? ∎

2.2.9. Commutative Pre-BCK-Algebras. By a *commutative pre-BCK-algebra* we mean a pre-BCK-algebra $\mathbf{A}$ such that $\mathbf{A}/\Xi \cong \mathbf{B}$ for some commutative BCK-algebra $\mathbf{B}$. By Yutani [245], the class of all commutative BCK-algebras is a variety, axiomatised relative to PBCK by the identity:

$$x \dot{-} (x \dot{-} y) \approx y \dot{-} (y \dot{-} x) \tag{2.32}$$

so by Theorem 2.2.3, the class cPBCK of all commutative pre-BCK-algebras is also a variety, axiomatised relative to PBCK by the identity:

$$\big(x \dot{-} (x \dot{-} y)\big) \dot{-} \big(y \dot{-} (y \dot{-} x)\big) \approx \mathbf{0} \tag{2.33}$$

since (2.32) is of the form of (2.29) and is symmetric in the individual variables x and y. Thus we have the following result:

Theorem 2.2.10. *An algebra $\langle A;\ \dot{-}, 0\rangle$ of type $\langle 2, 0\rangle$ is a commutative pre-BCK-algebra iff the following identities are satisfied:*

$$\big((x \dot{-} y) \dot{-} (x \dot{-} z)\big) \dot{-} (z \dot{-} y) \approx \mathbf{0} \tag{2.1}$$

$$\mathbf{0} \dot{-} x \approx \mathbf{0} \tag{2.3}$$

$$x \dot{-} \mathbf{0} \approx x \tag{2.6}$$

$$(x \dot{-} (x \dot{-} y)) \dot{-} (y \dot{-} (y \dot{-} x)) \approx 0. \tag{2.33}$$

Thus the class cPBCK *of commutative pre-BCK-algebras is a variety.*

Recall that, by definition, a BCK-algebra $\mathbf{A}$ is commutative if its underlying poset $\langle A; \leq \rangle$ is a meet semilattice, or equivalently, if $\mathrm{glb}\{a, b\}$ exists for any $a, b \in A$.

Proposition 2.2.11. *Let* $\mathbf{A}$ *be a pre-BCK-algebra. If* $\mathbf{A}$ *is commutative, then for every* $a, b \in A$*, both* $a \sqcap b$ *and* $b \sqcap a$ *are greatest lower bounds of* $\{a, b\}$ *with respect to the underlying pre-BCK quasiorder* $\preceq$*. Conversely, if for every* $a, b \in A$*, both* $a \sqcap b$ *and* $b \sqcap a$ *are greatest lower bounds of* $\{a, b\}$ *with respect to the underlying pre-BCK-quasiorder* $\preceq$*, then* $\mathbf{A}$ *is commutative.*

Proof. Let $\mathbf{A}$ be a pre-BCK-algebra and let $a, b \in A$.

Suppose $\mathbf{A}$ is commutative. By Lemma 2.1.42(2), $a \sqcap b$ is a lower bound of $\{a, b\}$. Suppose $c \preceq a$ and $c \preceq b$ for some $c \in A$. We have:

$$\begin{aligned} 0 &= \big(c(ca)\big)\big(a(ac)\big) && \text{by } (2.33) \\ &= (c0)\big(a(ac)\big) && \text{as } c \preceq a \\ &= c\big(a(ac)\big) && \text{by } (2.6) \end{aligned}$$

whence $c \preceq a \sqcap c$. Put $\alpha := a$, $\beta := ac$ and $\gamma := ab$. Since:

$$\begin{aligned} 0 &= \big((ab)(ac)\big)(cb) && \text{by } (2.1) \\ &= \big((ab)(ac)\big)0 && \text{as } c \preceq b \\ &= (ab)(ac) && \text{by } (2.6) \end{aligned} \tag{2.34}$$

we have:

$$\begin{aligned} 0 &= \big((\alpha\beta)(\alpha\gamma)\big)(\gamma\beta) && \text{by } (2.1) \\ &= \big((\alpha\beta)(\alpha\gamma)\big)\big((ab)(ac)\big) && \\ &= \big((\alpha\beta)(\alpha\gamma)\big)0 && \text{by } (2.34) \\ &= \big(a(ac)\big)\big(a(ab)\big) && \text{by } (2.6) \end{aligned}$$

whence $a \sqcap c \preceq a \sqcap b$. Thus $c \preceq a \sqcap c \preceq a \sqcap b$. By transitivity, $c \preceq a \sqcap b$, and so $a \sqcap b$ is a greatest lower bound of $\{a, b\}$. By (2.33), $a \sqcap b \Xi b \sqcap a$, so $b \sqcap a$ lies in the same clique as $a \sqcap b$. By Lemma 1.2.3(1), $b \sqcap a$ is a greatest lower bound of $\{a, b\}$.

Conversely, suppose that for every $a, b \in A$, both $a \sqcap b$ and $b \sqcap a$ are greatest lower bounds of $\{a, b\}$. By Lemma 1.2.3(2) we have that $a \sqcap b \Xi b \sqcap a$, whence $\big(a(ab)\big)\big(b(ba)\big) = 0$. Thus **A** is commutative. ■

Remark 2.2.12. Let **A** be a pre-BCK-algebra. For **A** to be commutative, the requirement that *both* $a \sqcap b$ and $b \sqcap a$ be greatest lower bounds of $a, b \in A$ cannot be weakened to the requirement that just one of $a \sqcap b$ or $b \sqcap a$ be a greatest lower bound for $\{a, b\}$. To see this, consider the pre-BCK-algebra **A** with operation table:

$\dot{-}^{\mathbf{A}}$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	2
3	3	0	0	0

An easy sequence of checks shows $a \sqcap b$ is a greatest lower bound of any $a, b \in \{0, 1, 2, 3\}$. However, **A** is not commutative, as $\big(1 \dot{-} (1 \dot{-} 2)\big) \dot{-} \big(2 \dot{-} (2 \dot{-} 1)\big) = (1 \dot{-} 0) \dot{-} (2 \dot{-} 2) = 1 \dot{-} 0 = 1 \neq 0$. ■

Let **A** be a commutative pre-BCK-algebra and let $m \in A$ be fixed. For $a, b \in A$ such that $a, b \preceq m$, we have $ab \preceq a \preceq m$ by Lemma 2.1.12(3). Thus the principal order ideal $(m]$ is a subuniverse of **A**. We write $(\boldsymbol{m}]$ for the subalgebra of **A** with subuniverse $(m]$. By definition, $a \preceq m$ for all $a \in (m]$. Thus m is a maximal element of $(\boldsymbol{m}]$, the restriction $\Xi|_{(m]}$ of Ξ to $(m]$ has a maximal class M and $(\boldsymbol{m}]$ induces a quasi-bounded pre-BCK-algebra $(\boldsymbol{m}]^{\mathbf{1}}$. In general, there may be many possible choices for an element $c \in M$ such that $1^{(\boldsymbol{m}]^{\mathbf{1}}} = c$. However, we will always fix $1^{(\boldsymbol{m}]^{\mathbf{1}}} := m$, whence we will not be careful to distinguish between $(\boldsymbol{m}]$ and $(\boldsymbol{m}]^{\mathbf{1}}$ in the sequel.

Let **A** be a commutative pre-BCK-algebra and let $m \in A$ be fixed. Since $(\boldsymbol{m}]$

is a quasi-bounded pre-BCK-algebra, it has (by the remarks of §2.1.40) both a derived 'complementation' operation * and a derived 'join' operation $\sqcup$. To signify these derived operations are local to $(m]$, we write $a^{*(m]}$ and $a \sqcup^{(m]} b$ for the complement of $a \in (m]$ and the join of $a, b \in (m]$ respectively. Further, to simplify notation in the sequel we write $a^*_{(m]}$ for $a^{*(m]}$, $a^{**}_{(m]}$ for $(a^*_{(m]})^*_{(m]}$ and $a^{***}_{(m]}$ for $((a^*_{(m]})^*_{(m]})^*_{(m]}$.

Lemma 2.2.13. *(cf. [126, Proposition 2]; cf. [126, Corollary 1]) Let* $\mathbf{A}^1$ *be a quasi-bounded commutative pre-BCK-algebra. For any* $a \in A$, $a^{**} \equiv a$.

Proof. Suppose $\mathbf{A}^1$ is a commutative pre-BCK-algebra. Let $a \in A$. By (2.2) we have $a^{**} = 1(1a) \preceq a$. By (2.33) we have also that $\big(a(a1)\big)\big(1(1a)\big) = 0$, whence $a\big(1(1a)\big) = 0$ since $a \preceq 1$. Thus $a \preceq 1(1a)$, so $a \preceq a^{**}$. Hence $a^{**} \equiv a$. ∎

If $\mathbf{A}$ is a commutative BCK-algebra then, by the remarks of §1.6.11, the underlying poset $\langle A; \leq\rangle$ is not merely a meet semilattice; it is in fact a (distributive) nearlattice. Let $\langle A; \leq\rangle$ be a meet semilattice. By definition, $\langle A; \leq\rangle$ is a nearlattice if, for every $m \in A$, the principal order ideal $(m]$ is a lattice. Equivalently, $\langle A; \leq\rangle$ is a nearlattice if it enjoys the *upper bound property*: that is, if the supremum $\mathrm{lub}\{a, b\}$ exists when $a, b \in A$ share a common upper bound [64, Section 3, p. 487].

Lemma 2.2.14. *(cf. [126, p. 9]) Let* $\mathbf{A}$ *be a commutative pre-BCK-algebra and let* $m \in A$ *be fixed. If* $a, b \preceq m$ *then* $a \sqcup^{(m]} b$ *is a-least upper bound of* a *and* b *in* $(m]$.

Proof. Assume $\mathbf{A}, a, b, m$ are as stated. By Proposition 2.2.11, $a^*_{(m]} \sqcap b^*_{(m]} \preceq a^*_{(m]}, b^*_{(m]}$, so $a^{**}_{(m]}, b^{**}_{(m]} \preceq (a^*_{(m]} \sqcap b^*_{(m]})^*_{(m]}$ by Lemma 2.1.43(3). Now $a \equiv a^{**}_{(m]}$ and $b \equiv b^{**}_{(m]}$ by Lemma 2.2.13, so $a \preceq a^{**}_{(m]} \preceq (a^*_{(m]} \sqcap b^*_{(m]})^*_{(m]}$ and $b \preceq b^{**}_{(m]} \preceq (a^*_{(m]} \sqcap b^*_{(m]})^*_{(m]}$. By transitivity, $a, b \preceq (a^*_{(m]} \sqcap b^*_{(m]})^*_{(m]}$. Thus $(a^*_{(m]} \sqcap b^*_{(m]})^*_{(m]}$ is an upper bound of $\{a, b\}$.

Let $c \preceq m$ for some $c \in A$ with $a, b \preceq c$. By Lemma 2.1.43(3) we have $c^*_{(m]} \preceq a^*_{(m]}, b^*_{(m]}$. Thus $c^*_{(m]} \preceq a^*_{(m]} \sqcap b^*_{(m]}$ by Proposition 2.2.11 and so $(a^*_{(m]} \sqcap b^*_{(m]})^*_{(m]} \preceq c^{**}_{(m]}$ by Lemma 2.1.43(3). Because $c^{**}_{(m]} \equiv c$ by Lemma 2.2.13, we have that $(a^*_{(m]} \sqcap b^*_{(m]})^*_{(m]} \preceq c^{**}_{(m]} \preceq c$. By transitivity, $(a^*_{(m]} \sqcap b^*_{(m]})^*_{(m]} \preceq c$, whence $(a^*_{(m]} \sqcap b^*_{(m]})^*_{(m]}$ is a least upper bound of $\{a, b\}$. ∎

Because of Proposition 2.2.11 and Lemma 2.2.14, it is natural to ask if each initial segment $(m]$, $m \in A$, of a commutative pre-BCK-algebra $\mathbf{A}$ supports (in some sense) a 'generalised lattice' structure, and in particular if $\langle (m];\ \sqcap, \sqcup^{(m]} \rangle$ is a non-commutative lattice (in the sense of §1.4.1). Let $\mathbf{A^1}$ be the 5-element quasi-bounded commutative pre-BCK-algebra (with derived pre-BCK meet $\sqcap^{\mathbf{A^1}}$) defined by the following operation tables:

$\dot{-}^{\mathbf{A^1}}$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	0	0	0
b	b	0	0	0	0
c	c	0	0	0	0
1	1	b	c	a	0

$\sqcap^{\mathbf{A^1}}$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	b	b	b	b
c	0	c	c	c	c
1	0	c	a	b	1

Observe $1 \sqcap (1 \sqcap a) = 1 \sqcap c = b \neq c = 1 \sqcap a = (1 \sqcap 1) \sqcap a$, whence $\langle A;\ \sqcap \rangle$ is not a band. Thus in particular $\langle A;\ \sqcap, \sqcup \rangle$ is not a double band and the induced algebra $\langle A;\ \sqcap, \sqcup \rangle$ is not a non-commutative lattice.

Remark 2.2.15. Let $\mathbf{A}$ be a commutative pre-BCK-algebra and let $m \in A$ be fixed. The preceding remarks notwithstanding, $(m]$ does support a 'generalised lattice' structure in a sense made precise as follows. Recall from [223, Section 10.1] that a *[quasi-bounded] quasi-lattice (in the sense of Ursini)* is a structure $[\langle A;\ \wedge, \vee, 0, 1;\ \preceq \rangle]$ $\langle A;\ \wedge, \vee;\ \preceq \rangle$ of type $[\langle 2, 2, 0, 0, 2 \rangle]$ $\langle 2, 2, 2 \rangle$ such that the following conditions hold for any $a, b \in A$:

QL1. $\langle A;\ \preceq \rangle$ is a quasiordered set;

QL2. The *quasi-meet* $a \wedge b$ of a and b satisfies:

(a) $a \wedge b \preceq a$;

(b) $a \wedge b \preceq b$;

(c) For all $c \in A$, $a \preceq c \wedge b \preceq c$ implies $a \wedge b \preceq c$;

QL3. The *quasi-join* $a \vee b$ of a and b satisfies:

(a) $a \preceq a \vee b$;

(b) $b \preceq a \vee b$;

(c) For all $c \in A$, $c \preceq a \wedge c \preceq b$ implies $c \preceq a \vee b$;

QL4. [For all $c \in A$, $0 \preceq c \preceq 1$.]

[Quasi-bounded] quasi-lattices were introduced by Ursini in his study of algebraic semantics for linear logic [223], in order to provide a model for the 'turbo monoids' of Girard [97, p. 24]. By remarks due to Ursini [223, Section 10.1], the equivalence $\approxeq$ on $A \times A$ induced by $\preceq$ in the sense of Lemma 1.2.2 is a congruence on any [quasi-bounded] quasi-lattice $[\langle A;\ \wedge, \vee, 0, 1;\ \preceq\rangle]$ $\langle A;\ \wedge, \vee;\ \preceq\rangle$ such that $[\langle A;\ \wedge, \vee, 0, 1;\ \preceq\rangle/\approxeq]$ $\langle A;\ \wedge, \vee;\ \preceq\rangle/\approxeq$ is a [bounded] lattice. Thus every [quasi-bounded] quasi-lattice has the global outline of a [bounded] lattice (compare this statement with the Clifford-McLean theorem for quasilattices). However, a quasi-lattice (in the sense of Ursini) is not in general a quasilattice in the sense of this thesis (that is, in the sense of §1.4.1); the converse does obtain.

By a *reduced* [*quasi-bounded*] *quasi-lattice* we mean a [quasi-bounded] quasi-lattice $[\langle A;\ \wedge, \vee, 0, 1;\ \preceq\rangle]$ $\langle A;\ \wedge, \vee;\ \preceq\rangle$ such that $a \approxeq b$ iff $a = b$ for any $a, b \in A$. Clearly every reduced [quasi-bounded] quasi-lattice is a [bounded] lattice. A *distributive* [*quasi-bounded*] *quasi-lattice* is a [quasi-bounded] quasi-lattice $[\langle A;\ \wedge, \vee, 0, 1;\ \preceq\rangle]$ $\langle A;\ \wedge, \vee;\ \preceq\rangle$ whose reduced image $[\langle A;\ \wedge, \vee, 0, 1;\ \preceq\rangle/\approxeq]$ $\langle A;\ \wedge, \vee;\ \preceq\rangle/\approxeq$ is distributive. Now let **A** be a commutative pre-BCK-algebra and let $m \in A$ be fixed. Then the induced structure $\langle (m];\ \sqcup^{(m)}, \sqcap, 0, m;\ \preceq|_{(m]}\rangle$ is a distributive quasi-bounded quasi-lattice, whose reduced image $\langle (m];\ \sqcup^{(m)}, \sqcap, 0, m;\ \preceq|_{(m]}\rangle/\Xi$ is precisely the bounded distributive sublattice $(\boldsymbol{m}/\Xi]_{\mathbf{A}/\Xi}$ of the commutative BCK-algebra $\mathbf{A}/\Xi$.

Proof. Let $\langle A;\ \dot{-}, 0\rangle$ be a commutative pre-BCK-algebra and let $m \in A$ be fixed. Let $a, b, c \in (m]$. Clearly the restriction $\preceq|_{(m]}$ of $\preceq$ to $(m]$ is a quasiorder on $(m]$, whence Condition (QL1) above is satisfied.

To see $\sqcap$ is a quasi-meet, observe first that $a \sqcap b \preceq a$ and $a \sqcap b \preceq b$, just because $a \sqcap b$ is a lower bound of $\{a, b\}$. Suppose $a \preceq c \sqcap b \preceq c$. Put $\alpha := a$, $\beta := c$ and $\gamma := c(cb)$. We have:

$$\begin{aligned}
0 &= \big((\alpha\beta)(\alpha\gamma)\big)(\gamma\beta) && \text{by (2.1)}\\
&= \big((\alpha\beta)(\alpha\gamma)\big)\big((c(cb))c\big)\\
&= \big((\alpha\beta)(\alpha\gamma)\big)0 && \text{by Lemma 2.1.12(3)}\\
&= \big((\alpha\beta)(\alpha\gamma)\big) && \text{by (2.6)}\\
&= (ac)\big(a(c(cb))\big)\\
&= (ac)0 && \text{since } a \preceq c \sqcap b\\
&= ac && \text{by (2.6).}
\end{aligned}$$

Thus $a \preceq c$. Since $a \sqcap b \preceq a$, we have $a \sqcap b \preceq a \preceq c$, whence $a \sqcap b \preceq c$ by transitivity. Thus $\sqcap$ is a quasi-meet and Condition (QL2) above is satisfied.

To see $\sqcup^{(m]}$ is a quasi-join, notice first that both $a \preceq a \sqcup^{(m]} b$ and $b \preceq a \sqcup^{(m]} b$, because $a \sqcup^{(m]} b$ is an upper bound of $\{a, b\}$ in $(m]$. Suppose $c \preceq a \sqcap c \preceq b$. Put $\alpha := c$, $\beta := a$ and $\gamma := a(ac)$. We have:

$$\begin{aligned}
0 &= \big((\alpha\beta)(\alpha\gamma)\big)(\gamma\beta) && \text{by (2.1)}\\
&= \big((\alpha\beta)(\alpha\gamma)\big)\big((a(ac))a\big)\\
&= \big((\alpha\beta)(\alpha\gamma)\big)0 && \text{by Lemma 2.1.12(3)}\\
&= (\alpha\beta)(\alpha\gamma) && \text{by (2.6)}\\
&= (ca)\big(c(a(ac))\big)\\
&= (ca)0 && \text{since } c \preceq a \sqcap c\\
&= ca && \text{by (2.6).}
\end{aligned}$$

Thus $c \preceq a$. Since $a \preceq a \sqcup^{(m]} b$, we have $c \preceq a \preceq a \sqcup^{(m]} b$, whence $c \preceq a \sqcup^{(m]} b$ by transitivity. Thus $\sqcup^{(m]}$ is a quasi-join and Condition (QL3) above is satisfied.

For Condition (QL4) above, just note $0 \preceq a \preceq m$ for all $a \in (m]$. Thus the induced structure $\langle (m];\ \sqcup^{(m]}, \sqcap, 0, m;\ \preceq|_{(m]} \rangle$ is a quasi-bounded quasi-lattice.

To complete the proof, just note the reduced image $\langle (m];\ \sqcap, \sqcup, 0, 1;\ \preceq|_{(m]}\rangle/\Xi$ of $\langle (m];\ \sqcup^{(m]}, \sqcap, 0, m;\ \preceq|_{(m]}\rangle$ may be identified with the bounded commutative BCK-subalgebra $(\boldsymbol{m}/\Xi]_{\mathbf{A}/\Xi}$, which is a bounded distributive sublattice of $\mathbf{A}/\Xi$ by the remarks of §1.6.11. ∎

2.2.16. Positive Implicative Pre-BCK-Algebras. A *positive implicative pre-BCK-algebra* is a pre-BCK-algebra $\mathbf{A}$ such that $\mathbf{A}/\Xi \cong \mathbf{B}$ for some positive implicative pre-BCK-algebra $\mathbf{B}$. Since the class pBCK of all positive implicative BCK-algebras is a variety, axiomatised relative to PBCK by the identity [38, pp. 294–295]:

$$(x \dot{-} (x \dot{-} y)) \dot{-} (y \dot{-} x) \approx (y \dot{-} (y \dot{-} x)) \dot{-} (x \dot{-} y) \tag{2.35}$$

the class pPBCK of all positive implicative pre-BCK-algebras is also a variety, by Theorem 2.2.3, axiomatised relative to PBCK by the identity:

$$((x \dot{-} (x \dot{-} y)) \dot{-} (y \dot{-} x)) \dot{-} ((y \dot{-} (y \dot{-} x)) \dot{-} (x \dot{-} y)) \approx \mathbf{0}$$

since (2.35) is of the form of (2.29) and is symmetric in the variables x and y. This characterisation of pPBCK notwithstanding, the following axiomatisation is often more useful in practice.

Theorem 2.2.17. *An algebra $\langle A;\ \dot{-}, 0\rangle$ of type $\langle 2, 0\rangle$ is a positive implicative pre-BCK-algebra iff the following identities are satisfied:*

$$((x \dot{-} y) \dot{-} (x \dot{-} z)) \dot{-} (z \dot{-} y) \approx \mathbf{0} \tag{2.1}$$

$$\mathbf{0} \dot{-} x \approx \mathbf{0} \tag{2.3}$$

$$x \dot{-} \mathbf{0} \approx x \tag{2.6}$$

$$(x \dot{-} y) \dot{-} ((x \dot{-} y) \dot{-} y) \approx \mathbf{0}. \tag{2.36}$$

Thus the class pPBCK *of positive implicative pre-BCK-algebras is a variety.*

Proof. Let $\mathbf{A}$ be a positive implicative pre-BCK-algebra. By definition we have that $\mathbf{A} \models (2.1), (2.3), (2.6)$, so it only remains to show $\mathbf{A} \models (2.36)$.

Since $\mathbf{A}$ is positive implicative, $\mathbf{A}/\Xi \cong \mathbf{B}$ for some positive implicative BCK-algebra $\mathbf{B}$. Because $\mathbf{B}$ is positive implicative, $\mathbf{B} \models (x \dot{-} y) \dot{-} y \approx x \dot{-} y$, so $\mathbf{B} \models$ (2.36) by (1.3). As identities are preserved by isomorphic copies, we have that $\mathbf{A}/\Xi \models$ (2.36), so $\mathbf{A} \models$ (2.36) by Corollary 2.1.16.

Conversely, suppose $\mathbf{A} := \langle A;\ \dot{-}, 0\rangle$ is an algebra of type $\langle 2, 0\rangle$ such that $\mathbf{A} \models (2.1), (2.3), (2.6), (2.36)$. By (2.1), (2.3), (2.6) and Lemma 2.1.6, $\mathbf{A}$ is a pre-BCK-algebra. By (2.36), $\mathbf{A}/\Xi \models (x \dot{-} y) \dot{-} ((x \dot{-} y) \dot{-} y) \approx \mathbf{0}$; also $\mathbf{A}/\Xi \models ((x \dot{-} y) \dot{-} y) \dot{-} (x \dot{-} y) \approx \mathbf{0}$ by (1.30). Hence $\mathbf{A}/\Xi \models (x \dot{-} y) \dot{-} y \approx x \dot{-} y$ by (1.5), so $\mathbf{A}$ is positive implicative. ∎

By Proposition 1.7.12, the variety pBCK of positive implicative BCK-algebras is precisely the class of all BCK-algebras with EDPI (witness $x \dot{-} y$). By Corollary 2.2.6(1), therefore, pBCK is precisely the class of all $\mathbf{0}$-regular pre-BCK-algebras with EDPI (witness $x \dot{-} y$). Since pBCK may be described alternatively as exactly the class of all $\mathbf{0}$-regular positive implicative pre-BCK-algebras (by Corollary 2.2.6(1) and Theorem 2.2.17), the preceding remarks suggest that pPBCK is a variety of pre-BCK-algebras with EDPI (witness $x \dot{-} y$) (but which is not $\mathbf{0}$-regular). This last motivates the study of the ideal theory of positive implicative pre-BCK-algebras, the key to which is Lemma 2.2.19 below. See also Agliano [6, Section 4, p. 14].

Lemma 2.2.18. *The variety of positive implicative pre-BCK-algebras satisfies the following identity:*

$$((x \dot{-} y) \dot{-} (z \dot{-} y)) \dot{-} ((x \dot{-} z) \dot{-} y) \approx \mathbf{0}. \tag{2.37}$$

Proof. Let $\mathbf{A}$ be a positive implicative pre-BCK-algebra. Then $\mathbf{A}/\Xi \cong \mathbf{B}$ for some positive implicative BCK-algebra $\mathbf{B}$. Since $\mathbf{B}$ is positive implicative, $\mathbf{B} \models ((x \dot{-} y) \dot{-} (z \dot{-} y)) \approx (x \dot{-} z) \dot{-} y$ by the remarks of §1.6.13, so $\mathbf{B} \models$ (2.37) by (1.3). Since identities are preserved by isomorphic copies, $\mathbf{A}/\Xi \models$ (2.37), whence $\mathbf{A} \models$ (2.37) by Corollary 2.1.16. ∎

Lemma 2.2.19. *(cf. [68, Lemma 2.3]; cf. [195, Theorem II§1.8]) Let* $\mathbf{A}$ *be a positive implicative pre-BCK-algebra. For any ideal* $I \in \mathrm{I}(\mathbf{A})$ *and fixed*

$a \in A$, the supremum $I \vee^{\mathbf{I}(\mathbf{A})} \langle a \rangle_{\mathbf{A}}$ of I and $\langle a \rangle_{\mathbf{A}}$ in the ideal lattice $\mathbf{I}(\mathbf{A})$ is $\{b \in A : ba \in I\}$. Consequently,

$$ba \in I \quad \textit{iff} \quad b \in I \vee^{\mathbf{I}(\mathbf{A})} \langle a \rangle_{\mathbf{A}}.$$

Proof. Let $\mathbf{A}$ be a positive implicative pre-BCK-algebra and let $I \in \mathrm{I}(\mathbf{A})$. Let $a \in A$ be fixed and let $J := \{b \in A : ba \in I\}$. Because $\mathrm{I}(\mathbf{A})$ is directed, $I \vee^{\mathbf{I}(\mathbf{A})} \langle a \rangle_{\mathbf{A}} = \langle I \cup \{a\} \rangle_{\mathbf{A}}$, so to prove the lemma it is sufficient to show:

(i) J is an ideal of $\mathbf{A}$;
(ii) J is the smallest ideal of $\mathbf{A}$ such that $a \in J$ and $I \subseteq J$.

For (i), we have that $0 \in J$ since $0a = 0 \in I$ by (2.4). Suppose $b, cb \in J$ for $b, c \in A$. Then $ba, (cb)a \in I$ by definition of J. As $\big((ca)(ba)\big)\big((cb)a\big) = 0 \in I$ (by (2.37)) we have that $ca \in I$ by Lemma 2.1.21(2). Hence $c \in J$ and J is an ideal.

For (ii), we have that $a \in J$ since $aa = 0 \in I$ by (2.3). Suppose $b \in I$. Because $ba \preceq b$ by Lemma 2.1.12(3) we have that $ba \in I$ by Lemma 2.1.21(1). Hence $b \in J$ and $I \subseteq J$. Therefore J is an ideal of $\mathbf{A}$ with $a \in J$ and $I \subseteq J$. Let $K \in \mathrm{I}(\mathbf{A})$ be such that $a \in K$ and $I \subseteq K$. Suppose $b \in J$. Then $ba \in I \subseteq K$. Since $ba, a \in K$, we have that $b \in K$ by definition of K as an ideal. Hence $J \subseteq K$ and J is the smallest ideal of $\mathbf{A}$ such that $a \in J$ and $a \subseteq J$. ∎

Let $\mathbf{A}$ be a pre-BCK-algebra, let $I \in \mathrm{I}(\mathbf{A})$ and let $a \in A$ be fixed. For ease of notation in the sequel, we write $\langle I, a \rangle_{\mathbf{A}}$ for the supremum $\langle I \cup \{a\} \rangle_{\mathbf{A}}$ of I and $\langle a \rangle_{\mathbf{A}}$ in the ideal lattice $\mathbf{I}(\mathbf{A})$. Also, we continue to write $(a]$ for $\{b : b \preceq a\}$, and we write $A \sqcap a$ for $\{b \sqcap a : b \in A\}$, the *principal left ideal generated by a*.

Theorem 2.2.20. *For a variety V of pre-BCK-algebras, the following are equivalent:*

1. *V is a variety of positive implicative pre-BCK-algebras;*

2. *V is weakly congruence orderable and the binary term $x \dot{-} y$ witnesses* both *subtractivity and EDPI for V in the sense of Theorem 1.7.9.*

If V *is a variety of positive implicative pre-BCK-algebras, then the following assertions also hold for any* $\mathbf{A} \in \mathsf{V}$ *and* $a, b \in A$*:*

3. $\langle a \rangle_{\mathbf{A}} = (a] = A \sqcap a$;

4. $a \in \langle b \rangle_{\mathbf{A}}$ *iff* $ab = 0$ *iff* $a \sqcap b = a$;

5. $\langle a \dot{-} b \rangle_{\mathbf{A}} = \langle a \rangle_{\mathbf{A}} * \langle b \rangle_{\mathbf{A}}$, *where* $*$ *denotes dual relative pseudocomplementation in the join semilattice* $\langle \mathrm{CI}(\mathbf{A});\ \vee, \langle 0 \rangle_{\mathbf{A}} \rangle$ *of compact ideals of* $\mathbf{A}$.

Proof. Let V be a variety of pre-BCK-algebras.

(1) $\Rightarrow$ (2) Suppose V is a variety of positive implicative pre-BCK-algebras. By Theorem 2.1.3, V is subtractive witness $x \dot{-} y$. To see $x \dot{-} y$ also witnesses EDPI for V, just note that $a \in \langle b \rangle_{\mathbf{A}}$ iff $a \in \langle \{0\}, b \rangle_{\mathbf{A}}$ iff $ab \in \langle 0 \rangle_{\mathbf{A}}$ (by Lemma 2.2.19) iff $ab = 0$ for any $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$. To see V is weakly congruence orderable, it is sufficient to show $\Theta^{\mathbf{A}}(0, a) \subseteq \Theta^{\mathbf{A}}(0, b)$ iff $a \preceq b$ for any $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$, just because of Lemma 1.7.17 and Theorem 2.1.14(4). So let $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$. Suppose $a \preceq b$. Then $a = a0 \equiv_{\Theta^{\mathbf{A}}(0,b)} ab = 0$, so $\Theta^{\mathbf{A}}(a, 0) \subseteq \Theta^{\mathbf{A}}(b, 0)$. Conversely, suppose $\Theta^{\mathbf{A}}(0, a) \subseteq \Theta^{\mathbf{A}}(0, b)$. Then $\langle a \rangle_{\mathbf{A}} \subseteq \langle b \rangle_{\mathbf{A}}$ by normality of ideals and so $a \in \langle b \rangle_{\mathbf{A}}$. Since V has EDPI witness $x \dot{-} y$ it follows that $ab = 0$; that is to say $a \preceq b$.

(2) $\Rightarrow$ (1) Suppose V is weakly congruence orderable and that the binary term $x \dot{-} y$ witnesses *both* subtractivity and EDPI for V in the sense of Theorem 1.7.9. Then V_ϵ is a congruence orderable subvariety of V by Proposition 1.7.19, and so also has EDPI witness $x \dot{-} y$. Moreover, V_ϵ is 0-regular witness $\{x \dot{-} y,\ y \dot{-} x\}$ by Proposition 1.7.18, and so is a variety of positive implicative BCK-algebras by Proposition 1.7.12. Let $\mathbf{A} \in \mathsf{V}$. By the preceding characterisation of V_ϵ, we have that $\mathbf{A}/\langle 0 \rangle^{\epsilon}_{\mathbf{A}} \cong \mathbf{B}$ for some positive implicative BCK-algebra $\mathbf{B}$. But this means that $\mathbf{A}/\Xi \cong \mathbf{B}$, because $\langle 0 \rangle^{\epsilon}_{\mathbf{A}} = \Xi_{\mathbf{A}}$ by Theorem 2.1.14(4). It follows that $\mathbf{A}$ is a positive implicative pre-BCK-algebra. Hence V is a variety of positive implicative pre-BCK-algebras.

Throughout the remainder of the proof, assume that V is a variety of positive implicative pre-BCK-algebras. Let $\mathbf{A} \in \mathsf{V}$ and $a, b, \in A$.

For (3), we have that $(a] = \langle a \rangle_{\mathbf{A}}$ by EDPI. Suppose $b \in (a]$. Then $ba = 0$ and so $b = b0 = b(ba) \in A \sqcap a$. Hence $(a] \subseteq A \sqcap a$. Conversely, suppose $b \in A \sqcap a$. Then $b = c(ca)$ for some $c \in A$. Since $c(ca) \preceq a$ by (2.2), we have that $c(ca) \in (a]$; that is to say $b \in (a]$. Hence $A \sqcap a \subseteq (a]$ and $(a] = A \sqcap a$. For (4), we have $a \in \langle b \rangle_{\mathbf{A}}$ iff $ab = 0$ (by EDPI) iff $a \sqcap b = a$ (by Lemma 2.1.42). For (5), let $I \in \mathrm{I}(\mathbf{A})$. Because of Lemma 2.2.19, we have $\langle ab \rangle_{\mathbf{A}} \subseteq I$ iff $\langle a \rangle_{\mathbf{A}} \subseteq \langle b \rangle_{\mathbf{A}} \vee I$, so $\langle ab \rangle_{\mathbf{A}}$ is the dual relative pseudocomplement of $\langle b \rangle_{\mathbf{A}}$ with respect to $\langle a \rangle_{\mathbf{A}}$ in $\langle \mathrm{CI}(\mathbf{A});\ \vee, \langle 0 \rangle_{\mathbf{A}} \rangle$. ∎

Remark 2.2.21. The variety of pre-BCK-algebras does not have EDPI. In particular, the class jBCK of all BCK-algebras satisfying the identity:

$$x \dot{-} (x \dot{-} (y \dot{-} (y \dot{-} x))) \approx y \dot{-} (y \dot{-} (x \dot{-} (x \dot{-} y)))$$

of Cornish [69] is a BCK-variety that does not enjoy EDPI. To see this, observe that jBCK is a variety (by Theorem 2.2.2 or Cornish [69, Lemma 1]) which is not contained in any of the varieties $\mathsf{e_n}$BCK, $n \in \omega$ (by Blok and Raftery [38, Proposition 16]). Since a variety of BCK-algebras has EDPI iff it is contained in some $\mathsf{e_n}$BCK, $n \in \omega$ (by Theorem 1.6.10, Theorem 1.6.15 and Proposition 1.7.10), jBCK does not have EDPI. (This argument was communicated to the author by Professor James Raftery [188].)

The construction used by Blok and Raftery to show jBCK is not contained in any $\mathsf{e_n}$BCK, $n \in \omega$, produces, for each $k > 1$, an algebra $\mathbf{A}_k$ in jBCK that is not in any $\mathsf{e_n}$BCK. However, the ideal lattice of $\mathbf{A}_k$ is, for each $k > 1$, isomorphic to the three-element dually relatively pseudocomplemented semilattice. Thus Blok and Raftery's construction cannot be used to exhibit an explicit example of a BCK-algebra $\mathbf{A}$ for which the join semilattice $\langle \mathrm{CI}(\mathbf{A});\ \vee, \langle 0 \rangle_{\mathbf{A}} \rangle$ of compact ideals is not dually relative pseudocomplemented. ∎

Problem 2.2.22. Exhibit an explicit example of a (pre-) BCK-algebra $\mathbf{A}$ for which the join semilattice $\langle \mathrm{CI}(\mathbf{A});\ \vee, \langle 0 \rangle_{\mathbf{A}} \rangle$ of compact ideals is not dually relatively pseudocomplemented. ∎

Because pre-BCK difference witnesses *both* subtractivity and EDPI for pPBCK in the sense of Theorem 1.7.9, Items (1) and (3) of the following corollary result as an immediate consequence of Theorem 1.7.15(1). Despite this remark, we give as an easy modification of a result of Agliano [7] a direct proof of the corollary, on the grounds that this is conceptually simpler than an appeal to Theorem 1.7.15.

Corollary 2.2.23. *(cf. [7, Lemma 3.3]) For any positive implicative pre-BCK-algebra* $\mathbf{A}$*, the following assertions hold:*

1. *The set* $\mathrm{PI}(\mathbf{A})$ *of principal ideals of* $\mathbf{A}$ *is closed under dual relative pseudocomplementation. Thus* $\langle \mathrm{PI}(\mathbf{A});\ *, \langle 0\rangle_{\mathbf{A}}\rangle$ *is a positive implicative BCK-algebra;*

2. *The map* $a \mapsto \langle a\rangle_{\mathbf{A}}$ *is a homomorphism from* $\mathbf{A}$ *onto* $\langle \mathrm{PI}(\mathbf{A});\ *, \langle 0\rangle_{\mathbf{A}}\rangle$*. Moreover,* $\ker f = \Xi$*;*

3. *The map* $[a]_{\Xi} \mapsto \langle a\rangle_{\mathbf{A}}$ *is an isomorphism from* $\mathbf{A}/\Xi$ *onto* $\langle \mathrm{PI}(\mathbf{A});\ *, \langle 0\rangle_{\mathbf{A}}\rangle$*.*

Proof. Let $\mathbf{A}$ be a positive implicative pre-BCK-algebra. For (1), just note the set $\mathrm{PI}(\mathbf{A})$ is closed under dual relative pseudocomplementation by Theorem 2.2.20(5), and hence that $\langle \mathrm{PI}(\mathbf{A});\ *, \langle 0\rangle_{\mathbf{A}}\rangle$ is a positive implicative BCK-algebra by Lemma 1.6.14. For (2) the map $f : A \to \mathrm{PI}(\mathbf{A})$ defined by $a \mapsto \langle a\rangle_{\mathbf{A}}$ is clearly onto. Moreover $f(a \dot{-}^{\mathbf{A}} b) = \langle a \dot{-}^{\mathbf{A}} b\rangle_{\mathbf{A}} = \langle a\rangle_{\mathbf{A}} *^{\langle \mathrm{PI}(\mathbf{A});\ *, \langle 0\rangle_{\mathbf{A}}\rangle} \langle b\rangle_{\mathbf{A}} = f(a) *^{\langle \mathrm{PI}(\mathbf{A});\ *, \langle 0\rangle_{\mathbf{A}}\rangle} f(b)$ by Theorem 2.2.20(5), so (ignoring issues of similarity type) f is a homomorphism from $\mathbf{A}$ onto $\langle \mathrm{PI}(\mathbf{A});\ *, \langle 0\rangle_{\mathbf{A}}\rangle$. Also $\langle a, b\rangle \in \ker f$ iff $\langle a\rangle_{\mathbf{A}} = \langle b\rangle_{\mathbf{A}}$ iff $[0]_{\Theta^{\mathbf{A}}(0,a)} = [0]_{\Theta^{\mathbf{A}}(0,b)}$ iff $a \equiv b \,(\mathrm{mod}\, \Xi)$ iff $\langle a, b\rangle \in \Xi$ by the proof of Theorem 2.2.20, so $\ker f = \Xi$. For (3), just note that $\mathbf{A}/\Xi$ is isomorphic to $\langle \mathrm{PI}(\mathbf{A});\ *, \langle 0\rangle_{\mathbf{A}}\rangle$ under the map $[a]_{\Xi} \mapsto \langle a\rangle_{\mathbf{A}}$ as an immediate consequence of (2) and the homomorphism theorem [99, Theorem 1§11.1]. ∎

Let $\mathbf{A}$ be a pre-BCK-algebra. Theorem 2.2.20(5) essentially provides an internal description of the dual relative pseudocomplement $I * J$ of principal ideals I and J in the join semilattice $\langle \mathrm{CI}(\mathbf{A});\ \vee, \langle 0\rangle_{\mathbf{A}}\rangle$ of compact ideals of $\mathbf{A}$. Theorem 2.2.26 in the sequel, which follows immediately from Proposition 2.2.25 below, extends this characterisation of dual relative pseudocomplementation to arbitrary members of $\langle \mathrm{CI}(\mathbf{A});\ \vee, \langle 0\rangle_{\mathbf{A}}\rangle$.

Lemma 2.2.24. *Let* $\mathbf{A}$ *be a pre-BCK-algebra. For any* $b, c \in A$ *and any* $a_1, \ldots, a_n \in A$,

$$\big(\cdots((bc)a_1)\cdots\big)a_n \equiv \big((\cdots(ba_1)\cdots)a_n\big)c \pmod{\Xi}.$$

Proof. The proof is by induction on n. We show only $\big(\cdots((bc)a_1)\cdots\big)a_n \preceq \big((\cdots(ba_1)\cdots)a_n\big)c$; the proof of the opposite inclusion is similar and is omitted. For $n = 1$, just note $(bc)a_1 \preceq (ba_1)c$ by (2.10). Suppose now that the claim holds for $k < n$. By the inductive hypothesis, $\big(\cdots((bc)a_1)\cdots\big)a_k \preceq \big((\cdots(ba_1)\cdots)a_k\big)c$, whence:

$$\Big(\big(\cdots((bc)a_1)\cdots\big)a_k\Big)a_{k+1} \preceq \Big(\big((\cdots(ba_1)\cdots)a_k\big)c\Big)a_{k+1} \tag{2.38}$$

by Lemma 2.1.12(2). Put $\alpha := \big(\cdots(ba_1)\cdots\big)a_k$, $\beta := c$ and $\gamma := a_{k+1}$. From (2.10) we have $(\alpha\beta)\gamma \preceq (\alpha\gamma)\beta$, whence:

$$\Big(\big((\cdots(ba_1)\cdots)a_k\big)c\Big)a_{k+1} \preceq \Big(\big((\cdots(ba_1)\cdots)a_k\big)a_{k+1}\Big)c. \tag{2.39}$$

From (2.38), (2.39) and transitivity we conclude:

$$\Big(\big(\cdots((bc)a_1)\cdots\big)a_k\Big)a_{k+1} \preceq \Big(\big((\cdots(ba_1)\cdots)a_k\big)a_{k+1}\Big)c$$

as required. ∎

Proposition 2.2.25. *Let* $\mathbf{A}$ *be a positive implicative pre-BCK-algebra. For any* $a_1, \ldots, a_n \in A$,

$$\langle a_1, \ldots, a_n\rangle_{\mathbf{A}} = \big\{c \in A : (\cdots((ca_1)a_2)\cdots)a_n = 0\big\}.$$

Proof. The proof is by induction on n. For $n = 1$, $\langle a_1\rangle_{\mathbf{A}} = \{c \in A : ca_1 = 0\}$ by EDPI, so the basis case holds. Suppose the claim holds for $k < n$. We are required to show:

$$\langle a_1, \ldots, a_k, a_{k+1}\rangle_{\mathbf{A}} = \big\{c \in A : ((\cdots((ca_1)a_2)\cdots)a_k)a_{k+1} = 0\big\}. \tag{2.40}$$

So let $I := \langle a_1, \ldots, a_k\rangle_{\mathbf{A}}$. By Lemma 2.2.19 and the inductive hypothesis, we have:

$$\begin{aligned}\langle a_1,\ldots,a_k,a_{k+1}\rangle_{\mathbf{A}} &= \langle I, a_{k+1}\rangle_{\mathbf{A}}\\ &= I \vee^{\mathbf{I}(\mathbf{A})} \langle a_{k+1}\rangle_{\mathbf{A}}\\ &= \{c\in A : ca_{k+1}\in I\}\\ &= \{c\in A : (\cdots((ca_{k+1})a_1)\cdots)a_k = 0\}.\end{aligned}$$

To complete the proof let $J := \{c \in A : (\cdots((ca_{k+1})a_1)\cdots)a_k = 0\}$ and let $K := \{c\in A : ((\cdots(ca_1)\cdots)a_k)a_{k+1} = 0\}$; we show $J = K$. Let $b \in J$. Then $(\cdots((ba_{k+1})a_1)\cdots)a_k = 0$, so $((\cdots(ba_1)\cdots)a_k)a_{k+1} = 0$ by Lemma 2.2.24 and Corollary 2.1.14(3). Thus $b \in K$ and $J \subseteq K$. For the converse, let $b \in K$. Then $((\cdots(ba_1)\cdots)a_k)a_{k+1} = 0$, so $(\cdots((ba_{k+1})a_1)\cdots)a_k = 0$, also by Lemma 2.2.24 and Corollary 2.1.14(3). Thus $b \in J$ and $K \subseteq J$. Hence $J = K$. This establishes (2.40), so the proof is complete. ∎

Theorem 2.2.26. *(cf. [68, Theorem 2.5]) Let* $\mathbf{A}$ *be a positive implicative pre-BCK-algebra and let* $I := \langle a_1,\ldots,a_t\rangle_{\mathbf{A}}$, $J := \langle b_1,\ldots,b_r\rangle_{\mathbf{A}}$ *be two finitely generated ideals of* $\mathbf{A}$. *For* $i = 1,\ldots,t$, *let* $d_i := (\cdots(a_ib_1)\cdots)b_r$. *Then the dual relative pseudocomplement* $I * J$ *of* I *and* J *in the join semilattice* $\langle \mathrm{CI}(\mathbf{A});\ \vee, \langle 0\rangle_{\mathbf{A}}\rangle$ *of compact ideals of* $\mathbf{A}$ *is the ideal* $\langle d_1,\ldots,d_t\rangle_{\mathbf{A}}$.

By a *quasi-bounded positive implicative pre-BCK-algebra* we mean any quasi-bounded pre-BCK-algebra induced from a positive implicative pre-BCK-algebra with a maximal class. Because of Theorem 2.2.17 and the remarks of §2.1.40, the class $\mathsf{pPBCK}^{\mathbf{1}}$ of all quasi-bounded positive implicative pre-BCK-algebras is a variety. We conclude this subsection with a technical lemma concerning $\mathsf{pPBCK}^{\mathbf{1}}$ that will be needed in the sequel.

Lemma 2.2.27. *The variety* $\mathsf{pPBCK}^{\mathbf{1}}$ *of quasi-bounded positive implicative pre-BCK-algebras coincides with* pPBCK^{+}, *the generic double-pointed expansion of the variety of positive implicative pre-BCK-algebras.*

Proof. Let $\mathbf{A}^{\mathbf{1}} \in \mathsf{pPBCK}^{\mathbf{1}}$. For any $a \in A$, $a = a \sqcap 1 \equiv_{\Theta^{\mathbf{A}^1}(0,1)} a \sqcap 0 = 0$, so $\Theta^{\mathbf{A}^1}(0,1) = \iota_{\mathbf{A}^1}$; that is to say $\mathbf{A}^{\mathbf{1}} \in \mathsf{pPBCK}^{+}$. Hence $\mathsf{pPBCK}^{\mathbf{1}} \subseteq \mathsf{pPBCK}^{+}$. For the opposite inclusion, let $\mathbf{A}^{+} \in \mathsf{pPBCK}^{+}$. We separate the proof into two cases:

(i) $0 = 1$;
(ii) $0 \neq 1$.

For (i), suppose $0 = 1$. Then $\Theta^{\mathbf{A}^+}(0,1) = \omega_{\mathbf{A}^+}$. But $\Theta^{\mathbf{A}^+}(0,1) = \iota_{\mathbf{A}^+}$ by hypothesis, so $\mathbf{A}^+$ is trivial and 1 is maximal. Hence $\mathbf{A}^+ \in \mathsf{pPBCK}^{\mathbf{1}}$.

For (ii), suppose $0 \neq 1$. Then $\mathbf{A}$ is non-trivial, so we may assume to the contrary that 1 is not maximal. Since 1 is not maximal, there exists $0 \neq a \in A$ such that either $1 \prec a$ or 1 and a are incomparable under the pre-BCK quasiorder. In either case it follows that $a \notin \langle 1 \rangle_{\mathbf{A}^-}$, where $\mathbf{A}^-$ denotes the positive implicative pre-BCK-algebra reduct of $\mathbf{A}^+$, because of the description of the principal ideals of $\mathbf{A}^-$ as the hereditary subsets of A. Thus $\langle 1 \rangle_{\mathbf{A}^-}$ is proper. Since $\mathbf{A}^+$ and $\mathbf{A}^-$ have the same congruences, $\langle 1 \rangle_{\mathbf{A}^+} = \langle 1 \rangle_{\mathbf{A}^-}$ by normality of ideals; that is to say $\langle 1 \rangle_{\mathbf{A}^+}$ is proper. But this implies $\langle 1 \rangle^{\delta}_{\mathbf{A}^+} < \iota_{\mathbf{A}^+}$, where $\langle 1 \rangle^{\delta}_{\mathbf{A}^+}$ is the least congruence on $\mathbf{A}^+$ whose 0-class is $\langle 1 \rangle_{\mathbf{A}^+}$, so $\Theta^{\mathbf{A}^+}(0,1) < \iota_{\mathbf{A}^+}$ since $\langle 1 \rangle^{\delta}_{\mathbf{A}^+} = \Theta^{\mathbf{A}^+}(0,1)$. Hence 0 and 1 are not residually distinct, a contradiction. Thus 1 is maximal, and $\mathbf{A}^+ \in \mathsf{pPBCK}^{\mathbf{1}}$.

By (i) and (ii), $\mathbf{A}^+ \in \mathsf{pPBCK}^{\mathbf{1}}$. Hence $\mathsf{pPBCK}^+ \subseteq \mathsf{pPBCK}^{\mathbf{1}}$. Therefore $\mathsf{pPBCK}^+ = \mathsf{pPBCK}^{\mathbf{1}}$, and the proof is complete. ∎

2.2.28. Implicative Pre-BCK-Algebras. By an *implicative pre-BCK-algebra* we mean a pre-BCK-algebra $\mathbf{A}$ such that $\mathbf{A}/\Xi \cong \mathbf{B}$ for some implicative BCK-algebra $\mathbf{B}$. Since the class iBCK of all implicative BCK-algebras is a variety, axiomatised relative to PBCK by the identity (*cf.* [38, pp. 294–295]):

$$\big(x \dot{-} (x \dot{-} y)\big) \dot{-} (y \dot{-} x) \approx y \dot{-} (y \dot{-} x) \tag{2.41}$$

the class iPBCK of all implicative pre-BCK-algebras is also a variety, by Theorem 2.2.3. In particular, iPBCK is axiomatised relative to PBCK by the pair of identities:

$$\big((x \dot{-} (x \dot{-} y)) \dot{-} (y \dot{-} x)\big) \dot{-} \big(y \dot{-} (y \dot{-} x)\big) \approx \mathbf{0}$$
$$\big(y \dot{-} (y \dot{-} x)\big) \dot{-} \big((x \dot{-} (x \dot{-} y)) \dot{-} (y \dot{-} x)\big) \approx \mathbf{0}$$

as (2.41) is of the form (2.29). The preceding characterisation of iPBCK

notwithstanding, the following axiomatisation often proves more useful in practice.

Theorem 2.2.29. *An algebra $\langle A;\ \dot{-}, 0\rangle$ of type $\langle 2,0\rangle$ is an implicative pre-BCK-algebra iff the following identities are satisfied:*

$$((x \dot{-} y) \dot{-} (x \dot{-} z)) \dot{-} (z \dot{-} y) \approx \mathbf{0} \tag{2.1}$$

$$\mathbf{0} \dot{-} x \approx \mathbf{0} \tag{2.3}$$

$$x \dot{-} \mathbf{0} \approx x \tag{2.6}$$

$$x \dot{-} \big(x \dot{-} (y \dot{-} x)\big) \approx \mathbf{0}. \tag{2.42}$$

Thus the class iPBCK *of implicative pre-BCK-algebras is a variety.*

Proof. Let $\mathbf{A}$ be an implicative pre-BCK-algebra. By definition we have that $\mathbf{A} \models (2.1), (2.3), (2.6)$, so it only remains to show $\mathbf{A} \models (2.42)$. Since $\mathbf{A}$ is implicative, $\mathbf{A}/\Xi \cong \mathbf{B}$ for some implicative BCK-algebra $\mathbf{B}$. Because $\mathbf{B}$ is implicative, $\mathbf{B} \models x \dot{-} (y \dot{-} x) \approx x$, so $\mathbf{B} \models (2.42)$ by (1.3). As identities are preserved by isomorphic copies, we have that $\mathbf{A}/\Xi \models (2.42)$, so $\mathbf{A} \models (2.42)$ by Corollary 2.1.16.

Conversely, suppose $\mathbf{A} := \langle A;\ \dot{-}, 0\rangle$ is an algebra of type $\langle 2,0\rangle$ such that $\mathbf{A} \models (2.1), (2.3), (2.6), (2.42)$. By (2.1), (2.3), (2.6) and Lemma 2.1.6, $\mathbf{A}$ is a pre-BCK-algebra. By (2.42), $\mathbf{A}/\Xi \models x \dot{-} \big(x \dot{-} (y \dot{-} x)\big) \approx \mathbf{0}$; also $\mathbf{A}/\Xi \models \big(x \dot{-} (y \dot{-} x)\big) \dot{-} x \approx \mathbf{0}$ by (1.30). Hence $\mathbf{A}/\Xi \models x \dot{-} (y \dot{-} x) \approx x$ by (1.5), so $\mathbf{A}$ is implicative. ∎

By the remarks of §1.6.16, a BCK-algebra is implicative iff it is both commutative and positive implicative. This observation yields the following alternative characterisation of implicative pre-BCK-algebras, which will be needed in the sequel.

Proposition 2.2.30. *A pre-BCK-algebra is implicative iff it is both commutative and positive implicative. Thus the variety of implicative pre-BCK-algebras* *is the intersection of the variety of commutative pre-BCK-algebras and the variety of positive implicative pre-BCK-algebras.*

Proof. Let $\mathbf{A}$ be a pre-BCK-algebra. Suppose $\mathbf{A}$ is implicative. Then $\mathbf{A}/\Xi \cong \mathbf{B}$ for some implicative BCK-algebra $\mathbf{B}$. Since $\mathbf{B}$ is both commutative and positive implicative, $\mathbf{A}$ is both commutative and positive implicative. Conversely, suppose $\mathbf{A}$ is both commutative and positive implicative. Then $\mathbf{A} \cong \mathbf{B}$ for some commutative and positive implicative BCK-algebra $\mathbf{B}$. Since $\mathbf{B}$ is implicative, $\mathbf{A}$ is implicative. The remaining assertion of the proposition now follows. ∎

Let V be a subtractive variety. Recall ([9, Section 2]) a term $t(\vec{x}, \vec{y}, \vec{z})$ is a *commutator term for* V *in* $\vec{y}$, $\vec{z}$ (where $\vec{y}, \vec{z}$ are disjoint sets of variables) if it is both an ideal term in $\vec{y}$ and an ideal term in $\vec{z}$; that is, if $t \in \mathrm{IT}_{\mathsf{V}}(\vec{y}) \cap \mathrm{IT}_{\mathsf{V}}(\vec{z})$. We denote the set of all commutator terms for V in $\vec{y}, \vec{z}$ by $\mathrm{CT}_{\mathsf{V}}(\vec{y}, \vec{z})$. For $\mathbf{A} \in \mathsf{V}$ and non-empty $H, K \subseteq A$, let:

$$[H, K]_{\mathbf{A}} := \left\{t^{\mathbf{A}}(\vec{a}, \vec{h}, \vec{k}) : t(\vec{x}, \vec{y}, \vec{z}) \in \mathrm{CT}_{\mathsf{V}}(\vec{y}, \vec{z});\ \vec{a} \in A, \vec{h} \in H, \vec{k} \in K\right\}.$$

$[H, K]_{\mathbf{A}}$ is called the *commutator* of H, K in $\mathbf{A}$ for V. The commutator of ideals was introduced for ideal determined varieties by Ursini in [220] (see also Gumm and Ursini [104, Section 2]) and for subtractive varieties in general by Ursini in [222, Section 2] (see also Agliano and Ursini [9, Section 2] and Ursini [225]). By [10, Proposition 2.1(i)], $[H, K]_{\mathbf{A}} \subseteq \langle H \rangle_{\mathbf{A}} \cap \langle K \rangle_{\mathbf{A}}$ always; when V is ideal distributive the opposite inclusion also holds, in which case $[H, K]_{\mathbf{A}} = \langle H \rangle_{\mathbf{A}} \cap \langle K \rangle_{\mathbf{A}}$ [9, Proposition 4.1(2)].

Let V be a subtractive variety and let $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$. A (proper) ideal I of $\mathbf{A}$ is said to be *prime* if whenever $[a, b]_{\mathbf{A}} \subseteq I$ then $a \in I$ or $b \in I$; this definition originates with Ursini [221] (see also Chajda and Halaš [57]). Therefore if V is ideal distributive, then I is prime if $\langle a \rangle_{\mathbf{A}} \cap \langle b \rangle_{\mathbf{A}} \subseteq I$ implies $a \in I$ or $b \in I$.

Proposition 2.2.31. *For any implicative pre-BCK-algebra* $\mathbf{A}$ *and* $a, b \in A$, $\langle a \rangle_{\mathbf{A}} \cap \langle b \rangle_{\mathbf{A}} = \langle a \sqcap b \rangle_{\mathbf{A}}$.

Proof. Let $\mathbf{A}$ be an implicative pre-BCK-algebra and let $a, b \in A$. By the proof of Corollary 2.1.29 we have $\langle a \sqcap b\rangle_{\mathbf{A}} \subseteq \langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}}$ always. For the opposite inclusion, $c \in \langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}}$ implies $c \in \langle a\rangle_{\mathbf{A}}$ and $c \in \langle b\rangle_{\mathbf{A}}$, which implies $c \preceq a, b$, which implies $c \preceq a \sqcap b$ (since $a \sqcap b$ is a greatest lower bound of $\{a, b\}$), which implies $c \in \langle a \sqcap b\rangle_{\mathbf{A}}$. Thus $\langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}} \subseteq \langle a \sqcap b\rangle_{\mathbf{A}}$ as required. ∎

Let $\mathbf{A}$ be an implicative pre-BCK-algebra. Because PBCK is ideal distributive (by Proposition 2.1.24(1)), from Proposition 2.2.31 we have that a proper ideal I of $\mathbf{A}$ is prime if $\langle a \sqcap b\rangle_{\mathbf{A}} \subseteq I$ implies either $a \in I$ or $b \in I$ for all $a, b \in A$. Equivalently, I is prime if:

$$a \sqcap b \in I \quad \text{implies either } a \in I \text{ or } b \in I \tag{2.43}$$

for all $a, b \in A$. For implicative BCK-algebras, an elegant theory of prime ideals (which exploits exactly the notion of primality expressed by (2.43)) has been developed by Rasiowa in [195, Chapter II§6]. This last, in conjunction with preceding remarks, motivates the study of prime ideals in implicative pre-BCK-algebras.

Lemma 2.2.32. *The variety of (positive) implicative pre-BCK-algebras satisfies the following identity:*

$$(x \dot{-} y) \dot{-} \big((x \dot{-} y) \dot{-} (y \dot{-} x)\big) \approx \mathbf{0}. \tag{2.44}$$

Proof. We have been unable to find a derivation of (2.44) (for (positive) implicative BCK-algebras) in the literature, so we do not invoke Corollary 2.1.16 to prove the lemma. Instead we provide a derivation of (2.44) (for (positive) implicative pre-BCK-algebras). Let $\mathbf{A}$ be a (positive) implicative pre-BCK-algebra and let $a, b \in A$. Put $\alpha := ab$, $\beta := b$ and $\gamma := ba$. We have:

$$\begin{aligned}
0 &= \big((\alpha\beta)(\alpha\gamma)\big)(\gamma\beta) && \text{by (2.1)}\\
&= \big((\alpha\beta)(\alpha\gamma)\big)\big((ba)b\big) \\
&= \big((\alpha\beta)(\alpha\gamma)\big)0 && \text{by Lemma 2.1.12(3)}\\
&= (\alpha\beta)(\alpha\gamma) && \text{by (2.6)}
\end{aligned}$$

$$= \big((ab)b\big)\big((ab)(ba)\big). \tag{2.45}$$

For (2.44), put $\alpha := ab$, $\beta := (ab)(ba)$ and $\gamma := (ab)b$. We have:

$$\begin{aligned}
0 &= \big((\alpha\beta)(\alpha\gamma)\big)(\gamma\beta) \\
&= \big((\alpha\beta)(\alpha\gamma)\big)\big(((ab)b)((ab)(ba))\big) \\
&= \big((\alpha\beta)(\alpha\gamma)\big)0 && \text{by (2.45)} \\
&= (\alpha\beta)(\alpha\gamma) && \text{by (2.6)} \\
&= (\alpha\beta)\big((ab)((ab)b)\big) \\
&= (\alpha\beta)0 && \text{by (2.36)} \\
&= (ab)\big((ab)(ba)\big) && \text{by (2.6).}
\end{aligned}$$

∎

Let **A** be an implicative pre-BCK-algebra and let I be a proper ideal of **A**. We say I is *irreducible* provided that:

$$I = J \cap K \quad \text{implies either } I = J \text{ or } I = K \tag{2.46}$$

for any two proper ideals $J, K \in \mathrm{I}(\mathbf{A})$. An ideal that is not irreducible is said to be *reducible.* The following proposition is a modification of a result due to Rasiowa [195, Chapter II].

Proposition 2.2.33. *(cf. [195, Theorem II§6.1]) Let* **A** *be an implicative pre-BCK-algebra and let I be an ideal of* **A**. *Then I is prime iff it is irreducible.*

Proof. Let **A** be an implicative pre-BCK-algebra and I be a proper ideal of **A**.

($\Rightarrow$) Suppose I is not irreducible. Then there exist proper ideals $J, K \in \mathrm{I}(\mathbf{A})$ such that $I = J \cap K$ but $I \neq J$ and $I \neq K$. Clearly $J \nsubseteq K$ and $K \nsubseteq J$. Indeed, $J \subseteq K$ implies $I = J \cap K = J$ while $K \subseteq J$ implies $I = J \cap K = K$, both of which contradict the reducibility of I. Thus there exist $a, b \in A$ such that $a \in J, b \in K$ but $a \notin K, b \notin J$. To complete the proof we show:

(i) $ab, ba \notin I$;
(ii) $ab \sqcap ba \in I$.

For (i), we show only that $ab \notin I$; the proof that $ba \notin I$ is analogous and is omitted. So let $\langle I, b\rangle_{\mathbf{A}}$ be the ideal generated by $I \cup \{b\}$. Clearly $I \subseteq \langle I, b\rangle_{\mathbf{A}} \subseteq K$. Hence $a \notin \langle I, b\rangle_{\mathbf{A}}$, since $a \in \langle I, b\rangle_{\mathbf{A}}$ implies $a \in K$, a contradiction. But this implies $ab \notin I$, because $\langle I, b\rangle_{\mathbf{A}} = \{c \in A : cb \in I\}$ by Lemma 2.2.19.

For (ii), simply note $ab \sqcap ba = 0 \in I$ by (2.44).

By (i) and (ii), $ab, ba \notin I$ but $ab \sqcap ba \in I$, which proves by (2.43) that I is not prime.

($\Leftarrow$) Suppose I is not prime. Then there exist $a, b \in A$ such that $a \neq b$ and $a \sqcap b \in I$, but $a \notin I$ and $b \notin I$. Let $\langle I, a\rangle_{\mathbf{A}}$ and $\langle I, b\rangle_{\mathbf{A}}$ be the ideals generated by $I \cup \{a\}$ and $I \cup \{b\}$ respectively. To complete the proof we show:

(i) $I = \langle I, a\rangle_{\mathbf{A}} \cap \langle I, b\rangle_{\mathbf{A}}$;
(ii) $\langle I, a\rangle_{\mathbf{A}}$ is a proper ideal such that $I \neq \langle I, a\rangle_{\mathbf{A}}$;
(iii) $\langle I, b\rangle_{\mathbf{A}}$ is a proper ideal such that $I \neq \langle I, b\rangle_{\mathbf{A}}$.

For (i), the inclusion $I \subseteq \langle I, a\rangle_{\mathbf{A}} \cap \langle I, b\rangle_{\mathbf{A}}$ is obvious. For the opposite inclusion, we have:

$$\begin{aligned}\langle I, a\rangle_{\mathbf{A}} \cap \langle I, b\rangle_{\mathbf{A}} &= \left(I \vee^{\mathbf{I(A)}} \langle a\rangle_{\mathbf{A}}\right) \cap \left(I \vee^{\mathbf{I(A)}} \langle b\rangle_{\mathbf{A}}\right) \\ &= I \vee^{\mathbf{I(A)}} \left(\langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}}\right) \\ &= I \vee^{\mathbf{I(A)}} \langle a \sqcap b\rangle_{\mathbf{A}}\end{aligned}$$

by Proposition 2.1.24(1) and Proposition 2.2.31, so $\langle I, a\rangle_{\mathbf{A}} \cap \langle I, b\rangle_{\mathbf{A}} = \{d \in A : d(a \sqcap b) \in I\}$ by Lemma 2.2.19. Let $c \in \langle I, a\rangle_{\mathbf{A}} \cap \langle I, b\rangle_{\mathbf{A}}$. Then $c(a \sqcap b) \in I$, and since $a \sqcap b \in I$ we have that $c \in I$ by the definition of I as an ideal. Hence $\langle I, a\rangle_{\mathbf{A}} \cap \langle I, b\rangle_{\mathbf{A}} \subseteq I$ and $I = \langle I, a\rangle_{\mathbf{A}} \cap \langle I, b\rangle_{\mathbf{A}}$.

For (ii), $I \neq \langle I, a\rangle_{\mathbf{A}}$ since $a \notin I$. To see $\langle I, a\rangle_{\mathbf{A}}$ is proper, assume to the contrary that $b \in \langle I, a\rangle_{\mathbf{A}}$. Then $ba \in I$ by Lemma 2.2.19. Since $(b \sqcap a)(a \sqcap b) = 0 \in I$ by (2.33) and $a \sqcap b \in I$ by hypothesis, we infer $b \in I$ by Lemma 2.1.21(2), which is a contradiction. Hence $b \notin \langle I, a\rangle_{\mathbf{A}}$ and $\langle I, a\rangle_{\mathbf{A}}$ is proper.

For (iii), $I \neq \langle I, b\rangle_{\mathbf{A}}$ since $b \notin I$. To see $\langle I, b\rangle_{\mathbf{A}}$ is proper, assume to the contrary that $a \in \langle I, b\rangle_{\mathbf{A}}$. Then $ab \in I$ by Lemma 2.2.19. Since $a \sqcap b \in I$

by hypothesis, we infer $a \in I$ by the definition of I as an ideal, which is a contradiction. Hence $a \notin \langle I, b\rangle_{\mathbf{A}}$ and $\langle I, b\rangle_{\mathbf{A}}$ is proper.

By (i), (ii) and (iii), $I = \langle I, a\rangle_{\mathbf{A}} \cap \langle I, b\rangle_{\mathbf{A}}$ where $\langle I, a\rangle_{\mathbf{A}}$ and $\langle I, b\rangle_{\mathbf{A}}$ are proper ideals such that $I \neq \langle I, a\rangle_{\mathbf{A}}$ and $I \neq \langle I, b\rangle_{\mathbf{A}}$, which proves by (2.46) that I is not irreducible. ∎

Let $\mathbf{A}$ be an implicative pre-BCK-algebra. A proper ideal I of $\mathbf{A}$ is said to be *maximal* provided it is not a proper subset of any proper ideal. The following proposition is an easy modification of a result due to Rasiowa [195, Chapter II].

Proposition 2.2.34. *(cf. [195, Theorem II§6.2]) Let* $\mathbf{A}$ *be an implicative pre-BCK-algebra and let* I *be an ideal of* $\mathbf{A}$*. Then* I *is prime iff it is maximal.*

Proof. Let $\mathbf{A}$ be an implicative pre-BCK-algebra and I be a proper ideal of $\mathbf{A}$.

($\Rightarrow$) Suppose I is prime and assume to the contrary that I is not maximal. Then I is a proper subset of some proper ideal $J \in \mathrm{I}(\mathbf{A})$ and so there exists $a \in A$ such that $a \notin I$ but $a \in J$. Let $\langle I, a\rangle_{\mathbf{A}}$ be the ideal generated by $I \cup \{a\}$. Clearly $I \subsetneq \langle I, a\rangle_{\mathbf{A}} \subseteq J$. To complete the proof it is sufficient to show that $\langle I, a\rangle_{\mathbf{A}}$ (and hence J) is improper. Since $ca \sqcap a = 0 \in I$ (by (2.36)) and $a \notin I$ we have that $ca \in I$ for all $c \in A$ by (2.43). But this implies $c \in \langle I, a\rangle_{\mathbf{A}}$ for all $c \in A$ because $\langle I, a\rangle_{\mathbf{A}} = \{b \in A : ba \in I\}$ by Lemma 2.2.19. Thus $\langle I, a\rangle_{\mathbf{A}} = A$ and so $J = A$, a contradiction.

($\Leftarrow$) Suppose I is maximal and assume to the contrary that I is not irreducible. Then there exist proper ideals $J, K \in \mathrm{I}(\mathbf{A})$ such that $I = J \cap K$ but $I \neq J$ and $I \neq K$. But this implies $I = J \cap K \subset J$ and $I = J \cap K \subset K$, whence $I = J \cap K$ is a proper ideal contained in both the proper ideals J and K. Thus I is not maximal, which is a contradiction. Hence I is irreducible, which shows by Proposition 2.2.33 that I is prime. ∎

The following result is also an easy modification of a theorem of Rasiowa [195, Chapter II].

Proposition 2.2.35. *(cf. [195, Theorem II§6.4]) Let* $\mathbf{A}$ *be an implicative pre-BCK-algebra and let* I *be an ideal of* $\mathbf{A}$*. Then* I *is prime iff* $\mathbf{A}/\phi_I$ *is* *isomorphic to the two-element flat implicative BCK-algebra* $\mathbf{C}_1$*, where* ϕ_I *is the congruence induced by* I *in the sense of Theorem 2.1.26.*

Proof. Let $\mathbf{A}$ be an implicative pre-BCK-algebra and let I be an ideal of $\mathbf{A}$.

($\Rightarrow$) Suppose I is prime. By Lemma 2.1.27 we have $a \in I$ iff $a \equiv 0 \,(\mathrm{mod}\, \phi_I)$, whence $[a]_{\phi_I} = [0]_{\phi_I}$ for $a \in I$. Suppose now that $a \notin I$ and $b \notin I$. From $a \notin I$ and $ba \sqcap a = 0 \in I$ (by (2.36)) we have that $ba \in I$ by (2.43). Likewise, from $b \notin I$ and $ab \sqcap b = 0 \in I$ (by (2.36)) we have that $ab \in I$ by (2.43). Since $ab, ba \in I$ we infer $a \equiv b \,(\mathrm{mod}\, \phi_I)$, or equivalently, $[a]_{\phi_I} = [b]_{\phi_I}$. Hence (the universe of) the quotient algebra $\mathbf{A}/\phi_I$ has exactly two elements. Since $\phi_I \geq \Xi$ we have that $\mathbf{A}/\phi_I$ is a BCK-algebra, and the result is now forced by the remarks of §1.6.16.

($\Leftarrow$) Suppose $\mathbf{A}/\phi_I$ is isomorphic to the 2-element flat implicative BCK-algebra $\mathbf{C}_1$. Then I is proper (because I improper implies $\mathbf{A}/\phi_I$ is trivial, a contradiction). Let $a \notin I$ and $b \notin I$. Since the equivalence class $[0]_{\phi_I}$ contains (by Lemma 2.1.27) exactly those elements of A belonging to I we infer that $[a]_{\phi_I} = [b]_{\phi_I}$, just because (the universe of) the quotient algebra $\mathbf{A}/\phi_I$ has only two elements. Hence $a \equiv b \,(\mathrm{mod}\, \phi_I)$, so $ab, ba \in I$. But then $a \sqcap b \notin I$, because $a \sqcap b \in I$ and $ab \in I$ implies $a \in I$ by the definition of I as an ideal, which is a contradiction. We have shown that $a \notin I$ and $b \notin I$ implies $a \sqcap b \notin I$, which proves by (2.43) that I is prime. ∎

Theorem 2.2.36. *(cf. [195, Theorem II§6.4]) Let* $\mathbf{A}$ *be an implicative pre-BCK-algebra. For any ideal* $I \in \mathrm{I}(\mathbf{A})$ *the following are equivalent:*

1. *I is prime;*
2. *I is maximal;*
3. *I is irreducible;*
4. *$\mathbf{A}/\phi_I$ is isomorphic to the two-element flat implicative BCK-algebra* $\mathbf{C}_1$*, where* ϕ_I *is the congruence induced by* I *in the sense of Theorem 2.1.26.*

Moreover, any proper ideal $I \in \mathrm{I}(\mathbf{A})$ is contained in a prime ideal. In particular,

$$I = \bigcap\{J : J \text{ is a prime ideal of } \mathbf{A} \text{ and } I \subseteq J\}.$$

Proof. It remains only to establish the final claim. By Proposition 2.2.33, the prime ideals of any implicative pre-BCK-algebra $\mathbf{A}$ are precisely the meet irreducible elements of $\mathrm{I}(\mathbf{A})$. The claim now follows, since any element of an algebraic lattice is the infimum of meet irreducible elements. ∎

2.3 Implicative BCS-Algebras

In this section we study the variety of implicative BCS-algebras, a class of pointed groupoids that more closely resemble implicative BCK-algebras than do implicative pre-BCK-algebras.

Implicative BCS-algebras proper are introduced in §2.3.1. We show the variety of implicative BCS-algebras is a subvariety of the variety of implicative pre-BCK-algebras, and also prove that iBCK is the only non-trivial subquasivariety of the variety of implicative BCS-algebras that is a non-trivial subquasivariety of BCK. Some examples showing that implicative BCS-algebras arise naturally in universal algebra in binary discriminator varieties (including pseudocomplemented semilattices) and in algebraic logic in 'pointed' fixedpoint discriminator varieties (including certain subvarieties of n-potent BCK-algebras) are presented.

The role of the pre-BCK-meet $\sqcap$ in the theory of implicative BCS-algebras is considered in §2.3.19. It is shown that the existence of a left normal band with zero polynomial reduct $\langle \sqcap, 0\rangle$ whose underlying natural band partial order $\leq_{\mathcal{H}}^{\langle \sqcap, 0\rangle}$ respects implicative pre-BCK difference in a certain precise sense distinguishes the implicative BCS-algebras among the implicative pre-BCK-algebras. A representation theorem for implicative BCS-algebras is proved: for suitable choices of objects and morphisms, it is shown that the categories of implicative BCS-algebras and left handed (equivalently, left regular) locally

Boolean bands are isomorphic.

In §2.3.42 we characterise (to within isomorphism) the subdirectly irreducible implicative BCS-algebras: they are precisely the 2-element implicative BCK-algebra and the algebras $\hat{\mathbf{B}}$ obtained from the non-trivial Boolean algebras $\mathbf{B}$ upon replacing the top element of each $\mathbf{B}$ with a two-element Ξ-class.

Quasi-bounded implicative BCS-algebras are studied in §2.3.57. For a quasi-bounded implicative BCS-algebra $\mathbf{A^1}$, the skeleton $\mathrm{S}(\mathbf{A^1})$ is the set $\{a^* : a \in A\}$. An internal description of the maximal bounded implicative BCS-algebra homomorphic image $\mathbf{A^1}/\Xi$ of a quasi-bounded implicative BCS-algebra $\mathbf{A^1}$ is given in terms of the skeleton $\mathrm{S}(\mathbf{A^1})$. We apply this characterisation to give a new and conceptually simple proof of the Glivenko-Frink theorem for pseudocomplemented semilattices.

In §2.3.70 the role played by the 3-element pre-BCK-algebra $\mathbf{B}_2$ of Example 2.1.5 in the theory of implicative BCS-algebras is investigated. We show $\mathbf{B}_2$ generates the class of implicative BCS-algebras (as a variety) and hence that the lattice of varieties of implicative BCS-algebras is a 3-element chain; the only non-trivial subvariety of the variety of implicative BCS-algebras is the variety of implicative BCK-algebras.

2.3.1. Implicative BCS-Algebras. An *implicative BCS-algebra* is an algebra $\langle A;\ \backslash, 0\rangle$ of type $\langle 2, 0\rangle$ such that the following identities hold:

$$x\backslash x \approx \mathbf{0} \tag{2.47}$$

$$(x\backslash y)\backslash z \approx (x\backslash z)\backslash y \tag{2.48}$$

$$(x\backslash z)\backslash(y\backslash z) \approx (x\backslash y)\backslash z \tag{2.49}$$

$$x\backslash(y\backslash x) \approx x. \tag{2.50}$$

Because of (2.47), the class iBCS of implicative BCS-algebras may be construed as a variety of pointed groupoids; consequently in the sequel we (informally) denote implicative BCS difference by juxtaposition when no confusion can arise. Identity (2.49) is an algebraic analogue of the S-combinator $\big(p \to (q \to r)\big) \to \big((p \to q) \to (p \to r)\big)$ of combinatory logic; this accounts for the origin

$\backslash^{\mathbf{A}}$	0	a
0	0	0
a	a	a

(a)

$\backslash^{\mathbf{A}}$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	b
b	b	0	0	a	b
c	c	c	d	0	0
d	d	c	d	0	0

(b)

$\backslash^{\mathbf{A}}$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	0	0	b
c	c	0	c	0

(c)

$\backslash^{\mathbf{A}}$	0	a
0	0	0
a	0	0

(d)

Figure 2.1. Independence tables for the implicative BCS identities.

of the term 'BCS-algebra'. The qualifier 'implicative' is intended to suggest (as per implicative BCK-algebras) that the binary operation in question is an algebraic analogue of (specifically) *classical* (that is, material) implication; recall (2.50) is an algebraic analogue of Peirce's law $\big((p \to q) \to p\big) \to p$.

Remark 2.3.2. In the variety of implicative BCS-algebras the identities (2.47)–(2.50) are independent, as the operation tables of Figure 2.1 confirm. Let $\mathbf{A}$ be the algebra defined by the operation table of Figure 2.1(a). Then $\mathbf{A} \models$ (2.48)–(2.50); however $\mathbf{A} \not\models$ (2.47) since $a\backslash a = a \neq 0$. Suppose instead that $\mathbf{A}$ is the algebra defined by the operation table of Figure 2.1(b). An easy sequence of checks shows $\mathbf{A} \models$ (2.47), (2.49)–(2.50) but that $\mathbf{A} \not\models$ (2.48), since $(a\backslash c)\backslash d = a\backslash d = b \neq a = b\backslash c = (a\backslash d)\backslash c$. Let $\mathbf{A}$ be the algebra defined by the operation table of Figure 2.1(c). It is readily verified that $\mathbf{A} \models$ (2.47)–(2.48), (2.50) but that $\mathbf{A} \not\models$ (2.49), just because $(b\backslash a)\backslash c = 0\backslash c = 0 \neq b = b\backslash 0 = (b\backslash c)\backslash(a\backslash c)$. Suppose now that $\mathbf{A}$ is the algebra defined by the operation table of Figure 2.1(d). Clearly $\mathbf{A} \models$ (2.47)–(2.49); however $\mathbf{A} \not\models$ (2.50) as $a\backslash(0\backslash a) = a\backslash 0 = 0 \neq a$. ∎

Lemma 2.3.3. *The variety of implicative BCS-algebras satisfies the following identities:*

$$(x\backslash y)\backslash y \approx x\backslash y \tag{2.51}$$

$$(x\backslash y)\backslash(z\backslash x) \approx x\backslash y \tag{2.52}$$

$$x\backslash\big(y\backslash(z\backslash x)\big) \approx x\backslash y \tag{2.53}$$

$$(x\backslash y)\backslash(y\backslash z) \approx x\backslash y \tag{2.54}$$

$$(x\backslash y)\backslash(z\backslash y) \approx (x\backslash z)\backslash(y\backslash z). \tag{2.55}$$

Proof. Let $\mathbf{A}$ be an implicative BCS-algebra and let $a, b, c \in A$. For (2.51) note $(ab)b = (ab)\big(b(ab)\big) = ab$ by (2.50) applied twice. For (2.52) we have $(ab)(ca) = \big(a(ca)\big)b = ab$ by (2.48) and (2.50). For (2.53) we have $a\big(b(ca)\big) = \big(a(ca)\big)\big(b(ca)\big) = (ab)(ca) = ab$ by (2.50), (2.49) and (2.52). For (2.54) we have $(ab)(bc) = (ab)\big((b(ab))c\big) = (ab)\big((bc)(ab)\big) = ab$ by (2.50), (2.48) and (2.50). For (2.55) we have $(ac)(bc) = (ab)c = (ac)b = (ab)(cb)$ by (2.49), (2.48) and (2.49). ∎

Proposition 2.3.4. *The variety of implicative BCS-algebras satisfies the following identity:*

$$x\backslash\big(x\backslash(x\backslash y)\big) \approx x\backslash y. \tag{2.56}$$

Proof. Let $\mathbf{A}$ be an implicative BCS-algebra and let $a, b \in A$. We have:

$$\begin{aligned}
ab &= (ab)\big(a(ab)\big) && \text{by (2.50)}\\
&= \big(a(a(ab))\big)\big(b(a(ab)\big) && \text{by (2.49)}\\
&= \big(a(a(ab)\big)(ba) && \text{by (2.53)}\\
&= \big(a(ba)\big)\big(a(ab)\big) && \text{by (2.48)}\\
&= a\big(a(ab)\big) && \text{by (2.50)}
\end{aligned}$$

which establishes (2.56) as required. ∎

In the statement and proof of the following two results we ignore issues of type.

Proposition 2.3.5. *If* $\mathbf{A}$ *is an implicative BCS-algebra then* $\mathbf{A}$ *is an implicative pre-BCK-algebra. Thus the variety of implicative BCS-algebras is a subvariety of the variety of implicative pre-BCK-algebras.*

Proof. Let $\mathbf{A}$ be an implicative BCS-algebra and let $a, b, c \in A$. By (2.50) and (2.47) we have $a = a(aa) = a0$, so $\mathbf{A} \models (2.6)$. From (2.50) it follows that

$0 = 0(a0) = 0a$, whence $\mathbf{A} \models (2.4)$. Also,

$$\begin{aligned}
\big((ab)(ac)\big)(cb) &= \big((ab)(cb)\big)(ac) && \text{by (2.48)}\\
&= \big((ac)b\big)(ac) && \text{by (2.49)}\\
&= \big((ac)(ac)\big)b && \text{by (2.48)}\\
&= 0b && \text{by (2.47)}\\
&= 0
\end{aligned}$$

so $\mathbf{A} \models (2.1)$. By Lemma 2.1.6 we conclude that $\mathbf{A}$ is a pre-BCK-algebra. To see $\mathbf{A}$ is implicative, just notice $0 = aa = a\big(a(ba)\big)$ by (2.47) and (2.50). ∎

Remark 2.3.6. The variety of implicative BCS-algebras is properly contained within the variety of implicative pre-BCK-algebras. To see this, consider the following algebra $\mathbf{A}$:

$\dot{-}^{\mathbf{A}}$	0	a	b	c
0	0	0	0	0
a	a	0	c	0
b	b	b	0	b
c	c	0	a	0

An easy sequence of checks shows that $\mathbf{A}$ is an implicative pre-BCK-algebra. However, $\mathbf{A}$ is not an implicative BCS-algebra, since $a(ba) = ab = c \neq a$. ∎

Proposition 2.3.7. *For the variety of implicative BCS-algebras the following assertions hold:*

1. iBCK *is contained in any non-trivial subquasivariety of* iBCS*;*

2. iBCK *is the only non-trivial subquasivariety of* iBCS *that is a non-trivial subquasivariety of* BCK*. Thus* iBCK *is axiomatised relative to* iBCS *by any identity of the form:*

$$x\backslash\prod_{i=1}^{n} u_i(x,y) \approx y\backslash\prod_{j=1}^{m} v_j(x,y)$$

where $n, m \in \omega$ and $u_1, \ldots, u_n, v_1, \ldots, v_m$ are $\langle \backslash, 0 \rangle$-terms such that BCK *satisfies:*

$$u_i(x,x) \approx 0 \approx v_j(x,x) \qquad i = 1, \ldots, n, \quad j = 1, \ldots, m.$$

In particular, iBCK *is axiomatised relative to* iBCS *by the identity:*

$$x \backslash (x \backslash y) \approx y \backslash (y \backslash x).$$

Proof. For (1) let K be a non-trivial subquasivariety of iBCS. Then K is a non-trivial subquasivariety of PBCK by Proposition 2.3.5 and so iBCK $\subseteq$ K by Example 2.1.4. For (2), let K be a non-trivial subquasivariety of iBCS such that K $\subseteq$ BCK. From K $\models x \backslash (y \backslash x) \approx x$ and K $\subseteq$ BCK we have that K $\subseteq$ iBCK; since there are no non-trivial subquasivarieties of iBCK (by [131, Theorem 2] and [39, Theorem 4.4]) we infer that K is iBCK. The remaining claims now follow from Theorem 2.2.2. ∎

Although PBCK has been encountered previously in the literature as a generalisation of BCK that fails in any sense to be point regular (recall Lemma 2.1.6 and Lemma 2.1.9), according to Iséki [124] the variety of implicative BCS-algebras has not previously been considered in the literature in any context. Nonetheless, individual members of iBCS have been employed in a number of studies of BCK-algebras and subtractive varieties; such algebras have invariably belonged to the following class of examples.

Example 2.3.8. Let $\mathbf{B}_n := \langle B_n; \backslash, 0 \rangle$ be an algebra of cardinality $n+1$ equipped with a distinguished element 0 and a binary operation $\backslash$ defined by:

$$ab := \begin{cases} a & \text{if } b = 0 \\ 0 & \text{otherwise} \end{cases}$$

for any $a, b \in B_n$. Then $\mathbf{B}_n$ is a *flat* implicative BCS-algebra (on $n+1$ elements). In particular, the algebra $\mathbf{B}_2$ of Example 2.1.5 is a flat implicative BCS-algebra (on 3 elements); of course, $\mathbf{B}_2$ is the simplest example of an implicative BCS-algebra that is not an (implicative) BCK-algebra. ∎

Remark 2.3.9. Let $\mathbf{B}_n := \langle B_n; \setminus, 0\rangle$ be a flat implicative BCS-algebra on $n+1$ elements. Let θ be an equivalence relation on $B_n - \{0\}$ and let $\phi := \theta \cup \{\langle 0, 0\rangle\}$. An easy inspection of $\mathbf{B}_n$ shows ϕ is a congruence on $\mathbf{B}_n$. Suppose now that $a \in B_n$ and $b \neq 0$. By inspection of $\mathbf{B}_n$ and Lemma 2.1.42 we have $a = a \sqcap b \equiv_{\Theta^{\mathbf{B}_n}(0,b)} a \sqcap 0 = 0$, whence $\iota_{\mathbf{B}_n} = \Theta^{\mathbf{B}_n}(b, 0)$. Therefore the lattice $\mathbf{Con}\,\mathbf{B}_n$ of congruences on $\mathbf{B}_n$ is isomorphic to the lattice of equivalence relations on $B_n - \{0\}$ together with a new largest element $\iota_{\mathbf{B}_n} = \Theta^{\mathbf{B}_n}(b, 0)$ for any $b \neq 0$ adjoined. (In essence this observation has been made previously by both Blok and Raftery [38, p. 74] and Agliano and Ursini [8, Example 6.2].)

In [65, Lemma 4.8(i)] Cornish considered the congruence structure of primitive left handed skew Boolean algebras and proved that for any primitive left handed skew Boolean algebra $\mathbf{A}$, the lattice $\mathbf{Con}\,\mathbf{A}$ of congruences on $\mathbf{A}$ is isomorphic to the lattice of equivalence relations of the set $A - \{0\}$, together with a new largest element $\iota_{\mathbf{A}} = \Theta^{\mathbf{A}}(a, 0)$ for any $a \in A - \{0\}$ adjoined. It follows that left handed skew Boolean algebra operations may be imposed on a flat implicative BCS-algebra $\mathbf{B}_n$, $n \in \omega$, without disturbing the congruence structure of $\mathbf{B}_n$, upon defining:

$$a \wedge b := \begin{cases} a & \text{if } b \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad a \vee b := \begin{cases} b & \text{if } b \neq 0 \\ a & \text{otherwise} \end{cases}$$

for any $a, b \in B_n$. ∎

Apropos preceding remarks, in [240] Wroński proved that the class BCK is not a variety by showing $\mathbf{B}_2 \in \mathbf{H}(\mathsf{BCK})$. Cornish extended Wroński's result in [71, Theorem 2.2], where he proved that any BCK-algebra with condition (S) can be embedded as a BCK-algebra into a BCK-algebra that has $\mathbf{B}_2$ as a homomorphic image. In [114] Idziak exploited properties of the algebra $\mathbf{B}_2$ in his study of the congruence n-permutability of BCK-varieties; various properties of the algebra $\mathbf{B}_2$ have also been exploited by Blok and Raftery [38, Theorem 8] and van Alten [229, Proposition 4.4] in obtaining results about BCK-algebras and related structures. Most recently Blok and La Falce [25] have employed a generalisation of the algebra $\mathbf{B}_2$ in their study of certain 'varietising' identities arising naturally in algebraic logic. Also, in [8, Example 6.2] Agliano and Ursini exhibited the variety $\mathbf{V}(\mathbf{B}_5)$ generated by the algebra $\mathbf{B}_5$ as an example of a subtractive variety that is not ideal determined, and in [11, Example 3.7] again used $\mathbf{B}_5$ in proving that the variety of MINI-algebras (recall Example 1.7.11) does not have EDPC. More generally, all the algebras $\mathbf{B}_n$, $n \in \omega$, have been considered briefly by Blok and Raftery in the context of their study of the lattice of subquasivarieties of BCK-algebras: see [38, Section 4] for details.

In the context of BCK-algebras, the following example, due in essence to Wroński [241], typifies many of the considerations encountered in the preceding applications and studies of (flat) implicative BCS-algebras.

Example 2.3.10. (*cf.* [241, Applications]) Recall from [125, Example 1] or [38, Example 1] that the set ω of all non-negative integers is the universe of a BCK-algebra $\boldsymbol{\omega} := \langle \omega; \dot{-}^{\omega}, 0\rangle$, where BCK difference is defined naturally by:

$$a \dot{-}^{\omega} b := \begin{cases} a - b & \text{if } a \geq b \\ 0 & \text{otherwise} \end{cases}$$

for any $a, b \in \omega$. In [241] Wroński employed $\boldsymbol{\omega}$ in constructing a family of BCK-algebras that generalise his example $\mathbf{W}_1(\omega)$ [240] showing that BCK-algebras do not form a variety. Let $R(\omega) := \{r_m : m \in \omega\}$ and let $I_n := \{0, \ldots, n-1\}$ for every $n = 2, 3, \ldots$. Let $\mathbf{N}_n := \langle (I_n \times R(\omega)) \cup \omega;\ \dot{-}^{\mathbf{N}_n}, 0\rangle$ be the algebra with distinguished element 0 and binary operation $\dot{-}^{\mathbf{N}_n}$ defined as follows [241, p. 222]:

$$a \dot{-}^{\mathbf{N}_n} b := a \dot{-}^{\omega} b$$

$$a \dot{-}^{\mathbf{N}_n} \langle i, r_b\rangle := 0$$

$$\langle i, r_b\rangle \dot{-}^{\mathbf{N}_n} a := \langle i, r_{b+a}\rangle$$

$$\langle i, r_a\rangle \dot{-}^{\mathbf{N}_n} \langle j, r_b\rangle := \big(b + |j - i|\big) \dot{-}^{\omega} a$$

for any $i, j \in I_n$ and $a, b \in \omega$. By Theorem 1 and Theorem 3 of [241] $\mathbf{N}_n$ is a BCK-algebra. Consider now the equivalence relation θ on the base set $(I_n \times R(\omega)) \cup \omega$ of $\mathbf{N}_n$ induced by the partition $\big\{\{0\} \times R(\omega), \ldots, \{n-1\} \times$

$R(\omega), \omega\}$. Clearly θ is a congruence relation on $\mathbf{N}_n$, and it is easy to see that the quotient algebra $\mathbf{N}_n/\theta$ is a flat implicative BCS-algebra on $n+1$ elements. By Remark 2.3.9 the congruences on $\mathbf{N}_n/\theta$ other than the universal congruence are in one-to-one correspondence with the partitions of $N_n/\theta - \{[0]_\theta\}$, while by Wroński [241, Theorem 4] the congruences of $\mathbf{N}_n$ other than the universal and identity congruences are in one-to-one correspondence with partitions of the set of indices I_n. It readily follows that the congruences of $\mathbf{N}_n$ other than the identity congruence are in one-to-one correspondence with the congruences of $\mathbf{N}_n/\theta$, and hence that $\langle \mathrm{Con}\,\mathbf{N}_n - \{\omega_{\mathbf{N}_n}\};\ \subseteq\rangle \cong \mathbf{Con}\,\mathbf{N}_n/\theta$. From this observation and Remark 2.3.9 we immediately obtain Wroński's results [241, Theorem 5, Theorem 6] that the congruence lattice of $\mathbf{N}_\omega$ obeys no special lattice identities at all, and that $\mathbf{N}_\omega$ is not congruence m-permutable for any $m \in \omega$; *cf.* [38, Section 1]. ∎

It is natural to ask if implicative BCS-algebras distinct from flat algebras occur readily in universal algebra and/or algebraic logic. The following example shows (non-flat) implicative BCS-algebras arise naturally from flat implicative BCS-algebras in a large class of varieties occurring in universal algebra.

Example 2.3.11. Let V be a binary discriminator variety with binary discriminator term $b(x, y)$ and let $\mathsf{K} \subseteq \mathsf{V}$ be a class of binary discriminator algebras generating V as a variety. Let $\mathbf{A} \in \mathsf{K}$. By definition of the binary discriminator,

$$b^{\mathbf{A}}(a, c) = \begin{cases} a & \text{if } c = 0 \\ 0 & \text{otherwise} \end{cases}$$

for any $a, c \in A$, whence $\langle A;\ b^{\mathbf{A}}, 0\rangle$ is a flat implicative BCS-algebra by Example 2.3.8. Since the identities satisfied by V are precisely those satisfied by K, it follows that any $\mathbf{B} \in \mathsf{V}$ has a canonical implicative BCS-algebra polynomial reduct $\langle B;\ b^{\mathbf{B}}, 0\rangle$. ∎

By way of illustration we give a practical application of Example 2.3.11.

Example 2.3.12. (*cf.* [10, Example 4.4]) The variety PCSL of pseudocomplemented semilattices is a binary discriminator variety with binary discriminator term $x \wedge y^*$. Thus for any $\mathbf{A} \in \mathsf{PCSL}$ the polynomial reduct $\langle A;\ \backslash, 0\rangle$ is an implicative BCS-algebra, where $a\backslash b := a \wedge b^*$ for any $a, b \in A$.

Proof. Let $\mathbf{3}$ be the 3-element chain (considered as a pseudocomplemented semilattice). An easy inspection of the induced algebra $\langle 3;\ \backslash, 0\rangle$ shows it to be a flat implicative BCS-algebra. Since $\mathbf{3}$ generates PCSL as a variety (by Theorem 1.3.8), we infer that the variety of pseudocomplemented semilattices is a binary discriminator variety with binary discriminator term $x \wedge y^*$. By Example 2.3.11 it follows that the polynomial reduct $\langle A;\ \backslash, 0\rangle$ of any $\mathbf{A} \in \mathsf{PCSL}$ is an implicative BCS-algebra.

This last may also be observed directly in an easy and instructive proof. Let $\mathbf{A} \in \mathsf{PCSL}$ and let $a, b, c \in A$. We verify that the defining identities for implicative BCS-algebras are satisfied. For (2.47) we have:

$$\begin{aligned} a\backslash a &= a \wedge a^* \\ &= 0 && \text{by (1.8).} \end{aligned}$$

For (2.48) we have:

$$\begin{aligned} (a\backslash b)\backslash c &= (a \wedge b^*) \wedge c^* \\ &= (a \wedge c^*) \wedge b^* \\ &= (a\backslash c)\backslash b. \end{aligned}$$

For (2.49) we have:

$$\begin{aligned} (a\backslash c)\backslash(b\backslash c) &= (a \wedge c^*) \wedge (b \wedge c^*)^* \\ &= a \wedge \left(c^* \wedge (b \wedge c^*)^*\right) \\ &= a \wedge \left(c^* \wedge (c^* \wedge b)^*\right) \\ &= a \wedge (c^* \wedge b^*) && \text{by (1.9)} \\ &= (a \wedge b^*) \wedge c^* \\ &= (a\backslash b)\backslash c. \end{aligned}$$

For (2.50) we have:

$$\begin{aligned}
a\backslash(b\backslash a) &= a \wedge (b \wedge a^*)^* \\
&= a \wedge \big(a \wedge (b \wedge a^*)\big)^* && \text{by (1.9)} \\
&= a \wedge \big((a \wedge a^*) \wedge b\big)^* \\
&= a \wedge (0 \wedge b)^* && \text{by (1.8)} \\
&= a \wedge 0^* \\
&= a && \text{by (1.10).}
\end{aligned}$$

Thus $\langle A;\ \backslash, 0\rangle \models$ (2.47)–(2.50), and the proof is complete. ∎

Implicative BCS-algebras distinct from flat algebras (but which nonetheless arise from such structures) also occur quite naturally in a fairly wide class of varieties occurring in algebraic logic, as the following example shows.

Example 2.3.13. (*cf.* [65, p. 290]; *cf.* [19, Theorem 4.7]) Let V be a fixedpoint discriminator variety with $\mathbf{0}$. Let $\mathsf{K} \subseteq \mathsf{V}$ be a class of fixedpoint discriminator algebras generating V as a variety and suppose that $\mathbf{0}^{\mathbf{A}}$ is the discriminating element on any $\mathbf{A} \in \mathsf{K}$. Let $f(x, y, z)$ be a discriminator term for V and let $x\backslash y := f(\mathbf{0}, y, x)$. Let $\mathbf{A} \in \mathsf{K}$. By definition of the fixedpoint discriminator,

$$\begin{aligned}
a\backslash^{\mathbf{A}} b &= f^{\mathbf{A}}(0, b, a) \\
&= \begin{cases} a & \text{if } b = 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

for any $a, b \in A$, whence $\langle A;\ \backslash^{\mathbf{A}}, 0\rangle$ is a flat implicative BCS-algebra by Example 2.3.8. Since the identities satisfied by V are precisely those satisfied by K it follows that any $\mathbf{B} \in \mathsf{V}$ has a canonical implicative BCS-algebra polynomial reduct $\langle B;\ \backslash^{\mathbf{B}}, 0\rangle$. ∎

To further illustrate Example 2.3.13 we give a concrete application.

Example 2.3.14. Let $\mathsf{ce_nBCK}$, $n \in \omega$, denote the intersection of the varieties cBCK and $\mathsf{e_nBCK}$. By the remarks of §1.6.13 the classes $\mathsf{ce_nBCK}$ are varieties that form an infinite strictly increasing chain. For each $n \in \omega$, let $\mathsf{lce_nBCK}$ denote the subclass of $\mathsf{ce_nBCK}$ satisfying the identity:

$$(x \dot{-} y) \cap (y \dot{-} x) \approx 0.$$

Concerning the classes $\mathsf{lce_nBCK}$, $n \in \omega$, the following assertions hold:

1. For each $n \in \omega$, $\mathsf{lce_nBCK}$ is a fixedpoint discriminator variety with fixedpoint discriminator term $f(x, y, z) := \big(z \dot{-} (x \dot{-} y)^n\big) \dot{-} (y \dot{-} x)^n$;

2. For each $n \in \omega$, $\mathsf{lce_nBCK}$ is generated by a single fixedpoint discriminator algebra $\mathbf{A}_n$, for which $0^{\mathbf{A}_n}$ is the discriminating element.

Thus for any $\mathbf{A} \in \mathsf{lce_nBCK}$, $n \in \omega$, the polynomial reduct $\langle A;\ \backslash, 0\rangle$ is an implicative BCS-algebra, where $a \backslash b := f^{\mathbf{A}}(0, b, a) = ab^n$ for any $a, b \in A$.

Proof. For (1), by Theorem 1.6.15(1) each variety $\mathsf{ce_nBCK}$, $n \in \omega$, has a commutative TD term $f(x, y, z) := \big(z \dot{-} (x \dot{-} y)^n\big) \dot{-} (y \dot{-} x)^n$, just because it is a subvariety of $\mathsf{e_nBCK}$. Since each $\mathsf{ce_nBCK}$ is semisimple (by Cornish [68, Corollary 3.2]), from Theorem 1.5.12 we have that $\mathsf{ce_nBCK}$ is a fixedpoint discriminator variety with fixedpoint discriminator term $f(x, y, z)$. By Cornish [70, Theorem 5.7; Section 3.6] the result now follows for $\mathsf{lce_nBCK}$, $n \in \omega$.

For (2) let A_ω be a chain of order type ω, say $0 < a_0 < a_1 \ldots < a_c < \ldots$. Let $\mathbf{A}_\omega := \langle A_\omega;\ \dot{-}^{\mathbf{A}_\omega}, 0\rangle$ where $a_i \dot{-}^{\mathbf{A}_\omega} a_j := a_{\max\{i-j,0\}}$ for any $a_i, a_j \in A_\omega$. Let $\mathbf{A}_k$ denote the subalgebra $(a_k]$ of $\mathbf{A}_\omega$. By [68, Theorem 3.5], $\mathsf{lce_kBCK} = \mathbf{V}(\mathbf{A}_k)$. Because $\mathbf{A}_k$ is simple (by Cornish [70, Section 3.6]), from Theorem 1.5.12 it follows that $\mathbf{A}_k$ is a fixedpoint discriminator algebra. Moreover, from the description of $\mathbf{A}_k$ we may infer additionally that $0^{\mathbf{A}_k}$ is the discriminating element. From these remarks it follows that, for each $n \in \omega$, the variety $\mathsf{lce_nBCK}$ is generated as a variety by the fixedpoint discriminator algebra $\mathbf{A}_n$, for which $0^{\mathbf{A}_n}$ is the discriminating element.

For the final claim, let $\mathbf{A} \in \mathsf{lce_nBCK}$, $n \in \omega$, and let $a, b \in A$. By Example 2.3.13 the polynomial reduct $\langle A;\ \backslash, 0\rangle$ is an implicative BCS-algebra.

Moreover,

$$\begin{aligned} a\backslash b &= f^{\mathbf{A}}(0,b,a) \\ &= \big(a(0b)^n\big)(b0)^n \\ &= (a0^n)b^n && \text{by (1.4), (1.7)} \\ &= ab^n && \text{by [68, Lemma 1.1(ii)]} \end{aligned}$$

and the result follows. ∎

Remark 2.3.15. The proof of Example 2.3.14(1) also shows that, for each $n \in \omega$, the class $\mathsf{ce_nBCK}$ is a fixedpoint discriminator variety with fixedpoint discriminator term $f(x,y,z) := \big(z \dot{-} (x \dot{-} y)^n\big) \dot{-} (y \dot{-} x)^n$. However, for no $\mathsf{ce_nBCK}$, $n \in \omega$, does any subclass of $\{\mathbf{A}_k : k \in \omega\}$ (where the $\mathbf{A}_k$ are as in the proof of Example 2.3.14(2)) generate $\mathsf{ce_nBCK}$ as a variety: see Cornish [70, Lemma 5.6] (and also Komori [137, Theorem 3.13] and Iséki and Tanaka [125, Example 5]). Therefore we may not conclude from the proof of Example 2.3.14(2) that each $\mathsf{ce_nBCK}$, $n \in \omega$, is generated as a variety by a class $\mathsf{K} \subseteq \mathsf{ce_nBCK}$ of fixedpoint discriminator algebras such that $0^{\mathbf{A}}$ is the discriminating element on any $\mathbf{A} \in \mathsf{K}$. However, see Example 3.2.30 in the sequel. ∎

The discussion of Example 2.3.14 can potentially be placed in the wider context of the classes $\mathsf{e_nBCK}$, $n \in \omega$. Let $\mathbf{A}$ be a positive implicative BCK-algebra and let $a, b \in A$. By Guzmán [105, Proposition 3.2(c)], the underlying BCK partial ordering $\leq_1$ on A is a meet semilattice ordering with greatest lower bound $a \sqcap_1 b := \big(a(ab)\big)(ba)$ such that every principal $\leq_1$-order ideal of $\langle A; \leq_1\rangle$ is a Boolean lattice. (In other words, $\langle A; \sqcap_1\rangle$ is semi-Boolean.) Since any BCK-algebra satisfying (E_n) also satisfies:

$$\big(x \dot{-} (x \dot{-} y)^n\big) \dot{-} (y \dot{-} x)^n \approx \big(y \dot{-} (y \dot{-} x)^n\big) \dot{-} (x \dot{-} y)^n$$

by Lemma 1.3 of Cornish [68], it is natural to pose the following problem.

Problem 2.3.16. Let $\mathbf{A} \in \mathsf{e_nBCK}$, $n \in \omega$. For any $a, b \in A$, let $a \sqcap_n b := \big(a(ab)^n\big)(ba)^n$. Is the derived algebra $\langle A; \sqcap_n\rangle$ always a meet semilattice? If so, let $\leq_n$ denote the underlying partial order of $\langle A;\ \cap_n\rangle$. Does $\langle A;\ \leq_n\rangle$ support in any sense a semi-Boolean or other 'locally Boolean' structure? ∎

Let **A** be a positive implicative BCK-algebra. Because the underlying $\leq_1$-ordering of **A** is semi-Boolean, **A** has an implicative BCK-algebra polynomial reduct $\langle A;\ /, 0\rangle$, where $a/b := a(a \cap_1 b)^1$ for any $a, b \in A$; for details, see Theorem 3.3 of Guzmán [105]. This remark prompts the following problem.

Problem 2.3.17. Let $\mathbf{A} \in \mathsf{e_nBCK}$, $n \in \omega$. Let $a\backslash b := a(a \cap_n b)^n$ for any $a, b \in A$. Is the induced algebra $\langle A;\ \backslash, 0\rangle$ always an implicative BCS-algebra? ∎

It is known and quite easy to see [68, p. 419] that any finite BCK-algebra satisfies (E_n) for some $n \in \omega$. This observation in conjunction with preceding remarks suggests the problem below.

Problem 2.3.18. Does every finite BCK-algebra have an implicative BCS-algebra polynomial reduct? ∎

In view of the preceding discussion, binary and 'pointed' fixedpoint discriminator varieties may not exhaust those classes of algebras in which implicative BCS-algebras arise naturally. We return to this point in Example 3.2.20 in the sequel.

2.3.19. Left Handed Locally Boolean Bands. Theorem 1.6.21 shows that the derived semilattice meet $\cap$ plays a fundamental role in the theory of implicative BCK-algebras, even to the point of determining some second-order properties (see Cornish [64, Section 3]). This observation, in conjunction with Proposition 2.3.5 and Proposition 2.2.31, prompts the study of the role played by the pre-BCK meet $\sqcap$ (hereafter, *implicative BCS meet* $\sqcap$) in implicative BCS-algebras.

Proposition 2.3.20. *The variety of implicative BCS-algebras satisfies the following identities:*

$$(x \sqcap y)\backslash(x \sqcap z) \approx (x\backslash z)\backslash(x\backslash y) \tag{2.57}$$

$$(x \sqcap y)\backslash z \approx (x\backslash z)\backslash(x\backslash y) \tag{2.58}$$

$$(x\backslash z) \sqcap y \approx (x\backslash z)\backslash(x\backslash y) \tag{2.59}$$

$$(x\backslash z) \sqcap (y\backslash z) \approx (x\backslash z)\backslash(x\backslash y) \tag{2.60}$$

$$x \sqcap (y\backslash z) \approx (x\backslash z)\backslash(x\backslash y) \tag{2.61}$$

$$(x \sqcap y)\backslash(z \sqcap y) \approx (x\backslash z)\backslash(x\backslash y). \tag{2.62}$$

Proof. Let **A** be an implicative BCS-algebra and let $a, b, c \in A$. For (2.57), we have:

$$\begin{aligned}
(a \sqcap b)(a \sqcap c) &= \big(a(ab)\big)\big(a(ac)\big) \\
&= \big(a(a(ac))\big)(ab) && \text{by (2.48)} \\
&= (ac)(ab) && \text{by (2.56).}
\end{aligned}$$

For (2.58), we have:

$$\begin{aligned}
(a \sqcap b)c &= \big(a(ab)\big)c \\
&= (ac)(ab) && \text{by (2.48).}
\end{aligned}$$

For (2.59), it is sufficient by (2.58) to show $(ac) \sqcap b = (a \sqcap b)c$. We have:

$$\begin{aligned}
(ac) \sqcap b &= (ac)\big((ac)b\big) \\
&= (ac)\big((ab)c\big) && \text{by (2.48)} \\
&= \big(a(ab)\big)c && \text{by (2.49)} \\
&= (a \sqcap b)c.
\end{aligned}$$

For (2.60), it is sufficient by (2.59) to show $(ac) \sqcap (bc) = (ac) \sqcap b$. We have:

$$\begin{aligned}
(ac) \sqcap (bc) &= (ac)\big((ac)(bc)\big) \\
&= (ac)\big((ab)c\big) && \text{by (2.49)} \\
&= (ac)\big((ac)b\big) && \text{by (2.48)} \\
&= (ac) \sqcap b.
\end{aligned}$$

For (2.61), it is sufficient by (2.60) to show $a \sqcap (bc) = (ac) \sqcap (bc)$. We have:

$$
\begin{aligned}
a \sqcap (bc) &= a\big(a(bc)\big) && \\
&= \big(a(ca)\big)\big(a(bc)\big) && \text{by (2.50)}\\
&= \big(a(a(bc))\big)(ca) && \text{by (2.48)}\\
&= \big(a(a(bc))\big)\big(c(a(bc))\big) && \text{by (2.53)}\\
&= (ac)\big(a(bc)\big) && \text{by (2.49)}\\
&= \big(a(a(bc))\big)c && \text{by (2.48)}\\
&= (ac)\big((a(bc))c\big) && \text{by (2.49)}\\
&= (ac)\big((ac)(bc)\big) && \text{by (2.48)}\\
&= (ac) \sqcap (bc).
\end{aligned}
$$

For (2.62), it is sufficient by (2.61) to show $(a \sqcap b)(c \sqcap b) = a \sqcap (bc)$. So let $\alpha := a$, $\beta := b$ and $\gamma := c \sqcap b$. We have:

$$
\begin{aligned}
(a \sqcap b)(c \sqcap b) &= (\alpha \sqcap \beta)\gamma && \\
&= (\alpha\gamma)(\alpha\beta) && \text{by (2.58)}\\
&= \alpha \sqcap (\beta\gamma) && \text{by (2.61)}\\
&= a \sqcap \big(b(c \sqcap b)\big) && \\
&= a \sqcap \big(b(c(cb))\big) && \\
&= a \sqcap (bc) && \text{by (2.53).}
\end{aligned}
$$

■

Proposition 2.3.21. *The variety of implicative BCS-algebras satisfies the following identities:*

$$x \sqcap (y \sqcap z) \approx (x \sqcap y) \sqcap z \tag{2.63}$$

$$x \sqcap (y \sqcap z) \approx x \sqcap (z \sqcap y). \tag{2.64}$$

Proof. Let $\mathbf{A}$ be an implicative BCS-algebra and let $a, b, c \in A$. For (2.63), we have:

$$\begin{aligned}(a \sqcap b) \sqcap c &= \big(a(ab)\big)\big((a(ab))c\big)\\ &= \big(a(ab)\big)\big((a \sqcap b)c\big)\\ &= \big(a(ab)\big)\big((ac)(ab)\big) && \text{by (2.58).}\end{aligned}$$

Put $\alpha := a$, $\beta := ab$ and $\gamma := (ac)(ab)$. We have:

$$\begin{aligned}\big(a(ab)\big)\big((ac)(ab)\big) &= (\alpha\beta)\gamma\\ &= (\alpha\gamma)\beta && \text{by (2.48)}\\ &= \big(a((ac)(ab))\big)(ab).\end{aligned}$$

Put $\alpha := a$, $\beta := b$ and $\gamma := c$. We have:

$$\begin{aligned}\big(a((ac)(ab))\big)(ab) &= \big(a((\alpha\gamma)(\alpha\beta))\big)(ab)\\ &= \big(a(\alpha \sqcap (\beta\gamma))\big)(ab) && \text{by (2.61)}\\ &= \big(a(a \sqcap (bc))\big)(ab)\\ &= \big(a(a(a(bc)))\big)(ab).\end{aligned}$$

Put $\alpha := a$ and $\beta := bc$. We have:

$$\begin{aligned}\big(a(a(a(bc)))\big)(ab) &= \big(\alpha(\alpha(\alpha\beta))\big)(ab)\\ &= (\alpha\beta)(ab) && \text{by (2.56)}\\ &= \big(a(bc)\big)(ab).\end{aligned}$$

Put $\alpha := a$, $\beta := b$ and $\gamma := bc$. We have:

$$\begin{aligned}\big(a(bc)\big)(ab) &= (\alpha\gamma)(\alpha\beta)\\ &= \alpha \sqcap (\beta\gamma) && \text{by (2.61)}\\ &= a \sqcap \big(b(bc)\big)\\ &= a \sqcap (b \sqcap c).\end{aligned}$$

For (2.64), put $\alpha := a$, $\beta := ab$ and $\gamma := c$. We have:

$$\begin{aligned}
a \sqcap (b \sqcap c) &= (a \sqcap b) \sqcap c \\
&= \big(a(ab)\big)\big((a(ab))c\big) \\
&= \big(a(ab)\big)\big((\alpha\beta)\gamma\big) \\
&= \big(a(ab)\big)\big((\alpha\gamma)\beta\big) && \text{by (2.48)} \\
&= \big(a(ab)\big)\big((ac)(ab)\big).
\end{aligned}$$

Put $\alpha := a$, $\beta := ab$ and $\gamma := ac$. We have:

$$\begin{aligned}
\big(a(ab)\big)\big((ac)(ab)\big) &= (\alpha\beta)(\gamma\beta) \\
&= (\alpha\gamma)(\beta\gamma) && \text{by (2.55)} \\
&= \big(a(ac)\big)\big((ab)(ac)\big).
\end{aligned}$$

Put $\alpha := a$, $\beta := b$ and $\gamma := ac$. We have:

$$\begin{aligned}
\big(a(ac)\big)\big((ab)(ac)\big) &= \big(a(ac)\big)\big((\alpha\beta)\gamma\big) \\
&= \big(a(ac)\big)\big(\alpha\gamma)\beta\big) && \text{by (2.48)} \\
&= \big(a(ac)\big)\big((a(ac))b\big) \\
&= (a \sqcap c) \sqcap b \\
&= a \sqcap (c \sqcap b).
\end{aligned}$$

■

Recall from [206, p. 295] that a *restrictive bisemigroup* is an algebra $\langle A;\ \triangle_L, \triangle_R\rangle$ of type $\langle 2, 2\rangle$ such that: (i) the reduct $\langle A;\ \triangle_L\rangle$ is a left normal band; (ii) the reduct $\langle A;\ \triangle_R\rangle$ is a right normal band; and (iii) the following associativity condition is satisfied:

$$(x\triangle_R y)\triangle_L z \approx x\triangle_R(y\triangle_L z). \tag{2.65}$$

Restrictive bisemigroups were introduced by Schein in [206] in connection with the theory of binary relations [228]. Let A and B be sets and let ρ and σ be binary relations from A into B. The *restrictive composition of the first kind* of ρ and σ is the binary relation $\rhd\ \subseteq A \times B$ defined by [207, p. 309]:

$$\rho \rhd \sigma := \big(\pi_1(\rho) \times B\big) \cap \sigma$$

where π_1 is the first projection map. Analogously the *restrictive composition of the second kind* of ρ and σ is the binary relation $\lhd\ \subseteq A \times B$ defined by [207, p. 309]:

$$\rho \lhd \sigma := \rho \cap \big(A \times \pi_2(\sigma)\big)$$

where π_2 is the second projection map. Upon denoting the set of all one-to-one binary relations from A to B by $\mathbb{R}(A, B)$, the structure $\big\langle \mathbb{R}(A, B);\ \rhd, \lhd\big\rangle$ is a restrictive bisemigroup, the restrictive subbisemigroups of which are known as *restrictive bisemigroups of invertible mappings.*

Corollary 2.3.22. *For any implicative BCS-algebra* $\mathbf{A}$*, the following assertions hold:*

1. $\mathbf{A}$ *has a left normal band with zero polynomial reduct* $\langle A;\ \sqcap_L, 0\rangle$*, where* $a \sqcap_L b := a \sqcap b$ *for any* $a, b \in A$*;*
2. $\mathbf{A}$ *has a right normal band with zero polynomial reduct* $\langle A;\ \sqcap_R, 0\rangle$*, where* $a \sqcap_R b := b \sqcap a$ *for any* $a, b \in A$*;*
3. $\mathbf{A}$ *has a restrictive bisemigroup polynomial reduct* $\langle A;\ \sqcap_L, \sqcap_R\rangle$*;*
4. *The restrictive bisemigroup polynomial reduct* $\langle A;\ \sqcap_L, \sqcap_R\rangle$ *of* (3) *is isomorphic to a restrictive bisemigroup of invertible mappings.*

Proof. Let $\mathbf{A}$ be an implicative BCS-algebra and let $a \sqcap_L b := a \sqcap b$ for any $a, b \in A$. For (1), just note that the identities (2.63)–(2.64) in conjunction with the identities (2.21)–(2.23) of Lemma 2.1.42 assert that the polynomial reduct $\langle A;\ \sqcap_L, 0\rangle$ is a left normal band with zero. For (2), let $a \sqcap_R b := b \sqcap a$ for any $a, b \in A$. An easy sequence of checks shows that the derived algebra $\langle A;\ \sqcap_R, 0\rangle$ is a right normal band with zero. For (3), recall from Schein [207, p. 313] that any left normal band $\langle A;\ \triangle_L\rangle$ induces a restrictive bisemigroup

$\langle A;\ \triangle_L, \triangle_R\rangle$ upon defining an inverse operation $\triangle_R$ by $a\triangle_R b := b\triangle_L a$ for any $a, b \in A$. For (4), just note by Schein [207, Theorem 5; pp. 313–314] that any restrictive bisemigroup induced by a left normal band in the above manner is isomorphic to a restrictive bisemigroup of invertible mappings. ∎

Remark 2.3.23. It is entirely arbitrary whether we consider in general the left normal band with zero polynomial reduct or the right normal band with zero polynomial reduct of an implicative BCS-algebra $\mathbf{A}$. In the sequel we will exclusively consider only the left normal band with zero polynomial reduct, denoting it simply $\langle A;\ \sqcap, 0\rangle$, bearing in mind that all results obtained extend to the right normal case. ∎

Because of Corollary 2.3.22(1), implicative BCS-algebras enjoy equationally definable properties of left normal bands with zero. Since the Green's quasiorderings $\preceq_{\mathcal{L}}$, $\preceq_{\mathcal{R}}$ and $\preceq_{\mathcal{D}}$ and the natural partial ordering $\leq_{\mathcal{H}}$ are term definable on any band, we have that they are also definable on any implicative BCS-algebra.

Lemma 2.3.24. *Let* $\mathbf{A}$ *be an implicative BCS-algebra and let* $\preceq$ *be the binary relation defined on* A *by* $a \preceq b$ *iff* $ab = 0$. *For any* $a, b \in A$, $a \preceq b$ *iff* $a \preceq_{\mathcal{D}}^{\langle A;\, \sqcap, 0\rangle} b$ *iff* $a \preceq_{\mathcal{L}}^{\langle A;\, \sqcap, 0\rangle} b$ *iff* $a \sqcap b = a$, *and so* $\langle A;\ \preceq\rangle$ *is a quasiordered set with least element* 0. *Moreover, the relation* $\preceq$ *satisfies the following conditions for any* $a, b, c \in A$*:*

1. *If* $a \preceq b$ *then* $cb \preceq ca$;
2. *If* $a \preceq b$ *then* $ac \preceq bc$;
3. *If* $a \preceq b$ *then* $a \sqcap c \preceq b \sqcap c$;
4. *If* $a \preceq b$ *then* $c \sqcap a \preceq c \sqcap b$.

Proof. For the first assertion, $\langle A;\ \preceq\rangle$ is a quasiordered set with 0 as least element (by Lemma 2.1.12) for which Lemma 2.1.42(1) ensures $ab = 0$ iff $a \sqcap b = a$, and for which the remarks of §1.3.11 and §1.3.15 ensure $a \sqcap b = a$ iff $a \preceq_{\mathcal{L}}^{\langle A;\, \sqcap, 0\rangle}$ iff $a \preceq_{\mathcal{D}}^{\langle A;\, \sqcap, 0\rangle} b$. It remains to prove Items (1)–(4). Items (1) and (2) have already been established in Lemma 2.1.12(1), (2) respectively.

Because $\preceq_{\mathcal{D}}^{\langle A;\, \sqcap, 0\rangle}$ and $\preceq$ coincide, Items (3) and (4) follow from a standard result of semigroup theory asserting that the $\mathcal{D}$-quasiorder on an arbitrary band is compatible with band multiplication: see Schein [208, Proposition 1]. ∎

Proposition 2.3.25. *Let* **A** *be an implicative BCS-algebra and let* $\leq$ *be the binary relation defined on* A *by* $a \leq b$ *iff* $a \sqcap b = a = b \sqcap a$. *For any* $a, b \in A$, $a \leq_{\mathcal{H}}^{\langle A;\, \sqcap, 0\rangle}$ *iff* $a \preceq_{\mathcal{R}}^{\langle A;\, \sqcap, 0\rangle} b$ *iff* $a \leq b$, $\langle A;\ \leq\rangle$ *is a partially ordered set with least element* 0, *and* $\preceq$ *is admissible with respect to* $\leq$. *Moreover,* $\leq$ *satisfies the following conditions for any* $a, b, c \in A$:

1. *If* $a \leq b$ *then* $cb \leq ca$;
2. *If* $a \leq b$ *then* $ac \leq bc$.
3. *If* $a \leq b$ *then* $a \sqcap c \leq b \sqcap c$;
4. *If* $a \leq b$ *then* $c \sqcap a \leq c \sqcap b$;

Proof. For the first assertion, $a \leq b$ iff $a \sqcap b = a = b \sqcap a$ iff $a \leq_{\mathcal{H}}^{\langle A;\, \sqcap, 0\rangle} b$, which implies $\langle A;\ \leq\rangle$ is a partially ordered set by the remarks of §1.3.11. Moreover, $0 \leq a$ for any $a \in A$ and $\preceq$ is admissible with respect to $\leq$ by Lemma 2.1.42. It remains to establish $a \leq_{\mathcal{H}}^{\langle A;\, \sqcap, 0\rangle} b$ iff $a \preceq_{\mathcal{R}}^{\langle A;\, \sqcap, 0\rangle} b$. It is clear by definition that $a \leq_{\mathcal{H}}^{\langle A;\, \sqcap, 0\rangle} b$ implies $a \preceq_{\mathcal{R}}^{\langle A;\, \sqcap, 0\rangle} b$. For the converse, assume $a \preceq_{\mathcal{R}}^{\langle A;\, \sqcap, 0\rangle} b$; it is sufficient to show $a \preceq_{\mathcal{L}}^{\langle A;\, \sqcap, 0\rangle} b$. By left normality, $a \sqcap b = a \sqcap b \sqcap a = a \sqcap (b \sqcap a) = a \sqcap a = a$, so $a \preceq_{\mathcal{L}}^{\langle A;\, \sqcap, 0\rangle} b$.

It remains to prove Items (1)–(4). Items (3) and (4) follow from left normality and Lemma 1.3.16. For (1), assume $a \leq b$. Then $a \preceq b$ and $cb \preceq ca$ by Lemma 2.3.24(1), whence:

$$\begin{aligned} (cb) \sqcap (ca) &= (cb)\big((cb)(ca)\big) \\ &= (cb)0 \\ &= cb && \text{by (2.6).} \end{aligned}$$

For the opposite inclusion, assume $a \leq b$. We have:

$$\begin{aligned}
(ca) \sqcap (cb) &= \big((ca)b\big)\big((ca)c\big) && \text{by (2.61)} \\
&= \big((ca)b\big)0 && \text{by Lemma 2.1.12(3)} \\
&= (ca)b && \text{by (2.6)} \\
&= (cb)a && \text{by (2.48)} \\
&= (cb)\big(b(ba)\big) && \text{since } b \sqcap a = a \\
&= cb && \text{by (2.54).}
\end{aligned}$$

Thus $cb \leq ca$, and (1) is proved. For (2), assume $a \leq b$. Then $a \preceq b$ and $ac \preceq bc$ by Lemma 2.3.24(2), whence:

$$\begin{aligned}
(ac) \sqcap (bc) &= (ac)\big((ac)(bc)\big) \\
&= (ac)0 \\
&= ac && \text{by (2.6).}
\end{aligned}$$

For the opposite inclusion, assume $a \leq b$. We have:

$$\begin{aligned}
(bc) \sqcap (ac) &= \big((bc)c\big)\big((bc)a\big) && \text{by (2.61)} \\
&= (bc)\big((bc)a\big) && \text{by (2.51)} \\
&= (bc) \sqcap a \\
&= (bc)(ba) && \text{by (2.59)} \\
&= \big(b(ba)\big)c && \text{by (2.48)} \\
&= ac && \text{since } b \sqcap a = a.
\end{aligned}$$

Thus $ac \leq bc$, and the proof is complete. ∎

By Lemma 2.3.24 and Proposition 2.3.25, small finite implicative BCS-algebras may be depicted graphically using the Hasse diagramming conventions of §1.4.18. We provide a concrete example by way of illustration.

Example 2.3.26. Using Lemma 2.3.3 and Proposition 2.3.4 the free implicative BCS-algebra $\mathbf{F}(\overline{x}, \overline{y})$ on two free generators $\overline{x}, \overline{y}$ may be determined simply by computing all products involving $\overline{x}$ and $\overline{y}$. The resulting operation tables for $\mathbf{F}(\overline{x},\overline{y})$ and its left normal band with zero polynomial reduct are shown below (where for simplicity of notation in the tables, we denote implicative BCS difference by juxtaposition, and also write x and y for $\overline{x}$ and $\overline{y}$ respectively (and similarly for products of $\overline{x}$ and $\overline{y}$)).

$\setminus^{\mathbf{F}(\overline{x},\overline{y})}$	0	x	y	xy	yx	$x \sqcap y$	$y \sqcap x$
0	0	0	0	0	0	0	0
x	x	0	xy	$x \sqcap y$	x	xy	xy
y	y	yx	0	y	$y \sqcap x$	yx	yx
xy	xy	0	xy	0	xy	xy	xy
yx	yx	yx	0	yx	0	yx	yx
$x \sqcap y$	$x \sqcap y$	0	0	$x \sqcap y$	$x \sqcap y$	0	0
$y \sqcap x$	$y \sqcap x$	0	0	$y \sqcap x$	$y \sqcap x$	0	0

$\sqcap^{\mathbf{F}(\overline{x},\overline{y})}$	0	x	y	xy	yx	$x \sqcap y$	$y \sqcap x$
0	0	0	0	0	0	0	0
x	0	x	$x \sqcap y$	xy	0	$x \sqcap y$	$x \sqcap y$
y	0	$y \sqcap x$	y	0	yx	$y \sqcap x$	$y \sqcap x$
xy	0	xy	0	xy	0	0	0
yx	0	0	yx	0	yx	0	0
$x \sqcap y$	0	$x \sqcap y$	$x \sqcap y$	0	0	$x \sqcap y$	$x \sqcap y$
$y \sqcap x$	0	$y \sqcap x$	$y \sqcap x$	0	0	$y \sqcap x$	$y \sqcap x$

The Hasse diagram of $\mathbf{F}(\overline{x},\overline{y})$ is shown in Figure 2.2. For notational reasons, implicative BCS difference is denoted by juxtaposition in the figure; also the free generators $\overline{x}$, $\overline{y}$ are denoted simply by x, y respectively (and like remarks apply to products of $\overline{x}$, $\overline{y}$). ∎

Let $\mathbf{A}$ be a pre-BCK-algebra with an underlying partial order $\leq$ (which need not necessarily coincide with $\preceq$). If $ab \leq a$ for any $a, b \in A$ we say $\leq$ *respects* pre-BCK difference. In Theorem 2.3.29 below we show that the existence of a left normal band with zero polynomial reduct $\langle \sqcap, 0 \rangle$ whose underlying natural band partial order respects implicative pre-BCK difference distinguishes the

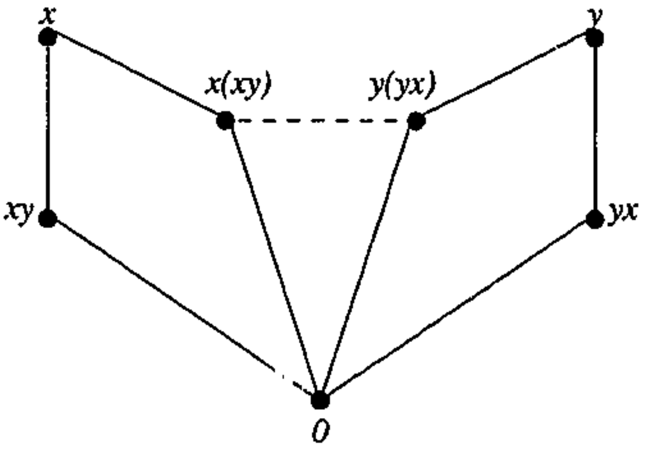

Figure 2.2. The iBCS-free algebra on two free generators $\overline{x}, \overline{y}$.

implicative BCS-algebras among the implicative pre-BCK-algebras. But first, two auxilliary lemmas; Theorem 2.3.29 shows the hypotheses of the second of these lemmas are not artificial.

Lemma 2.3.27. *The variety of (positive) implicative pre-BCK-algebras satisfies the identity:*

$$\big((x \dot{-} y) \dot{-} ((x \dot{-} y) \dot{-} z)\big) \dot{-} (z \dot{-} y) \approx \mathbf{0}. \tag{2.66}$$

Proof. Let $\mathbf{A}$ be a (positive) implicative pre-BCK-algebra and let $a, b, c \in A$. We have $ab \preceq (ab)b$, so $(ab)\big((ab)c\big) \preceq \big((ab)b\big)\big((ab)c\big)$ (by Lemma 2.1.12(2)), so $\big((ab)((ab)c)\big)(cb) \preceq \big(((ab)b)((ab)c)\big)(cb)$ (by Lemma 2.1.12(2)) $= 0$ by (2.1), so $\big((ab)((ab)c)\big)(cb) = 0$ by Lemma 2.1.12. ∎

Lemma 2.3.28. *Let* $\mathbf{A}$ *be an implicative pre-BCK-algebra such that the polynomial reduct* $\langle A;\ \sqcap, 0\rangle$ *is a left normal band with zero. Suppose further that the underlying natural band partial order* $\leq_{\mathcal{H}}^{\langle A;\,\sqcap,0\rangle}$ *on the polynomial reduct* $\langle A;\ \sqcap, 0\rangle$ *respects implicative pre-BCK difference; that is, for any* $a, b \in A$,

$$ab \leq_{\mathcal{H}}^{\langle A;\,\sqcap,0\rangle} a. \tag{2.67}$$

Then $\mathbf{A}$ *satisfies the following identities:*

$$(x \dot{-} y) \sqcap (z \dot{-} y) \approx (x \dot{-} y) \sqcap z \tag{2.68}$$

$$x \sqcap (y \dot{-} z) \approx (x \dot{-} z) \sqcap y \tag{2.69}$$

$$(x \dot{-} y) \dot{-} z \approx (x \dot{-} z) \sqcap (x \dot{-} y). \tag{2.70}$$

Proof. Let $\mathbf{A}$ be an implicative pre-BCK-algebra for which the polynomial reduct $\langle A;\ \sqcap, 0\rangle$ is a left normal band with zero and suppose further that $\mathbf{A}$ satisfies (2.67) for any $a, b \in A$. Let $a, b, c \in A$. For (2.68) we have:

$$\begin{aligned}
(ab) \sqcap (cb) &= (ab) \sqcap \big(c \sqcap (cb)\big) && \text{by (2.67)}\\
&= \big((ab) \sqcap c\big) \sqcap (cb) &&\\
&= \big((ab) \sqcap c\big)\big(((ab) \sqcap c)(cb)\big) &&\\
&= \big((ab) \sqcap c\big)\big(((ab)((ab)c))(cb)\big) &&\\
&= \big((ab) \sqcap c\big)0 && \text{by (2.66)}\\
&= (ab) \sqcap c && \text{by (2.6).}
\end{aligned}$$

For (2.69), we have:

$$\begin{aligned}
a \sqcap (bc) &= a \sqcap \big((bc) \sqcap a\big) && \text{by left normality}\\
&= a \sqcap \big((bc) \sqcap (ac)\big) && \text{by (2.68)}\\
&= a \sqcap \big((ac) \sqcap (bc)\big) && \text{by left normality}\\
&= \big(a \sqcap (ac)\big) \sqcap (bc) &&\\
&= (ac) \sqcap (bc) && \text{by (2.67)}\\
&= (ac) \sqcap b && \text{by (2.68).}
\end{aligned}$$

For (2.70), we have:

$$\begin{aligned}
(ab)c &= (ab) \sqcap \big((ab)c\big) && \text{by (2.67)}\\
&= \big(a \sqcap (ab)\big) \sqcap \big((ab)c\big) && \text{by (2.67)}\\
&= a \sqcap \big((ab) \sqcap ((ab)c)\big) &&\\
&= a \sqcap \big((ab)c\big) && \text{by (2.67)}\\
&= (ac) \sqcap (ab) && \text{by (2.69).}
\end{aligned}$$

∎

Theorem 2.3.29. *For any implicative pre-BCK-algebra* **A** *the following are equivalent:*

1. **A** *is an implicative BCS-algebra;*

2. *The polynomial reduct* $\langle A;\ \sqcap, 0\rangle$ *is a left normal band with zero, and* any one *of the following conditions is satisfied:*

   (a) *Right [left] multiplication by a fixed element of* A *is isotone [antitone] with respect to the underlying natural band partial order* $\leq_{\mathcal{H}}^{\langle A;\,\sqcap,0\rangle}$. *That is, for any* $a, b, c \in A$,

      i. *If* $a \leq_{\mathcal{H}}^{\langle A;\,\sqcap,0\rangle} b$ *then* $cb \leq_{\mathcal{H}}^{\langle A;\,\sqcap,0\rangle} ca$;

      ii. *If* $a \leq_{\mathcal{H}}^{\langle A;\,\sqcap,0\rangle} b$ *then* $ac \leq_{\mathcal{H}}^{\langle A;\,\sqcap,0\rangle} bc$.

   (b) *The underlying natural band partial order* $\leq_{\mathcal{H}}^{\langle A;\,\sqcap,0\rangle}$ *respects implicative pre-BCK difference. That is, for any* $a, b \in A$,

   $$ab \leq_{\mathcal{H}}^{\langle A;\,\sqcap,0\rangle} a. \tag{2.67}$$

   (c) **A** *satisfies the identities:*

   $$x \dot{-} (x \sqcap y) \approx x \dot{-} y \approx x \sqcap (x \dot{-} y). \tag{2.71}$$

Proof. Let **A** be an implicative pre-BCK-algebra. To prove the theorem we show (1) ⇒ (2)(a) ⇒ (2)(b) ⇔ (2)(c) and (2)(b), (2)(c) ⇒ (1).

(1) ⇒ (2)(a) Suppose **A** is an implicative BCS-algebra. Then the polynomial reduct $\langle A;\ \sqcap, 0\rangle$ of **A** is a left normal band with zero (by Corollary 2.3.22(1)) such that right [left] multiplication by a fixed element of A is isotone [antitone] with respect to the underlying natural band partial order $\leq_{\mathcal{H}}^{\langle A;\,\sqcap,0\rangle}$ (by Proposition 2.3.25(1),(2)).

Throughout the remainder of the proof, assume that the polynomial reduct $\langle A;\ \sqcap, 0\rangle$ of **A** is a left normal band with zero.

(2)(a) $\Rightarrow$ (2)(b) Let $a \in A$ and suppose right [left] multiplication by a fixed element of A is isotone [antitone] with respect to the underlying natural band partial order $\leq_{\mathcal{H}}^{\langle A;\,\sqcap,0\rangle}$. Since $\langle A;\ \sqcap, 0\rangle$ is a band with zero, $0 \leq_{\mathcal{H}}^{\langle A;\,\sqcap,0\rangle} b$ for any $b \in A$, whence $ab \leq_{\mathcal{H}}^{\langle A;\,\sqcap,0\rangle} a0 = a$ for any $a \in A$. Thus $\leq_{\mathcal{H}}^{\langle A;\,\sqcap,0\rangle}$ respects implicative pre-BCK difference.

(2)(b) $\Leftrightarrow$ (2)(c) Suppose $\leq_{\mathcal{H}}^{\langle A;\,\sqcap,0\rangle}$ respects implicative pre-BCK difference. Then $ab \leq_{\mathcal{H}}^{\langle A;\,\sqcap,0\rangle} a$ for any $a, b \in A$, so $a(a \sqcap b) = a\big(a(ab)\big) = a \sqcap (ab) = ab$. Thus $\mathbf{A} \models (2.71)$. Conversely, suppose $\mathbf{A} \models (2.71)$. Let $a, b \in A$. By hypothesis, $a \sqcap (ab) = ab$; also $(ab) \sqcap a = (ab)\big((ab)a\big) = (ab)0 = ab$ by Lemma 2.1.12(3) and (2.6). Thus $ab \leq_{\mathcal{H}}^{\langle A;\,\sqcap,0\rangle} a$.

(2)(b), (2)(c) $\Rightarrow$ (1) Suppose both $\mathbf{A} \models (2.71)$ and $ab \leq_{\mathcal{H}}^{\langle A;\,\sqcap,0\rangle} a$ for any $a, b \in A$. We verify directly that the defining identities (2.47)–(2.50) for implicative BCS-algebras are satisfied. So let $a, b, c \in A$. It is clear that (2.47) is satisfied, because $\mathbf{A}$ is a pre-BCK-algebra. For (2.48), we have:

$$\begin{aligned}
(ab)c &= (ac) \sqcap (ab) && \text{by (2.70)}\\
&= a \sqcap (ac) \sqcap (ab) && \text{by (2.67)}\\
&= a \sqcap (ab) \sqcap (ac) && \text{by left normality}\\
&= (ab) \sqcap (ac) && \text{by (2.67)}\\
&= (ac)b && \text{by (2.70).}
\end{aligned}$$

For (2.49), put $\alpha := a$, $\beta := ac$ and $\gamma := (ac \sqcap b)$. We have:

$$\begin{aligned}
(ac)(bc) &= \big(a(bc)\big) \sqcap (ac) && \text{by (2.70)}\\
&= \big(a \sqcap (a(bc))\big) \sqcap (ac) && \text{by (2.67)}\\
&= \big(a(a \sqcap (bc))\big) \sqcap (ac) && \text{by (2.71)}\\
&= \big(a(ac \sqcap b)\big) \sqcap (ac) && \text{by (2.69)}\\
&= \big(\alpha\gamma\big) \sqcap \beta &&\\
&= \alpha \sqcap (\beta\gamma) && \text{by (2.69)}\\
&= a \sqcap \big((ac)((ac) \sqcap b)\big) &&
\end{aligned}$$

$$
\begin{aligned}
&= a \sqcap \big((ac) \sqcap ((ac)b)\big) && \text{by (2.71)}\\
&= a \sqcap \big((ac)b\big) && \text{by (2.67)}\\
&= (ab) \sqcap (ac) && \text{by (2.69)}\\
&= (ac)b && \text{by (2.70)}\\
&= (ab)c.
\end{aligned}
$$

For (2.50), we have:

$$
\begin{aligned}
a(ba) &= a \sqcap \big(a(ba)\big) && \text{by (2.67)}\\
&= a\big(a \sqcap (ba)\big) && \text{by (2.71)}\\
&= a\big(a(a(ba))\big)\\
&= a0 && \text{by (2.42)}\\
&= a && \text{by (2.6).}
\end{aligned}
$$

■

Remark 2.3.30. The hypothesis that the natural band partial order respect implicative pre-BCK difference cannot be omitted in the assertion of Theorem 2.3.29. To see this, consider the 4-element implicative pre-BCK-algebra **A** of Remark 2.3.6, whose polynomial reduct $\langle A;\ \sqcap, 0\rangle$ has the following operation table:

$\sqcap^{\mathbf{A}}$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	0
c	0	c	0	c

An easy sequence of checks shows that $\langle A;\ \sqcap, 0\rangle$ is a left normal band with zero. However, the natural band partial order $\leq_{\mathcal{H}}^{\langle A;\, \sqcap, 0\rangle}$ does not respect implicative pre-BCK difference, since $ab = c \not\leq_{\mathcal{H}}^{\langle A;\, \sqcap, 0\rangle} a$; and **A** is not an implicative BCS-algebra by the remarks of Example 2.3.6. ■

Recall from Theorem 1.6.21 that every implicative BCK-algebra $\langle A;\ /, 0\rangle$ has a semi-Boolean algebra polynomial reduct $\langle A;\ \cap, 0\rangle$, where $a \cap b := a/(a/b)$ for any $a, b \in A$, and conversely that every semi-Boolean algebra $\langle A;\ \cap, 0\rangle$ induces an implicative BCK-algebra $\langle A;\ /, 0\rangle$, where $a/b := (a \cap b)'_{(a]}$. In a sense, this representation result characterises implicative BCK-algebras entirely in semilattice-theoretic terms. This remark, in conjunction with Theorem 2.3.29, begs the question of whether implicative BCS-algebras may be characterised similarly in semigroup-theoretic terms, and in particular in terms of left normal bands with zero. We devote the remainder of this subsection to the study and solution of this problem. To begin, let **A** be an implicative BCS-algebra with left normal band with zero polynomial reduct $\langle A;\ \sqcap, 0\rangle$. Recall from §1.3.11 that for any $m \in A$, $m \sqcap A := \{m \sqcap a : a \in A\}$.

Proposition 2.3.31. *Let* **A** *be an implicative BCS-algebra and let $m \in A$ be fixed. Then $m \sqcap A = \{a \in A : a \leq m\} = \{a \in A : m \sqcap a = a\}$. Moreover, $m \sqcap A$ is closed under $\backslash$, and on $m \sqcap A$ the partial order $\leq$ and the quasiorder $\preceq$ coincide. Thus the principal subalgebra $(\boldsymbol{m}] := \langle m \sqcap A;\ \sqcap^{\mathbf{A}}|_{m \sqcap A}\rangle$ generated by m is a Boolean lattice.*

Proof. Let **A** be an implicative BCS-algebra and let $m \in A$ be fixed. By left normality and Lemma 1.3.13(1), $m \sqcap A = \{a : a \leq m\}$. Since $m \sqcap a = a$ iff $a \leq m$, the first statement of the proposition holds. For the second assertion, suppose $a, b \in m \sqcap A$. Then $a \leq m$, so $ab \leq mb$ by Proposition 2.3.25(2). But $mb \leq m$, so $ab \leq m$ by transitivity, whence $m \sqcap A$ is closed under $\backslash$. Also, $a \leq b$ implies $a \preceq b$ since $\preceq$ is admissible with respect to $\leq$. Conversely, $a \preceq b$ implies $ab = 0$, which implies $a \sqcap b = a$ (by Lemma 2.1.42(1)), which implies $b \sqcap a = a$ (since $(\boldsymbol{m}]$ is a subsemilattice of $\langle A;\ \sqcap, 0\rangle$ by Lemma 1.3.16). Thus $a \leq b$, and the partial order $\leq$ and the quasiorder $\preceq$ coincide on $m \sqcap A$. Therefore for any $a, b \in m \sqcap A$, $ab = 0 = ba$ implies $a = b$, so $\langle m \sqcap A;\ \backslash, 0\rangle$ is an implicative BCK-algebra such that the implicative BCK partial ordering coincides with the semilattice partial ordering on $m \sqcap A$. Since $(\boldsymbol{m}]$ is bounded (by Lemma 1.3.13(1)), $\langle m \sqcap A;\ \backslash, 0\rangle$ is bounded, so $\langle m \sqcap A;\ \leq\rangle$ is a Boolean lattice by Corollary 1.6.22; that is to say $(\boldsymbol{m}]$ is a Boolean lattice. This establishes the final assertion of the proposition, and the proof is complete. ∎

Recall from §1.3.15 that a band with zero **A** is locally Boolean if for every

$a \in A$, the principal subalgebra $(a]$ generated by a is a Boolean lattice. If **A** is a locally Boolean band and **A** is also left regular (equivalently, left normal), then by analogy with non-commutative lattice theory we call **A** a *left handed locally Boolean band.* (Right handed locally Boolean bands may be defined dually, though our concern here is with left handed locally Boolean bands.) As an immediate consequence of Corollary 2.3.22(1) and Proposition 2.3.31 we have the following result.

Theorem 2.3.32. *Every implicative BCS-algebra* $\langle A;\ \backslash, 0\rangle$ *has a left handed locally Boolean band polynomial reduct* $\langle A;\ \sqcap, 0\rangle$, *where* $a \sqcap b := a\backslash(a\backslash b)$ *for any* $a, b \in A$.

Let A be an implicative BCS-algebra. Call **A** *bounded* if there exists $1 \in A$ such that $a \leq 1$ for any $a \in A$. As usual, by abuse of language and notation we confuse a bounded implicative BCS-algebra **A** with its expansion to $\langle A;\ \backslash, 0, 1\rangle$, where **1** is a new nullary operation symbol adjoined to the language of **A** whose canonical interpretation on $\langle A;\ \backslash, 0, 1\rangle$ is $1 \in A$. The following corollary to Theorem 2.3.32 may be inferred immediately from Corollary 1.6.22 and Proposition 2.3.31.

Corollary 2.3.33. *(cf. [126, Theorem 12]; cf. [2, Theorem 8]) The underlying poset* $\langle A;\ \leq\rangle$ *of a bounded implicative BCS-algebra* $\langle A;\ /, 0, 1\rangle$ *is a Boolean lattice. For any* $a, b \in A$,

$$a \wedge b = a \sqcap b$$
$$a \vee b = 1\backslash\big((1\backslash a) \sqcap (1\backslash b)\big).$$

Let $\langle A;\ \sqcap, 0\rangle$ be a left handed locally Boolean band. Because $a \sqcap b \leq_{\mathcal{H}} a$ for any $a, b \in A$, locally Boolean bands possess an induced difference operation $\backslash$. In more detail: given $a, b \in A$, the difference $a\backslash b$ is defined to be $(a \sqcap b)^*_{(a]}$, namely the complement of $a \sqcap b$ in the principal subalgebra $(a]$ generated by a. In the following two results, we denote this induced difference by juxtaposition.

Lemma 2.3.34. *For a left handed locally Boolean band* **A** *and* $a, b \in A$ *the following assertions hold:*

1. *$a \sqcap (ab) = ab = (ab) \sqcap a$;*

2. *$c^*_{(ab]} = (ab) \sqcap c^*_{(a]}$ for any $c \in A$ such that $c \leq_{\mathcal{H}} ab$.*

Proof. For (1), just note $ab = (a \sqcap b)^*_{(a]} \leq_{\mathcal{H}} a$. For (2), assume $c \leq_{\mathcal{H}} ab$. Since $(ab) \sqcap c^*_{(a]} \leq_{\mathcal{H}} ab$ by left normality, $(ab) \sqcap c^*_{(a]} \in (ab]$. It remains to show $(ab) \sqcap c^*_{(a]}$ is the complement of c in $(ab]$. For this, observe:

$$\begin{aligned}\left((ab) \sqcap c^*_{(a]}\right) \sqcap c &= (ab) \sqcap (c^*_{(a]} \sqcap c)\\ &= 0\end{aligned}$$

and also:

$$\begin{aligned}\left((ab) \sqcap c^*_{(a]}\right) \sqcup^{(ab]} c &= \left((ab) \sqcap c^*_{(a]}\right) \sqcup^{(a]} c && \text{as } (\mathbf{ab}] \leq (\mathbf{a}]\\ &= \left((ab) \sqcup^{(a]} c\right) \sqcap (c^*_{(a]} \sqcup^{(a]} c) && \text{by distributivity}\\ &= (ab) \sqcap (c^*_{(a]} \sqcup^{(a]} c) && \text{as } c \leq_{\mathcal{H}} ab,\ (\mathbf{ab}] \leq (\mathbf{a}]\\ &= (ab) \sqcap a\\ &= ab && \text{by (1).}\end{aligned}$$

■

Theorem 2.3.35. *Every left handed locally Boolean band $\langle A;\ \sqcap, 0\rangle$ induces an implicative BCS-algebra $\langle A;\ \backslash, 0\rangle$ under the operation $a \backslash b := (a \sqcap b)^*_{(a]}$ for any $a, b \in A$, where $(a \sqcap b)^*_{(a]}$ denotes the complement of $a \sqcap b$ in the principal subalgebra $(\mathbf{a}]$ generated by a.*

Proof. Let $\mathbf{A}$ be a left handed locally Boolean band and let $a,'\ c \in A$. To see the derived algebra $\langle A;\ \backslash, 0\rangle$ is an implicative BCS-algebra, we verify $\langle A;\ \backslash, 0\rangle$ satisfies the defining identities (2.47)–(2.50).

For (2.47), simply notice $aa = (a \sqcap a)^*_{(a]} = a^*_{(a]} = 0$.

For (2.48), observe $(ab) \sqcap c \leq_{\mathcal{H}} ab$ by left normality, and hence that:

$$\begin{aligned}(ab)c &= \left((ab) \sqcap c\right)^*_{(ab]}\\ &= (ab) \sqcap \left((ab) \sqcap c\right)^*_{(a]} && \text{by Lemma 2.3.34(2)}\end{aligned}$$

$$
\begin{aligned}
&= (ab) \sqcap \big((ab) \sqcap a \sqcap c\big)^*_{(a]} && \text{by Lemma 2.3.34(1)}\\
&= (ab) \sqcap \big((ab)^*_{(a]} \sqcup^{(a]} (a \sqcap c)^*_{(a]}\big) && \text{by De Morgan's laws}\\
&= (ab) \sqcap \big((ab)^*_{(a]} \sqcup^{(a]} (ac)\big) \\
&= \big((ab) \sqcap (ab)^*_{(a]}\big) \sqcup^{(a]} \big((ab) \sqcap (ac)\big) && \text{by distributivity}\\
&= 0 \sqcup^{(a]} \big((ab) \sqcap (ac)\big)\\
&= (ab) \sqcap (ac).
\end{aligned}
$$

By symmetry in the elements b and c we deduce also that $(ac)b = (ac) \sqcap (ab)$. But this implies $(ab)c = (ac)b$, since:

$$
\begin{aligned}
(ab) \sqcap (ac) &= a \sqcap (ab) \sqcap (ac) && \text{by Lemma 2.3.34(1)}\\
&= a \sqcap (ac) \sqcap (ab) && \text{by left normality}\\
&= (ac) \sqcap (ab) && \text{by Lemma 2.3.34(1).}
\end{aligned}
$$

For (2.49) it is sufficient to show $(ab)c = (ab)(cb)$. By definition $(ab)c = \big((ab) \sqcap c\big)^*_{(ab]}$ and $(ab)(cb) = \big((ab) \sqcap (cb)\big)^*_{(ab]}$; we claim $(ab) \sqcap c \,\mathcal{D}\, (ab) \sqcap (cb)$. Because $\mathbf{A}$ is locally Boolean, $\mathbf{A}/\mathcal{D}$ is semi-Boolean by Lemma 1.3.17. Since $(x/y)/\big((x/y)/z\big) \approx (x/y)/\big((x/y)/(z/y)\big)$ is an identity of implicative BCK-algebras (by (2.59) and (2.60)), from Theorem 1.6.21 we may infer $\big[(ab) \sqcap c\big]_{\mathcal{D}} = \big[(ab) \sqcap (cb)\big]_{\mathcal{D}}$ in $\mathbf{A}/\mathcal{D}$; that is to say $(ab) \sqcap c \,\mathcal{D}\, (ab) \sqcap (cb)$. Now $(ab) \sqcap c, (ab) \sqcap (cb) \leq_{\mathcal{H}} ab$ by left normality, so $(ab) \sqcap c, (ab) \sqcap (cb) \in (ab]$. But this implies (by Proposition 2.3.31) that the equivalence $(ab) \sqcap c \,\mathcal{D}\, (ab) \sqcap (cb)$ collapses in $(\boldsymbol{ab}]$ to the equality $(ab) \sqcap (cb) = (ab) \sqcap c$. Because $(\boldsymbol{ab}]$ is Boolean, it is uniquely complemented, and so we deduce $\big((ab) \sqcap c\big)^*_{(ab]} = \big((ab) \sqcap (cb)\big)^*_{(ab]}$; that is to say $(ab)c = (ab)(cb)$.

For (2.50), observe first that $a \sqcap (ba) = 0$. Indeed,

$$
\begin{aligned}
0 &= (b \sqcap a) \sqcap (b \sqcap a)^*_{(b]}\\
&= b \sqcap a \sqcap (ba)\\
&= b \sqcap (ba) \sqcap a && \text{by left normality}\\
&= (ba) \sqcap a && \text{by Lemma 2.3.34(1)}
\end{aligned}
$$

which implies $a \sqcap (ba) = 0$ by Lemma 1.4.13, just because $a \sqcap (ba) \mathcal{D}(ba) \sqcap a$. To complete the proof, simply notice:

$$\begin{aligned} a(ba) &= \big(a \sqcap (ba)\big)^*_{(a]} \\ &= 0^*_{(a]} \\ &= a. \end{aligned}$$

■

Theorem 2.3.36. *In the variety of implicative BCS-algebras and the class of left handed locally Boolean bands, the following assertions hold:*

1. *Every implicative BCS-algebra $\langle A;\ \backslash, 0\rangle$ induces a left handed locally Boolean band $\langle A;\ \sqcap, 0\rangle$ upon defining $a \sqcap b := a\backslash(a\backslash b)$ for any $a, b \in A$;*

2. *Every left handed locally Boolean band $\langle A;\ \sqcap, 0\rangle$ determines an implicative BCS-algebra $\langle A;\ \backslash, 0\rangle$ under the operation $a\backslash b := (a \sqcap b)^*_{(a]}$ for any $a, b \in A$, where $(a \sqcap b)^*_{(a]}$ denotes the complement of $a \sqcap b$ in the principal subalgebra $(\mathbf{a}]$ generated by a.*

Moreover, the correspondences of (1) *and* (2) *are inverse to each other. In more detail, if $\langle A;\ \backslash, 0\rangle$ is an implicative BCS-algebra with left handed locally Boolean band polynomial reduct $\langle A;\ \sqcap, 0\rangle$, then implicative BCS difference on $\langle A;\ \sqcap, 0\rangle$ as induced by the operation $\backslash^{\langle A;\, \sqcap, 0\rangle}$ of* (2) *coincides with implicative BCS difference $\backslash^{\langle A;\, \backslash, 0\rangle}$ on $\langle A;\ \backslash, 0\rangle$; that is, $a\backslash^{\langle A;\, \sqcap, 0\rangle} b = a\backslash^{\langle A;\, \backslash, 0\rangle} b$ for any $a, b \in A$. Conversely, if $\langle A;\ \sqcap, 0\rangle$ is a left handed locally Boolean band with induced implicative BCS-algebra $\langle A;\ \backslash, 0\rangle$, then the implicative BCS meet on $\langle A;\ \backslash, 0\rangle$ as determined by the operation $\sqcap^{\langle A;\, \backslash, 0\rangle}$ of* (1) *coincides with the band operation $\sqcap^{\langle A;\, \sqcap, 0\rangle}$ on $\langle A;\ \sqcap, 0\rangle$; that is, $a \sqcap^{\langle A;\, \backslash, 0\rangle} b = a \sqcap^{\langle A;\, \sqcap, 0\rangle} b$ for any $a, b \in A$.*

Proof. It remains only to prove the final assertion. Let $\langle A;\ \backslash, 0\rangle$ be an implicative BCS-algebra with left handed locally Boolean band polynomial reduct $\langle A;\ \sqcap, 0\rangle$. For any $a, b, \in A$,

$$a\backslash^{\langle A;\, \sqcap, 0\rangle} b = (a \sqcap b)^*_{(a]}$$

$$\begin{aligned}
&= \left(a\backslash^{\langle A;\,\backslash,0\rangle}(a\backslash^{\langle A;\,\backslash,0\rangle}b)\right)^*_{(a]} \\
&= a\backslash^{\langle A;\,\backslash,0\rangle}\left(a\backslash^{\langle A;\,\backslash,0\rangle}(a\backslash^{\langle A;\,\backslash,0\rangle}b)\right) && \text{by Corollary 2.3.33} \\
&= a\backslash^{\langle A;\,\backslash,0\rangle}b && \text{by (2.56).}
\end{aligned}$$

Conversely, let $\langle A;\ \sqcap, 0\rangle$ be a left handed locally Boolean band with induced implicative BCS-algebra $\langle A;\ \backslash, 0\rangle$. For any $a, b \in A$,

$$\begin{aligned}
a \sqcap^{\langle A;\,\backslash,0\rangle} b &= a\backslash(a\backslash b) \\
&= \left(a \sqcap^{\langle A;\,\sqcap,0\rangle} (a \sqcap^{\langle A;\,\sqcap,0\rangle} b)^*_{(a]}\right)^*_{(a]} \\
&= a^*_{(a]} \sqcup^{(a]} (a \sqcap^{\langle A;\,\sqcap,0\rangle} b)^{**}_{(a]} && \text{by De Morgan's laws} \\
&= 0 \sqcup^{(a]} (a \sqcap^{\langle A;\,\sqcap,0\rangle} b)^{**}_{(a]} \\
&= (a \sqcap^{\langle A;\,\sqcap,0\rangle} b)^{**}_{(a]} \\
&= a \sqcap^{\langle A;\,\sqcap,0\rangle} b.
\end{aligned}$$

■

Remark 2.3.37. Let $\mathbf{A} := \langle A;\ \cdot, 0\rangle$ be a locally Boolean band. For any $a, b \in A$, $aba \leq_{\mathcal{H}} a$, and so $\mathbf{A}$ possesses an induced difference operation $\backslash^{\langle A;\,\cdot,0\rangle}$, where $a\backslash^{\langle A;\,\cdot,0\rangle}b := (aba)^*_{(a]_{\mathbf{A}}}$, the complement of aba in the principal subalgebra $(a]_{\mathbf{A}}$ generated by a. As with left handed locally Boolean bands, the induced algebra $\langle A;\ \backslash^{\langle A;\,\cdot,0\rangle}, 0\rangle$ is an implicative BCS-algebra. (For a justification of this assertion, see the proof below.) In spite of this observation, however, there seems little profit in studying locally Boolean bands as a generalisation of left handed locally Boolean bands, because the correspondences of Theorem 2.3.36 are not preserved: while every (not necessarily left or right handed) locally Boolean band induces an implicative BCS-algebra, the locally Boolean band polynomial reduct of any implicative BCS-algebra is always left (or right) handed.

Proof. Let $\mathbf{A} := \langle A;\ \cdot, 0\rangle$ be a locally Boolean band. For any $a, b \in A$, let $a\backslash^{\langle A;\,\cdot,0\rangle}b := (aba)^*_{(a]_{\mathbf{A}}}$, the complement of aba in the principal subalgebra $(a]_{\mathbf{A}}$ generated by a. To see the induced algebra $\langle A;\ \backslash^{\langle A;\,\cdot,0\rangle}, 0\rangle$ is an implicative BCS-algebra, let $a \sqcap b := aba$ for any $a, b \in A$. We claim:

(i) The polynomial reduct $\langle A;\ \sqcap, 0\rangle$ is a left normal band with zero;
(ii) For any $a \in A$, $(a]_{\langle A;\ \cdot,0\rangle} = (a]_{\langle A;\ \sqcap,0\rangle}$;
(iii) If $b, c \in (a]_{\mathbf{A}}$ then $b \cdot c = b \sqcap c$.

For (i), easy but tedious computations show the polynomial reduct $\langle A;\ \sqcap, 0\rangle$ is a left normal band with zero, just because $\langle A;\ \cdot, 0\rangle$ is normal (by Lemma 1.3.16).

For (ii), it sufficient to show $a \leq^{\langle A;\ \cdot,0\rangle} b$ iff $a \leq^{\langle A;\ \sqcap,0\rangle} b$ for any $a, b \in A$. So let $a, b \in A$. Clearly $a \leq^{\langle A;\ \cdot,0\rangle} b$ implies $a \leq^{\langle A;\ \sqcap,0\rangle} b$. Conversely, $a \leq^{\langle A;\ \sqcap,0\rangle} b$ implies $aba = a = bab$, so $ab = (bab)b = bab = a$ and $ba = b(bab) = bab = a$, whence $a \leq^{\langle A;\ \cdot,0\rangle} b$.

For (iii), let $b, c \in (a]_{\mathbf{A}}$. Since $(a]_{\mathbf{A}}$ is a subsemilattice of $\langle A;\ \cdot, 0\rangle$ we have $b \cdot c = b \cdot c \cdot b = b \sqcap c$ as desired.

To complete the proof, notice (ii)–(iii) imply the principal subalgebra $(a]_{\langle A;\ \sqcap,0\rangle}$ of $\langle A;\ \sqcap, 0\rangle$ coincides with the principal subalgebra $(a]_{\mathbf{A}}$ of $\mathbf{A}$ for any $a \in A$, whence $\langle A;\ \sqcap, 0\rangle$ is a left handed locally Boolean band by (i). By Theorem 2.3.35 we have that the induced algebra $\langle A;\ \backslash^{\langle A;\ \sqcap,0\rangle}, 0\rangle$ is an implicative BCS-algebra, where for any $a, b \in A$, $a\backslash^{\langle A;\ \sqcap,0\rangle}b := (a \sqcap b)^*_{(a]_{\langle A;\ \sqcap,0\rangle}}$. But $(a \sqcap b)^*_{(a]_{\langle A;\ \sqcap,0\rangle}} = (aba)^*_{(a]_{\mathbf{A}}}$ for any $a, b \in A$, so $a\backslash^{\langle A;\ \cdot,0\rangle}b = a\backslash^{\langle A;\ \sqcap,0\rangle}b$. Thus $\langle A;\ \backslash^{\langle A;\ \cdot,0\rangle}, 0\rangle$ is an implicative BCS-algebra as asserted. ∎

We cannot hope for a further sharpening (in purely algebraic terms) of the relationship between the class of implicative BCS-algebras and the class of left handed locally Boolean bands on at least two counts. On the one hand, the class of all left handed locally Boolean bands is not even a quasivariety, since it is not closed under the formation of subalgebras. (To see this, just consider the 4-element Boolean lattice $\mathbf{4}$ (with universe $\{0, a, b, 1\}$, least element 0 and greatest element 1) as a semi-Boolean algebra. Clearly $\langle\{0, a, 1\};\ \sqcap, 0\rangle$ is a subalgebra of $\mathbf{4}$ such that $0 < a < 1$.) On the other hand, the natural morphisms between left handed locally Boolean bands (namely those left normal band with zero homomorphisms that preserve Boolean sublattices) cannot be given a purely algebraic description. This is the subject of Theorem 2.3.39 below. But first, the following lemma, which is an easy modification of a result due to Cornish [64].

Lemma 2.3.38. *(cf. [64, Lemma 3.3]) Let* $\mathbf{A}$ *be a left handed locally Boolean band and let* $a \in A$ *be fixed. Let* θ *be a congruence on the underlying left normal band with zero polynomial reduct* $\langle A; \sqcap, 0\rangle$ *such that for each* $b \in A$, *the restriction* $\theta|_{(b]}$ *of* θ *to the principal subalgebra* $(\mathbf{b}]$ *generated by* b *is a lattice congruence on* $(\mathbf{b}]$. *Then* $\mathbf{A}/\theta$ *is a left handed locally Boolean band and the restriction of the canonical map* $\nu : \mathbf{A} \to \mathbf{A}/\theta$ *is a lattice homomorphism of* $(a]_{\mathbf{A}}$ *onto* $(\nu(a)]_{\mathbf{A}/\theta}$.

Proof. Let r denote the restriction of ν. Notice that for $\nu(b) \in A/\theta$, $\nu(b) \in (\nu(a)]_{\mathbf{A}/\theta}$ implies $\nu(b) \leq^{(\nu(a)]_{\mathbf{A}/\theta}} \nu(a)$, which implies $\nu(b) = \nu(a) \sqcap^{(\nu(a)]_{\mathbf{A}/\theta}} \nu(b) = \nu(c)$ for some $c \in A$, whereby $c = a \sqcap^{\mathbf{A}} b \in (a]_{\mathbf{A}}$. Hence r is onto, and we can regard each element of $(\nu(a)]_{\mathbf{A}/\theta}$ to be of the form $\nu(c)$ for a suitable $c \leq^{\mathbf{A}} a$.

Let $b, c \in (a]$ and let $d := b \sqcup^{(a]} c$. Then $r(b), r(c) \leq^{(\nu(a)]_{\mathbf{A}/\theta}} r(d)$. Suppose $r(b), r(c) \leq^{(\nu(a)]_{\mathbf{A}/\theta}} r(e)$ also for some $e \in (a]$. Then $b \equiv b \sqcap e \,(\mathrm{mod}\, \theta)$ and $c \equiv c \sqcap e \,(\mathrm{mod}\, \theta)$. Therefore $d = b \sqcup^{(a]} c \equiv (b \sqcap e) \sqcup^{(a]} (c \sqcap e) \,(\mathrm{mod}\, \theta)$. In other words, $r(d) = r\big((b \sqcap e) \sqcup^{(a]} (c \sqcap e)\big) \leq^{(\nu(a)]_{\mathbf{A}/\theta}} r(e)$. Therefore $r(d) = r(b) \sqcup^{(\nu(a)]_{\mathbf{A}/\theta}} r(c)$. Thus $(\nu(\mathbf{a})]_{\mathbf{A}/\theta}$ is a lattice and the quotient $\mathbf{A}/\theta$ is a left handed locally Boolean band. ∎

Theorem 2.3.39. *Let* $\mathbf{A}$ *be an implicative BCS-algebra with left handed locally Boolean band polynomial reduct* $\langle A; \sqcap, 0\rangle$. *The following are equivalent:*

1. $\theta \in \mathrm{Con}\,\mathbf{A}$;
2. $\theta \in \mathrm{Con}\,\langle A; \sqcap, 0\rangle$ *and* $\theta|_{(\mathbf{a}]} \in \mathrm{Con}\,(\mathbf{a}]$ *for each* $a \in A$, *where* $\theta|_{(\mathbf{a}]}$ *denotes the restriction of* θ *to the principal subalgebra* $(\mathbf{a}]$ *generated by* a;
3. $\theta \in \mathrm{Con}\,\langle A; \sqcap, 0\rangle$ *and* $\theta|_{(\mathbf{a}]}$ *is a lattice congruence on* $(\mathbf{a}]$ *for each* $a \in A$.

Proof. (1) $\Rightarrow$ (2). Clearly θ is a congruence on the polynomial reduct $\langle A; \sqcap, 0\rangle$. Moreover, because $(\mathbf{a}]$ is a subalgebra of $\mathbf{A}$ for each $a \in A$, the restriction $\theta|_{(\mathbf{a}]}$ must be a congruence on $(\mathbf{a}]$.

(2) $\Rightarrow$ (3). Let $a \in A$ be fixed. As the infimum $b \sqcap c$ is a derived operation for any $b, c \in (a]$, we deduce that θ is a semilattice congruence. Moreover, since

$(a]$ is a bounded BCK-subalgebra of $\mathbf{A}$, for any $b, c \in A$ the lattice supremum $b \sqcup c$ is also a derived operation, given by the formula of Corollary 2.3.33. Thus the restriction of θ to $(a]$ is a lattice congruence.

(3) $\Rightarrow$ (1). Suppose that for each $a \in A$, $\theta|_{(a]}$ is a lattice congruence on $(a]$. Then $\mathbf{A}/\theta$ is left handed locally Boolean by Lemma 2.3.38. Let $\nu : \mathbf{A} \to \mathbf{A}/\theta$ be the canonical homomorphism and let $a, a_1, b, b_1 \in A$ be such that $a \equiv a_1 \pmod{\theta|_{(a]}}$ and $b \equiv b_1 \pmod{\theta|_{(a]}}$. Now in $\mathbf{A}$, $a \backslash b$ is the complement of $a \sqcap b$ in $(a]$. Since $\mathbf{A}/\theta$ is left handed locally Boolean, $\nu(a \backslash b)$ is the complement of $\nu(a) \sqcap \nu(b)$ in the Boolean lattice $(\nu(a)]_{\mathbf{A}/\theta}$. As $\nu(a) = \nu(a_1)$ and $\nu(b) = \nu(b_1)$, $\nu(a_1 \backslash b_1)$ is also a complement of $\nu(a) \sqcap \nu(b)$ in $(\nu(a)]_{\mathbf{A}/\theta}$. Because complements are unique in $(\nu(a)]_{\mathbf{A}/\theta}$, we conclude $\nu(a \backslash b) = \nu(a_1 \backslash b_1)$, which implies θ is a congruence on $\mathbf{A}$ as required. ∎

Although the correspondence between implicative BCS-algebras and left handed locally Boolean bands cannot be given a purely algebraic description, Theorem 2.3.39 immediately suggests a category-theoretic formalisation. Let **IBCS** denote the category for which:

- The objects of **IBCS** are the implicative BCS-algebras;
- The morphisms of **IBCS** are the implicative BCS homomorphisms.

Also, let **LLBB** denote the category for which:

- The objects of **LLBB** are the left handed locally Boolean bands;
- The morphisms of **LLBB** are the Boolean sublattice preserving homomorphisms, namely those homomorphisms $h : \mathbf{A} \to \mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are left handed locally Boolean bands, such that for each $a \in A$, the restriction $(\ker h)|_{(a]}$ of the relation kernel $\ker h$ to the principal subalgebra $(a]$ generated by a is a lattice congruence on $(a]$.

Recall from category theory [160, Section 3.6] that categories $\mathbf{C}$ and $\mathbf{D}$ are *isomorphic* iff there exists a one-to-one functor F mapping $\mathbf{C}$ onto $\mathbf{D}$. Since a functor $F : \mathbf{C} \to \mathbf{D}$ is an isomorphism iff there exists a functor $G : \mathbf{D} \to \mathbf{C}$ such that $F \circ G$ is the identity map on $\mathbf{D}$ and $G \circ F$ is the identity map on

C (where C [D] is the underlying class of objects in the category **C** [**D**]) [160, Section 3.6], the following result is clear upon combining Theorem 2.3.36 and Theorem 2.3.39.

Theorem 2.3.40. **IBCS** *and* **LLBB** *are isomorphic as categories.*

In light of the preceding result, it is natural to anticipate that structural results concerning left normal bands with zero transfer to the setting of implicative BCS-algebras. Certainly this is the case in relation to the Clifford-McLean theorem for bands, for which we have the following natural analogue.

Theorem 2.3.41 (Clifford-McLean Theorem for Implicative BCS-Algebras). *Let* $\mathbf{A}$ *be an implicative BCS algebra with left handed locally Boolean band polynomial reduct* $\langle A;\ \sqcap, 0\rangle$. *Then* $\mathcal{D}_{\langle A;\ \sqcap,0\rangle}$*-equivalence is a congruence relation on* both $\langle A;\ \sqcap, 0\rangle$ *and* $\mathbf{A}$. *The quotient algebra* $\mathbf{A}/\mathcal{D}_{\langle A;\ \sqcap,0\rangle}$ *is the maximal implicative BCK-algebra homomorphic image of* $\mathbf{A}$ *and the quotient algebra* $\langle A;\ \sqcap, 0\rangle/\mathcal{D}_{\langle A;\ \sqcap,0\rangle}$ *is the maximal semi-Boolean algebra homomorphic image of* $\langle A;\ \sqcap, 0\rangle$, *whilst the* $\mathcal{D}_{\langle A;\ \sqcap,0\rangle}$*-congruence classes of both* $\mathbf{A}$ *and* $\langle A;\ \sqcap, 0\rangle$ *are the maximal left zero semigroups of* $\langle A;\ \sqcap, 0\rangle$. *For all* $a, b \in A$, *the following are equivalent:*

1. $a \Xi_{\mathbf{A}}\, b$;
2. $a\mathcal{D}_{\langle A;\ \sqcap,0\rangle} b$;
3. $a\mathcal{L}_{\langle A;\ \sqcap,0\rangle} b$.

Proof. Let $\mathbf{A}$ be an implicative BCS-algebra with left handed locally Boolean band polynomial reduct $\langle A;\ \sqcap, 0\rangle$. Observe first that $\mathcal{D}_{\langle A;\ \sqcap,0\rangle}$ and $\mathcal{L}_{\langle A;\ \sqcap,0\rangle}$ coincide by left normality and the remarks of §1.3.15, and also that $\mathcal{D}_{\langle A;\ \sqcap,0\rangle}$ and $\Xi_{\mathbf{A}}$ coincide by Lemma 2.3.24 (since they are respectively the equivalences induced by the quasiorderings $\preceq_{\mathcal{D}}^{\langle A;\ \sqcap,0\rangle}$ and $\preceq$ in the sense of Lemma 1.2.2). Thus Items (1)–(3) are equivalent, and $\mathcal{D}_{\langle A;\ \sqcap,0\rangle}$-equivalence is a congruence relation on *both* $\langle A;\ \sqcap, 0\rangle$ and $\mathbf{A}$. By Theorem 2.1.14 the quotient algebra $\mathbf{A}/\mathcal{D}_{\langle A;\ \sqcap,0\rangle}$ is the maximal implicative BCK-algebra homomorphic image of $\mathbf{A}$, while by Lemma 1.3.17 $\langle A;\ \sqcap, 0\rangle/\mathcal{D}_{\langle A;\ \sqcap,0\rangle}$ is the maximal semi-Boolean algebra homomorphic image of $\langle A;\ \sqcap, 0\rangle$. Finally, left normality and the Clifford-McLean theorem for bands ensure the maximal left zero semigroups of $\langle A;\ \sqcap, 0\rangle$ are the $\mathcal{D}_{\langle A;\,\sqcap,0\rangle}$-congruence classes of $\langle A;\ \sqcap, 0\rangle$, and hence (by previous remarks) of $\mathbf{A}$. ∎

Theorem 2.3.41 justifies dubbing the congruence Ξ on any implicative BCS-algebra the 'Clifford-McLean congruence'. We will (sometimes) employ this vocabulary in the sequel.

2.3.42. Subdirectly Irreducible Implicative BCS-Algebras. The complete description (to within isomorphism) of both the subdirectly irreducible left normal bands with zero and the subdirectly irreducible implicative BCK-algebras (recall Corollary 1.3.19 and Theorem 1.6.19 respectively), in conjunction with the affinity of implicative BCS-algebras to both implicative BCK-algebras and left normal bands with zero, suggests that the possibility of completely characterising the subdirectly irreducible implicative BCS-algebras (to within isomorphism) may be strong. This subsection is devoted to the study and solution of this problem. Our first order of business is to isolate a family of subdirectly irreducible implicative BCS-algebras. This is the subject of the following lemma, which is due to the author's Ph.D. supervisor.

Lemma 2.3.43. *(Bignall) Let* $\mathbf{B} := \langle B;\ \wedge, \vee, ', 0, 1\rangle$ *be a non-trivial Boolean algebra with bounded implicative BCK-algebra polynomial reduct* $\langle B;\ /, 0, 1\rangle$, *where* $a/b = a \wedge b'$ *for any* $a, b \in B$. *Let* $B' := B - \{1\}$ *and let* $\{m_1, m_2\}$ *be disjoint from* B. *Let* $\hat{B} := B' \cup \{m_1, m_2\}$ *and let the difference* $\backslash$ *be defined on* $\hat{B}$ *as follows:*

$$a\backslash b := \begin{cases} a/b & \text{if } a, b \in B' \\ 0 & \text{if } b \in \{m_1, m_2\} \text{ and } a \in \hat{B} \\ a & \text{if } a \in \{m_1, m_2\} \text{ and } b = 0 \\ b' & \text{if } a \in \{m_1, m_2\} \text{ and } 0 \neq b \in B'. \end{cases}$$

Then the derived algebra $\hat{\mathbf{B}} := \langle \hat{B};\ \backslash, 0\rangle$ *is an implicative BCS-algebra.*

Proof. Let $\mathbf{B}$ be a non-trivial Boolean algebra with greatest element 1. Let $\check{\mathbf{B}}$ be the subdirectly irreducible pseudocomplemented semilattice with greatest element $m \in \breve{B}$ induced by $\mathbf{B}$ as per Theorem 1.3.7. Let $\overline{\mathbf{B}}$ denote the canonical implicative BCS-algebra polynomial reduct of $\breve{\mathbf{B}}$ (arising as per Example 2.3.12). It is trivial to check that the operation $\backslash^{\overline{\mathbf{B}}}$ defined on $\breve{B}$ is the same as the operation $\backslash^{\hat{\mathbf{B}}}$ defined on $\hat{B}$ when the element 1 is renamed as m_1 and the element m is renamed as m_2. Hence $\hat{\mathbf{B}}$ is an implicative BCS-algebra. ∎

Lemma 2.3.44. *Let* $\mathbf{B}$ *be a non-trivial Boolean algebra. Then the implicative BCS-algebra* $\hat{\mathbf{B}}$ *induced from* $\mathbf{B}$ *as per Lemma 2.3.43 is subdirectly irreducible with monolith* Ξ.

Proof. Observe first that $\omega_{\hat{\mathbf{B}}} \cup \{\langle m_1, m_2\rangle, \langle m_2, m_1\rangle\} = \Xi \in \operatorname{Con}\hat{\mathbf{B}}$. Therefore to see $\hat{\mathbf{B}}$ is subdirectly irreducible with monolith Ξ it is sufficient to show $m_1 \equiv m_2 \pmod{\theta}$ for any $\theta \in \operatorname{Con}\hat{\mathbf{B}}$ where $\theta \neq \omega_{\hat{\mathbf{B}}}$. So let $\theta \in \operatorname{Con}\hat{\mathbf{B}}$ be such that $\theta \neq \omega_{\hat{\mathbf{B}}}$. There are two cases to consider:

(i) $[0]_\theta = \{0\}$;

(ii) $[0]_\theta \neq \{0\}$.

For Case (i), suppose $[0]_\theta = \{0\}$. Then $\theta \subseteq \Xi = \omega_{\hat{\mathbf{B}}} \cup \{\langle m_1, m_2\rangle, \langle m_2, m_1\rangle\}$ by Theorem 2.1.14(4). Assume to the contrary that $m_1 \not\equiv_\theta m_2$. Then $\theta = \omega_{\hat{\mathbf{B}}}$ which is a contradiction. Thus $m_1 \equiv_\theta m_2$.

For Case (ii), suppose $[0]_\theta \neq \{0\}$. Then there exists $0 \neq a \in \hat{B}$ such that $0 \equiv a \pmod{\theta}$. If $a \notin \{m_1, m_2\}$ then $m_1 = m_1 0 \equiv_\theta m_1 a = m_2 a \equiv_\theta m_2 0 = m_2$ by definition of $\backslash^{\hat{\mathbf{B}}}$. So suppose $a \in \{m_1, m_2\}$. Let $b \in \{m_1, m_2\}$ be such that $b \neq a$. Then $b = b0 \equiv_\theta ba = 0 \equiv_\theta a$ by definition of $\backslash^{\hat{\mathbf{B}}}$ as required. ∎

Implicative BCS-algebras arising as per Lemma 2.3.43 may be most easily envisaged as Boolean algebras in which the unit element has been replaced by a 2-element clique. To see this more clearly, let $\hat{\mathbf{B}}_0 := \mathbf{C}_1$, where $\mathbf{C}_1$ is the 2-element flat implicative BCK-algebra (recall Example 1.6.18). For $1 \leq n < \omega$, let $\hat{\mathbf{B}}_n$ denote the implicative BCS-algebra induced as per Lemma 2.3.43 from the non-trivial finite Boolean algebra $\mathbf{B}$ of cardinality 2^n. For $n = 0, 1, 2, 3$, the algebras $\hat{\mathbf{B}}_n$ are depicted in Figure 2.3.

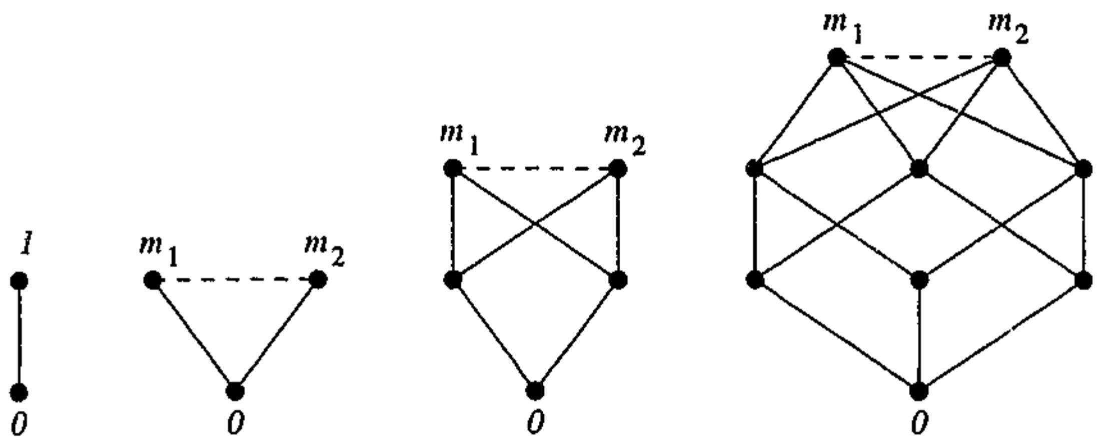

Figure 2.3. The subdirectly irreducible implicative BCS-algebras $\hat{\mathbf{B}}_n$ for $n = 0, 1, 2, 3$.

We devote the remainder of this subsection to proving that the family of algebras $\{\hat{\mathbf{B}}_0\} \cup \{\hat{\mathbf{B}} : \mathbf{B}$ a non-trivial Boolean algebra$\}$ comprises, to within isomorphism, all subdirectly irreducible members of the variety of implicative BCS-algebras. With this aim in mind we identify and briefly study some standard congruences of implicative BCS-algebras in the following two lemmas.

Lemma 2.3.45. *For any implicative BCS-algebra* $\mathbf{A}$ *the following assertions hold:*

1. *For a fixed* $c \in A$*, the maps:*

   (a) $a \mapsto ac$;

   (b) $a \mapsto a \sqcap c$;

   (c) $a \mapsto c \sqcap a$

   are endomorphisms of $\mathbf{A}$*. Respectively, the associated congruences are defined by:*

   (a)′ $a \equiv b \,(\mathrm{mod}\, \vartheta_c)$ *iff* $ac = bc$;

   (b)′ $a \equiv b \,(\mathrm{mod}\, \varrho_c)$ *iff* $a \sqcap c = b \sqcap c$;

   (c)′ $a \equiv b \,(\mathrm{mod}\, \varsigma_c)$ *iff* $c \sqcap a = c \sqcap b$.

2. *For a fixed* $c \in A$*,*

(a) $\vartheta_c = \langle c \rangle^\delta_{\mathbf{A}} = \Theta^{\mathbf{A}}(0, c)$;

(b) $\Xi \leq \varsigma_c$;

(c) $\bigcap\{\varsigma_c\} = \Xi$.

Proof. Let $\mathbf{A}$ be an implicative BCS-algebra. Let c be a fixed element of A and let $a, b \in A$. For (1) we have that $(ab)c = (ac)(bc)$ by (2.49), whence the map $a \mapsto ac$ is an endomorphism. Also $(ab) \sqcap c = (ab)(ac) = (a \sqcap c)(b \sqcap c)$ by (2.59) and (2.62), so the map $a \mapsto a \sqcap c$ is also an endomorphism. And, $c \sqcap (ab) = (cb)(ca) = (c \sqcap a)(c \sqcap b)$ by (2.61) and (2.57), which implies the map $a \mapsto c \sqcap a$ is an endomorphism as well. The remaining assertions follow immediately from [55, Theorem II§6.8].

For (2)(a) let ϑ_c be as stated. Since $\langle c \rangle^\delta_{\mathbf{A}} = \left([0]_{\Theta^{\mathbf{A}}(0,c)}\right)^\delta = \Theta^{\mathbf{A}}(0, c)$, it only remains to show $\vartheta_c = \Theta^{\mathbf{A}}(0, c)$. From $cc = 0 = 0c$ we have $0 \equiv c \,(\mathrm{mod}\, \vartheta_c)$ whence $\Theta^{\mathbf{A}}(0, c) \subseteq \vartheta_c$. For the opposite inclusion just notice $a \equiv b \,(\mathrm{mod}\, \vartheta_c)$ implies $a = a0 \equiv_{\Theta^{\mathbf{A}}(0,c)} ac = bc \equiv_{\Theta^{\mathbf{A}}(0,c)} b0 = b$. Thus $\vartheta_c \subseteq \Theta^{\mathbf{A}}(0, c)$ and so $\vartheta_c = \Theta^{\mathbf{A}}(0, c)$.

For (2)(b) let $a \equiv b \,(\mathrm{mod}\, \Xi)$. Then $a \sqcap b = a$ and $b \sqcap a = b$ by Lemma 2.1.42(1), whence $c \sqcap a = c \sqcap a \sqcap b = c \sqcap b \sqcap a = c \sqcap b$ by left normality. Thus $a \equiv b \,(\mathrm{mod}\, \varsigma_c)$ and $\Xi \leq \varsigma_c$.

For (2)(c) suppose $a \equiv b \left(\mathrm{mod} \bigcap_{c \in A}\{\varsigma_c\}\right)$. Then $c \sqcap a = c \sqcap b$ for any $c \in A$, and in particular $a \sqcap a = a \sqcap b$ and $b \sqcap a = b \sqcap b$. Thus $a = a \sqcap b$ and $b \sqcap a = b$, so $a \preceq b$ and $b \preceq a$ by Lemma 2.1.42(1); that is to say $a \equiv b \,(\mathrm{mod}\, \Xi)$. Conversely, $\Xi \leq \varsigma_c$ for any $c \in A$ by (2)(b), so $\Xi \leq \bigcap_{c \in A}\{\varsigma_c\}$. Thus $\bigcap_{c \in A}\{\varsigma_c\} = \Xi$. ∎

Lemma 2.3.46. *Let* $\mathbf{A}$ *be an implicative BCS-algebra and let* $m \in A$ *be fixed. The following assertions hold:*

1. $\omega_{\mathbf{A}} = \varrho_m$ *iff* $a \preceq m$ *for any* $a \in A$*, where* ϱ_m *denotes the congruence of Lemma 2.3.45*(1)*(b)′;*

2. $\Xi = \varsigma_m$ *iff* $a \preceq m$ *for any* $a \in A$*, where* ς_m *denotes the congruence of Lemma 2.3.45*(1)*(c)′;*

Proof. Let $\mathbf{A}$ be an implicative BCS-algebra and let $m \in A$ be fixed. For (1), suppose $\omega_{\mathbf{A}} = \varrho_m$. Notice $a \sqcap m = (a \sqcap m) \sqcap m$ for any $a \in A$ and hence that $a \equiv a \sqcap m \pmod{\varrho_m}$. Now $\varrho_m \leq \Xi$ since ϱ_m is the identity congruence, so $a \equiv a \sqcap m \pmod{\Xi}$. Thus $a \preceq a \sqcap m$. But $\big(a(am)\big)m = 0$ by (2.2) and so $a \sqcap m \preceq m$. By transitivity we conclude $a \preceq m$ as desired. Conversely, suppose $c \preceq m$ for any $c \in A$ and note this implies $c \sqcap m = c$. Suppose $a \equiv b \pmod{\varrho_m}$ for $a, b \in A$. Then $a \sqcap m = b \sqcap m$, so $a = a \sqcap m = b \sqcap m = b$. Thus $a \equiv b \pmod{\omega_{\mathbf{A}}}$ and hence $\varrho_m = \omega_{\mathbf{A}}$.

For (2), suppose $\Xi = \varsigma_m$. By left normality $m \sqcap a = m \sqcap (a \sqcap m)$ for any $a \in A$, so $a \equiv a \sqcap m \pmod{\varsigma_m}$, which implies $a \equiv a \sqcap m \pmod{\Xi}$. Thus $a \preceq a \sqcap m$. But $\big(a(am)\big)m = 0$ by (2.2) and so $a \sqcap m \preceq m$. By transitivity we conclude $a \preceq m$ as desired. Conversely, suppose $c \preceq m$ for any $c \in A$ and note this implies $c \sqcap m = c$. Suppose $a \equiv b \pmod{\varsigma_m}$ for $a, b \in A$. Then $m \sqcap a = m \sqcap b$, whence $a = a \sqcap m = a \sqcap m \sqcap a = a \sqcap m \sqcap b = a \sqcap b \sqcap m = a \sqcap b$ and $b = b \sqcap m = b \sqcap m \sqcap b = b \sqcap m \sqcap a = b \sqcap a \sqcap m = b \sqcap a$ by left normality. By Lemma 2.1.42(1) we conclude $a \preceq b$ and $b \preceq a$. Thus $a \equiv b \pmod{\Xi}$ and $\varsigma_m \subseteq \Xi$. The opposite inclusion follows immediately from Lemma 2.3.45(2)(b). ∎

Recall that an element $m \in A$ of a pre-BCK-algebra $\mathbf{A}$ is maximal if $a \preceq m$ for all $a \in A$. Recall also that all maximal elements (where they exist) of a pre-BCK-algebra $\mathbf{A}$ lie in a unique Ξ-class (the maximal class).

Lemma 2.3.47. *Let* $\mathbf{A}$ *be a subdirectly irreducible implicative BCS-algebra and let* $\{m_1, m_2\}$ *be the pair of elements identified under every non-trivial congruence relation of* $\mathbf{A}$. *Then either* $\mathbf{A} \cong \hat{\mathbf{B}}_0$ *or* $\mathbf{A}$ *has a maximal class* M *such that* $\{m_1, m_2\} \subseteq M$.

Proof. Let $\mathbf{A}$ be a subdirectly irreducible implicative BCS-algebra. We separate the proof into two cases:

(i) $\mathbf{A} \in \mathsf{iBCK}$;
(ii) $\mathbf{A} \notin \mathsf{iBCK}$.

For Case (i), suppose $\mathbf{A} \in \mathsf{iBCK}$. From Theorem 1.6.19 we deduce $\mathbf{A}$ is isomorphic to the 2-element implicative BCK-algebra $\mathbf{C}_1$, which implies by definition that $\mathbf{A}$ is isomorphic to $\hat{\mathbf{B}}_0$.

For Case (ii), suppose $\mathbf{A} \notin \mathsf{iBCK}$. Let $\{m_1, m_2\}$ be the pair of elements identified under every non-trivial congruence. Because $\mathbf{A} \notin \mathsf{iBCK}$ there exists a subalgebra of $\mathbf{A}$ isomorphic to $\mathbf{B}_2$ by Proposition 2.2.5, so the Clifford-McLean congruence Ξ on $\mathbf{A}$ is non-trivial. Thus $m_1 \equiv m_2 \,(\mathrm{mod}\,\Xi)$ and in particular $m_2 \preceq m_1$, whence $m_2 \sqcap m_1 = m_2$ by Lemma 2.1.42(1). Let now ϱ_{m_1} be the congruence relation of Lemma 2.3.45(1)(b)$'$ defined by $a \equiv b \,(\mathrm{mod}\,\varrho_{m_1})$ iff $a \sqcap m_1 = b \sqcap m_1$ for any $a, b \in A$. Because $\mathbf{A}$ is subdirectly irreducible ϱ_{m_1} is the identity congruence. For suppose to the contrary that ϱ_{m_1} is not the identity congruence. Then $m_1 \equiv m_2 \,(\mathrm{mod}\,\varrho_{m_1})$, so $m_1 \sqcap m_1 = m_2 \sqcap m_1$; that is to say $m_1 = m_2 \sqcap m_1$. But $m_2 \sqcap m_1 = m_2$. Thus $m_1 = m_2$, which contradicts the subdirect irreducibility of $\mathbf{A}$. Since ϱ_{m_1} is the identity congruence, we deduce from Lemma 2.3.46(1) that m_1 is maximal. Since m_2 lies in the same Ξ-class as m_1 we have also that m_2 is maximal. We have shown $\mathbf{A}$ has a maximal class M such that $\{m_1, m_2\} \subseteq M$, and the proof is complete. ∎

Lemma 2.3.47 demands attention be focussed on congruences of implicative BCS-algebras with a maximal class, and in particular on the role played by the Clifford-McLean congruence in such algebras.

Lemma 2.3.48. *Let $\mathbf{A}$ be an implicative BCS-algebra with maximal class M. For a fixed $m \in M$, $m \equiv ma \,(\mathrm{mod}\,\Xi)$ iff $a = 0$ for any $a \in A$. Therefore if $a \neq 0$ then $ma \notin M$.*

Proof. Let $\mathbf{A}$ be an implicative BCS-algebra with maximal class M. Let $m \in M$ be fixed and let $a \in A$. If $a = 0$ then $ma = m0 = m$, so $ma \equiv m \,(\mathrm{mod}\,\Xi)$. Conversely, if $ma \equiv m \,(\mathrm{mod}\,\Xi)$ then $m \preceq ma$, so $m(ma) = 0$. But then $a = a0 = a\big(m(ma)\big) = am = 0$ by (2.53) and the maximality of m. Thus $m \equiv ma \,(\mathrm{mod}\,\Xi)$ iff $a = 0$. Suppose now that $a \neq 0$ and assume to the contrary that $ma \in M$. Then $ma \equiv m \,(\mathrm{mod}\,\Xi)$ and so $a = 0$, a contradiction. ∎

The remaining results of this subsection, including the following lemma, are due jointly to the author and the author's Ph.D. supervisor.

Lemma 2.3.49. *Let* $\mathbf{A}$ *be an implicative BCS-algebra with maximal class* M*. Let* $\mathbf{B}$ *be the subalgebra of* $\mathbf{A}$ *with universe* $B := A - M$ *and let* $\Xi_{\mathbf{B}}$ *denote the Clifford-McLean congruence on* $\mathbf{B}$*. Then the relation:*

$$\lfloor \Xi \rfloor := \Xi_{\mathbf{B}} \cup \{\langle m, m\rangle : m \in M\}$$

is a congruence relation on $\mathbf{A}$.

Proof. Let $\mathbf{A}$ be an implicative BCS-algebra with maximal class M. Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ with universe $B := A - M$ and let $\Xi_{\mathbf{B}}$ denote the Clifford-McLean congruence on $\mathbf{B}$. Clearly $\lfloor \Xi \rfloor$ is an equivalence relation on A such that $\Xi_{\mathbf{B}} \subseteq \lfloor \Xi \rfloor \subseteq \Xi_{\mathbf{A}}$, where $\Xi_{\mathbf{A}}$ denotes the Clifford-McLean congruence on $\mathbf{A}$. Suppose $a_1 \equiv b_1 \,(\mathrm{mod}\, \lfloor \Xi \rfloor)$ and $a_2 \equiv b_2 \,(\mathrm{mod}\, \lfloor \Xi \rfloor)$ for $a_1, b_1, a_2, b_2 \in A$. To see $\lfloor \Xi \rfloor$ is a congruence on $\mathbf{A}$ we consider four cases:

(i) $a_1, b_1 \in B$ and $a_2, b_2 \in B$; (iii) $a_1 = b_1 \in M$ and $a_2, b_2 \in B$;
(ii) $a_1, b_1 \in B$ and $a_2 = b_2 \in M$; (iv) $a_1 = b_1 \in M$ and $a_2 = b_2 \in M$.

Of these cases, Cases (i) and (iii) are non-trivial. For Case (i), suppose $a_1, b_1 \in B$ and $a_2, b_2 \in B$. Then $a_1 \equiv b_1 \,(\mathrm{mod}\, \Xi_{\mathbf{B}})$ and $a_2 \equiv b_2 \,(\mathrm{mod}\, \Xi_{\mathbf{B}})$, so $a_1 a_2 \equiv b_1 b_2 \,(\mathrm{mod}\, \Xi_{\mathbf{B}})$, which implies $a_1 a_2 \equiv b_1 b_2 \,(\mathrm{mod}\, \lfloor \Xi \rfloor)$.

For Case (iii), we distinguish two subcases:

(iii)(a) $a_1 = b_1 \in M$ and $a_2 = b_2 = 0 \in B$;
(iii)(b) $a_1 = b_1 \in M$ and $0 \neq a_2 \in B$, $0 \neq b_2 \in B$.

Of these subcases, only Subcase (iii)(b) is non-trivial. So suppose $a_1 = b_1 \in M$ and $0 \neq a_2 \in B$, $0 \neq b_2 \in B$. Then $a_1 \equiv b_1 \,(\mathrm{mod}\, \Xi_{\mathbf{A}})$ and $a_2 \equiv b_2 \,(\mathrm{mod}\, \Xi_{\mathbf{A}})$ (since $a_2 \equiv b_2 \,(\mathrm{mod}\, \Xi_{\mathbf{B}})$), so $a_1 a_2 \equiv b_1 b_2 \,(\mathrm{mod}\, \Xi_{\mathbf{A}})$. Because $a_1 a_2 \notin M$ and $b_1 b_2 \notin M$ (by Lemma 2.3.48) we have that $a_1 a_2 \equiv b_1 b_2 \,(\mathrm{mod}\, \Xi_{\mathbf{B}})$, which implies $a_1 a_2 \equiv b_1 b_2 \,(\mathrm{mod}\, \lfloor \Xi \rfloor)$. ∎

Lemma 2.3.50. *Let* $\mathbf{A}$ *be an implicative BCS-algebra with maximal class* M *and let* $\lfloor \Xi \rfloor$ *be the congruence relation on* $\mathbf{A}$ *of Lemma 2.3.49. Then the relation:*

$$\lfloor \Xi \rfloor_n^m := \lfloor \Xi \rfloor \cup \{\langle m, n\rangle, \langle n, m\rangle\}$$

is a congruence relation on $\mathbf{A}$ *for any pair of maximal elements* $\{m, n\} \subseteq M$.

Proof. Let $\mathbf{A}$ be an implicative BCS-algebra with maximal class M. Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ with universe $B := A - M$ and let $\Xi_{\mathbf{B}}$ denote the Clifford-McLean congruence on $\mathbf{B}$. Let $\lfloor \Xi \rfloor$ be the congruence relation on $\mathbf{A}$ of Lemma 2.3.49 and let $m, n \in M$. Clearly $\lfloor \Xi \rfloor_n^m$ is an equivalence relation on A such that $\Xi_{\mathbf{B}} \subsetneq \lfloor \Xi \rfloor \subseteq \lfloor \Xi \rfloor_n^m \subseteq \Xi_{\mathbf{A}}$, where $\Xi_{\mathbf{A}}$ denotes the Clifford-McLean congruence on $\mathbf{A}$. Suppose $a_1 \equiv b_1 \left(\operatorname{mod} \lfloor \Xi \rfloor_n^m\right)$ and $a_2 \equiv b_2 \left(\operatorname{mod} \lfloor \Xi \rfloor_n^m\right)$ for $a_1, b_1, a_2, b_2 \in A$. To see $\lfloor \Xi \rfloor_n^m$ is a congruence on $\mathbf{A}$ we consider nine cases:

(i) $a_1, b_1 \in B$ and $a_2, b_2 \in B$;
(ii) $a_1, b_1 \in B$ and $a_2 = b_2 \in M$;
(iii) $a_1, b_1 \in B$ and $a_2, b_2 \in \{m, n\}$;
(iv) $a_1 = b_1 \in M$ and $a_2, b_2 \in B$;
(v) $a_1 = b_1 \in M$ and $a_2 = b_2 \in M$;
(vi) $a_1 = b_1 \in M$ and $a_2, b_2 \in \{m, n\}$;
(vii) $a_1, b_1 \in \{m, n\}$ and $a_2, b_2 \in B$;
(viii) $a_1, b_1 \in \{m, n\}$ and $a_2 = b_2 \in M$;
(ix) $a_1, b_1 \in \{m, n\}$ and $a_2, b_2 \in \{m, n\}$.

Of these cases, Cases (ii), (iii), (v), (vi), (viii) and (ix) are trivial, while Cases (i) and (iv) are covered by Lemma 2.3.49 (since $\lfloor \Xi \rfloor \subsetneq \lfloor \Xi \rfloor_n^m$). For Case (vii), we distinguish two subcases:

(vii)(a) $a_1, b_1 \in \{m, n\}$ and $a_2 = b_2 = 0 \in B$;
(vii)(b) $a_1, b_1 \in \{m, n\}$ and $0 \neq a_2 \in B$, $0 \neq b_2 \in B$.

Of these subcases, only Subcase (vii)(b) is non-trivial. So suppose $a_1, b_1 \in \{m, n\}$ and $0 \neq a_2 \in B$, $0 \neq b_2 \in B$. Then $a_1 \equiv b_1 (\operatorname{mod} \Xi_{\mathbf{A}})$ (since $a_1, b_1 \in M$) and $a_2 \equiv b_2 (\operatorname{mod} \Xi_{\mathbf{A}})$ (since $a_2 \equiv b_2 (\operatorname{mod} \Xi_{\mathbf{B}})$), so $a_1 a_2 \equiv b_1 b_2 (\operatorname{mod} \Xi_{\mathbf{A}})$. Because $a_1 a_2 \notin M$ and $b_1 b_2 \notin M$ (by Lemma 2.3.48) we have that $a_1 a_2 \equiv b_1 b_2 (\operatorname{mod} \Xi_{\mathbf{B}})$, which implies $a_1 a_2 \equiv b_1 b_2 \left(\operatorname{mod} \lfloor \Xi \rfloor\right)$, which implies $a_1 a_2 \equiv b_1 b_2 \left(\operatorname{mod} \lfloor \Xi \rfloor_n^m\right)$. ■

We are now in a position to complete the characterisation of the subdirectly irreducible implicative BCS-algebras. The proof is via three lemmas.

Lemma 2.3.51. *Let* $\mathbf{A}$ *be a subdirectly irreducible implicative BCS-algebra such that* $|A| > 2$. *Then the relation* $\lfloor \Xi \rfloor$ *of Lemma 2.3.49 is a congruence relation on* $\mathbf{A}$*; moreover* $\lfloor \Xi \rfloor = \omega_{\mathbf{A}}$.

Proof. Let $\mathbf{A}$ be a subdirectly irreducible implicative BCS-algebra such that $|A| > 2$. Then the relation $\lfloor \Xi \rfloor$ is a congruence relation on $\mathbf{A}$ by Lemma 2.3.47 and Lemma 2.3.49. Let $\{m_1, m_2\}$ be the pair of maximal elements identified under every non-trivial congruence on $\mathbf{A}$ as per Lemma 2.3.47. Since $\lfloor \Xi \rfloor$ does not identify m_1 and m_2 it must be trivial; that is to say $\lfloor \Xi \rfloor = \omega_{\mathbf{A}}$. ∎

Lemma 2.3.52. *Let* $\mathbf{A}$ *be a subdirectly irreducible implicative BCS-algebra such that* $|A| > 2$*. Then* $M = \{m_1, m_2\}$*, where* M *is the maximal class of* $\mathbf{A}$ *and* $\{m_1, m_2\}$ *is the pair of maximal elements identified under every non-trivial congruence relation on* $\mathbf{A}$*.*

Proof. Let $\mathbf{A}$ be a subdirectly irreducible implicative BCS-algebra such that $|A| > 2$. By Lemma 2.3.47, $\mathbf{A}$ has a maximal class M such that $\{m_1, m_2\} \subseteq M$, where $\{m_1, m_2\}$ is the pair of elements identified under every non-trivial congruence on $\mathbf{A}$. Assume to the contrary that there exists $n \in A$ such that $n \neq m_1, m_2$ but $n \in M$. Let $\lfloor \Xi \rfloor_n^{m_1} := \lfloor \Xi \rfloor \cup \{\langle m_1, n\rangle, \langle n, m_1\rangle\}$ and let $\lfloor \Xi \rfloor_n^{m_2} := \lfloor \Xi \rfloor \cup \{\langle m_2, n\rangle, \langle n, m_2\rangle\}$. By Lemma 2.3.50 both $\lfloor \Xi \rfloor_n^{m_1}$ and $\lfloor \Xi \rfloor_n^{m_2}$ are congruences on $\mathbf{A}$. Moreover, $\lfloor \Xi \rfloor_n^{m_1} = \omega_{\mathbf{A}} \cup \{\langle m_1, n\rangle, \langle n, m_1\rangle\}$ and $\lfloor \Xi \rfloor_n^{m_2} = \omega_{\mathbf{A}} \cup \{\langle m_2, n\rangle, \langle n, m_2\rangle\}$ by Lemma 2.3.51. Thus $\lfloor \Xi \rfloor_n^{m_1}$ and $\lfloor \Xi \rfloor_n^{m_2}$ are non-trivial congruences on $\mathbf{A}$ whose intersection $\lfloor \Xi \rfloor_n^{m_1} \cap \lfloor \Xi \rfloor_n^{m_2}$ is the identity congruence $\omega_{\mathbf{A}}$. Since this contradicts the subdirect irreducibility of $\mathbf{A}$, the only maximal elements of $\mathbf{A}$ are the elements m_1, m_2. Thus $M = \{m_1, m_2\}$. ∎

Lemma 2.3.53. *Let* $\mathbf{A}$ *be a subdirectly irreducible implicative BCS-algebra such that* $|A| > 2$*. Then* $\mu = \omega_{\mathbf{A}} \cup \{(m_1, m_2), (m_2, m_1)\} = \Xi$*, where* μ *denotes the monolith on* $\mathbf{A}$*.*

Proof. Let $\mathbf{A}$ be a subdirectly irreducible implicative BCS-algebra such that $|A| > 2$. Let $\lfloor \Xi \rfloor_{m_2}^{m_1} := \lfloor \Xi \rfloor \cup \{\langle m_1, m_2\rangle, \langle m_2, m_1\rangle\}$. By Lemma 2.3.47 and Lemma 2.3.50 $\lfloor \Xi \rfloor_{m_2}^{m_1}$ is a congruence on $\mathbf{A}$ and by Lemma 2.3.51 $\lfloor \Xi \rfloor_{m_2}^{m_1} = \omega_{\mathbf{A}} \cup \{\langle m_1, m_2\rangle, \langle m_2, m_1\rangle\}$. Since $m_1 \equiv m_2 \pmod{\mu}$ we have $\lfloor \Xi \rfloor_{m_2}^{m_1} \subseteq \mu$; conversely $\mu \subseteq \lfloor \Xi \rfloor_{m_2}^{m_1}$ because the monolith on $\mathbf{A}$ is contained in any non-trivial congruence on $\mathbf{A}$. Thus $\mu = \lfloor \Xi \rfloor_{m_2}^{m_1} = \omega_{\mathbf{A}} \cup \{\langle m_1, m_2\rangle, \langle m_2, m_1\rangle\}$.

For the remaining equivalence, let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ with universe $B := A - \{m_1, m_2\}$ and let $\Xi_{\mathbf{B}}$ be the Clifford-McLean congruence on $\mathbf{B}$.

Let $\Xi_{\mathbf{A}}$ denote the Clifford-McLean congruence on $\mathbf{A}$ and let $\Theta := \Xi_{\mathbf{B}} \cup \{\langle m_1, m_1\rangle, \langle m_1, m_2\rangle, \langle m_2, m_1\rangle, \langle m_2, m_2\rangle\}$. From Lemma 2.3.52 we infer $\Theta = \Xi_{\mathbf{A}}$, because maximal elements always lie in their own distinct $\Xi_{\mathbf{A}}$-class. But clearly $\Theta = \lfloor \Xi \rfloor \cup \{\langle m_1, m_2\rangle, \langle m_2, m_1\rangle\} = \lfloor \Xi \rfloor_{m_2}^{m_1} = \mu$. Therefore $\Xi_{\mathbf{A}} = \mu = \omega_{\mathbf{A}} \cup \{\langle m_1, m_2\rangle, \langle m_2, m_1\rangle\}$ and the lemma is proved. ∎

Let $\mathbf{A}$ be a subdirectly irreducible implicative BCS-algebra such that $|A| > 2$. Then the underlying poset $\langle A/\Xi;\ \leq^{\mathbf{A}/\Xi}\rangle$ of the maximal implicative BCK-algebra homomorphic image $\mathbf{A}/\Xi$ is a Boolean lattice by Lemma 2.3.47 and Corollary 1.6.22. From Proposition 2.3.31 and Lemma 2.3.53 we deduce that the underlying poset $\langle A;\ \leq\rangle$ of $\mathbf{A}$ is order isomorphic to a Boolean lattice with its unit element replaced by a two-element clique. This forces the following result.

Theorem 2.3.54. *A non-trivial implicative BCS-algebra* $\mathbf{A}$ *is subdirectly irreducible iff* $\mathbf{A}$ *is isomorphic to* $\hat{\mathbf{B}}_0$ *or* $\mathbf{A}$ *is isomorphic to* $\hat{\mathbf{B}}$ *for some non-trivial Boolean algebra* $\mathbf{B}$.

Corollary 2.3.55. *An implicative BCS-algebra is subdirectly irreducible iff it is isomorphic to the canonical implicative BCS-algebra polynomial reduct of a subdirectly irreducible pseudocomplemented semilattice.*

Recall that for a class K of similar algebras $\mathsf{K}_{\mathsf{FIN}}$ denotes the subclass of finite members of K.

Corollary 2.3.56. *(cf. [70, Corollary 6.2]) The variety of implicative BCS-algebras is locally finite. Thus* $\mathsf{iBCS} = \mathbf{V}(\mathsf{iBCS}_{\mathsf{FIN}})$*; that is,* iBCS *is generated as a variety by its finite members.*

Proof. Let $\mathbf{A}$ be a finitely generated (say n generated) implicative BCS-algebra. Then each subdirectly irreducible homomorphic image of $\mathbf{A}$ is finitely generated. By Theorem 2.3.54 these finitely generated subdirectly irreducible homomorphic images are all finite; moreover to within isomorphism there are only finitely many such images that are generated by n or fewer elements. Let V be the variety generated by this finite set of finite and subdirectly irreducible algebras. Then $\mathbf{A} \in \mathsf{V}$ and V is locally finite, since any variety generated by a finite set of finite algebras is locally finite (by [55, Theorem II§10.16]). Thus $\mathbf{A}$ is locally finite, and in particular finite. Since every finitely generated implicative BCS-algebra is finite, iBCS is locally finite as asserted. The second statement now follows, since any locally finite variety is generated as a variety by its finite members (see van Alten [229, p. 13]). ∎

2.3.57. Quasi-Bounded Implicative BCS-Algebras. A *quasi-bounded implicative BCS-algebra* is an implicative BCS-algebra that is quasi-bounded. By the remarks of §2.1.40 and §2.3.1 the class $\mathsf{iBCS}^{\mathbf{1}}$ of all quasi-bounded implicative BCS-algebras is a variety, which coincides with the generic double-pointed expansion iBCS^{+} of the variety of implicative BCS-algebras by Lemma 2.2.27. The study of quasi-bounded implicative algebras is prompted by Lemma 2.3.47, which asserts that any subdirectly irreducible implicative BCS-algebra $\hat{\mathbf{B}}$ with $|\hat{B}| > 2$ has a maximal class, and thus gives rise to a quasi-bounded algebra $\hat{\mathbf{B}}^{\mathbf{1}}$. Our investigation of quasi-bounded implicative BCS-algebras begins with the following three results, which summarise some elementary properties of these algebras.

Lemma 2.3.58. *The variety of quasi-bounded implicative BCS-algebras satisfies the following identities:*

$$x \sqcap x^{*} \approx \mathbf{0} \tag{2.72}$$

$$x^{*} \approx x^{***} \tag{2.73}$$

$$(x\backslash y)^{**} \approx x^{**}\backslash y^{**} \tag{2.74}$$

$$x^{*} \sqcap y^{*} \approx y^{*} \sqcap x^{*} \tag{2.75}$$

$$x \sqcap (x \sqcup y) \approx x. \tag{2.76}$$

Moreover, for any quasi-bounded implicative BCS-algebra $\mathbf{A}^{\mathbf{1}}$ and $a, b \in A$,

$$a \leq b \quad \textit{implies} \quad b^{*} \leq a^{*}.$$

Proof. Let $\mathbf{A}^{\mathbf{1}}$ be a quasi-bounded implicative BCS-algebra and let $a, b \in A$. For (2.72), $a \sqcap a^{*} = a\big(a(1a)\big) = aa = 0$ by (2.50) and (2.47). For (2.73), $a^{*} = 1a = 1\big(1(1a)\big) = a^{***}$ by (2.56). For (2.74), $(ab)^{**} = 1 \sqcap (ab) =$

$(1b)(1a) = (1 \sqcap a)(1 \sqcap b) = a^{**}b^{**}$ by (2.25), (2.61) and (2.57). For (2.75), $a^* \sqcap b^* = (a^*)^{**} \sqcap b^* = 1 \sqcap a^* \sqcap b^* = 1 \sqcap b^* \sqcap a^* = (b^*)^{**} \sqcap a^* = b^* \sqcap a^*$ by (2.73), (2.25), left normality, (2.25) and (2.73). For (2.76), put $\alpha := a$, $\beta := 1$ and $\gamma := a^* \sqcap b^*$. We have:

$$\begin{aligned}
a \sqcap (a \sqcup b) &= a \sqcap \big(1(a^* \sqcap b^*)\big) \\
&= \alpha \sqcap (\beta\gamma) \\
&= (\alpha\gamma)(\alpha\beta) && \text{by (2.61)} \\
&= (\alpha \sqcap \beta)(\alpha \sqcap \gamma) && \text{by (2.57)} \\
&= (a \sqcap 1)\big(a \sqcap (a^* \sqcap b^*)\big) \\
&= (a \sqcap 1)\big((a \sqcap a^*) \sqcap b^*\big) \\
&= (a \sqcap 1)(0 \sqcap b^*) && \text{by (2.72)} \\
&= (a \sqcap 1)0 && \text{by (2.22)} \\
&= (a \sqcap 1) && \text{by (2.6)} \\
&= a && \text{by (2.24).}
\end{aligned}$$

For the final assertion of the lemma, suppose $a \leq b$. We have $a^* \sqcap b^* = (1a) \sqcap (1b) = (1a)\big(1(1b)\big) = \big(1(1(1b))\big)a = (1b)a = (1b)\big(b(ba)\big) = 1b = b^*$ by (2.59), (2.48), (2.56) and (2.54). Since $b^* \sqcap a^* = a^* \sqcap b^*$ (by (2.75)) we deduce $b^* \leq a^*$ as required. ∎

Lemma 2.3.59. *cf. ([10, Proposition 4.1]) The variety of quasi-bounded implicative BCS-algebras satisfies the following identities:*

$$x \sqcup y \approx y \sqcup x \tag{2.77}$$

$$x \sqcup (y \sqcup z) \approx (x \sqcup y) \sqcup z. \tag{2.78}$$

Moreover, for any quasi-bounded implicative BCS-algebra $\mathbf{A}^1$ and $a, b, c \in A$,

$$b \leq c \quad \textit{implies} \quad a \sqcup b \leq a \sqcup c.$$

Proof. Let $\mathbf{A}^1$ be a quasi-bounded implicative BCS-algebra and let $a, b, c \in A$. For (2.77) we have $a \sqcup b = (a^* \sqcap b^*)^* = (b^* \sqcap a^*)^* = b \sqcup a$ by (2.75). For (2.78) we have:

$$\begin{aligned}
a \sqcup (b \sqcup c) &= \left(a^* \sqcap ((b^* \sqcap c^*)^*)^*\right)^* \\
&= \left(a^* \sqcap (b^* \sqcap c^*)^{**}\right)^* \\
&= \left(a^* \sqcap (1 \sqcap (b^* \sqcap c^*))\right)^* && \text{by (2.25)} \\
&= \left((a^* \sqcap 1) \sqcap (b^* \sqcap c^*)\right)^* \\
&= (a^* \sqcap b^* \sqcap c^*)^* && \text{by (2.24).}
\end{aligned}$$

But,

$$\begin{aligned}
(a \sqcup b) \sqcup c &= \left(((a^* \sqcap b^*)^*)^* \sqcap c^*\right)^* \\
&= \left((a^* \sqcap b^*)^{**} \sqcap c^*\right)^* \\
&= \left((1 \sqcap (a^* \sqcap b^*)) \sqcap c^*\right)^* && \text{by (2.25)} \\
&= \left((1 \sqcap a^*) \sqcap (b^* \sqcap c^*)\right)^* \\
&= ((a^*)^{**} \sqcap (b^* \sqcap c^*))^* && \text{by (2.25)} \\
&= (a^* \sqcap b^* \sqcap c^*)^* && \text{by (2.73),}
\end{aligned}$$

so we conclude $a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c$ as required.

For the final assertion of the lemma, suppose $b \leq c$. Then $c^* \leq b^*$ by Lemma 2.3.58, so $a^* \sqcap c^* \leq a^* \sqcap b^*$ by Proposition 2.3.25(4), so $(a^* \sqcap b^*)^* \leq (a^* \sqcap c^*)^*$ by Lemma 2.3.58; that is to say $a \sqcup b \leq a \sqcup c$. ∎

Proposition 2.3.60. *For any quasi-bounded implicative BCS-algebra $\mathbf{A^1}$, the following assertions hold:*

1. *The polynomial reduct $\langle A;\ \sqcup, 1\rangle$ is a commutative semigroup with identity whose operation $\sqcup$ is isotone with respect to the underlying natural band partial order of $\mathbf{A^1}$;*

2. *For any $a, b \in A$, $\langle a\rangle_{\mathbf{A^1}} \vee^{\mathbf{I}(\mathbf{A^1})} \langle b\rangle_{\mathbf{A^1}} = \langle a \sqcup b\rangle_{\mathbf{A^1}}$.*

Proof. Item (1) follows as an immediate consequence of Lemma 2.3.59, (2.27) and (2.28). For Item (2), it is sufficient to show $x \sqcup y$ is a join generator term for iBCS^1. Let $\mathbf{A^1}$ be a quasi-bounded implicative BCS-algebra and let $a, b \in A$.

Put $r(x,y,z) := (x \sqcap z)\backslash(y\backslash x)$. Then $r^{\mathbf{A}^1}(a,b,0) = (a \sqcap 0)(ba) = 0(ba) = 0$ by (2.21) and (2.4) and $r^{\mathbf{A}^1}(a,b,a \sqcup b) = \big(a \sqcap (a \sqcup b)\big)(ba) = a(ba) = a$ by (2.76) and (2.50). Similarly $t^{\mathbf{A}^1}(a,b,0) = 0$ and $t^{\mathbf{A}^1}(a,b,a \sqcup b) = b$ for $t(x,y,z) := (y \sqcap z)\backslash(x\backslash y)$. Since $0 \sqcup 0 = 0^{**} = 1 \sqcap 0 = 0$ by (2.26) and (2.25), we have that $\sqcup$ is a join generator term for iBCS^1 by Proposition 1.7.13. ∎

Example 2.3.61. (*cf.* [10, Example 4.4]) Let $\mathbf{A} := \langle A;\ \wedge, ^*, 0\rangle$ be a pseudocomplemented semilattice with greatest element $1 := 0^*$. For any $a, b \in A$, let $a\backslash b := a \wedge b^*$ as per Example 2.3.12. The following assertions hold:

1. The polynomial reduct $\langle A;\ \backslash, 0, 1\rangle$ is a quasi-bounded implicative BCS-algebra;

2. For any $a, b \in A$,

$$\begin{aligned} a \sqcap b &= a \wedge b^{**}, \\ (a^*)^{\langle A;\,\backslash,0,1\rangle} &= (a^*)^{\mathbf{A}}, \\ a \sqcup b &= (a^* \wedge b^*)^*. \end{aligned}$$

Proof. For (1), by Example 2.3.12 it is sufficient to show $a \preceq 1$ for any $a \in A$. Let $a \in A$. We have $a\backslash 1 = a \wedge 1^* = a \wedge 0^{**} = a \wedge 0 = 0$ by (1.11), so $a \preceq 1$ and $\langle A;\ \backslash, 0, 1\rangle$ is quasi-bounded.

For (2) let $a, b \in A$. We have $a \sqcap b = a\backslash(a\backslash b) = a \wedge (a \wedge b^*)^* = a \wedge b^{**}$ by (1.9). Also $(a^*)^{\langle A;\,\backslash,0,1\rangle} = 1\backslash a = 1 \wedge (a^*)^{\mathbf{A}} = (a^*)^{\mathbf{A}}$. Hence $a \sqcup b = (a^* \sqcap b^*)^* = (a^* \wedge b^{***})^* = (a^* \wedge b^*)^*$ by (2.73). ∎

Example 2.3.61 and the theory of pseudocomplemented semilattices motivate the following definitions. For any quasi-bounded implicative BCS-algebra $\mathbf{A}^1$ let the *skeleton* of $\mathbf{A}^1$ be:

$$\mathrm{S}(\mathbf{A}^1) := \{a^* : a \in A\}.$$

Also, define the *dense set* of $\mathbf{A}^1$ to be:

$$\mathrm{D}(\mathbf{A}^1) := \{a : a^* = 0\}.$$

Call $\mathbf{A}^1$ *dense* if $\mathrm{D}(\mathbf{A}^1) = A - \{0\}$. The following two technical lemmas collect together some useful properties of the skeleton $\mathrm{S}(\mathbf{A}^1)$ and dense set $\mathrm{D}(\mathbf{A}^1)$.

Lemma 2.3.62. *For any quasi-bounded implicative BCS-algebra* $\mathbf{A}^1$ *and* $a, b \in A$*, the following assertions hold:*

1. $0 \in \mathrm{S}(\mathbf{A}^1)$ *and* $1 \in \mathrm{S}(\mathbf{A}^1)$*;*

2. $a \in \mathrm{S}(\mathbf{A}^1)$ *iff* $a = a^{**}$*;*

3. $a, b \in \mathrm{S}(\mathbf{A}^1)$ *implies* $ab \in \mathrm{S}(\mathbf{A}^1)$.

Proof. Let $\mathbf{A}^1$ be a quasi-bounded implicative BCS-algebra and let $a, b \in A$. For (1) just note $0 = 11 = 1^* \in \mathrm{S}(\mathbf{A}^1)$ by (2.3) and $1 = 10 = 0^* \in \mathrm{S}(\mathbf{A}^1)$ by (2.6). For (2), suppose $a \in \mathrm{S}(\mathbf{A}^1)$. Then $a = c^*$ for some $c \in A$, and $a^{**} = (c^*)^{**} = c^* = a$ by (2.73). Conversely, suppose $a = a^{**}$. Then $a = c^*$ with $c := a^*$ so $a \in \mathrm{S}(\mathbf{A}^1)$. For (3) suppose $a, b \in \mathrm{S}(\mathbf{A}^1)$. Then $a = a^{**}$ and $b = b^{**}$ by (2) and so $ab = a^{**}b^{**} = (ab)^{**}$ by (2.74); from (2) we conclude $ab \in \mathrm{S}(\mathbf{A}^1)$ as desired. ∎

Lemma 2.3.63. *For any quasi-bounded implicative BCS-algebra* $\mathbf{A}^1$ *and* $a, b \in A$*, the following assertions hold:*

1. $1 \in \mathrm{D}(\mathbf{A}^1)$*;*

2. $a \in \mathrm{D}(\mathbf{A}^1)$ *iff* $c \preceq a$ *for any* $c \in A$*. Thus* $\mathbf{A}^1$ *is dense iff* $\langle A; \backslash, 0\rangle$ *is flat;*

3. $a^* = b^*$ *iff* $a \equiv b \,(\mathrm{mod}\, \Xi)$.

Proof. Let $\mathbf{A}^1$ be a quasi-bounded implicative BCS-algebra and let $a, b \in A$. For (1), just note $1^* = 11 = 0$ by (2.3). For (2), suppose $a \in \mathrm{D}(\mathbf{A}^1)$ and $c \in A$. Since $a^* = 0$, we have $ca = (ca)0 = (ca)a^* = (ca)(1a) = (c1)a = 0a$ (as $c \preceq 1$) $= 0$ by (2.49). Thus $c \preceq a$. Conversely, $c \preceq a$ for any $c \in A$ implies $1 \preceq a$ in particular. Thus $1a = 0$ and $a^* = 0$. Hence $a \in \mathrm{D}(\mathbf{A}^1)$. The second assertion follows immediately.

For (3), suppose $a \equiv b \,(\mathrm{mod}\, \Xi)$. Then $ab = 0 = ba$, whence $a^* = 1a = (1a)0 = (1a)(ba) = (1b)(ab) = (1b)0 = 1b = b^*$ by (2.6), (2.55) and (2.6).

Conversely, suppose $a^* = b^*$. Then $ab = (ab)(1a) = (ab)a^* = (ab)b^* = (ab)(1b) = (a1)b = 0b$ (as $a \preceq 1$) $= 0$ by (2.52), (2.49) and (2.6). Likewise $ba = (ba)(1b) = (ba)b^* = (ba)a^* = (ba)(1a) = (b1)a = 0a$ (as $b \preceq 1$) $= 0$ by (2.52), (2.49) and (2.6). Thus $a \equiv b \,(\mathrm{mod}\, \Xi)$. ∎

For any quasi-bounded implicative BCS-algebra $\mathbf{A^1}$, the following two theorems show the skeleton $\mathrm{S}(\mathbf{A^1})$ gives an internal description of the maximal bounded implicative BCK-algebra homomorphic image $\mathbf{A^1}/\Xi$, in the sense that $\mathrm{S}(\mathbf{A^1})$ has the structure of a Boolean lattice order isomorphic to the underlying Boolean lattice of $\mathbf{A^1}/\Xi$ (recall Corollary 1.6.22).

Theorem 2.3.64. *Let $\mathbf{A^1}$ be a quasi-bounded implicative BCS-algebra with skeleton $\mathrm{S}(\mathbf{A^1})$. Then $\mathrm{S}(\mathbf{A^1})$ is a subuniverse of $\mathbf{A^1}$. Thus the underlying natural band partial ordering of $\mathbf{A^1}$ partially orders $\mathrm{S}(\mathbf{A^1})$ and makes $\mathrm{S}(\mathbf{A^1})$ into a Boolean lattice. For any $a, b \in \mathrm{S}(\mathbf{A^1})$, the meet and join in $\mathrm{S}(\mathbf{A^1})$ are respectively given by:*

$$
\begin{aligned}
a \wedge b &= a \sqcap b \\
a \vee b &= (a^* \sqcap b^*)^*.
\end{aligned}
$$

Proof. Let $\mathbf{A^1}$ be a quasi-bounded implicative BCS-algebra with skeleton $\mathrm{S}(\mathbf{A^1})$. By Lemma 2.3.62(1), $\{0,1\} \subseteq \mathrm{S}(\mathbf{A^1})$, while $a, b \in \mathrm{S}(\mathbf{A^1})$ implies $ab \in \mathrm{S}(\mathbf{A^1})$ by Lemma 2.3.62(3). Thus $\mathrm{S}(\mathbf{A^1})$ is a subuniverse of $\mathbf{A^1}$ and so inherits the underlying natural band partial ordering of $\mathbf{A^1}$. Let $a, b \in \mathrm{S}(\mathbf{A^1})$. By Lemma 2.3.62(2) $a = a^{**}$ and $b = b^{**}$, whence:

$$
\begin{aligned}
a \sqcap b &= a^{**} \sqcap b^{**} \\
&= (1 \sqcap a) \sqcap (1 \sqcap b) && \text{by (2.25)} \\
&= (1 \sqcap b) \sqcap (1 \sqcap a) && \text{by left normality} \\
&= b^{**} \sqcap a^{**} && \text{by (2.25)} \\
&= b \sqcap a,
\end{aligned}
$$

so the quasi-bounded subalgebra $\langle \mathrm{S}(\mathbf{A^1}); \backslash, 0, 1\rangle$ is a bounded implicative BCK-algebra. From Corollary 1.6.22 we have that $\langle \mathrm{S}(\mathbf{A^1}); \leq\rangle$ is a Boolean lattice, where $a \wedge b = a \sqcap b$ and $a \vee b = (a^* \sqcap b^*)^*$ for any $a, b \in \mathrm{S}(\mathbf{A^1})$. ∎

Let $\mathbf{A^1}$ be a quasi-bounded implicative BCS-algebra with skeleton $\mathrm{S}(\mathbf{A^1})$. In the statement of the following theorem and in the sequel, let $\mathrm{S}(\mathbf{A^1})$ denote the bounded implicative BCK-algebra $\langle \mathrm{S}(\mathbf{A^1});\ \backslash, 0, 1\rangle$.

Theorem 2.3.65. *(cf. [10, Proposition 4.4]) Let* $\mathbf{A^1}$ *be a quasi-bounded implicative BCS-algebra. Then* $\mathrm{S}(\mathbf{A^1})$ *is isomorphic to the maximal bounded implicative BCK-algebra homomorphic image* $\mathbf{A^1}/\Xi$ *of* $\mathbf{A^1}$ *under the map* $a \overset{h}{\mapsto} [a]_\Xi$ $(a \in \mathrm{S}(\mathbf{A^1}))$.

Proof. Let $\mathbf{A^1}$ be a quasi-bounded implicative BCS-algebra. Suppose $\overline{b} \in A/\Xi$. Then $\overline{b} = [a]_\Xi$ for some $a \in A$. Now $a^{**} \in \mathrm{S}(\mathbf{A^1})$ and $a^{**} \equiv a\,(\mathrm{mod}\,\Xi)$ by Lemma 2.2.13, so $[a^{**}]_\Xi = [a]_\Xi = \overline{b}$. Thus h is onto. Let $a, b \in \mathrm{S}(\mathbf{A^1})$ and suppose $a \equiv b\,(\mathrm{mod}\,\Xi)$. From Lemma 2.3.63(3) we have that $a^* = b^*$. Therefore $a^{**} = b^{**}$ and $a = a^{**} = b^{**} = b$ since $a = a^{**}$ and $b = b^{**}$ by Lemma 2.3.62(2). Thus h is one-to-one, and hence is a bijection between $\mathrm{S}(\mathbf{A^1})$ and A/Ξ.

Let $a, b \in \mathrm{S}(\mathbf{A})^{\mathbf{1}}$. By Lemma 2.3.62(2) we have that $a = a^{**}$ and $b = b^{**}$, so $h(a\backslash^{\mathrm{S}(\mathbf{A^1})}b) = h(a\backslash^{\mathbf{A^1}}b)$ (by Theorem 2.3.64) $= h(a^{**}\backslash^{\mathbf{A^1}}b^{**}) = [a^{**}\backslash^{\mathbf{A^1}}b^{**}]_\Xi = [a^{**}]_\Xi\backslash^{\mathbf{A^1}/\Xi}[b^{**}]_\Xi = [a]_\Xi\backslash^{\mathbf{A^1}/\Xi}[b]_\Xi = h(a)\backslash^{\mathbf{A^1}/\Xi}h(b)$. Moreover, $h(\mathbf{0}^{\mathrm{S}(\mathbf{A^1})}) = h(\mathbf{0}^{\mathbf{A^1}}) = [\mathbf{0}^{\mathbf{A^1}}]_\Xi = \mathbf{0}^{\mathbf{A^1}}/\Xi$ and $h(\mathbf{1}^{\mathrm{S}(\mathbf{A^1})}) = h(\mathbf{1}^{\mathbf{A^1}}) = [\mathbf{1}^{\mathbf{A^1}}]_\Xi = \mathbf{1}^{\mathbf{A^1}}/\Xi$. Thus h is a map from $\mathrm{S}(\mathbf{A^1})$ into $\mathbf{A^1}/\Xi$ preserving $\backslash^{\mathrm{S}(\mathbf{A^1})}$, $\mathbf{0}^{\mathrm{S}(\mathbf{A^1})}$ and $\mathbf{1}^{\mathrm{S}(\mathbf{A^1})}$; that is to say h is an isomorphism. ∎

As an application of Theorem 2.3.64 and Theorem 2.3.65 in a very natural setting, we give below a new and conceptually simple proof of the Glivenko-Frink theorem for pseudocomplemented semilattices.

Theorem 2.3.66 (Glivenko-Frink Theorem). *Let* $\mathbf{A}$ *be a pseudocomplemented semilattice with canonical quasi-bounded implicative BCS-algebra polynomial reduct* $\langle A;\ \backslash, 0, 1\rangle$. *Then* $\mathrm{S}(\mathbf{A})$ *is a subuniverse of* $\mathbf{A}$ *and so inherits the underlying semilattice partial ordering of* $\mathbf{A}$. *Moreover* $\mathrm{S}(\mathbf{A}) = \mathrm{S}(\langle A;\ \backslash, 0, 1\rangle)$ *and the semilattice partial ordering on* $\mathrm{S}(\mathbf{A})$ *and the natural band partial ordering on* $\mathrm{S}(\langle A;\ \backslash, 0, 1\rangle)$ *coincide on* $\mathrm{S}(\mathbf{A})$. *Thus the semilattice partial ordering* *on* $\mathrm{S}(\mathbf{A})$ *makes* $\mathrm{S}(\mathbf{A})$ *into a Boolean lattice, and this Boolean lattice is order isomorphic to the underlying poset of the maximal bounded implicative BCK-algebra homomorphic image of* $\langle A;\ \backslash, 0, 1\rangle$. *For any* $a, b \in \mathrm{S}(\mathbf{A})$ *we have* $a \wedge b \in \mathrm{S}(\mathbf{A})$, *and the join in* $\mathrm{S}(\mathbf{A})$ *is described by:*

$$a \vee b = (a^* \wedge b^*)^*.$$

Proof. Let $\mathbf{A}$ and $\langle A;\ \backslash, 0, 1\rangle$ be as in the statement of the theorem. Because of Theorem 2.3.64 and Theorem 2.3.65, to prove the theorem it is sufficient to show:

(i) $\mathrm{S}(\mathbf{A})$ is a subuniverse of $\mathbf{A}$;
(ii) $\mathrm{S}(\mathbf{A}) = \mathrm{S}(\langle A;\ \backslash, 0, 1\rangle)$;
(iii) The semilattice partial ordering on $\mathrm{S}(\mathbf{A})$ and the natural band partial ordering on $\mathrm{S}(\langle A;\ \backslash, 0, 1\rangle)$ coincide on $\mathrm{S}(\mathbf{A})$.

For (i), by the remarks of §1.3.5 we have both $0 \in \mathrm{S}(\mathbf{A})$ and $a, b \in \mathrm{S}(\mathbf{A})$ implies $a \wedge b \in \mathrm{S}(\mathbf{A})$. Since $\mathrm{S}(\mathbf{A})$ is closed under * by definition we have that $\mathrm{S}(\mathbf{A})$ is a subuniverse of $\mathbf{A}$ (this observation is also implicit in the original statement of the Glivenko-Frink theorem (Theorem 1.3.10)).

For (ii) just note $(a^*)^{\langle A;\ \backslash,0,1\rangle} = (a^*)^{\mathbf{A}}$ for any $a \in A$ by Example 2.3.61(2), and hence that $\mathrm{S}(\mathbf{A}) = \{(a^*)^{\mathbf{A}} : a \in A\} = \{(a^*)^{\langle A;\ \backslash,0,1\rangle}\} = \mathrm{S}(\langle A;\ \backslash, 0, 1\rangle)$.

For (iii), let $a, b \in \mathrm{S}(\mathbf{A})$. We have $a \leq^{\mathbf{A}} b$ iff $a \wedge b = a = b \wedge a$ iff $a \wedge b^{**} = a = b^{**} \wedge a$ (since $b^{**} = b$ by Lemma 2.3.62(2)) iff $a \sqcap b = a = b \sqcap a$ iff $a \leq^{\langle A;\ \backslash,0,1\rangle} b$, so the semilattice partial ordering on $\mathrm{S}(\mathbf{A})$ and the natural band partial ordering on $\mathrm{S}(\langle A;\ \backslash, 0, 1\rangle)$ coincide on $\mathrm{S}(\mathbf{A})$. ■

Remark 2.3.67. The Glivenko-Frink theorem for pseudocomplemented semilattices has been established by at least three methods different from the above: see the remark prior to [101, Theorem 1§6.4]. The present proof is direct and would seem in principle to be the most elementary, simpler even than the short proof of Katriňák [132]. ■

Let $\mathbf{A}$ be an implicative BCS-algebra. A *multiplier* on $\mathbf{A}$ is a function $f : A \to A$ such that $f(a \sqcap b) = f(a) \sqcap b$ for all $a, b \in A$. The set of all multipliers is denoted $\mathrm{M}(\mathbf{A})$. Multiplier extensions of implicative BCK-algebras have been studied by Cornish [66]; what follows here is a version of a construction due to Bignall [17, pp. 49–51]. For any functions $f, g : A \to A$, define $f\backslash g : A \to A$ and $f \sqcap g : A \to A$ pointwise by $(f\backslash g)(a) := f(a)\backslash g(a)$ and $(f \sqcap g)(a) := f(a) \sqcap g(a)$ for all $a \in A$ respectively. Now $f\backslash g \in \mathrm{M}(\mathbf{A})$ since $(f\backslash g)(a \sqcap b) = f(a \sqcap b)\backslash g(a \sqcap b) = (f(a) \sqcap b)\backslash(g(a) \sqcap b) = (f(a)\backslash g(a)) \sqcap b$ (by (2.62)) $= (f\backslash g)(a) \sqcap b$. Also $f \sqcap g \in \mathrm{M}(\mathbf{A})$, because $(f \sqcap g)(a \sqcap b) = f(a \sqcap b) \sqcap g(a \sqcap b) = f(a) \sqcap b \sqcap g(a) \sqcap b = f(a) \sqcap g(a) \sqcap b$ (by left normality) $= (f \sqcap g)(a) \sqcap b$. Define $0 \in \mathrm{M}(\mathbf{A})$ by $0(a) := 0$ and $1 \in \mathrm{M}(\mathbf{A})$ by $1(a) := a$ for all $a \in A$. Put $\mathbf{M}(\mathbf{A}) := \langle \mathrm{M}(\mathbf{A});\ \backslash, 0, 1\rangle$. Because the operations on $\mathrm{M}(\mathbf{A})$ are defined pointwise, the reduct $\langle \mathrm{M}(\mathbf{A});\ \backslash, 0\rangle$ of $\mathbf{M}(\mathbf{A})$ is an implicative BCS-algebra. Moreover, because $(f \sqcap g)(a) = f(a) \sqcap g(a) = f(a)\backslash(f(a)\backslash g(a)) = f(a)\backslash((f\backslash g)(a)) = (f\backslash(f\backslash g))(a)$, $f \sqcap g$ is the implicative BCS meet in $\langle \mathrm{M}(\mathbf{A});\ \backslash, 0\rangle$. Since $(f \sqcap 1)(a) = f(a) \sqcap 1(a) = f(a) \sqcap a = f(a \sqcap a) = f(a)$, from Lemma 2.1.42(1) we deduce $f \preceq 1$ for any $f \in \mathrm{M}(\mathbf{A})$; that is to say $\mathbf{M}(\mathbf{A})$ is a quasi-bounded implicative BCS-algebra.

Let $\mu_a : A \to A$ denote the map defined by $\mu_a(b) := a \sqcap b$. Since $\mu_a(b \sqcap c) = a \sqcap (b \sqcap c) = (a \sqcap b) \sqcap c = \mu_a(b) \sqcap c$ for all $b, c \in A$ we have $\mu(a) \in \mathrm{M}(\mathbf{A})$ for each $a \in A$. Define $\mu : A \to \mathrm{M}(\mathbf{A})$ by $\mu(a) := \mu_a$. Then μ is a homomorphism, since $\mu(a\backslash^{\mathbf{A}} b)(c) = \mu_{(a\backslash^{\mathbf{A}} b)}(c) = (a\backslash^{\mathbf{A}} b) \sqcap c = (a \sqcap c)\backslash^{\mathbf{A}}(b \sqcap c)$ (by (2.62)) $= \mu_a(c)\backslash^{\mathbf{M}(\mathbf{A})} \mu_b(c)$ and $\mu(0^{\mathbf{A}})(c) = \mu_{0^{\mathbf{A}}}(c) = 0^{\mathbf{A}} \sqcap c = 0^{\mathbf{A}} = 0(c) = 0^{\mathbf{M}(\mathbf{A})}$. Moreover, μ is one-to-one, since $\mu(a) = \mu(b)$ implies $\mu_a(c) = \mu_b(c)$ for all $c \in A$, which implies in particular that $\mu_a(a) = \mu_b(a)$ and $\mu_a(b) = \mu_b(b)$. But then $a = b \sqcap a$ and $b = a \sqcap b$, so $b = a \sqcap b = b \sqcap a \sqcap b = b \sqcap b \sqcap a$ (by left normality) $= b \sqcap a = a$. Hence μ is an isomorphism from $\mathbf{A}$ onto the subalgebra $\mu[\mathbf{A}]$ of $\mathbf{M}(\mathbf{A})$, where $\mu[\mathbf{A}]$ denotes the image of $\mathbf{A}$ under μ.

Lemma 2.3.68. *Every implicative BCS-algebra* $\mathbf{A}$ *embeds (as an implicative BCS-algebra) into its canonical quasi-bounded implicative BCS-algebra multiplier extension* $\mathbf{M}(\mathbf{A})$.

By a $\langle\backslash, 0\rangle$-*subreduct* of a pseudocomplemented semilattice $\mathbf{A} := \langle A;\ \wedge, ^*, 0\rangle$ we mean a subalgebra of the canonical implicative BCS-algebra polynomial reduct $\langle A;\ \backslash, 0\rangle$ of $\mathbf{A}$. We conclude this subsection with the following problem, suggested by the preceding lemma and Theorem 1.6.20.

Problem 2.3.69. Is an algebra $\langle A;\ \backslash, 0\rangle$ of type $\langle 2,0\rangle$ an implicative BCS-algebra iff it is a $\langle\backslash, 0\rangle$-subreduct of a pseudocomplemented semilattice? ∎

2.3.70. The Lattice of Varieties of Implicative BCS-Algebras. Problem 2.3.69 and the results of the preceding subsections suggest that the theory of implicative BCS-algebras may bear the same relationship to the theory of pseudocomplemented semilattices as the theory of implicative BCK-algebras bears to the theory of Boolean algebras. Because the canonical implicative BCS-algebra polynomial reduct of the 3-element chain $\mathbf{3}$ (considered as a pseudocomplemented semilattice) is flat, Theorem 1.3.8 and the preceding remarks call for a closer examination of the role played by the algebra $\mathbf{B}_2 := \langle\{0,1,2\};\ \backslash, 0\rangle$ of Example 2.1.5 in the variety of implicative BCS-algebras.

Lemma 2.3.71. *Let $\mathbf{B}_2^k$, $k \in \omega$, be the k-th direct power of $\mathbf{B}_2$; that is, the direct product of k copies of $\mathbf{B}_2$. Let:*

$$M := \{(c_1, \ldots, c_k) : c_i \neq 0;\ i = 1, \ldots, k\}$$

be the maximal class of $\mathbf{B}_2^k$ and let:

$$M_1 := \{c \in M : \pi_1(c) = 1\} \qquad \text{and} \qquad M_2 := \{c \in M : \pi_1(c) = 2\},$$

where π_1 denotes the first projection map. Let $\mathbf{B}$ be the subalgebra of $\mathbf{B}_2^k$ with universe $B := B_2^k - M$ and let $\Xi_{\mathbf{B}}$ denote the Clifford-McLean congruence on $\mathbf{B}$. Then the relation:

$$\Theta := \Xi_{\mathbf{B}} \cup (M_1 \times M_1) \cup (M_2 \times M_2)$$

is a congruence on $\mathbf{B}_2^k$.

Proof. Let $\mathbf{B}_2^k, M, M_1, M_2, \mathbf{B}, \Xi_{\mathbf{B}}$ and Θ be as in the statement of the lemma. Since $\{B, M_1, M_2\}$ partitions B_2^k we infer Θ is an equivalence relation on B_2^k. Moreover, clearly $\Xi_{\mathbf{B}} \subseteq \Theta \subseteq \Xi_{\mathbf{B}_2^k}$, where $\Xi_{\mathbf{B}_2^k}$ denotes the Clifford-McLean congruence on $\mathbf{B}_2^k$. Suppose $a_1 \equiv b_1 \pmod{\Theta}$ and $a_2 \equiv b_2 \pmod{\Theta}$ for $a_1, b_1, a_2, b_2 \in A$. To see Θ is a congruence we consider nine cases:

(i) $a_1, b_1 \in B$ and $a_2, b_2 \in B$;
(ii) $a_1, b_1 \in B$ and $a_2, b_2 \in M_1$;
(iii) $a_1, b_1 \in B$ and $a_2, b_2 \in M_2$;
(iv) $a_1, b_1 \in M_1$ and $a_2, b_2 \in B$;
(v) $a_1, b_1 \in M_1$ and $a_2, b_2 \in M_1$;
(vi) $a_1, b_1 \in M_1$ and $a_2, b_2 \in M_2$;
(vii) $a_1, b_1 \in M_2$ and $a_2, b_2 \in B$;
(viii) $a_1, b_1 \in M_2$ and $a_2, b_2 \in M_1$;
(ix) $a_1, b_1 \in M_2$ and $a_2, b_2 \in M_2$.

Of these cases, Cases (i), (iv) and (vii) are non-trivial. For Case (i), suppose $a_1, b_1 \in B$ and $a_2, b_2 \in B$. Then $a_1 \equiv b_1 \pmod{\Xi_{\mathbf{B}}}$ and $a_2 \equiv b_2 \pmod{\Xi_{\mathbf{B}}}$, so $a_1 a_2 \equiv b_1 b_2 \pmod{\Xi_{\mathbf{B}}}$, which implies $a_1 a_2 \equiv b_1 b_2 \pmod{\Theta}$.

Cases (iv) and (vii) are analogous, so we show only Case (iv). For Case (iv), we distinguish two subcases:

(iv)(a) $a_1, b_1 \in M_1$ and $a_2 = b_2 = 0 \in B$;
(iv)(b) $a_1, b_1 \in M_1$ and $0 \neq a_2 \in B$, $0 \neq b_2 \in B$.

Of these subcases, only Subcase (iv)(b) is non-trivial. So suppose $a_1, b_1 \in M_1$ and $0 \neq a_2 \in B$, $0 \neq b_2 \in B$. Then $a_1 \equiv b_1 \pmod{\Xi_{\mathbf{B}_2^k}}$ (since $a_1, b_1 \in M$) and $a_2 \equiv b_2 \pmod{\Xi_{\mathbf{B}_2^k}}$ (since $a_2 \equiv b_2 \pmod{\Xi_{\mathbf{B}}}$), so $a_1 a_2 \equiv b_1 b_2 \pmod{\Xi_{\mathbf{B}_2^k}}$. Because $a_1 a_2 \notin M$ and $b_1 b_2 \notin M$ (by Lemma 2.3.48) we have that $a_1 a_2 \equiv b_1 b_2 \pmod{\Xi_{\mathbf{B}}}$, which implies $a_1 a_2 \equiv b_1 b_2 \pmod{\Theta}$. ∎

Lemma 2.3.72. *Let $\mathbf{B}_2^k$, $k \in \omega$, be the k-th direct power of $\mathbf{B}_2$ and let Θ be the congruence relation on $\mathbf{B}_2^k$ of Lemma 2.3.71. Then the quotient algebra $\mathbf{B}_2^k/\Theta$ is isomorphic to $\hat{\mathbf{B}}_k$, the finite and subdirectly irreducible implicative BCS-algebra of cardinality $2^k + 1$.*

Proof. Let $\mathbf{B}_2^k$ and Θ be an in the statement of the lemma. To prove the lemma it is sufficient to show:

(i) $\mathbf{B}_2^k/\Theta$ is subdirectly irreducible;
(ii) $|B_2^k/\Theta| = 2^k + 1$.

For (i), for ease of notation let $\mathbf{B} := \mathbf{B}_2^k/\Theta$ and let $\Xi_{\mathbf{B}}$ denote the Clifford-McLean congruence on $\mathbf{B}$. Now $\Xi_{\mathbf{B}} = \Xi_{\mathbf{B}_2^k/\Theta} = \Xi_{\mathbf{B}_2^k}/\Theta$ (where $\Xi_{\mathbf{B}_2^k/\Theta}$ and $\Xi_{\mathbf{B}_2^k}$ denote the Clifford-McLean congruence on $\mathbf{B}_2^k/\Theta$ and $\mathbf{B}_2^k$ respectively), so

$\Xi_{\mathbf{B}} = \omega_{\mathbf{B}} \cup \{\langle M_1, M_2\rangle, \langle M_2, M_1\rangle\}$, where M_1 and M_2 are as in Lemma 2.3.71. Therefore to see $\mathbf{B}$ is subdirectly irreducible (with monolith $\Xi_{\mathbf{B}}$) it is sufficient to show $M_1 \equiv_\theta M_2$ for any $\theta \in \operatorname{Con}\mathbf{B}$ such that $\theta \neq \omega_{\mathbf{B}}$. So let $\theta \in \operatorname{Con}\mathbf{B}$ be such that $\theta \neq \omega_{\mathbf{B}}$. There are two cases to consider:

(1) $[0^{\mathbf{B}}]_\theta = \{0^{\mathbf{B}}\}$;
(2) $[0^{\mathbf{B}}]_\theta \neq \{0^{\mathbf{B}}\}$.

For Case (1), suppose $[0^{\mathbf{B}}]_\theta = \{0^{\mathbf{B}}\}$. Then $\theta \subseteq \Xi_{\mathbf{B}} = \omega_{\mathbf{B}} \cup \{\langle M_1, M_2\rangle, \langle M_2, M_1\rangle\}$ by Theorem 2.1.14(4). Assume to the contrary that $M_1 \not\equiv_\theta M_2$. Then $\theta = \omega_{\mathbf{B}}$ which is a contradiction. Thus $M_1 \equiv_\theta M_2$.

For Case (2), suppose $[0^{\mathbf{B}}]_\theta \neq \{0^{\mathbf{B}}\}$. Then there exists $0^{\mathbf{B}} \neq A \in B$ (where $B := B_2^k/\Theta$) such that $0^{\mathbf{B}} \equiv_\theta A$. We consider two subcases:

(2)(a) $A \notin \{M_1, M_2\}$;
(2)(b) $A \in \{M_1, M_2\}$.

For Subcase (2)(a), suppose $A \notin \{M_1, M_2\}$. From $M_1 \equiv M_2 \pmod{\Xi_{\mathbf{B}}}$ and Lemma 2.1.12(1) we have $M_1 \backslash^{\mathbf{B}} A \equiv M_2 \backslash^{\mathbf{B}} A \pmod{\Xi_{\mathbf{B}}}$. But $M_1 \backslash^{\mathbf{B}} A \notin M$ and $M_2 \backslash^{\mathbf{B}} A \notin M$ by Lemma 2.3.48, where M is the maximal class of $\mathbf{B}$, which implies by the description of the Clifford-McLean congruence $\Xi_{\mathbf{B}}$ that $M_1 \backslash^{\mathbf{B}} A = M_2 \backslash^{\mathbf{B}} A$. But then $M_1 = M_1 \backslash^{\mathbf{B}} 0^{\mathbf{B}} \equiv_\theta M_1 \backslash^{\mathbf{B}} A = M_2 \backslash^{\mathbf{B}} A \equiv_\theta M_2 \backslash^{\mathbf{B}} 0^{\mathbf{B}} = M_2$, whence $M_1 \equiv_\theta M_2$.

For Subcase (2)(b), suppose $A \in \{M_1, M_2\}$. Let $C \in \{M_1, M_2\}$ be such that $C \neq A$. From $C \equiv A \pmod{\Xi_{\mathbf{B}}}$ we have $C = C\backslash^{\mathbf{B}} 0^{\mathbf{B}} \equiv_\theta C\backslash^{\mathbf{B}} A = 0^{\mathbf{B}} \equiv_\theta A$ as required. This completes the proof that $\mathbf{B}$ is subdirectly irreducible.

For (ii), just note $|B| = |B_2^k/\Theta| = |B_2^k/\Xi_{\mathbf{B}_2^k}| - 1 + 2$. But $\mathbf{B}_2^k/\Xi_{\mathbf{B}_2^k} \cong (\mathbf{B}_2/\Xi_{\mathbf{B}_2})^k$, so $|B_2^k/\Xi_{\mathbf{B}_2^k}| = 2^k$, which implies $|B| = |B_2^k/\Theta| = 2^k - 1 + 2 = 2^k + 1$. Thus $\mathbf{B}$ has cardinality $2^k + 1$ as asserted. ∎

Theorem 2.3.73. *The 3-element flat implicative BCS-algebra* $\mathbf{B}_2$ *generates the class of implicative BCS-algebras (as a variety). In symbols,* $\mathsf{iBCS} = \mathbf{V}(\mathbf{B}_2)$.

Proof. Since $\mathbf{B}_2 \in \mathsf{iBCS}$ we have $\mathbf{V}(\mathbf{B}_2) \subseteq \mathsf{iBCS}$. Conversely, by Theorem 2.3.54 we have that any finite and subdirectly irreducible implicative

BCS-algebra is isomorphic either to $\hat{\mathbf{B}}_0$ or to some $\hat{\mathbf{B}}_n$, $n \in \omega$. Now $\mathbf{B}_2/\Xi$ is isomorphic to $\hat{\mathbf{B}}_0$, so $\hat{\mathbf{B}}_0 \in \mathsf{H}(\mathbf{B}_2)$. For any $k \geq 1$, let $\mathbf{B}_2^k$ be the k-th direct power of $\mathbf{B}_2$ and let Θ be the relation defined on B_2^k of Lemma 2.3.71. By Lemma 2.3.71 and Lemma 2.3.72 we have that Θ is a congruence relation on $\mathbf{B}_2^k$ such that $\mathbf{B}_2^k/\Theta$ is isomorphic to $\hat{\mathbf{B}}_k$, so $\hat{\mathbf{B}}_k \in \mathsf{HP}(\mathbf{B}_2)$. Suppose now that $\mathbf{A}$ is a finite implicative BCS-algebra. Then $\mathbf{A}$ is isomorphic to a subdirect product of finite and subdirectly irreducible implicative BCS-algebras, and so $\mathbf{A} \in \mathsf{IP_sHPH}(\mathbf{B}_2)$. But $\mathsf{IP_sHPH} \leq \mathsf{IP_sHHP}$ (by [160, Lemma 4.92]) $= \mathsf{IP_sHP} \leq \mathsf{HP_sHP} \leq \mathsf{HHP_sP}$ (by [160, Lemma 4.92]) $= \mathsf{HP_sP} = \mathsf{HP_s}$ (by [160, Lemma 4.92]) $\leq \mathsf{HSP}$, so $\mathbf{A} \in \mathsf{V}(\mathbf{B}_2)$. Thus $\mathsf{V}(\mathbf{B}_2)$ contains every finite implicative BCS-algebra; that is to say $\mathsf{iBCS_{FIN}} \subseteq \mathsf{V}(\mathbf{B}_2)$. But this implies $\mathsf{V}(\mathsf{iBCS_{FIN}}) \subseteq \mathsf{V}(\mathbf{B}_2)$, which implies $\mathsf{iBCS} \subseteq \mathsf{V}(\mathbf{B}_2)$ by Corollary 2.3.56. ∎

Corollary 2.3.74. *The following assertions hold in the variety of implicative BCS-algebras:*

1. *The equational theory of the variety of implicative BCS-algebras is decidable;*

2. *The first-order theory of the variety of implicative BCS-algebras is undecidable.*

Proof. Item (1) follows from local finiteness (Corollary 2.3.56) and a well known result due to Harrop [107] (see also Blok and Ferreirim [27, Lemma 3.13]) to the effect that a variety V of algebras over a finite language has a decidable equational theory if V is finitely axiomatisable and is generated (as a variety) by its finite members. Item (2) follows immediately from the remarks of Example 2.1.5 and Theorem 2.3.73. ∎

Let V be a variety. Denote by $\Lambda^V(\mathsf{V})$ the 'lattice of varieties' of V, namely the dual of the lattice of corresponding equational theories (see for example Grätzer [99, p. 172]). For pseudocomplemented semilattices, it follows easily from the fact that the 3-element chain $\mathbf{3}$ (considered as a pseudocomplemented semilattice) generates PCSL as a variety (recall Theorem 1.3.8) that $\Lambda^V(\mathsf{PCSL})$ is a 3-element chain, whose unique atom $\{\mathbf{A} \in \mathsf{PCSL} : \mathbf{A} \models x^{**} \approx x\}$ is termwise definitionally equivalent to the variety of Boolean algebras (recall Theorem 1.3.9). This remark, in conjunction with Theorem 2.3.73, calls for a study of $\Lambda^V(\mathsf{iBCS})$, the lattice of varieties of implicative BCS-algebras. But first, concerning the lattice of varieties of pre-BCK-algebras $\Lambda^V(\mathsf{PBCK})$, recall iBCK is the unique atom of $\Lambda^V(\mathsf{PBCK})$ by the remarks of Example 2.1.4.

Theorem 2.3.75. *$\Lambda^V(\mathsf{iBCS})$ is a three-element chain. The only non-trivial subvariety of iBCS is iBCK, the unique atom of $\Lambda^V(\mathsf{PBCK})$. Thus iBCS is a cover of iBCK in $\Lambda^V(\mathsf{PBCK})$ (in fact, is the only cover of iBCK in $\Lambda^V(\mathsf{PBCK})$ that is not a variety of BCK-algebras).*

Proof. Let $\mathsf{V} \subseteq \mathsf{iBCS}$. If $\mathsf{V} \subseteq \mathsf{BCK}$ then $\mathsf{V} = \mathsf{iBCK}$ by Proposition 2.3.7(2). If $\mathsf{V} \not\subseteq \mathsf{BCK}$ then $\mathbf{Q}(\mathbf{B}_2) \subseteq \mathsf{V}$ by Proposition 2.2.5, so $\mathsf{iBCS} = \mathbf{V}(\mathbf{B}_2) \subseteq \mathsf{V}$ by Theorem 2.3.73. Thus iBCS is a cover of iBCK in $\Lambda^V(\mathsf{PBCK})$ (in fact, is the only cover of iBCK in $\Lambda^V(\mathsf{PBCK})$ that is not a variety of BCK-algebras). ∎

By Example 2.1.4 iBCK is also the unique atom of $\Lambda^Q(\mathsf{PBCK})$, the lattice of quasivarieties of pre-BCK-algebras. However, Theorem 2.3.75 cannot be generalised to the assertion that iBCS covers iBCK in $\Lambda^Q(\mathsf{PBCK})$, in view of the following result of Blok and Raftery [38]. The proof given here is new.

Proposition 2.3.76. *[38, Proposition 6] The quasivariety $\mathbf{Q}(\mathbf{B}_2)$ is not a variety. Thus iBCS is not a cover of iBCK in $\Lambda^Q(\mathsf{PBCK})$.*

Proof. Assume to the contrary that $\mathbf{Q}(\mathbf{B}_2)$ is a variety. By Theorem 2.3.73 and hypothesis we have $\mathsf{iBCS} = \mathbf{V}(\mathbf{B}_2) = \mathbf{Q}(\mathbf{B}_2) = \mathbf{ISP}(\mathbf{B}_2)$, which implies $\mathsf{iBCS}_{\mathrm{SI}} \subseteq \mathbf{IS}(\mathbf{B}_2)$, a contradiction. The remaining assertion now follows. ∎

Blok and Raftery first proved Proposition 2.3.76 by showing that the algebra $\mathbf{B}_2 \times \mathbf{C}_1$ has a homomorphic image isomorphic to $\hat{\mathbf{B}}_2$, and hence that $\mathbf{H}(\mathbf{B}_2 \times \mathbf{C}_1) \not\subseteq \mathbf{IPs}(\mathbf{B}_2, \mathbf{C}_1)$. Thus they do not exhibit a quasi-identity satisfied by $\mathbf{Q}(\mathbf{B}_2)$ but not by iBCS. We also have been unable to exhibit such a quasi-identity.

Problem 2.3.77. Exhibit a quasi-identity satisfied by $\mathbf{Q}(\mathbf{B}_2)$ but not by iBCS.
∎

For BCK-algebras, the partially ordered 'set' $\mathrm{P}^V(\mathsf{BCK})$ of all subvarieties of BCK has been investigated by several authors, including Wroński [239], Wroński and Kabziński [243], Blok and Raftery [38] and Kowalski [142] (see also Van Alten [229, Chapter 6] and van Alten and Raftery [231]), and in particular it is known that $\mathbf{P}^V(\mathsf{BCK}) := \langle \mathrm{P}^V(\mathsf{BCK}); \subseteq \rangle$ is a (distributive) lattice with unique atom iBCK [38, Theorem 11]. Let $\mathbf{Ł}_3 := \langle \{0,1,2\}; \dot{-}, 0 \rangle$ be the BCK-algebra with $0 \leq 1 \leq 2$ and $2 \dot{-} 1 := 1$ and let $\mathbf{H}_3 := \langle \{0,1,2\}; \dot{-}, 0 \rangle$ be the BCK-algebra with $0 \leq 1 \leq 2$ and $2 \dot{-} 1 := 2$. The algebras $\mathbf{Ł}_3$ and $\mathbf{H}_3$ are dually isomorphic to the implicational reducts of the 3-element Lukasiewicz algebra and the 3-element linearly ordered Heyting algebra, respectively, and by Jónsson's lemma $\mathsf{V}(\mathbf{Ł}_3)$ and $\mathsf{V}(\mathbf{H}_3)$ are covers of iBCK in the lattice $\mathbf{P}^V(\mathsf{BCK})$. By results due to Kowalski [142] the converse also holds, and so $\mathsf{V}(\mathbf{Ł}_3)$, $\mathsf{V}(\mathbf{H}_3)$ are the *only* covers of iBCK in $\mathbf{P}^V(\mathsf{BCK})$.

Theorem 2.3.78. *The varieties* $\mathsf{V}(\mathbf{Ł}_3)$, $\mathsf{V}(\mathbf{H}_3)$ *and* iBCS *are the only covers of* iBCK *in* $\mathbf{\Lambda}^V(\mathsf{PBCK})$.

Proof. Let V be a cover of iBCK in $\mathbf{\Lambda}^V(\mathsf{PBCK})$. By Proposition 2.2.5 either $\mathsf{V} \in \mathrm{P}^V(\mathsf{BCK})$ or $\mathbf{Q}(\mathbf{B}_2) \subseteq \mathsf{V}$. If $\mathbf{Q}(\mathbf{B}_2) \subseteq \mathsf{V}$ then V is iBCS by Theorem 2.3.75. So suppose $\mathsf{V} \in \mathrm{P}^V(\mathsf{iBCK})$. Since $\mathbf{P}^V(\mathsf{BCK})$ is a sublattice of $\mathbf{\Lambda}^V(\mathsf{PBCK})$ (by [38, Theorem 11]) we must have that V is a cover of iBCK in $\mathbf{P}^V(\mathsf{BCK})$. Thus V is either $\mathsf{V}(\mathbf{Ł}_3)$ or $\mathsf{V}(\mathbf{H}_3)$. ∎

Chapter 3

Applications to Universal Algebra and Algebraic Logic

In this chapter we consider applications of the theory of pre-BCK-algebras to universal algebra and algebraic logic. In particular, we study three classes of algebras arising naturally in both universal algebra and algebraic logic, namely: subtractive varieties with EDPI; binary (and dual binary) discriminator varieties; and (pointed ternary discriminator varieties *qua*) pre-BCK-algebras structurally enriched with band operations. Our motivation for studying subtractive varieties with EDPI stems from Theorem 2.2.20 and the fundamental role played by MINI-algebras in such varieties (recall the remarks of §1.1.1). Our study of binary discriminator varieties is stimulated by Example 2.3.11 and Example 2.3.13; recall these results collectively assert that implicative BCS-algebras arise naturally as polynomial reducts of members of binary and pointed ternary discriminator varieties. Our investigation of pre-BCK-algebras structurally enriched with band operations is prompted by Theorem 1.4.39 and Corollary 1.4.40, which show that, in a sense, the study of pointed ternary discriminator varieties reduces to the study of skew Boolean $\cap$-algebras.

In investigating each of the three classes described above, our main aim is to establish the role played (if any), at a structural level, by the theory of pre-BCK-algebras. In particular, our object (with occasional diversions) is to ascertain the extent to which the ideal theory and/or congruence structure of subtractive varieties with EDPI, binary discriminator varieties, and pointed ternary discriminator varieties, may be reduced to a study of the ideal theory and/or congruence structure of the varieties of MINI-algebras and implicative BCS-algebras (possibly structurally enriched with additional operations) respectively.

3.1 Subtractive Varieties with EDPI

By the remarks of Example 1.7.11, the variety of MINI-algebras is a natural generalisation of the variety of Hilbert algebras to the subtractive but not point-regular case, and hence is a natural example of a subtractive variety with EDPI. Recently Agliano and Ursini have shown [10, Corollary 3.8] that the variety of MINI-algebras is in fact a paradigm for subtractive varieties with EDPI in the sense that a variety V is subtractive with EDPI iff every member of V has a MINI-algebra polynomial reduct satisfying a certain weak 'ideal compatibility property'. Insofar as the results of §2.2.16 show that the variety of positive implicative pre-BCK-algebras is both a natural generalisation of the variety of positive implicative BCK-algebras to the subtractive but not point regular case and a natural example of a subtractive variety with EDPI, the central role played by MINI-algebras in the theory of subtractive varieties with EDPI calls for a study of the role played by positive implicative pre-BCK-algebras in the theory of subtractive varieties with EDPI.

In §3.1.1 positive implicative pre-BCK-algebras, MINI-algebras and subtractive varieties with EDPI are studied. It is shown that the variety of positive implicative pre-BCK-algebras is termwise definitionally equivalent to (in fact, is dually isomorphic to) Agliano and Ursini's variety of MINI-algebras. We show that a variety V is subtractive with EDPI iff every algebra $\mathbf{A} \in \mathsf{V}$ has a MINI-algebra polynomial reduct whose ideals coincide with those of $\mathbf{A}$, sharpening the result of Agliano and Ursini alluded to above. A representation theorem for weakly congruence orderable subtractive varieties with EDPI is also proved: for a suitable notion of weakly compatible operation, a variety is weakly congruence orderable and subtractive with EDPI iff it is termwise definitionally equivalent to a variety of MINI-algebras with weakly compatible operations.

Subtractive WBSO varieties are studied in §3.1.22. Such varieties arise naturally in algebraic logic and are subtractive (by definition) and have EDPI (since they have EDPC). The subtractive WBSO varieties are characterised: they are precisely the subtractive, strongly point regular varieties with EDPC. We also show that any such variety V is distinguished as a WBSO variety by the presence of a weak relative pseudocomplementation $\rightarrow$ such that the polynomial reduct $\langle A;\ \rightarrow^{\mathbf{A}}, 1\rangle$ is a MINI-algebra for any $\mathbf{A} \in \mathsf{V}$. An interesting example of a subtractive WBSO variety is the variety N of Nelson algebras, which arises in the first instance from the algebraisation of constructive logic with strong negation. It is shown that N has a commutative (but not regular) TD term and is congruence permutable. An explicit QD term for N is also given. In consequence we infer that, for any variety V of Nelson algebras, the class of implicative subreducts of V is a subvariety of the variety of MINI-algebras. The results answer a question of Blok and Pigozzi.

3.1.1. Positive Implicative Pre-BCK Algebras and MINI-Algebras. Recall from Example 1.7.11 that a MINI-algebra is an algebra $\langle A;\ \rightarrow, 1\rangle$ of type $\langle 2, 0\rangle$ satisfying the following identities:

$$x \rightarrow \mathbf{1} \approx \mathbf{1} \tag{1.45}$$

$$\mathbf{1} \rightarrow x \approx x \tag{1.46}$$

$$\big(x \rightarrow (y \rightarrow z)\big) \rightarrow \big((x \rightarrow y) \rightarrow (x \rightarrow z)\big) \approx \mathbf{1} \tag{1.47}$$

$$x \rightarrow (y \rightarrow x) \approx \mathbf{1} \tag{1.48}$$

while a Hilbert algebra is a MINI-algebra satisfying the quasi-identity:

$$x \rightarrow y \approx \mathbf{1} \,\&\, y \rightarrow x \approx \mathbf{1} \supset x \approx y. \tag{1.49}$$

By definition, the class MINI of all MINI-algebras is a variety. By the remarks of §1.6.13, the quasivariety HI of all Hilbert algebras is also a variety, which is termwise definitionally equivalent to (in fact, is dually isomorphic to) the variety of positive implicative BCK-algebras.

By the preceding remarks, Agliano and Ursini's variety of MINI-algebras is a natural generalisation of the variety of Hilbert algebras to the subtractive but not point regular case, in the sense that HI is precisely the subquasivariety of MINI axiomatised by the quasi-identity (1.49). On the other hand, Theorem 2.2.17 shows the variety of positive implicative pre-BCK-algebras is a natural generalisation of the variety of positive implicative BCK-algebras to the subtractive but not point regular case, in the sense that pPBCK is precisely the subquasivariety of pBCK axiomatised by the quasi-identity (2.5). Hence pPBCK stands in relation to pBCK as MINI stands in relation to HI. Because the varieties HI and pBCK are dually isomorphic, this remark suggests that the varieties MINI and pPBCK may themselves be dually isomorphic.

To clarify the relationship between the variety of positive implicative pre-BCK-algebras and the variety of MINI-algebras, let the class MINI^D of *dual MINI-algebras* be the variety of algebras with language $\langle \dot{-}, \mathbf{0} \rangle$ of type $\langle 2, 0 \rangle$ axiomatised by the following identities:

$$\mathbf{0} \dot{-} x \approx \mathbf{0} \tag{3.1}$$

$$x \dot{-} \mathbf{0} \approx x \tag{3.2}$$

$$\big((x \dot{-} y) \dot{-} (z \dot{-} y)\big) \dot{-} \big((x \dot{-} z) \dot{-} y\big) \approx \mathbf{0} \tag{3.3}$$

$$(x \dot{-} y) \dot{-} x \approx \mathbf{0}. \tag{3.4}$$

Let $\mathrm{T}_{\mathcal{L}}(\mathrm{X})$ denote the term algebra of type $\mathcal{L}$ over X, where $\mathcal{L}$ is the language of MINI-algebras. Also, let $\mathrm{T}_{\mathcal{L}^D}(\mathrm{X})$ denote the term algebra of type $\mathcal{L}^D$ over X, where $\mathcal{L}^D$ is the language of dual MINI-algebras. Consider the maps $\xi : \mathrm{T}_{\mathcal{L}}(\mathrm{X}) \to \mathrm{T}_{\mathcal{L}^D}(\mathrm{X})$ and $\eta : \mathrm{T}_{\mathcal{L}^D}(\mathrm{X}) \to \mathrm{T}_{\mathcal{L}}(\mathrm{X})$ defined respectively by:

$$\begin{aligned} \xi(\mathbf{1}) &:= \mathbf{0} && \\ \xi(x) &:= x && x \in \mathrm{X} \\ \xi(p \to q) &:= \xi(q) \dot{-} \xi(p) && p, q \in \mathrm{T}_{\mathcal{L}}(\mathrm{X}) \end{aligned}$$

and:

$$\begin{aligned} \eta(0) &:= \mathbf{1} & & \\ \eta(x) &:= x & & x \in \mathrm{X} \\ \eta(r \mathbin{\dot{-}} s) &:= \eta(s) \to \eta(r) & & r, s \in \mathrm{T}_{\mathcal{L}^D}(X). \end{aligned}$$

(The maps ξ and η so defined should not be confused with the similar maps of §2.1.33 in the prequel.) Because of the axiomatisation of MINI by (1.45)–(1.48) and the axiomatisation of MINI^D by (3.1)–(3.4), the proof of the following lemma is trivial and so is omitted.

Lemma 3.1.2. *For $p, q \in \mathrm{T}_{\mathcal{L}}(\mathrm{X})$ and $r, s \in \mathrm{T}_{\mathcal{L}^D}(\mathrm{X})$ the following assertions hold:*

1. *If* $\mathsf{MINI} \models p \approx q$ *then* $\mathsf{MINI}^D \models \xi(p) \approx \xi(q)$;
2. *If* $\mathsf{MINI}^D \models r \approx s$ *then* $\mathsf{MINI} \models \eta(r) \approx \eta(s)$.

Moreover, $\eta \circ \xi = \omega_{\mathrm{T}_{\mathcal{L}}(\mathbf{X})}$ *and* $\xi \circ \eta = \omega_{\mathrm{T}_{\mathcal{L}^D}(\mathbf{X})}$.

By the preceding lemma, the variety of MINI-algebras is termwise definitionally equivalent to (in fact, is dually isomorphic to) the variety of dual MINI-algebras. When coupled with the following proposition, this result yields Theorem 3.1.4 below, which confirms that the varieties of positive implicative pre-BCK-algebras and MINI-algebras are indeed dually isomorphic.

Proposition 3.1.3. *An algebra $\langle A;\ \dot{-}, 0\rangle$ of type $\langle 2, 0\rangle$ is a dual MINI-algebra iff it is a positive implicative pre-BCK-algebra. Thus the variety of dual MINI-algebras coincides with the variety of positive implicative pre-BCK-algebras.*

Proof. ($\Rightarrow$) Let $\mathbf{A}$ be a dual MINI-algebra and let $a, b \in A$. Throughout the proof we denote $\dot{-}^{\mathbf{A}}$ by juxtaposition for ease of notation. By definition we have that $\mathbf{A} \models (2.4), (2.6)$, so to see $\mathbf{A}$ is a pre-BCK-algebra we have only to show (by Proposition 2.1.11) that $\mathbf{A} \models (2.7), (2.2)$. For (2.7), put $\alpha := (ab)(cb), \beta := ac$ and $\gamma := (ac)b$. We have:

$$0 = \big((\alpha\beta)(\gamma\beta)\big)\big((\alpha\gamma)\beta\big) \qquad \text{by (3.3)}$$

$$
\begin{aligned}
&= ((\alpha\beta)(\gamma\beta))\big((((ab)(cb))((ac)b))\beta\big) && \\
&= ((\alpha\beta)(\gamma\beta))(0\beta) && \text{by (3.3)} \\
&= ((\alpha\beta)(\gamma\beta))0 && \text{by (3.1)} \\
&= (\alpha\beta)(\gamma\beta) && \text{by (3.2)} \\
&= (((ab)(cb))(ac))(((ac)b)(ac)). &&
\end{aligned}
$$

Put $\alpha := ac$ and $\beta := b$. We have:

$$
\begin{aligned}
&(((ab)(cb))(ac))(((ac)b)(ac)) && \\
&\quad = (((ab)(cb))(ac))((\alpha\beta)\alpha) && \\
&\quad = (((ab)(cb))(ac))0 && \text{by (3.4)} \\
&\quad = ((ab)(cb))(ac) && \text{by (3.2).}
\end{aligned}
$$

For (2.2), put $\alpha := a(ab), \beta := b$ and $\gamma := b(ab)$. We have:

$$
\begin{aligned}
0 &= ((\alpha\beta)(\gamma\beta))(\alpha\gamma) && \text{by (2.7)} \\
&= ((\alpha\beta)(\gamma\beta))((a(ab))(b(ab))) && \\
&= ((\alpha\beta)(\gamma\beta))(((a(ab))(b(ab)))0) && \text{by (3.2)} \\
&= (((a(ab))b)((b(ab))b))(((a(ab))(b(ab)))0). &&
\end{aligned}
$$

Put $\alpha := ab$ and $\beta := 0$. We have:

$$
\begin{aligned}
&(((a(ab))b)((b(ab))b))(((a(ab))(b(ab)))0) && \\
&\quad = (((a(ab))b)((b(ab))b))(((a(ab))(b(ab)))((\alpha\beta)\alpha)) && \\
&\quad = (((a(ab))b)((b(ab))b))(((a(ab))(b(ab)))(((ab)0)(ab))) && \text{by (3.4)} \\
&\quad = (((a(ab))b)((b(ab))b))(((a(ab))(b(ab)))((ab)(ab))) && \text{by (3.2).}
\end{aligned}
$$

Put $\alpha := a, \beta := ab$ and $\gamma := b$. We have:

$$
(((a(ab))b)((b(ab))b))(((a(ab))(b(ab)))((ab)(ab)))
$$

$$
\begin{aligned}
&= \big(((a(ab))b)((b(ab))b)\big)\big(((\alpha\beta)(\gamma\beta))((\alpha\gamma)\beta)\big) & \\
&= \big(((a(ab))b)((b(ab))b)\big)0 & \text{by (3.3)}\\
&= \big((a(ab))b\big)\big((b(ab))b\big) & \text{by (3.2).}
\end{aligned}
$$

Put $\alpha := b$ and $\beta := ab$. We have:

$$
\begin{aligned}
&\big((a(ab))b\big)\big((b(ab))b\big) & \\
&\quad = \big((a(ab))b\big)\big((\alpha\beta)\alpha\big) & \\
&\quad = \big((a(ab))b\big)0 & \text{by (3.4)}\\
&\quad = \big(a(ab)\big)b & \text{by (3.2).}
\end{aligned}
$$

Hence $\mathbf{A} \models (2.2), (2.7)$ and so is a pre-BCK-algebra. To see $\mathbf{A}$ is positive implicative it is sufficient to note:

$$
\begin{aligned}
0 &= \big((ab)(bb)\big)\big((ab)b\big) & \text{by (3.3)}\\
&= \big((ab)((b0)b)\big)\big((ab)b\big) & \text{by (3.2)}\\
&= \big((ab)0\big)\big((ab)b\big) & \text{by (3.4)}\\
&= (ab)\big((ab)b\big) & \text{by (3.2).}
\end{aligned}
$$

($\Leftarrow$) Let $\mathbf{A}$ be a positive implicative pre-BCK-algebra. By definition we have that $\mathbf{A} \models (3.1), (3.2)$. Moreover $\mathbf{A} \models (3.3)$ by Lemma 2.2.18; since $\mathbf{A} \models (3.4)$ by Lemma 2.1.12(3) we have that $\mathbf{A}$ is a dual MINI-algebra. ∎

Theorem 3.1.4. *The variety of MINI-algebras is termwise definitionally equivalent to the variety of positive implicative pre-BCK algebras. Given a positive implicative pre-BCK-algebra $\langle A;\ \dot{-}, 0\rangle$, MINI-algebra operations are defined on A by:*

$$a \to b := b \dot{-} a$$

$$1 := 0$$

for any $a, b \in A$. Conversely, given a MINI-algebra $\langle A;\ \to, 1\rangle$, positive im-

plicative pre-BCK-algebra operations are defined on A by:

$$a \dot{-} b := b \to a$$

$$0 := 1$$

for any $a, b \in A$.

Because the variety of positive implicative pre-BCK-algebras has EDPI witness $x \dot{-} y$ (by Theorem 2.2.20), from Theorem 3.1.4 it follows immediately that the variety of MINI-algebras is subtractive with EDPI witness $y \to x$ (compare this remark with Example 1.7.11). For the sake of developments in the sequel we find it convenient to present this result here explicitly as a corollary.

Corollary 3.1.5. *(cf. Example 1.7.11) The variety of MINI-algebras is subtractive with EDPI. Moreover, the binary term* $y \to x$ *witnesses* both *subtractivity and EDPI for* MINI *in the sense of Theorem 1.7.9. That is, for any MINI-algebra* $\mathbf{A}$ *and* $a, b \in A$,

$$a \to^{\mathbf{A}} a = 1$$

$$1 \to^{\mathbf{A}} a = a$$

$$a \in \langle b \rangle_{\mathbf{A}} \quad \textit{iff} \quad b \to^{\mathbf{A}} a = 1.$$

The variety of MINI-algebras is more than just a natural example of a subtractive variety with EDPI; by Agliano and Ursini [11, Corollary 3.8] it is a paradigm for such varieties in that a variety V is subtractive with EDPI iff every $\mathbf{A} \in \mathsf{V}$ has a MINI-algebra polynomial reduct for which any ideal term $t(\vec{x}, y) \in \mathrm{IT}_{\mathsf{V}}(y)$ is *compatible* with V in the sense that for any $a, \vec{b} \in A$ [11, p. 375],

$$a \to^{\mathbf{A}} t^{\mathbf{A}}(\vec{b}, a) = 1.$$

The following theorem, whose proof is included for the sake of completeness, presents a variant on this result.

Theorem 3.1.6. *(cf. [11, Corollary 3.8]) For a variety* V *with* $\mathbf{1}$*, the following are equivalent for a binary term* $x \to y$ *of* V*:*

1. *The term* $y \to x$ *witnesses* both *subtractivity and EDPI for* V *in the sense of Theorem 1.7.9;*

2. *For any* $\mathbf{A} \in \mathsf{V}$*, the polynomial reduct* $\langle A;\ \to^{\mathbf{A}}, 1\rangle$ *is a MINI-algebra, and* either one *of the following conditions is satisfied:*

   (a) Any ideal term $t(\vec{x}, y) \in \mathrm{IT}_{\mathsf{V}}(y)$ *is compatible with* V*;*

   (b) For any $a \in A$*,* $\langle a\rangle_{\mathbf{A}} = \langle a\rangle_{\langle A;\ \to^{\mathbf{A}}, 1\rangle}$*.*

Proof. Let V be as stated and let $x \to y$ be a binary term of V.

(1) $\Rightarrow$ (2)(a) Suppose $y \to x$ witnesses *both* subtractivity and EDPI for V in the sense of Theorem 1.7.9. Then for any $\mathbf{A} \in \mathsf{V}$, $\langle A;\ \to^{\mathbf{A}}, 1\rangle/\approx_{\mathbf{A}}$ is a Hilbert algebra dually isomorphic with $\langle \mathrm{PI}(\mathbf{A});\ *, \langle 1\rangle_{\mathbf{A}}\rangle$ by Theorem 1.7.15(1). Throughout the remainder of the proof to simplify notation we write $\overline{a}$ for the equivalence class $[a]_{\approx_{\mathbf{A}}}$ in $A/\approx_{\mathbf{A}}$ containing $a \in A$. As (1.45)-(1.47) are identities in the language $\langle \to, \mathbf{1}\rangle$ that hold in the variety of Hilbert algebras, for any $\overline{a}, \overline{b}, \overline{c} \in A/\approx_{\mathbf{A}}$ with $a, b, c \in A$ we have:

$$
\begin{aligned}
&\overline{a} \to^{\mathbf{A}/\approx_{\mathbf{A}}} \overline{1} = \overline{1}\\
&\overline{1} \to^{\mathbf{A}/\approx_{\mathbf{A}}} \overline{a} = \overline{a}\\
&\big(\overline{a} \to^{\mathbf{A}/\approx_{\mathbf{A}}} (\overline{b} \to^{\mathbf{A}/\approx_{\mathbf{A}}} \overline{c})\big) \to^{\mathbf{A}/\approx_{\mathbf{A}}} \big((\overline{a} \to^{\mathbf{A}/\approx_{\mathbf{A}}} \overline{b}) \to^{\mathbf{A}/\approx_{\mathbf{A}}} (\overline{a} \to^{\mathbf{A}/\approx_{\mathbf{A}}} \overline{c})\big) = \overline{1}\\
&\overline{a} \to^{\mathbf{A}/\approx_{\mathbf{A}}} (\overline{b} \to^{\mathbf{A}/\approx_{\mathbf{A}}} \overline{a}) = \overline{1}.
\end{aligned}
$$

Because $\overline{1} = \{1\}$, we infer:

$$
\begin{aligned}
&a \to^{\mathbf{A}} 1 = 1\\
&1 \to^{\mathbf{A}} a = a\\
&\big(a \to^{\mathbf{A}} (b \to^{\mathbf{A}} c)\big) \to^{\mathbf{A}} \big((a \to^{\mathbf{A}} b) \to^{\mathbf{A}} (a \to^{\mathbf{A}} c)\big) = 1\\
&a \to^{\mathbf{A}} (b \to^{\mathbf{A}} a) = 1
\end{aligned}
$$

in $\mathbf{A}$, whence $\langle A;\ \to^{\mathbf{A}}, 1\rangle$ is a MINI-algebra. To see any ideal term $t(\vec{x}, y) \in \mathrm{IT}_{\mathsf{V}}(y)$ is compatible with V let $t(\vec{x}, y) \in \mathrm{IT}_{\mathsf{V}}(y)$ and suppose $\vec{b}, a \in A$. Then

$t^{\mathbf{A}}(\vec{b}, a) \in \langle a \rangle_{\mathbf{A}}$ by the description of $\langle a \rangle_{\mathbf{A}}$ given in §1.7.1. Since $y \to x$ witnesses EDPI for V we have that $a \to t^{\mathbf{A}}(\vec{b}, a) = 1$ as required.

(2)(a) $\Rightarrow$ (2)(b) Suppose that for any algebra $\mathbf{A} \in \mathsf{V}$, the polynomial reduct $\langle A;\ \to^{\mathbf{A}}, 1 \rangle$ is a MINI-algebra and that any ideal term $t(\vec{x}, y) \in \mathrm{IT}_{\mathsf{V}}(y)$ is compatible with V. For any $a \in A$, trivially $\langle a \rangle_{\langle A;\ \to^{\mathbf{A}}, 1 \rangle} \subseteq \langle a \rangle_{\mathbf{A}}$. For the opposite inclusion, let $c \in \langle a \rangle_{\mathbf{A}}$. Then $c = t^{\mathbf{A}}(\vec{b}, a)$ where $t(\vec{x}, y)$ is an ideal term in y and $\vec{b}, a \in A$, just because of the description of $\langle a \rangle_{\mathbf{A}}$ given in §1.7.1. Since $t(\vec{x}, y)$ is compatible with V we have $a \to t^{\mathbf{A}}(\vec{b}, a) = 1$, whence $t^{\mathbf{A}}(\vec{b}, a) \in \langle a \rangle_{\langle A;\ \to^{\mathbf{A}}, 1 \rangle}$ by Corollary 3.1.5. Thus $c \in \langle a \rangle_{\langle A;\ \to^{\mathbf{A}}, 1 \rangle}$ and $\langle a \rangle_{\mathbf{A}} \subseteq \langle a \rangle_{\langle A;\ \to^{\mathbf{A}}, 1 \rangle}$.

(2)(b) $\Rightarrow$ (1) Suppose that for any $\mathbf{A} \in \mathsf{V}$ the polynomial reduct $\langle A;\ \to^{\mathbf{A}}, 1 \rangle$ is a MINI-algebra and $\langle a \rangle_{\mathbf{A}} = \langle a \rangle_{\langle A;\ \to^{\mathbf{A}}, 1 \rangle}$ for any $a \in A$. Since $y \to x$ witnesses *both* subtractivity and EDPI for $\langle A;\ \to^{\mathbf{A}}, 1 \rangle$ in the sense of Theorem 1.7.9 (by Corollary 3.1.5), $y \to x$ witnesses *both* subtractivity and EDPI for $\mathbf{A}$ in this sense also (by hypothesis). Thus $y \to x$ witnesses *both* subtractivity and EDPI for V in the sense of Theorem 1.7.9, and the proof is complete. ∎

Let V be a subtractive variety let $\mathbf{A} \in \mathsf{V}$. Since any V-ideal of $\mathbf{A}$ is a directed union of principal V-ideals, from Theorem 3.1.6 we may infer:

Corollary 3.1.7. *For a variety* V *with* $\mathbf{1}$ *and a binary term* $x \to y$ *of* V, *the term* $y \to x$ *witnesses* both *subtractivity and EDPI for* V *in the sense of Theorem 1.7.9 iff every algebra* $\mathbf{A} \in \mathsf{V}$ *has a MINI-algebra polynomial reduct* $\langle A;\ \to^{\mathbf{A}}, 1 \rangle$ *whose* MINI*-ideals coincide with the* V*-ideals of* $\mathbf{A}$.

For a variety V with $\mathbf{1}$ and a binary term $x \to y$ of V, the preceding corollary cannot be strengthened to the assertion that $y \to x$ witnesses *both* subtractivity and EDPI for V in the sense of Theorem 1.7.9 iff every $\mathbf{A} \in \mathsf{V}$ has a MINI-algebra polynomial reduct $\langle A;\ \to^{\mathbf{A}}, 1 \rangle$ whose *congruences* coincide with those of A. If every $\mathbf{A} \in \mathsf{V}$ has a MINI-algebra polynomial reduct $\langle A;\ \to^{\mathbf{A}}, 1 \rangle$ whose congruences coincide with those of A, then certainly $y \to x$ witnesses both subtractivity and EDPI for V in the sense of Theorem 1.7.9. However, as an immediate consequence of the following proposition we have that the converse does not hold.

Proposition 3.1.8. *The binary term $x\backslash y$ witnesses* both *subtractivity and EDPI for the variety* lhSBA *of left handed skew Boolean algebras in the sense of Theorem 1.7.9. Hence the* lhSBA*-ideals of any left handed skew Boolean algebra* $\mathbf{A}$ *coincide with the* MINI*-ideals of its canonical MINI-algebra polynomial reduct* $\langle A;\ \rightarrow^{\mathbf{A}}, \mathbf{1}^{\mathbf{A}}\rangle$, *where* $a \rightarrow^{\mathbf{A}} b := b\backslash a$ *for any* $a, b \in A$ *and* $\mathbf{1}^{\mathbf{A}} := 0$. *Nonetheless, there exists a left handed skew Boolean algebra* $\mathbf{S}$ *that has* no *MINI-algebra polynomial reduct whose* congruences *coincide with those of* $\mathbf{S}$.

Proof. To establish the first two assertions of the theorem, it is sufficient to show that for any skew Boolean algebra $\mathbf{A}$ and $a \in A$,

(i) The reduct $\langle A;\ \backslash, 0\rangle$ is an implicative BCS-algebra;
(ii) $\langle a\rangle_{\mathbf{A}} = \langle a\rangle_{\langle A;\ \backslash, 0\rangle}$.

For (i), an easy inspection of the subdirectly irreducible skew Boolean algebras $\mathbf{2}^p$, $\mathbf{3}^p_L$ and $\mathbf{3}^p_R$ shows that the reduct $\langle A;\ \backslash, 0\rangle$ of any skew Boolean algebra $\mathbf{A}$ is an implicative BCS-algebra.

For (ii), let $\mathbf{A}$ be a skew Boolean algebra. From remarks due to Leech [150, Section 4.6] it is known that a non-empty subset $\{0\} \subseteq I \subseteq A$ is an SBA-ideal iff the following conditions are satisfied [19, Definition 3.3]:

$$a, b \in I \text{ implies } a \vee b \in I; \text{ and} \tag{3.5}$$

$$a \in I, b \in A \text{ implies } b \wedge a \wedge b \in I. \tag{3.6}$$

Notice (3.6) above is equivalent to:

$$a \in I, b \preceq_{\mathcal{D}} a, \text{ implies } b \in I. \tag{3.7}$$

For assume (3.6) and let $a \in I$ and $b \preceq_{\mathcal{D}} a$. Since $b \preceq_{\mathcal{D}} a$, $b \wedge a \wedge b = b$. By (3.6), $b \in I$. Conversely, assume (3.7) and let $a \in I$ and $b \in A$. Since $b \wedge a \wedge b \preceq_{\mathcal{D}} a$, by (3.7) we have $b \wedge a \wedge b \in I$.

Let $a \in A$. From (3.5) and (3.7) it follows easily that $\langle a\rangle_{\mathbf{A}} = \{b : b \preceq_{\mathcal{D}} a\}$. On the other hand, $\langle a\rangle_{\langle A;\ \backslash,0\rangle} = \{b : b \preceq^{\langle A;\ \backslash,0\rangle} a\}$ by Theorem 2.2.20(4), Proposition 2.2.30 and Proposition 2.3.5. Therefore to see $\langle a\rangle_{\mathbf{A}} = \langle a\rangle_{\langle A;\ \backslash,0\rangle}$ it is sufficient to show $c \preceq_{\mathcal{D}} d$ iff $c \preceq^{\langle A;\ \backslash,0\rangle} d$ for any $c, d \in A$. So let $c, d \in A$.

Suppose $c \preceq_{\mathcal{D}} d$. Then $c \wedge d \wedge c = c$, so $0 = (c\backslash d) \wedge (c \wedge d \wedge c) = (c\backslash d) \wedge c = c\backslash d$ by (1.25) and (1.28). Thus $c \preceq^{\langle A;\, \backslash, 0\rangle} d$. Conversely, suppose $c \preceq^{\langle A;\, \backslash, 0\rangle} d$. Then $c\backslash d = 0$, so $c = (c\backslash d) \vee (c \wedge d \wedge c) = 0 \vee (c \wedge d \wedge c) = c \wedge d \wedge c$ by (1.23). Thus $c \preceq_{\mathcal{D}} d$ and so $c \preceq_{\mathcal{D}} d$ iff $c \preceq^{\langle A;\, \backslash, 0\rangle} d$. Hence $\langle a\rangle_{\mathbf{A}} = \langle a\rangle_{\langle A;\, \backslash, 0\rangle}$.

To establish the remaining assertion of the theorem, it is sufficient to show that for any left handed skew Boolean algebra $\mathbf{A}$,

(iii) The only polynomial reduct of $\mathbf{A}$ term equivalent to a MINI-algebra is $\langle A;\ \backslash, 0\rangle$;

and also that:

(iv) There exists a left handed skew Boolean algebra $\mathbf{S}$ with $\mathrm{Con}\,\langle S;\ \backslash, 0\rangle \neq \mathrm{Con}\,\mathbf{S}$.

For (iii), let $\mathbf{F}(\overline{x}, \overline{y})$ denote the lhSBA-free algebra on two free generators $\overline{x}, \overline{y}$. From Example 1.4.23, the remarks of §1.4.24, Example 2.3.26 and (i) we see at once that $\mathbf{F}(\overline{x}, \overline{y})$ has $\mathcal{D}$-equivalence classes:

$$\begin{array}{c} \{\overline{0}\}, \quad \{\overline{x}, (\overline{x} \vee \overline{y}) \wedge \overline{x}\}, \quad \{\overline{y}, (\overline{y} \vee \overline{x}) \wedge \overline{y}\} \\ \{\overline{x}\backslash\overline{y}\}, \quad \{\overline{y}\backslash\overline{x}\}, \quad \{\overline{x} \wedge \overline{y}, \overline{y} \wedge \overline{x}\}, \quad \{\overline{x} \vee \overline{y}, \overline{y} \vee \overline{x}\} \end{array}$$

with:

$$\begin{array}{ll} \overline{x} \wedge \overline{y} \leq \overline{x},\ (\overline{y} \vee \overline{x}) \wedge \overline{y} & \overline{y} \wedge \overline{x} \leq \overline{y},\ (\overline{x} \vee \overline{y}) \wedge \overline{x} \\ \overline{x},\ (\overline{y} \vee \overline{x}) \wedge \overline{y} \leq \overline{x} \vee \overline{y} & \overline{y},\ (\overline{x} \vee \overline{y}) \wedge \overline{x} \leq \overline{y} \vee \overline{x} \\ \overline{x}\backslash\overline{y} \leq \overline{x},\ (\overline{x} \vee \overline{y}) \wedge \overline{x} & \overline{y}\backslash\overline{x} \leq \overline{y},\ (\overline{y} \vee \overline{x}) \wedge \overline{y}. \end{array}$$

Routine computations (by inspection of $\mathbf{2}^p$ and $\mathbf{3}_L^p$) show that $\mathbf{F}(\overline{x}, \overline{y})$ has exactly one other $\mathcal{D}$-class distinct from those listed above, namely:

$$\{(\overline{x} \vee \overline{y})\backslash(\overline{y} \wedge \overline{x})\}$$

with:

$$\overline{x}\backslash\overline{y},\ \overline{y}\backslash\overline{x} \leq (\overline{x} \vee \overline{y})\backslash(\overline{x} \wedge \overline{y}) \leq \overline{x} \vee \overline{y},\ \overline{y} \vee \overline{x}$$

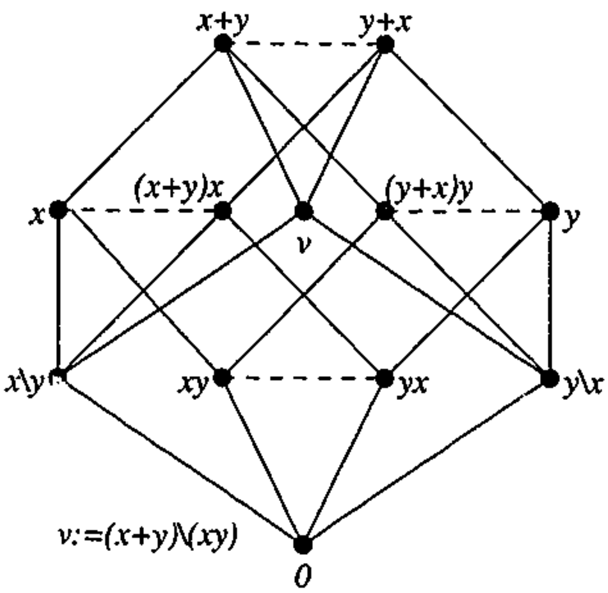

Figure 3.1. The lhSBA-free algebra on two free generators $\overline{x}, \overline{y}$.

whence $\mathbf{F}(\overline{x}, \overline{y})$ has the Hasse diagram of Figure 3.1 (where for notational purposes the free generators $\overline{x}$, $\overline{y}$ are denoted in the figure simply by x, y respectively; like remarks apply to products of $\overline{x}$, $\overline{y}$). Conversely, the diagram of Figure 3.1 completely determines $\mathbf{F}(\overline{x}, \overline{y})$, just because $\mathbf{F}(\overline{x}, \overline{y})$ is left handed. From this characterisation of $\mathbf{F}(\overline{x}, \overline{y})$ it is now easy to see (by inspection of $\mathbf{2}^p$ and $\mathbf{3}^p_L$) that there exists no term function $\to^{\mathbf{A}}$ (other than that induced by the term $y \backslash x$) definable in terms of the fundamental left handed skew Boolean algebra operations $\wedge, \vee, \backslash$ such that for any left handed skew Boolean algebra $\mathbf{A}$, the polynomial reduct $\langle A; \to^{\mathbf{A}}, 0\rangle$ is a MINI-algebra. Hence the only polynomial reduct of $\mathbf{A}$ that is term equivalent to a MINI-algebra is $\langle A; \backslash, 0\rangle$.

For (iv), let $\mathbf{S}$ denote the 6-element algebra defined by the following operation tables:

$\wedge^{\mathbf{S}}$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	b	b	b	b
c	0	a	b	c	b	c
d	0	0	d	d	d	d
e	0	a	d	e	d	e

$\vee^{\mathbf{S}}$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	a	c	c	e	e
b	b	c	b	c	d	e
c	c	c	c	c	e	e
d	d	e	b	c	d	e
e	e	e	c	c	e	e

$\backslash^{\mathbf{S}}$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	a	0	a	0
b	b	b	0	0	0	0
c	c	b	a	0	a	0
d	d	d	0	0	0	0
e	e	d	a	0	a	0

It is readily verified that $\mathbf{S}$ is a left handed skew Boolean algebra: see Figure 3.2(a). To complete the proof it is sufficient by (iii) to show $\mathrm{Con}\,\langle S;\ \backslash, 0\rangle \neq \mathrm{Con}\,\langle S;\ \wedge, \vee, 0\rangle$, just because $\mathrm{Con}\,\mathbf{S} = \mathrm{Con}\,\langle S;\ \wedge, \vee, 0\rangle$ by Proposition 1.4.27. To this end, let θ be the equivalence relation on $S \times S$ induced by the partition $\big\{\{0\}, \{a\}, \{b, d\}, \{c\}, \{e\}\big\}$. It is tedious but straightforward to check that θ is a congruence relation on $\langle S;\ \backslash, 0\rangle$ (for a complete description of the congruence structure of $\langle S;\ \backslash, 0\rangle$, see Figure 3.2(c); notice that for ease of notation, congruences on $\langle S;\ \backslash, 0\rangle$ are represented in the figure by their corresponding partitions (with all parentheses dropped)). However, θ is not a congruence on $\langle S;\ \wedge, \vee, 0\rangle$. Indeed, suppose $\theta \in \mathrm{Con}\,\langle S;\ \wedge, \vee, 0\rangle$. Because $e \equiv_\theta e$ and $b \equiv_\theta d$, we must have that $e \vee b \equiv_\theta e \vee d$, which implies $c \equiv_\theta e$, a contradiction. Hence $\theta \notin \mathrm{Con}\langle S;\ \wedge, \vee, 0\rangle$ (for a complete description of the congruence structure of $\langle S;\ \wedge, \vee, 0\rangle$, see Figure 3.2(b)) and $\mathrm{Con}\,\langle S;\ \backslash, 0\rangle \neq \mathrm{Con}\,\langle S;\ \wedge, \vee, 0\rangle$. ∎

Corollary 3.1.9. *Let* V *be a variety with* $\mathbf{1}$ *and let* $x \to y$ *be a binary term of* V *such that* $y \to x$ *witnesses* both *subtractivity and EDPI for* V *in the sense of Theorem 1.7.9. In general, the congruences on the canonical MINI-algebra polynomial reduct* $\langle A;\ \to^{\mathbf{A}}, 1\rangle$ *of* $\mathbf{A} \in \mathsf{V}$ *need not coincide with those on* $\mathbf{A}$.

Let A be a set and let f be an n-ary operation on A. The *slice* $\widetilde{f}_i(a)$, $a \in A$, is the unary operation obtained from f upon defining:

$$\widetilde{f}_i(a) := f(b_1, \ldots, b_{i-1}, a, b_{i+1}, \ldots, b_n)$$

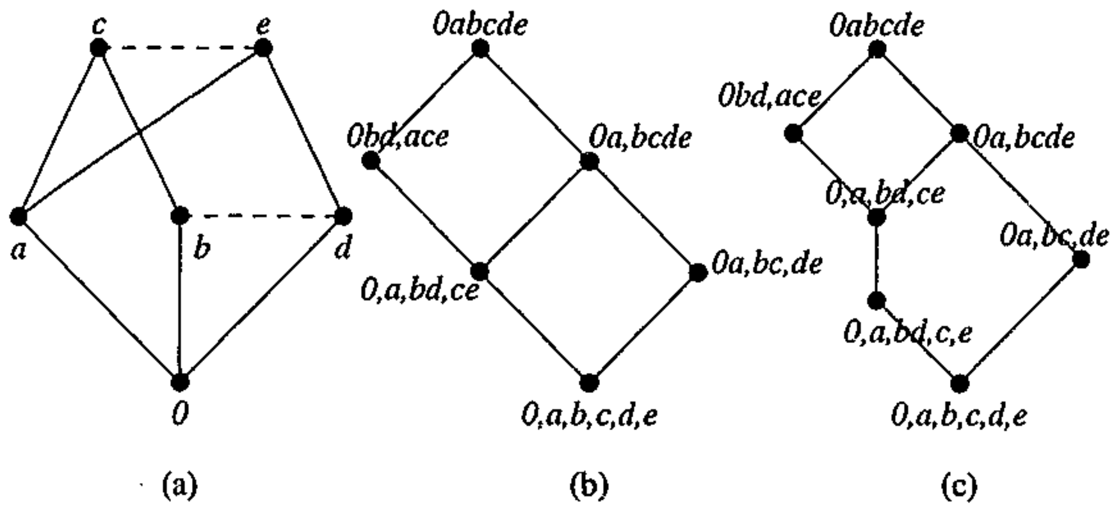

Figure 3.2. (a) The left handed skew Boolean algebra S of Proposition 3.1.8; (b) Con $\langle S;\ \wedge,\vee,0\rangle$; (c) Con $\langle S;\ \backslash,0\rangle$.

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for fixed $b_1,\ldots,b_{i-1},b_{i+1},\ldots,b_n \in A$. In other words, the slice $\widetilde{f}_i(a)$ is the unary operation obtained from f by fixing all but one of its arguments. Given an algebra $\mathbf{A}$, an n-ary operation f of $\mathbf{A}$ is said to be *compatible* with a congruence θ on $\mathbf{A}$ if θ has the substitution property with respect to f; that is, if:

$$a \equiv b \pmod{\theta} \quad \text{implies} \quad \widetilde{f}_i(a) \equiv \widetilde{f}_i(b) \pmod{\theta}$$

for all $a, b \in A$ and slices $\widetilde{f}_i(a), \widetilde{f}_i(b)$, $1 \leq i \leq n$. An n-ary operation f of $\mathbf{A}$ is said to be *compatible* if it is compatible with every congruence on $\mathbf{A}$.

A *MINI-algebra [Hilbert algebra] with compatible operations* is an algebra $\mathbf{A} := \langle A;\ \rightarrow,1,f_j\rangle_{j\in J}$ of type $\langle 2,0,\ldots\rangle$ such that: (i) the reduct $\langle A;\ \rightarrow,1\rangle$ is a MINI-algebra [Hilbert algebra]; and (ii) each additional operation f_j is compatible with $\langle A;\ \rightarrow,1\rangle$. Clearly an algebra $\mathbf{A} := \langle A;\ \rightarrow,1,f_j\rangle_{j\in J}$ of type $\langle 2,0,\ldots\rangle$ is a MINI-algebra [Hilbert algebra] with compatible operations iff $\mathrm{Con}\,\mathbf{A} = \mathrm{Con}\,\langle A;\ \rightarrow,1\rangle$. A version of the following lemma is presented without proof in Agliano [7, Section 3].

Lemma 3.1.10. *(cf. [7, Section 3, p. 9]) Let $\mathbf{A}$ be an algebra with $\mathbf{1}$, say $\langle A;\ 1,f_j\rangle_{j\in J}$, and let $x \rightarrow y$ be a binary term of $\mathbf{A}$ such that the polynomial reduct $\langle A;\ \rightarrow^{\mathbf{A}},1\rangle$ is a Hilbert algebra. The following are equivalent:*

1. $\mathbf{A}$ *is term equivalent to a Hilbert algebra with compatible operations* $\langle A;\ \to^{\mathbf{A}}, 1, f_j\rangle_{j\in J}$;

2. *For any n-ary fundamental operation* $f := f_j$ *of* $\mathbf{A}$,

$$(a \to^{\mathbf{A}} b) \to^{\mathbf{A}} \big((b \to^{\mathbf{A}} a) \to (\widetilde{f}_i(a) \to^{\mathbf{A}} \widetilde{f}_i(b))\big) = 1$$

for all $a, b \in A$ *and slices* $\widetilde{f}_i(a), \widetilde{f}_i(b)$, $1 \leq i \leq n$.

Proof. Let $\mathbf{A}$ be a Hilbert algebra and let $a, b, c, d \in A$. By Theorem 1.6.15(2),

$$c \equiv d \left(\operatorname{mod} \Theta^{\mathbf{A}}(a, b)\right) \quad \text{iff} \quad (a \to b) \to \big((b \to a) \to (c \to d)\big) = (a \to b) \to \big((b \to a) \to (d \to c)\big);$$

the lemma follows directly from this description of the principal congruences. ∎

A variety V is said to be a *variety of MINI-algebras [Hilbert algebras] with compatible operations* if every member of V is a MINI-algebra [Hilbert algebra] with compatible operations. Given Corollary 3.1.9 and the remarks immediately preceding Proposition 3.1.8, it is natural to ask if varieties of MINI-algebras with compatible operations admit a relevant structure theorem. For the special case of varieties of Hilbert algebras with compatible operations, a positive answer to this question has been obtained by Agliano in [7, Section 3]. For the sake of both completeness and developments in the sequel, we reproduce Agliano's result in Theorem 3.1.11 below.

Theorem 3.1.11. *[7, Theorem 3.4] For a variety* V *with* $\mathbf{1}$ *and a binary term* $x \to y$ *of* V, *the following are equivalent:*

1. V *is termwise definitionally equivalent to a variety of Hilbert algebras with compatible operations. In particular, any algebra* $\mathbf{A} := \langle A;\ 1, f_j\rangle_{j\in J} \in$ V *is term equivalent to a Hilbert algebra with compatible operations* $\langle A;\ \to^{\mathbf{A}}, 1, f_j\rangle_{j\in J}$;

2. V *is congruence orderable and subtractive with EDPC. In particular,* V *is congruence orderable and the binary term* $y \to x$ *witnesses* both *subtractivity and EDPI for* V *in the sense of Theorem 1.7.9.*

Proof. Let V be a variety with $\mathbf{1}$ and let $x \to y$ be a binary term of V.

(1) $\Rightarrow$ (2) Suppose V is termwise definitionally equivalent to a variety of Hilbert algebras with compatible operations. For the first assertion, if $\mathbf{A} := \langle A;\ 1, f_j\rangle_{j\in J} \in \mathsf{V}$ is term equivalent to a Hilbert algebra with compatible operations $\langle A;\ \to^{\mathbf{A}}, 1, f_j\rangle_{j\in J}$, then $\operatorname{Con}\mathbf{A} = \operatorname{Con}\langle A;\ \to^{\mathbf{A}}, 1, f_j\rangle_{j\in J} = \operatorname{Con}\langle A;\ \to^{\mathbf{A}}, 1\rangle$, so $\mathbf{A}$ is congruence orderable and subtractive with EDPC. It follows that V is congruence orderable and subtractive with EDPC. For the second assertion, if $\mathbf{A} := \langle A;\ 1, f_j\rangle_{j\in J} \in \mathsf{V}$ is term equivalent to a Hilbert algebra with compatible operations $\langle A;\ \to^{\mathbf{A}}, 1, f_j\rangle_{j\in J}$, then $\operatorname{Con}\mathbf{A} = \operatorname{Con}\langle A;\ \to^{\mathbf{A}}, 1\rangle$, so in particular $\mathrm{I}(\mathbf{A}) = \mathrm{I}(\langle A;\ \to^{\mathbf{A}}, 1\rangle)$, whence $a \in \langle b\rangle_{\mathbf{A}}$ iff $a \in \langle b\rangle_{\langle A;\ \to^{\mathbf{A}}, 1\rangle}$ iff $b \to^{\mathbf{A}} a = 1$. It follows that $\to$ witnesses *both* subtractivity and EDPI for V in the sense of Theorem 1.7.9, and the proof is complete.

(2) $\Rightarrow$ (1) Suppose V is congruence orderable and subtractive with EDPC, and in particular that the binary term $y \to x$ witnesses *both* subtractivity and EDPI for V in the sense of Theorem 1.7.9. Let $\mathbf{A} := \langle A;\ 1, f_j\rangle_{j\in J} \in \mathsf{V}$. Clearly $\mathbf{A}$ is term equivalent to the algebra $\langle A;\ \to^{\mathbf{A}}, 1, f_j\rangle_{j\in J}$ obtained from $\mathbf{A}$ by enriching the type of $\mathbf{A}$ with a binary operation symbol $\to$ whose canonical interpretation is the term function $\to^{\mathbf{A}}$. We claim $\langle A;\ \to^{\mathbf{A}}, 1, f_j\rangle_{j\in J}$ is a Hilbert algebra with compatible operations. To see this, it is sufficient by Lemma 3.1.10 to show:

(i) The polynomial reduct $\langle A;\ \to^{\mathbf{A}}, 1\rangle$ is a Hilbert algebra;
(ii) For any n-ary fundamental operation $f := f_j$ of $\mathbf{A}$,

$$(a \to^{\mathbf{A}} b) \to^{\mathbf{A}} \left((b \to^{\mathbf{A}} a) \to (\widetilde{f_i}(a) \to^{\mathbf{A}} \widetilde{f_i}(b))\right) = 1$$

for all $a, b \in A$ and slices $\widetilde{f_i}(a), \widetilde{f_i}(b)$, $1 \leq i \leq n$.

For (i), by Theorem 3.1.6 we have that the polynomial reduct $\langle A;\ \to^{\mathbf{A}}, 1\rangle$ is a MINI-algebra. By Theorem 3.1.4 and Corollary 2.2.23(2), the map $f : a \mapsto \langle a\rangle_{\mathbf{A}}$ is a homomorphism and $\langle A;\ \to^{\mathbf{A}}, 1\rangle / \ker f$ is a Hilbert algebra, where $\ker f = \{(a, b) : a \to^{\mathbf{A}} b = 1 = b \to^{\mathbf{A}} a\}$. Because V is congruence orderable, it is $\mathbf{1}$-regular (in the sense of Proposition 1.2.6) witness $d_1(x, y) := x \to y$, $d_2(x, y) := y \to x$ by Proposition 1.7.16, from which it follows that $\ker f = \omega_{\mathbf{A}}$.

Hence $\langle A;\ \rightarrow^{\mathbf{A}}, 1\rangle$ is a Hilbert algebra.

For (ii), let $f(x)$ be a fundamental operation of $\mathbf{A}$, which we may take to be unary without loss of generality. Let $a, b \in A$ and let:

$$\langle a \rightarrow^{\mathbf{A}} b\rangle \vee^{\mathbf{I}(\mathbf{A})} \langle b \rightarrow^{\mathbf{A}} a\rangle = [1]_\theta$$

for some $\theta \in \operatorname{Con}\mathbf{A}$. Then of course $[a]_\theta \rightarrow^{\mathbf{A}/\theta} [b]_\theta = [1]_\theta = [b]_\theta \rightarrow^{\mathbf{A}/\theta} [a]_\theta$, so $(a, b) \in \theta$. It follows that $\big(f(a), f(b)\big) \in \theta$ and by properties of $\rightarrow$, $f(a) \rightarrow^{\mathbf{A}} f(b) \in [1]_\theta$. But $[1]_\theta = \langle a \rightarrow^{\mathbf{A}} b\rangle_{\mathbf{A}} \vee^{\mathbf{I}(\mathbf{A})} \langle b \rightarrow^{\mathbf{A}} a\rangle_{\mathbf{A}}$, so Theorem 3.1.4 and Lemma 2.2.19 applied twice gives:

$$(a \rightarrow^{\mathbf{A}} b) \rightarrow^{\mathbf{A}} \big((b \rightarrow^{\mathbf{A}} a) \rightarrow^{\mathbf{A}} (f(a) \rightarrow^{\mathbf{A}} f(b))\big) = 1$$

as desired. ∎

Let $\mathbf{A}$ be an algebra with $\mathbf{1}$ and let $\operatorname{Con}_\epsilon \mathbf{A} := \{\theta \in \operatorname{Con}\mathbf{A} : \theta \geq \langle 1\rangle^\epsilon_{\mathbf{A}}\}$. An n-ary operation f of $\mathbf{A}$ is said to be *weakly compatible* if it is compatible with every congruence $\theta \in \operatorname{Con}_\epsilon \mathbf{A}$. By analogy with the theory of MINI-algebras with compatible operations, an algebra $\mathbf{A} := \langle A;\ \rightarrow, 1, f_j\rangle_{j\in J}$ of type $\langle 2, 0, \ldots\rangle$ is a *MINI-algebra with weakly compatible operations* if: (i) the reduct $\langle A;\ \rightarrow, 1\rangle$ is a MINI-algebra; and (ii) each additional operation f_j is weakly compatible with $\langle A;\ \rightarrow, 1\rangle$.

Lemma 3.1.12. *Let* $\mathbf{A} := \langle A;\ \rightarrow, 1, f_j\rangle_{j\in J}$ *be a MINI-algebra with weakly compatible operations. Then* $\langle 1\rangle^\epsilon_{\mathbf{A}} = \langle 1\rangle^\epsilon_{\langle A;\ \rightarrow,1\rangle} = \approx_{\langle A;\ \rightarrow,1\rangle}$.

Proof. Let $\mathbf{A} := \langle A;\ \rightarrow, 1, f_j\rangle_{j\in J}$ be a MINI-algebra with weakly compatible operations. By hypothesis, $\operatorname{Con}_\epsilon \mathbf{A} = \operatorname{Con}_\epsilon\langle A;\ \rightarrow, 1\rangle$, so $\langle 1\rangle^\epsilon_{\mathbf{A}} \in \operatorname{Con}_\epsilon \mathbf{A} \subseteq \operatorname{Con}_\epsilon\langle A;\ \rightarrow, 1\rangle$. Hence $\langle 1\rangle^\epsilon_{\mathbf{A}} \in \operatorname{Con}_\epsilon\langle A;\ \rightarrow, 1\rangle$; that is to say $\langle 1\rangle^\epsilon_{\mathbf{A}} \geq \langle 1\rangle^\epsilon_{\langle A;\ \rightarrow,1\rangle}$. The opposite inclusion is handled similarly. Thus $\langle 1\rangle^\epsilon_{\mathbf{A}} = \langle 1\rangle^\epsilon_{\langle A;\ \rightarrow,1\rangle}$. Since $\langle A;\ \rightarrow, 1\rangle$ is weakly congruence orderable, by Lemma 1.7.17 we have that $\langle 1\rangle^\epsilon_{\langle A;\ \rightarrow,1\rangle} = \approx_{\langle A;\ \rightarrow,1\rangle}$. Hence $\langle 1\rangle^\epsilon_{\mathbf{A}} = \langle 1\rangle^\epsilon_{\langle A;\ \rightarrow,1\rangle} = \approx_{\langle A;\ \rightarrow,1\rangle}$. ∎

Lemma 3.1.13. *Let* $\mathbf{A} := \langle A;\ \rightarrow, 1, f_j\rangle_{j\in J}$ *be a MINI-algebra with weakly compatible operations. Then* $\operatorname{Con}\mathbf{A}/\langle 1\rangle^\epsilon_{\mathbf{A}} = \operatorname{Con}\langle A;\ \rightarrow, 1\rangle/\langle 1\rangle^\epsilon_{\mathbf{A}}$.

Proof. Let $\mathbf{A} := \langle A;\ \rightarrow, 1, f_j\rangle_{j\in J}$ be a MINI-algebra with weakly compatible operations. To prove the lemma, just observe that $\theta \in \operatorname{Con}\mathbf{A}/\langle 1\rangle^{\epsilon}_{\mathbf{A}}$ iff $\theta = \psi/\langle 1\rangle^{\epsilon}_{\mathbf{A}}$ for some $\psi \in \operatorname{Con}_{\epsilon}\mathbf{A}$ (since $\langle 1\rangle^{\epsilon}_{\mathbf{A}}$ is the least element of $\operatorname{Con}_{\epsilon}\mathbf{A}$) iff $\theta = \psi/\langle 1\rangle^{\epsilon}_{\langle A;\ \rightarrow, 1\rangle}$ with $\psi \in \operatorname{Con}_{\epsilon}\langle A;\ \rightarrow, 1\rangle$ (by Lemma 3.1.12 and hypothesis) iff $\theta \in \operatorname{Con}\langle A;\ \rightarrow, 1\rangle/\langle 1\rangle^{\epsilon}_{\langle A;\ \rightarrow, 1\rangle}$ (since $\langle 1\rangle^{\epsilon}_{\langle A;\ \rightarrow,1\rangle}$ is the least element of $\operatorname{Con}_{\epsilon}\langle A;\ \rightarrow, 1\rangle$) iff $\theta \in \operatorname{Con}\langle A;\ \rightarrow, 1\rangle/\langle 1\rangle^{\epsilon}_{\mathbf{A}}$ (by Lemma 3.1.12). Hence $\operatorname{Con}\mathbf{A}/\langle 1\rangle^{\epsilon}_{\mathbf{A}} = \operatorname{Con}\langle A;\ \rightarrow, 1\rangle/\langle 1\rangle^{\epsilon}_{\mathbf{A}}$ as desired. ∎

Lemma 3.1.14. *Let* $\mathbf{A}$ *be an algebra with* **1**, *say* $\langle A;\ 1, f_j\rangle_{j\in J}$, *and let* $x \rightarrow y$ *be a binary term of* $\mathbf{A}$ *such that the polynomial reduct* $\langle A;\ \rightarrow^{\mathbf{A}}, 1\rangle$ *is a MINI-algebra. The following are equivalent:*

1. $\mathbf{A}$ *is term equivalent to a MINI-algebra with weakly compatible operations* $\langle A;\ \rightarrow^{\mathbf{A}}, 1, f_j\rangle_{j\in J}$;

2. *For any* n*-ary fundamental operation* $f := f_j$ *of* $\mathbf{A}$,

$$(a \rightarrow^{\mathbf{A}} b) \rightarrow^{\mathbf{A}} \left((b \rightarrow^{\mathbf{A}} a) \rightarrow (\widetilde{f}_i(a) \rightarrow^{\mathbf{A}} \widetilde{f}_i(b))\right) = 1$$

for all $a, b \in A$ *and slices* $\widetilde{f}_i(a), \widetilde{f}_i(b)$, $1 \le i \le n$.

Proof. Let $\mathbf{A} := \langle A;\ 1, f_j^{\mathbf{A}}\rangle_{j\in J}$ and $\rightarrow$ be as in the statement of the lemma.

(1) ⇒ (2) Suppose $\mathbf{A}$ is term equivalent to a MINI-algebra with weakly compatible operations $\langle A;\ \rightarrow^{\mathbf{A}}, 1, f_j^{\mathbf{A}}\rangle_{j\in J}$. Because of Lemma 3.1.13, $\mathbf{A}/\langle 1\rangle^{\epsilon}_{\mathbf{A}}$ is term equivalent to a Hilbert algebra with compatible operations $\langle A;\ \rightarrow^{\mathbf{A}}, 1, f_j^{\mathbf{A}}\rangle_{j\in J}/\langle 1\rangle^{\epsilon}_{\mathbf{A}}$. To simplify notation, throughout the remainder of the proof we write $\overline{a}$ for the equivalence class $[a]_{\langle 1\rangle^{\epsilon}_{\mathbf{A}}}$ in $A/\langle 1\rangle^{\epsilon}_{\mathbf{A}}$ containing $a \in A$. Let $f^{\mathbf{A}} := f_j^{\mathbf{A}}$ be an n-ary fundamental operation of $\mathbf{A}$. Since the operation $f^{\mathbf{A}}/\langle 1\rangle^{\epsilon}_{\mathbf{A}}$ of $\mathbf{A}/\langle 1\rangle^{\epsilon}_{\mathbf{A}}$ is compatible with the Hilbert algebra polynomial reduct $\langle A;\ \rightarrow^{\mathbf{A}}, 1\rangle/\langle 1\rangle^{\epsilon}_{\mathbf{A}}$, we have that:

$$\begin{aligned}(\overline{a} \rightarrow^{\mathbf{A}/\langle 1\rangle^{\epsilon}_{\mathbf{A}}} \overline{b}) \rightarrow^{\mathbf{A}/\langle 1\rangle^{\epsilon}_{\mathbf{A}}} \Big((\overline{b} \rightarrow^{\mathbf{A}/\langle 1\rangle^{\epsilon}_{\mathbf{A}}} \overline{a}) \rightarrow^{\mathbf{A}/\langle 1\rangle^{\epsilon}_{\mathbf{A}}} \\ (\widetilde{f}_i^{\mathbf{A}/\langle 1\rangle^{\epsilon}_{\mathbf{A}}}(\overline{a}) \rightarrow^{\mathbf{A}/\langle 1\rangle^{\epsilon}_{\mathbf{A}}} \widetilde{f}_i^{\mathbf{A}/\langle 1\rangle^{\epsilon}_{\mathbf{A}}}(\overline{b}))\Big) = \overline{1}\end{aligned}$$

for all $\overline{a}, \overline{b} \in A/\langle 1\rangle^{\epsilon}_{\mathbf{A}}$ and slices $\widetilde{f}_i^{\mathbf{A}/\langle 1\rangle^{\epsilon}_{\mathbf{A}}}(\overline{a}), \widetilde{f}_i^{\mathbf{A}/\langle 1\rangle^{\epsilon}_{\mathbf{A}}}(\overline{b})$, $1 \le i \le n$. Since

$[1]_{\langle 1\rangle^\epsilon_{\mathbf{A}}} = \{1\}$, we infer:

$$(a \to^{\mathbf{A}} b) \to^{\mathbf{A}} \big((b \to^{\mathbf{A}} a) \to^{\mathbf{A}} (\widetilde{f_i}^{\mathbf{A}}(a) \to^{\mathbf{A}} \widetilde{f_i}^{\mathbf{A}}(b))\big) = 1$$

for all $a, b \in A$ and slices $\widetilde{f_i}^{\mathbf{A}}(a), \widetilde{f_i}^{\mathbf{A}}(b)$, $1 \le i \le n$, as required.

(2) $\Rightarrow$ (1) Clearly $\mathbf{A}$ is term equivalent to the algebra $\langle A;\ \to^{\mathbf{A}}, 1, f_j^{\mathbf{A}}\rangle_{j\in J}$ obtained from $\mathbf{A}$ by enriching the type of $\mathbf{A}$ with a binary operation symbol $\to$ whose canonical interpretation is the term function $\to^{\mathbf{A}}$. We claim $\langle A;\ \to^{\mathbf{A}}, 1, f_j^{\mathbf{A}}\rangle_{j\in J}$ is a MINI-algebra with weakly compatible operations. By assumption, $\langle A;\ \to^{\mathbf{A}}, 1\rangle$ is a MINI-algebra. To complete the proof it is sufficient to show $\mathrm{Con}_\epsilon\,\mathbf{A} = \mathrm{Con}_\epsilon\,\langle A;\ \to^{\mathbf{A}}, 1\rangle$. If $\theta \in \mathrm{Con}_\epsilon\,\mathbf{A}$ then certainly $\theta \in \mathrm{Con}_\epsilon\langle A;\ \to^{\mathbf{A}}, 1\rangle$, so $\mathrm{Con}_\epsilon\,\mathbf{A} \subseteq \mathrm{Con}_\epsilon\langle A;\ \to^{\mathbf{A}}, 1\rangle$. For the converse, let $\theta \in \mathrm{Con}_\epsilon\langle A;\ \to^{\mathbf{A}}, 1\rangle$. From the remarks of [11, p. 315] we have that $\theta = I^\epsilon$ for some $I \in \mathrm{I}\big(\langle A;\ \to^{\mathbf{A}}, 1\rangle\big)$, whence $\theta = \phi_I$ by Theorem 3.1.4 and Proposition 2.1.35, where:

$$\phi_I := \big\{(a, b) \in A \times A : a \to^{\mathbf{A}} b, b \to^{\mathbf{A}} a \in I\big\}.$$

To complete the proof we show:

(i) ϕ_I is a congruence on $\mathbf{A}$;
(ii) $\phi_I \ge \langle 1\rangle^\epsilon_{\mathbf{A}}$.

For (i), by the proof of Theorem 2.1.26, ϕ_I is an equivalence relation on $A \times A$. Let $f := f_j^{\mathbf{A}}$ be an n-ary fundamental operation of $\mathbf{A}$ and suppose $a_i \approx_{\mathbf{A}} b_i$ for $i = 1, \ldots, n$. For $i = 1, \ldots, n$ and any $c \in A$, let $\widetilde{f_i}(c)$ denote the slice:

$$f(b_1, \ldots, b_{i-1}, c, a_{i+1}, \ldots, a_n).$$

In what follows we write $\to$ for $\to^{\mathbf{A}}$ to simplify notation. By hypothesis, we have that:

$$(a_1 \to b_1) \to \big((b_1 \to a_1) \to (\widetilde{f_1}(a_1) \to \widetilde{f_1}(b_1))\big) = 1$$
$$\vdots$$

$$(a_i \to b_i) \to \big((b_i \to a_i) \to (\widetilde{f}_i(a_i) \to \widetilde{f}_i(b_i))\big) = 1$$
$$\vdots$$
$$(a_n \to b_n) \to \big((b_n \to a_n) \to (\widetilde{f}_n(a_n) \to \widetilde{f}_n(b_n))\big) = 1$$

and by symmetry, also:

$$(b_1 \to a_1) \to \big((a_1 \to b_1) \to (\widetilde{f}_1(b_1) \to \widetilde{f}_1(a_1))\big) = 1$$
$$\vdots$$
$$(b_i \to a_i) \to \big((a_i \to b_i) \to (\widetilde{f}_i(b_i) \to \widetilde{f}_i(a_i))\big) = 1$$
$$\vdots$$
$$(b_n \to a_n) \to \big((a_n \to b_n) \to (\widetilde{f}_n(b_n) \to \widetilde{f}_n(a_n))\big) = 1.$$

That is to say,

$$\begin{aligned}(a_1 \to b_1) \to \big((b_1 \to a_1) \to (f(a_1, a_2, \ldots, a_n) \\ \to f(b_1, a_2, \ldots, a_n))\big) = 1\end{aligned}$$
$$\vdots$$
$$\begin{aligned}(a_i \to b_i) \to \big((b_i \to a_i) \to (f(b_1, \ldots, b_{i-1}, a_i, a_{i+1}, \ldots, a_n) \\ \to f(b_1, \ldots, b_{i-1}, b_i, a_{i+1}, \ldots, a_n))\big) = 1\end{aligned}$$
$$\vdots$$
$$\begin{aligned}(a_n \to b_n) \to \big((b_n \to a_n) \to (f(b_1, \ldots, b_{n-1}, a_n) \\ \to f(b_1, \ldots, b_{n-1}, b_n))\big) = 1.\end{aligned}$$

and:

$$\begin{aligned}(b_1 \to a_1) \to \big((a_1 \to b_1) \to (f(b_1, a_2, \ldots, a_n) \\ \to f(a_1, a_2, \ldots, a_n))\big) = 1\end{aligned}$$
$$\vdots$$
$$(b_i \to a_i) \to \big((a_i \to b_i) \to (f(b_1, \ldots, b_{i-1}, b_i, a_{i+1}, \ldots, a_n)$$

$$\rightarrow f(b_1, \ldots, b_{i-1}, a_i, a_{i+1}, \ldots, a_n))\big) = 1$$

$$\vdots$$

$$(b_n \rightarrow a_n) \rightarrow \big((a_n \rightarrow b_n) \rightarrow (f(b_1, \ldots, b_{n-1}, b_n) \rightarrow f(b_1, \ldots, b_{n-1}, a_n))\big) = 1.$$

Since $a_i \approx b_i$ implies $a_i \rightarrow b_i, b_i \rightarrow a_i \in I$ for $i = 1, \ldots, n$, from (the dual of) Lemma 2.1.21(2) we infer that:

$$f(a_1, a_2, \ldots, a_n) \rightarrow f(b_1, a_2, \ldots, a_n) \in I$$

$$\vdots$$

$$f(b_1, \ldots, b_{i-1}, a_i, a_{i+1}, \ldots, a_n) \rightarrow f(b_1, \ldots, b_{i-1}, b_i, a_{i+1}, \ldots, a_n) \in I$$

$$\vdots$$

$$f(b_1, \ldots, b_{n-1}, a_n) \rightarrow f(b_1, \ldots, b_{n-1}, b_n) \in I$$

and also that:

$$f(b_1, a_2, \ldots, a_n) \rightarrow f(a_1, a_2, \ldots, a_n) \in I$$

$$\vdots$$

$$f(b_1, \ldots, b_{i-1}, b_i, a_{i+1}, \ldots, a_n) \rightarrow f(b_1, \ldots, b_{i-1}, a_i, a_{i+1}, \ldots, a_n) \in I$$

$$\vdots$$

$$f(b_1, \ldots, b_{n-1}, b_n) \rightarrow f(b_1, \ldots, b_{n-1}, a_n) \in I.$$

Hence:

$$f(a_1, a_2, \ldots, a_n) \equiv f(b_1, a_2, \ldots, a_n) \pmod{\phi_I}$$

$$\vdots$$

$$f(b_1, \ldots, b_{i-1}, a_i, a_{i+1}, \ldots, a_n) \equiv f(b_1, \ldots, b_{i-1}, b_i, a_{i+1}, \ldots, a_n) \pmod{\phi_I}$$

$$\vdots$$

$$f(b_1, \ldots, b_{n-1}, a_n) \equiv f(b_1, \ldots, b_{n-1}, b_n) \pmod{\phi_I}$$

which implies $f(a_1, \ldots, a_n) \equiv f(b_1, \ldots, b_n) \pmod{\phi_I}$ by transitivity. Hence ϕ_I is a congruence on $\mathbf{A}$.

For (ii), just note that $\phi_I \geq \langle 1 \rangle^{\epsilon}_{\langle A;\, \to, 1\rangle}$ (because of Proposition 2.1.35) $\geq \langle 1 \rangle^{\epsilon}_{\mathbf{A}}$ (by Lemma 3.1.12). Hence $\phi_I \geq \langle 1 \rangle^{\epsilon}_{\mathbf{A}}$.

By (i) and (ii), ϕ_I is a congruence on $\mathbf{A}$ such that $\phi_I \geq \langle 1 \rangle^{\epsilon}_{\mathbf{A}}$. Therefore $\phi_I \in \mathrm{Con}_\epsilon\, \mathbf{A}$ and hence $\theta \in \mathrm{Con}_\epsilon\, \mathbf{A}$ (since $\theta = \phi_I$). Thus $\mathrm{Con}_\epsilon \langle A;\, \to^{\mathbf{A}}, 1\rangle \subseteq \mathrm{Con}_\epsilon\, \mathbf{A}$, and the proof is complete. ∎

A variety V is called a *variety of MINI-algebras with weakly compatible operations* if every $\mathbf{A} \in \mathsf{V}$ is a MINI-algebra with weakly compatible operations. If V is a variety of MINI-algebras with compatible operations, then certainly V is a variety of MINI-algebras with weakly compatible operations. In general, however, the converse does not hold. This is shown by Corollary 3.1.16 below, which obtains as an immediate consequence of the following proposition.

Proposition 3.1.15. *The variety of left handed skew Boolean algebras is termwise definitionally equivalent to a variety of MINI-algebras with weakly compatible operations. However, there exists a left handed skew Boolean algebra* $\mathbf{S}$ *that has* no *MINI-algebra polynomial reduct whose* congruences *coincide with those of* $\mathbf{S}$. *Hence the variety of left handed skew Boolean algebras is not termwise definitionally equivalent to a variety of MINI-algebras with compatible operations.*

Proof. For the first assertion, by the proof of Proposition 3.1.8 we have that any skew Boolean algebra $\mathbf{A} := \langle A;\, \wedge, \vee, \backslash, 0\rangle$ has a MINI-algebra polynomial reduct $\langle A;\, \to^{\mathbf{A}}, \mathbf{1}^{\mathbf{A}}\rangle$, where $a \to^{\mathbf{A}} b := b \backslash a$ for any $a, b \in A$ and $\mathbf{1}^{\mathbf{A}} := 0$. Further, an easy inspection of the subdirectly irreducible skew Boolean algebras $\mathbf{2}^p, \mathbf{3}^p_L$ and $\mathbf{3}^p_R$ shows SBA satisfies the identities:

$$(x \to y) \to \big((y \to x) \to ((x \circ z) \to (y \circ z))\big) \approx \mathbf{1}$$
$$(x \to y) \to \big((y \to x) \to ((z \circ x) \to (z \circ y))\big) \approx \mathbf{1}$$

where $\circ$ denotes either the term corresponding to the skew lattice meet $\wedge$ or the term corresponding skew lattice join $\vee$. From these remarks it follows that $\mathbf{A}$ is term equivalent to a MINI-algebra with weakly compatible operations $\langle A;\ \rightarrow, \dot{-}, 1^{\mathbf{A}}, \wedge, \vee\rangle$. Hence SBA (and in particular, IhSBA) is termwise definitionally equivalent to a variety of MINI-algebras with weakly compatible operations. The remaining statements of the proposition may be inferred from Proposition 3.1.8. ∎

Corollary 3.1.16. *Let* V *be a variety termwise definitionally equivalent to a variety of MINI-algebras with weakly compatible operations. In general,* V *is not termwise definitionally equivalent to a variety of MINI-algebras with compatible operations.*

Varieties of MINI-algebras with weakly compatible operations were introduced by Agliano in [6, Section 4] (under the name *MINI-algebras with compatible operations*) in the context of his study of weakly congruence orderable subtractive varieties with EDPI. Because of Lemma 3.1.13, the class of all varieties of MINI-algebras with weakly compatible operations may be understood as a natural generalisation of the class of all varieties of Hilbert algebras with compatible operations; that this generalisation is essential follows from Corollary 3.1.16. Inasmuch as varieties of MINI-algebras with weakly compatible operations generalise varieties of Hilbert algebras with compatible operations, Theorem 3.1.11 lends one to ask if varieties of MINI-algebras with weakly compatible operations also admit a relevant structure theorem. The following result answers this question in the affirmative.

Theorem 3.1.17. *For a variety* V *with* $\mathbf{1}$ *and a binary term* $x \rightarrow y$ *of* V*, the following are equivalent:*

1. V *is weakly congruence orderable and the binary term* $y \rightarrow x$ *witnesses* both *subtractivity and EDPI for* V *in the sense of Theorem 1.7.9;*

2. V *is termwise definitionally equivalent to a variety of MINI-algebras with weakly compatible operations. In particular, any* $\mathbf{A} := \langle A;\ 1, f_j\rangle_{j\in J} \in \mathsf{V}$ *is term equivalent to a MINI-algebra with weakly compatible operations* $\langle A;\ \rightarrow^{\mathbf{A}}, 1, f_j\rangle_{j\in J}$.

Remark 3.1.18. The statement but not the proof of Theorem 3.1.17 is due to Agliano [6, Section 4]. In more detail, in [6, Theorem 4.5] Agliano asserted that:

> 'A pointed variety is weakly congruence orderable, subtractive and has EDPI iff it is termwise definitionally equivalent to a variety of MINI-algebras with [weakly] compatible operations.'

Although Agliano's proof of the sufficiency of the above assertion is valid, his proof of the necessity of the assertion is not. For his proof of necessity Agliano simply asserts that [6, p. 16]:

> 'In a MINI-algebra with [weakly] compatible operations the congruences (and hence the ideals) depend only on the MINI-algebra operation. Therefore any such variety is weakly congruence orderable, subtractive and has EDPI.'

Because a variety of MINI-algebras with weakly compatible operations need not have compatible operations (by Corollary 3.1.16), this argument is not sufficient to establish the necessary direction of the preceding assertion. To correct this error, we provide new proofs below of both the necessity and the sufficiency of Theorem 3.1.17. ∎

Proof (of Theorem 3.1.17). Let V be a variety with $\mathbf{1}$ and let $x \to y$ be a binary term of V.

(1) $\Rightarrow$ (2) Suppose V is weakly congruence orderable and that the binary term $y \to x$ witnesses *both* subtractivity and EDPI for V in the sense of Theorem 1.7.9. Let $\mathbf{A} := \langle A;\ 1, f_j^{\mathbf{A}}\rangle_{j\in J} \in \mathsf{V}$. Clearly $\mathbf{A}$ is term equivalent to the algebra $\langle A;\ \to^{\mathbf{A}}, 1, f_j^{\mathbf{A}}\rangle_{j\in J}$ obtained from $\mathbf{A}$ by enriching the type of $\mathbf{A}$ with a binary operation symbol $\to$ whose canonical interpretation on $\langle A;\ \to^{\mathbf{A}}, 1, f_j^{\mathbf{A}}\rangle_{j\in J}$ is the term function $\to^{\mathbf{A}}$. We claim $\langle A;\ \to^{\mathbf{A}}, 1, f_j^{\mathbf{A}}\rangle_{j\in J}$ is a MINI-algebra with weakly compatible operations. Since $y \to x$ witnesses *both* subtractivity and EDPI for V in the sense of Theorem 1.7.9, from Theorem 3.1.6 we have that the polynomial reduct $\langle A;\ \to^{\mathbf{A}}, 1\rangle$ is a MINI-algebra. Also, since V is weakly congruence orderable, from Proposition 1.7.19 we have

that V_ϵ is a subtractive congruence orderable variety with EDPC. From Theorem 3.1.11 it follows that $\langle A;\ \to^{A}, 1, f_j^{\mathbf{A}}\rangle_{j\in J}/{\approx_{\mathbf{A}}}$ is a Hilbert algebra with compatible operations. That is to say, $\langle A;\ \to^{\mathbf{A}}, 1, f_j^{\mathbf{A}}\rangle_{j\in J}/\langle 1\rangle^{\epsilon}_{\mathbf{A}}$ is a Hilbert algebra with compatible operations, just because $\approx_{\mathbf{A}} = \langle 1\rangle^{\epsilon}_{\mathbf{A}}$ by Lemma 1.7.17. The proof of Lemma 3.1.14 now shows that for any n-ary fundamental operation $f^{\mathbf{A}} := f_j^{\mathbf{A}}$ of $\langle A;\ \to^{\mathbf{A}}, 1, f_j^{\mathbf{A}}\rangle_{j\in J}$,

$$(a \to^{\mathbf{A}} b) \to^{\mathbf{A}} \left((b \to^{\mathbf{A}} a) \to^{\mathbf{A}} (\widetilde{f_i}^{\mathbf{A}}(a) \to^{\mathbf{A}} \widetilde{f_i}^{\mathbf{A}}(b))\right) = 1$$

for all $a, b \in A$ and slices $\widetilde{f_i}^{\mathbf{A}}(a), \widetilde{f_i}^{\mathbf{A}}(b)$, $1 \le i \le n$. Thus $\langle A;\ \to^{\mathbf{A}}, 1, f_j^{\mathbf{A}}\rangle_{j\in J}$ is a MINI-algebra with weakly compatible operations. Hence V is termwise definitionally equivalent to a variety of MINI-algebras with weakly compatible operations, and the proof is complete.

(2) $\Rightarrow$ (1) Let $\mathbf{A} := \langle A;\ 1, f_j\rangle_{j\in J} \in \mathsf{V}$. By hypothesis, $\mathbf{A}$ is term equivalent to a MINI-algebra with weakly compatible operations $\langle A;\ \to^{\mathbf{A}}, 1, f_j^{\mathbf{A}}\rangle_{j\in J}$. Therefore V is subtractive (witness $y \to x$). To see V has EDPI (witness $y \to x$), it is sufficient by Corollary 3.1.7 to show $\mathrm{I}(\mathbf{A}) = \mathrm{I}\big(\langle A;\ \to^{\mathbf{A}}, 1\rangle\big)$. As $\mathrm{Con}\,\mathbf{A} \subseteq \mathrm{Con}\,\langle A;\ \to^{\mathbf{A}}, 1\rangle$, we have that $\mathrm{N}(\mathbf{A}) \subseteq \mathrm{N}\big(\langle A;\ \to^{\mathbf{A}}, 1\rangle\big)$, and hence that $\mathrm{I}(\mathbf{A}) \subseteq \mathrm{I}\big(\langle A;\ \to^{\mathbf{A}}, 1\rangle\big)$. For the converse, let $I \in \mathrm{I}\big(\langle A;\ \to^{\mathbf{A}}, 1\rangle\big)$ and let ϕ_I be the relation on $A \times A$ defined by:

$$\phi_I := \big\{(a, b) \in A \times A : a \to^{\mathbf{A}} b, b \to^{\mathbf{A}} a \in I\big\}.$$

By the proof of Lemma 3.1.14, ϕ_I is a congruence on $\mathbf{A}$. Moreover, $I = [0]_{\phi_I}$, just because of Theorem 3.1.4 and Lemma 2.1.27. Hence $I \in \mathrm{I}_{\mathsf{V}}(\mathbf{A})$ and $\mathrm{I}\big(\langle A;\ \to^{\mathbf{A}}, 1\rangle\big) \subseteq \mathrm{I}(\mathbf{A})$. Thus $\mathrm{I}(\mathbf{A}) = \mathrm{I}\big(\langle A;\ \to^{\mathbf{A}}, 1\rangle\big)$ and V has EDPI (witness $y \to x$).

It remains to show V is weakly congruence orderable. Since V is subtractive with EDPI (witness $y \to x$), from Lemma 1.7.14 we have that $a \approx_{\mathbf{A}} b$ iff $a \to^{\mathbf{A}} b = 1 = b \to^{\mathbf{A}} a$ iff $a \equiv b \pmod{\phi_{\{1\}}}$. Since $\phi_{\{1\}} \in \mathrm{Con}\,\mathbf{A}$ by the proof of Lemma 3.1.14 we have that $\approx_{\mathbf{A}}$ is a congruence on $\mathbf{A}$, so $\mathbf{A}$ is weakly congruence orderable. Hence V is weakly congruence orderable, and the proof is complete. ∎

Remark 3.1.19. In [6, Theorem 4.15] Agliano extended the characterisation of weakly congruence orderable subtractive varieties with EDPI of Theorem 3.1.17 to subtractive weakly congruence orderable varieties with EDPI and both a meet generator and a join generator term. Agliano's proof of [6, Theorem 4.15] relies on the assumption that if $\mathbf{A}$ is an algebra term equivalent to a (certain notion of) L-algebra with compatible operations $\langle A;\ \sqcap^{\mathbf{A}}, \sqcup^{\mathbf{A}}, \rightarrow^{\mathbf{A}}, 1, f_j\rangle_{j\in J}$, then $\operatorname{Con}\mathbf{A} = \operatorname{Con}\langle A;\ \rightarrow^{\mathbf{A}}, 1\rangle$. From the definition of an L-algebra given by Agliano in [6, Section 4, p. 20], it is readily verified by inspection of the subdirectly irreducible skew Boolean algebras $\mathbf{2}^p$, $\mathbf{3}^p_L$ and $\mathbf{3}^p_R$ that any skew Boolean algebra $\langle B;\ \wedge, \vee, \backslash, 0\rangle$ is term equivalent to an L-algebra $\langle B;\ \sqcap, \sqcup, \rightarrow, 1\rangle$. From Proposition 3.1.15 it follows that, in general, $\operatorname{Con}\mathbf{A} \neq \operatorname{Con}\langle A;\ \rightarrow^{\mathbf{A}}, 1\rangle$. Hence the argument employed by Agliano to characterise subtractive weakly congruence orderable varieties with EDPI and both a meet generator and a join generator term does not hold in general, whence the problem of characterising such varieties remains open. ∎

Example 3.1.20. By Example 2.3.12, any pseudocomplemented semilattice $\mathbf{A}$ has a canonical implicative BCS-algebra polynomial reduct $\langle A;\ \backslash, 0\rangle$, where $a\backslash b := a \wedge b^*$ for any $a, b \in A$. For any $a, b \in A$, let $a \rightarrow^{\mathbf{A}} b := b\backslash a$ and let $1^{\mathbf{A}} := 0$. Then for any $a, b, c \in A$, we have that:

$$\begin{aligned}
(a \rightarrow^{\mathbf{A}} b) &\rightarrow^{\mathbf{A}} \big((b \rightarrow^{\mathbf{A}} a) \rightarrow^{\mathbf{A}} (a^* \rightarrow^{\mathbf{A}} b^*)\big) \\
&= \big((b^*\backslash a^*)\backslash(a\backslash b)\big)\backslash(b\backslash a) \\
&= b^* \wedge a^{**} \wedge (a \wedge b^*)^* \wedge (b \wedge a^*)^* \\
&= \big(a^{**} \wedge (a^* \wedge b)^*\big) \wedge \big(b^* \wedge (b^* \wedge a)^*\big) \\
&= \big(a^{**} \wedge (a^* \wedge b)^*\big) \wedge (b^* \wedge a^*) && \text{by (1.9)} \\
&= (a^{**} \wedge a^*) \wedge \big((a^* \wedge b)^* \wedge b^*\big) \\
&= 0 \wedge \big((a^* \wedge b)^* \wedge b^*\big) && \text{by (1.8)} \\
&= 0 \\
&= \mathbf{1}^{\mathbf{A}}
\end{aligned}$$

and:

$$\begin{aligned}
&(a \to^{\mathbf{A}} b) \to^{\mathbf{A}} \big((b \to^{\mathbf{A}} a) \to^{\mathbf{A}} ((a \wedge c) \to^{\mathbf{A}} (b \wedge c))\big) \\
&\quad = \big(((b \wedge c)\backslash(a \wedge c))\backslash(a\backslash b)\big)\backslash(b\backslash a) \\
&\quad = (b \wedge c) \wedge (a \wedge c)^* \wedge (a \wedge b^*)^* \wedge (b \wedge a^*)^* \\
&\quad = \big(c \wedge (c \wedge a)^*\big) \wedge \big(b \wedge (b \wedge a^*)^*\big) \wedge (a \wedge b^*)^* \\
&\quad = (c \wedge a^*) \wedge (b \wedge a^{**}) \wedge (a \wedge b^*)^* && \text{by (1.9)} \\
&\quad = (c \wedge b) \wedge (a^* \wedge a^{**}) \wedge (a \wedge b^*)^* \\
&\quad = (c \wedge b) \wedge 0 \wedge (a \wedge b^*)^* && \text{by (1.8)} \\
&\quad = 0 \\
&\quad = \mathbf{1}^{\mathbf{A}}
\end{aligned}$$

whence $\mathbf{A}$ is term equivalent to a MINI-algebra with weakly compatible operations $\langle A;\, \to^{\mathbf{A}}, \mathbf{1}^{\mathbf{A}}, \wedge, {}^* \rangle$ (by Theorem 3.1.4). Hence PCSL is termwise definitionally equivalent to a variety of MINI-algebras with weakly compatible operations; from Theorem 3.1.17 we conclude that PCSL is weakly congruence orderable and subtractive with EDPI.

Conversely, from Example 2.3.12 we have that the variety PCSL of pseudocomplemented semilattices is subtractive witness $x\backslash y$. Also, by Agliano and Ursini [11, p. 387] it is known that $\langle a \rangle_{\mathbf{A}} = \{b : b \leq a^{**}\}$ for any pseudocomplemented semilattice $\mathbf{A}$ and $a \in A$. Suppose $b \leq a^{**}$. Then $b \wedge a^{**} = b$, whence $b \wedge a^* = (b \wedge a^{**}) \wedge a^* = b \wedge (a^{**} \wedge a^*) = 0$ by (1.8). Hence $b \preceq^{\langle A;\, \backslash, 0\rangle} a$. On the other hand, suppose $b \preceq^{\langle A;\, \backslash, 0\rangle} a$. Then $b \wedge a^* = 0$, whence $b \wedge a^{**} = b \wedge (b \wedge a^*)^* = b \wedge 0^* = b$ by (1.9). Hence $b \leq a^{**}$ and so $b \leq a^{**}$ iff $b \preceq^{\langle A;\, \backslash, 0\rangle} a$. Thus $\langle a \rangle_{\mathbf{A}} = \langle a \rangle_{\langle A;\, \backslash, 0\rangle}$ and PCSL has EDPI (we remark that this result has been obtained independently by Agliano and Ursini [11, Example 5.9]). Moreover, $\approx_{\mathbf{A}}$ is an equivalence relation on $A \times A$ (recall the remarks of §1.7.8) and for any $a, b \in A$ we have that $a \approx_{\mathbf{A}} b$ iff $a\backslash b = 0 = b\backslash a$ (by Lemma 1.7.14) iff $a \equiv b \,(\mathrm{mod}\, \Xi_{\langle A;\, \backslash, 0\rangle})$ iff $a^* = b^*$ (by Example 2.3.61(2) and Lemma 2.3.63(3)). Suppose $a_1 \approx_{\mathbf{A}} b_1$ and $a_2 \approx_{\mathbf{A}} b_2$ for $a_1, a_2, b_1, b_2 \in A$. Since $a_1^{**} = (a_1^*)^* = (b_1^*)^* = b_1^{**}$ we have that $\approx_{\mathbf{A}}$ has the substitution property for the * operation. Also,

$$\begin{aligned}
(a_1 \wedge a_2)^* &= (a_1^{**} \wedge a_2^{**})^* && \text{by Jones [128, p. 2]} \\
&= \left((a_1^*)^* \wedge (a_2^*)^*\right)^* \\
&= \left((b_1^*)^* \wedge (b_2^*)^*\right)^* \\
&= (b_1^{**} \wedge b_2^{**})^* \\
&= (b_1 \wedge b_2)^* && \text{by Jones [128, p. 2]}
\end{aligned}$$

and so $\approx_{\mathbf{A}}$ also has the substitution property for the $\wedge$ operation. Thus $\approx_{\mathbf{A}}$ is a congruence on $\mathbf{A}$ and so PCSL is weakly congruence orderable. Since PCSL is weakly congruence orderable and subtractive with EDPI, from Theorem 3.1.17 we conclude that PCSL is termwise definitionally equivalent to a variety of MINI-algebras with weakly compatible operations. ∎

Theorem 3.1.17 notwithstanding, we do not know if varieties of MINI-algebras with compatible operations admit a coherent structure theory in general. Hence we conclude this subsection with the following problem.

Problem 3.1.21. Do varieties of MINI-algebras with compatible operations admit a coherent structure theory? ∎

3.1.22. Subtractive WBSO Varieties and Nelson Algebras. Recall the definition of a WBSO variety from §1.5.3. By a *subtractive WBSO variety* we mean a WBSO variety that is subtractive. Subtractive WBSO varieties have been characterised in the literature: by [11, Theorem 5.4] a pointed variety is a subtractive WBSO variety iff it is ideal determined, has EDPI and a join generator term. Let V be a subtractive WBSO variety. By hypothesis, V is subtractive; moreover V is strongly point regular with EDPC by Lemma 1.5.4, Proposition 1.5.5 and Proposition 1.5.6. Suppose now that V is a variety with $\mathbf{1}$ that is subtractive and strongly $\mathbf{1}$-regular with EDPC. Subtractivity and $\mathbf{1}$-regularity imply V is ideal determined (by Proposition 1.7.3); also subtractivity and EDPC imply V has EDPI (by Proposition 1.7.10). Let $\mathbf{A} \in \mathsf{V}$ and let $a, b \in A$. Since V is strongly $\mathbf{1}$-regular there exists $c \in A$ such that $\Theta^{\mathbf{A}}(a, 1) \vee^{\mathbf{Cp\,A}}$

$\Theta^{\mathbf{A}}(b,1) = \Theta^{\mathbf{A}}(c,1)$, whence:

$$\begin{aligned}\langle a\rangle_{\mathbf{A}} \vee^{\mathbf{CI}(\mathbf{A})} \langle b\rangle_{\mathbf{A}} &= [1]_{\Theta^{\mathbf{A}}(a,1)} \vee^{\mathbf{CI}(\mathbf{A})} [1]_{\Theta^{\mathbf{A}}(b,1)}\\ &= [1]_{\Theta^{\mathbf{A}}(a,1) \vee^{\mathbf{Cp\,A}} \Theta^{\mathbf{A}}(b,1)}\\ &= [1]_{\Theta^{\mathbf{A}}(c,1)}\\ &= \langle c\rangle_{\mathbf{A}}\end{aligned}$$

by subtractivity and the dual Brouwerian semilattice isomorphism $\theta \mapsto [1]_\theta$ between $\mathbf{Cp\,A}$ and $\mathbf{CI}(\mathbf{A})$ of Proposition 1.7.10. Thus the join on $\mathbf{I}(\mathbf{A})$ of two principal ideals of $\mathbf{A}$ is always principal; from Proposition 1.7.13 it follows that V has a join generator term. As V is ideal determined with both EDPI and a join generator term, we have that V is a subtractive WBSO variety. That is, we have proved:

Proposition 3.1.23. *For a variety* V *with* $\mathbf{1}$, *the following are equivalent:*

1. V *is a subtractive WBSO variety;*

2. V *is ideal determined, has EDPI and a join generator term;*

3. V *is subtractive, strongly* $\mathbf{1}$*-regular and has EDPC.*

Let V be a WBSO variety. Recall from [29, Section 2] that the following identities and quasi-identity are satisfied for any weak relative pseudocomplementation $\rightarrow$ of V:

$$x \rightarrow x \approx \mathbf{1} \tag{3.8}$$

$$\mathbf{1} \rightarrow x \approx x \supset x \approx \mathbf{1}. \tag{3.9}$$

From (3.8) and (3.9) it is clear that for V to be subtractive the quasi-identity (3.9) need only be strengthened to the identity $\mathbf{1} \rightarrow x \approx x$. By a *WBSO*$^{\#}$ *variety* we mean a WBSO variety V such that V $\models \mathbf{1} \rightarrow x \approx x$ for a weak relative pseudocomplementation $\rightarrow$. Clearly any WBSO$^{\#}$ variety is subtractive; in [4] Agliano investigated subtractive WBSO varieties and proved that the converse also holds. That is, in a subtractive WBSO variety V it is always possible to choose a weak relative pseudocomplementation $\rightarrow$ in such a way that $\mathsf{V} \models \mathbf{1} \to x \approx x$; we call such a term a *subtractive weak relative pseudocomplementation* in the sequel. The following proposition, which sharpens Agliano's characterisation of subtractive WBSO varieties, is implicit in results due to Agliano [4] and Agliano and Ursini [11].

Proposition 3.1.24. *For a variety* V *with* $\mathbf{1}$ *the following are equivalent:*

1. V *is a subtractive WBSO variety;*
2. V *is a WBSO*$^{\#}$ *variety;*
3. V *is a WBSO variety and there exists a binary term* $x \to y$ *of* V *such that:*
   (a) *For any* $\mathbf{A} \in \mathsf{V}$*, the polynomial reduct* $\langle A;\ \to^{\mathbf{A}}, 1\rangle$ *is a MINI-algebra;*
   (b) $x \to y$ *is a weak relative pseudocomplementation for* V.

Proof. The equivalence of (1) and (2) is proved by Agliano in [4, Theorem 4]. (3) $\Rightarrow$ (2) is trivial. It therefore remains to show (2) $\Rightarrow$ (3). So let V be a WBSO$^{\#}$ variety. Then there exists a binary term $x \to y$ of V such that $\mathsf{V} \models \mathbf{1} \to x \approx x$ and $x \to y$ is a weak relative pseudocomplementation, whence Condition (3)(b) is satisfied. Let $\mathbf{A} \in \mathsf{V}$ and let $a, b \in A$. Since $x \to y$ is a weak relative pseudocomplementation for V,

$$
\begin{array}{llll}
a \to^{\mathbf{A}} b = 1 & \text{iff} & a \preceq b & \text{where } \preceq \text{ is the quasiorder of §1.5.3} \\
 & \text{iff} & \Theta^{\mathbf{A}}(b,1) \subseteq \Theta^{\mathbf{A}}(a,1) & \text{by [29, Lemma 2.5(i)]} \\
 & \text{iff} & [1]_{\Theta^{\mathbf{A}}(b,1)} \subseteq [1]_{\Theta^{\mathbf{A}}(a,1)} & \\
 & \text{iff} & \langle b\rangle_{\mathbf{A}} \subseteq \langle a\rangle_{\mathbf{A}} & \\
 & \text{iff} & b \in \langle a\rangle_{\mathbf{A}}. &
\end{array}
$$

Hence $y \to x$ witnesses *both* subtractivity and EDPI for V in the sense of Theorem 1.7.9. By Theorem 3.1.6 we conclude that the polynomial reduct $\langle A;\ \to^{\mathbf{A}}, 1\rangle$ is a MINI-algebra; thus Condition (3)(a) is satisfied also and the proof is complete. ∎

WBSO$^\#$ varieties arise very naturally in algebraic logic as arbitrary congruence permutable point regular varieties with EDPC. Indeed, suppose V is a congruence permutable point regular (say $\mathbf{1}$-regular) variety with EDPC. Then V has a QD term $q(x, y, z, w)$ by Theorem 1.5.8 and binary terms $d_1(x, y), \ldots, d_n(x, y)$ witnessing the $\mathbf{1}$-regularity of V by Proposition 1.2.6. By Proposition 1.5.11(2),

$$\begin{aligned} x \to y &:= q(x, \mathbf{1}, y, \mathbf{1}) \\ x \cdot y &:= q(x, \mathbf{1}, y, x) \\ x \Delta y &:= \big(\cdots(d_1(x, y) \cdot d_2(x, y)) \cdot \cdots\big) \cdot d_n(x, y) \end{aligned}$$

are respectively subtractive weak relative pseudocomplementation, weak meet and Gödel equivalence terms for V, whence V is a WBSO$^\#$ variety.

To within clone equivalence, the term operations induced by a weak meet $x \cdot y$ and weak relative pseudocomplementation $x \to y$ in a WBSO variety faithfully reflect conjunction and implication on the Brouwerian semilattice of compact congruences of any member of the variety: see Blok and Pigozzi [34, p. 547]. In view of this remark it is easy to see that $\cdot$ and $\to$ are respectively a conjunction and implication for the intrinsic assertional logic $\mathbb{S}(\mathsf{V}, \mathbf{1})$ of V. The following result, which is essentially well known, formalises and extends this observation.

Proposition 3.1.25. *Let V be a WBSO$^\#$ variety with assertional logic $\mathbb{S}(\mathsf{V}, \mathbf{1})$. Then $\mathbb{S}(\mathsf{V}, \mathbf{1})$ is algebraisable and its equivalent algebraic semantics is exactly V. Moreover, for binary terms $\cdot, +, \Delta, \to$ of V the following assertions hold (upon identifying the individual variables x, y with the propositional variables p, q respectively):*

1. *[181, Theorem 2.1] $x \cdot y$ is a weak meet for V iff $p \cdot q$ is a conjunction for $\mathbb{S}(\mathsf{V}, \mathbf{1})$;*
2. *[181, Theorem 2.2] $x \Delta y$ is a Gödel equivalence term for V iff $p \Delta q$ is a G-identity for $\mathbb{S}(\mathsf{V}, \mathbf{1})$;*
3. *[181, Theorem 2.3] $x \to y$ is a weak relative pseudocomplementation for V iff $p \to q$ is a conditional for $\mathbb{S}(\mathsf{V}, \mathbf{1})$;*

4. *[181, p. 485]* $(p \to q) \cdot (q \to p)$ *is a biconditional for* $\mathbb{S}(\mathsf{V}, \mathbf{1})$ *iff* $x \cdot y$ *and* $x \to y$ *are respectively a weak meet and weak relative pseudocomplementation for* V;

5. *[224, Theorem 5.1]* $x + y$ *is a weak join for* V *iff* $p + q$ *is a disjunction for* $\mathbb{S}(\mathsf{V}, \mathbf{1})$;

6. *If* V *is double-pointed (say with* $\{\mathbf{0}, \mathbf{1}\}$*) and* $x \to y$ *is a subtractive weak relative pseudocomplementation for* V *then* $p \to \mathbf{0}$ *is a weak negation for* $\mathbb{S}(\mathsf{V}, \mathbf{1})$.

Proof. Let V be a WBSO$^{\#}$ variety with assertional logic $\mathbb{S}(\mathsf{V}, \mathbf{1})$. By Theorem 1.8.15 $\mathbb{S}(\mathsf{V}, \mathbf{1})$ is algebraisable and its equivalent algebraic semantics is exactly V. Of the remaining statements, only (6) is not explicit in the literature. So suppose V is with $\{\mathbf{0}, \mathbf{1}\}$ and that $\to$ is a subtractive weak relative pseudocomplementation for V. Let $\neg\, p := p \to \mathbf{0}$. To see $\neg$ is a weak negation we show (CN) and (RA$_J$) are respectively rules of $\mathbb{S}(\mathsf{V}, \mathbf{1})$, *viz.*:

(i) $\varphi, \neg\, \varphi \vdash_{\mathbb{S}(\mathsf{V},\mathbf{1})} \psi$

(ii) $\dfrac{\Gamma, \psi \vdash_{\mathbb{S}(\mathsf{V},\mathbf{1})} \neg\, \psi}{\Gamma \vdash_{\mathbb{S}(\mathsf{V},\mathbf{1})} \neg\, \psi}$

Throughout the remainder of the proof we identify the individual variables $\vec{x}$ with the propositional variables $\vec{p}$ and the terms $s(\vec{x}), \{s_i(\vec{x}) : i = 1, \ldots, n\}, t(\vec{x})$ with the formulas $\varphi(\vec{p}), \{\varphi_i(\vec{p}) : i = 1, \ldots, n\}, \psi(\vec{p})$ respectively. To ease notation, for a given formula $\varphi(\vec{p})$ we also write simply φ.

For (i), it is sufficient to show $\{s(\vec{x}) \approx \mathbf{1}, \neg\, s(\vec{x}) \approx \mathbf{1}\} \models_{\mathsf{V}} t(\vec{x}) \approx \mathbf{1}$. So let $\mathbf{A} \in \mathsf{V}$ and $\vec{a} \in A$. If $s^{\mathbf{A}}(\vec{a}) = 1$ and $\neg\, s^{\mathbf{A}}(\vec{a}) = 1$ then $1 \to 0 = 1$, which implies $0 = 1$ as $\to$ is a subtractive weak relative pseudocomplementation. But then 0 and 1 are not residually distinct, so $\mathbf{A}$ is trivial and $t^{\mathbf{A}}(\vec{a}) = 1$.

For (ii), by the remarks of §1.8.9 we may assume without loss of generality that Γ is finite, say $\{\varphi_1, \ldots, \varphi_n\}$. Moreover, by (3) $\to$ is a conditional for $\mathbb{S}(\mathsf{V}, \mathbf{1})$, so the entailment $\varphi_1, \ldots, \varphi_n, \psi \vdash_{\mathbb{S}(\mathsf{V},\mathbf{1})} \neg\, \psi$ is equivalent to the entailment $\vdash_{\mathbb{S}(\mathsf{V},\mathbf{1})} \varphi_1 \to (\varphi_2 \to (\cdots \to (\varphi_n \to (\psi \to \neg\, \psi)) \cdots))$, and likewise the entailment $\varphi_1, \ldots, \varphi_n \vdash_{\mathbb{S}(\mathsf{V},\mathbf{1})} \neg\, \psi$ is equivalent to the entailment

$\vdash_{\mathbb{S}(\mathsf{V},\mathbf{1})} \varphi_1 \to (\varphi_2 \to (\cdots \to (\varphi_n \to \neg\, \psi)\cdots))$. To complete the proof it is therefore sufficient to show:

$$\mathsf{V} \models s_1(\vec{x}) \to (s_2(\vec{x}) \to (\cdots \to (s_n(\vec{x}) \to (t(\vec{x}) \to \neg\, t(\vec{x})))\cdots)) \approx \mathbf{1}$$

implies:

$$\mathsf{V} \models s_1(\vec{x}) \to (s_2(\vec{x}) \to (\cdots \to (s_n(\vec{x}) \to \neg\, t(\vec{x}))\cdots)) \approx \mathbf{1}.$$

Let $\mathbf{A} \in \mathsf{V}$ and $\vec{a} \in A$. Since the polynomial reduct $\langle A;\, \to, 1\rangle$ is a MINI-algebra, $(c \to (c \to b)) \to (c \to b) = 1$ for any $b, c \in A$ by Theorem 3.1.4 and (2.36). Thus in particular $t^{\mathbf{A}}(\vec{a}) \to (t^{\mathbf{A}}(\vec{a}) \to 0) \preceq t^{\mathbf{A}}(\vec{a}) \to 0$ (where $\preceq$ is the underlying quasiorder on the MINI-algebra polynomial reduct $\langle A;\, \to, 1\rangle$), whence repeated application of (the dual of) Lemma 2.1.12(2) shows:

$$s_1^{\mathbf{A}}(\vec{a}) \to (s_2^{\mathbf{A}}(\vec{a}) \to (\cdots \to (s_n^{\mathbf{A}}(\vec{a}) \to (t^{\mathbf{A}}(\vec{a}) \to (t^{\mathbf{A}}(\vec{a}) \to 0)))\cdots)) \preceq$$
$$s_1^{\mathbf{A}}(\vec{a}) \to (s_2^{\mathbf{A}}(\vec{a}) \to (\cdots \to (s_n^{\mathbf{A}}(\vec{a}) \to (t^{\mathbf{A}}(\vec{a}) \to 0))\cdots)).$$

Since $s_1^{\mathbf{A}}(\vec{a}) \to (s_2^{\mathbf{A}}(\vec{a}) \to (\cdots \to (s_n^{\mathbf{A}}(\vec{a}) \to (t^{\mathbf{A}}(\vec{a}) \to (t^{\mathbf{A}}(\vec{a}) \to 0)))\cdots)) = 1$ by hypothesis, we conclude $s_1^{\mathbf{A}}(\vec{a}) \to (s_2^{\mathbf{A}}(\vec{a}) \to (\cdots \to (s_n^{\mathbf{A}}(\vec{a}) \to (t^{\mathbf{A}}(\vec{a}) \to 0))\cdots)) = 1$ (because $[1]_{\equiv} = \{1\}$ by (the dual of) Theorem 2.1.14(3)), which implies the required implication holds.

By (i) and (ii), both (CN) and (RA_J) are rules of $\mathbb{S}(\mathsf{V},\mathbf{1})$. Therefore $\neg$ is a weak negation for $\mathbb{S}(\mathsf{V},\mathbf{1})$, and the proof is complete. ∎

Remark 3.1.26. In general, the proofs of Items (1)–(4) of Proposition 3.1.25 do not require the hypothesis of subtractivity. We have been unable to formulate proofs of Items (5)–(6) without this assumption; however, we have no proof that this additional condition is necessary. More generally, it is not known to what extent satisfaction of the identity $\mathbf{1} \to x \approx x$ (where $\to$ is a weak relative pseudocomplementation) is reflected in special properties of WBSO varieties: see Blok and Pigozzi [29, p. 365; Theorem 3.7]. ∎

Problem 3.1.27. Comprehensively characterise those properties of WBSO$^{\#}$ varieties that do not extend to arbitrary WBSO varieties. ∎

Recall from [14, Definition XI§2.1] that a *De Morgan algebra* is an algebra $\langle A;\ \wedge, \vee, \sim, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ such that the reduct $\langle A;\ \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and moreover the following identities are satisfied:

$$\sim\sim x \approx x \tag{3.10}$$

$$\sim(x \wedge y) \approx \sim x \vee \sim y \tag{3.11}$$

$$\sim(x \vee y) \approx \sim x \wedge \sim y. \tag{3.12}$$

A *Nelson algebra* (or *quasi-pseudo Boolean algebra* in the terminology of Rasiowa [196, pp. 75 ff.]) is an algebra $\langle A;\ \wedge, \vee, \rightarrow, \sim, 0, 1\rangle$ of type $\langle 2, 2, 2, 1, 0, 0\rangle$ such that the following conditions are satisfied for all $a, b, c \in A$ [209, Section 0]:

N1. The reduct $\langle A;\ \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra with greatest element 1, least element 0 and lattice ordering $\leq$;

N2. The relation $\preceq$ defined by $a \preceq b$ iff $a \rightarrow b = 1$ is a quasiordering on A;

N3. $a \wedge b \preceq c$ iff $a \preceq b \rightarrow c$;

N4. $a \leq b$ iff $a \preceq b$ and $\sim b \preceq \sim a$;

N5. $a \preceq c$ and $b \preceq c$ implies $a \vee b \preceq c$;

N6. $a \preceq b$ and $a \preceq c$ implies $a \preceq b \wedge c$;

N7. $a \wedge \sim b \preceq \sim(a \rightarrow b)$ and $\sim(a \rightarrow b) \preceq a \wedge \sim b$;

N8. $\sim(a \rightarrow 0) \preceq a$ and $a \preceq \sim(a \rightarrow 0)$;

N9. $a \wedge \sim a \preceq b$.

Nelson algebras were introduced by Rasiowa in [194] (under the name $\mathcal{N}$-*lattices*) as the algebraic counterpart of a particular (non-axiomatic) extension of the intuitionistic sentential calculus called *constructive logic with strong negation*, which latter was introduced independently by Nelson in [174] and Markov in [157] in response to certain philosophical objections concerning the non-constructive nature of falsity in IPC (for a discussion see Rasiowa [195, Chapter XII] or Wójcicki [238, Section 5.3.0]; see also Vorob'ev [233, 234] and Thomason [215]). More particularly, by results of Rasiowa [195, Chapter XII] and Blok and Pigozzi [31, Section 5.2.1] constructive logic with strong negation is precisely the inherent assertional logic $\mathbb{S}(\mathsf{N}, \mathbf{1})$ of the class N of Nelson algebras. Since their introduction Nelson algebras have been studied by a number of authors, including Brignole [47], Rasiowa [195, Chapter V], Sendlewski [209] and, in a recent major study, by Viglizzo [232]. By Brignole [47] or Rasiowa [195, Theorem V§2.1] the class N is a variety; by Blok and Pigozzi [29, pp. 357–358] N is a WBSO variety with weak meet $x \wedge y$, weak relative pseudocomplementation $x \to y$ and Gödel equivalence term:

$$x \Leftrightarrow y := (x \Rightarrow y) \wedge (y \Rightarrow x)$$

where:

$$x \Rightarrow y := (x \to y) \wedge ({\sim}\, y \to {\sim}\, x).$$

Results due to Rasiowa [195, Theorem V§1.3] show also that the variety of Nelson algebras satisfies the identities:

$$x \to x \approx \mathbf{1} \tag{3.13}$$

$$\mathbf{1} \to x \approx x, \tag{3.14}$$

whence $\to$ witnesses both subtractivity and weak relative pseudocomplementation for N in the sense of Theorem 3.1.24. Thus N is a WBSO$^{\#}$ variety.

There exist a number of open problems and erroneous communications in the literature concerning Nelson algebras *qua* weak Brouwerian semilattices with filter preserving operations [29, 34, 4]. In particular, in their paper [34] on the structure of varieties with EDPC, Blok and Pigozzi posed the following problem [34, Problem 7.4]: *Does the variety of Nelson algebras have a commutative, regular TD term, or even a TD term?* Notice that although N has EDPC (by Proposition 1.5.6, since it is a WBSO$^{\#}$ variety) this problem is nontrivial, because EDPC in and of itself does not imply the existence of a term

$e(x, y, z)$ of N such that both $\mathsf{N} \models e(x, x, z) \approx z$ and $e^{\mathbf{A}}(a, b, c) = e^{\mathbf{A}}(a, b, d)$ if $c \equiv d \left(\operatorname{mod} \Theta^{\mathbf{A}}(a, b)\right)$ for any $\mathbf{A} \in \mathsf{N}$ and $a, b, c, d \in A$. See Blok and Pigozzi [34, p. 570; Problem 7.1].

Lemma 3.1.28. *The variety of Nelson algebras satisfies the following identities:*

$$\sim x \to (x \to y) \approx \mathbf{1} \tag{3.15}$$

$$x \wedge (x \to y) \approx x \wedge (\sim x \vee y) \tag{3.16}$$

$$x \to (y \to z) \approx y \to (x \to z) \tag{3.17}$$

$$(x \wedge y) \to z \approx x \to (y \to z) \tag{3.18}$$

$$x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z) \tag{3.19}$$

$$x \to (y \wedge z) \approx (x \to y) \wedge (x \to z) \tag{3.20}$$

$$(x \vee y) \to z \approx (x \to z) \wedge (y \to z) \tag{3.21}$$

$$(x \Rightarrow y) \to \big((y \Rightarrow x) \to y\big) \approx (y \Rightarrow x) \to \big((x \Rightarrow y) \to x\big). \tag{3.22}$$

Proof. Identity (3.15) is proved in Rasiowa [195, Theorem V§1.3], while identity (3.16) is established in Brignole [47, Theorem 3]. Identities (3.17) and (3.18) are proved in Rasiowa [195, Theorem V§1.3]. Identity (3.19) follows trivially from (N1). Identity (3.20) is proved in Brignole [47, Theorem 3]. Identity (3.21) is established by Monteiro in [169, 170] (see also Viglizzo [232, (1.9), pp. 6–7]). Identity (3.22) is Theorem V§1.4 of Rasiowa [195]. ∎

Lemma 3.1.29. *The variety of Nelson algebras satisfies the following identities:*

$$x \to (\sim x \to y) \approx \mathbf{1} \tag{3.23}$$

$$x \to (x \to y) \approx x \to y \tag{3.24}$$

$$\sim(x \wedge y) \to z \approx (\sim x \to z) \wedge (\sim y \to z). \tag{3.25}$$

Proof. Let $\mathbf{A}$ be a Nelson algebra and let $a, b, c \in A$. For (3.23), we have $1 = \sim\sim a \to (\sim a \to b) = a \to (\sim a \to b)$ by (3.15) and (3.10). For (3.24), we have $a \to (a \to b) = (a \wedge a) \to b = a \to b$ by (3.18). For (3.25), we have $\sim(a \wedge b) \to c = (\sim a \vee \sim b) \to c = (\sim a \to c) \wedge (\sim b \to c)$ by (3.11) and (3.21). ∎

Lemma 3.1.30. *The variety of Nelson algebras satisfies the following identities:*

$$x \wedge (x \to y) \approx x \wedge \sim(x \wedge \sim y) \tag{3.26}$$

$$x \to \sim(x \wedge \sim y) \approx x \to y \tag{3.27}$$

$$x \to \sim(x \wedge y) \approx x \to \sim y. \tag{3.28}$$

Proof. Let $\mathbf{A}$ be a Nelson algebra and let $a, b \in A$. By (3.11), (3.10) and (3.16) we have $a \wedge \sim(a \wedge \sim b) = a \wedge (\sim a \vee \sim\sim b) = a \wedge (\sim a \vee b) = a \wedge (a \to b)$, which establishes (3.26). For (3.27), put $\alpha := a$, $\beta := a$ and $\gamma := a \to b$. We have:

$$\begin{aligned}
a \to b &= a \to (a \to b) && \text{by (3.24)}\\
&= 1 \wedge \big(a \to (a \to b)\big)\\
&= (a \to a) \wedge \big(a \to (a \to b)\big) && \text{by (3.13)}\\
&= (\alpha \to \beta) \wedge (\alpha \to \gamma)\\
&= \alpha \to (\beta \wedge \gamma) && \text{by (3.20)}\\
&= a \to \big(a \wedge (a \to b)\big)\\
&= a \to \big(a \wedge (\sim a \vee b)\big) && \text{by (3.16).}
\end{aligned}$$

Put $\alpha := a$, $\beta := a$ and $\gamma := \sim a \vee b$. We have:

$$\begin{aligned}
&a \to \big(a \wedge (\sim a \vee b)\big)\\
&\qquad = \alpha \to (\beta \wedge \gamma)\\
&\qquad = (\alpha \to \beta) \wedge (\alpha \to \gamma) && \text{by (3.20)}\\
&\qquad = (a \to a) \wedge \big(a \to (\sim a \vee b)\big)\\
&\qquad = 1 \wedge \big(a \to (\sim a \vee b)\big) && \text{by (3.13)}\\
&\qquad = a \to (\sim a \vee b)\\
&\qquad = a \to (\sim a \vee \sim\sim b) && \text{by (3.10)}
\end{aligned}$$

$$= a \to {\sim}(a \wedge {\sim} b) \qquad \text{by (3.11).}$$

For (3.28) just note $a \to {\sim}(a \wedge b) = a \to {\sim}(a \wedge {\sim}{\sim} b) = a \to {\sim} b$ by (3.10) and (3.27). ∎

Lemma 3.1.31. *The variety of Nelson algebras satisfies the following identities:*

$$(x \Leftrightarrow y) \to x \approx (x \Leftrightarrow y) \to y \tag{3.29}$$

$$x \to {\sim} y \approx x \to {\sim}(x \to y) \tag{3.30}$$

$$x \to (y \to z) \approx (x \to y) \to (x \to z) \tag{3.31}$$

$$x \to (y \vee z) \approx x \to \big((x \to y) \vee (x \to z)\big). \tag{3.32}$$

Proof. Let $\mathbf{A}$ be a Nelson algebra and let $a, b, c \in A$. For (3.29) we have:

$$\begin{aligned}
(a \Leftrightarrow b) \to a &= \big((a \Rightarrow b) \wedge (b \Rightarrow a)\big) \to a \\
&= (a \Rightarrow b) \to \big((b \Rightarrow a) \to a\big) && \text{by (3.18)} \\
&= (b \Rightarrow a) \to \big((a \Rightarrow b) \to a\big) && \text{by (3.17)} \\
&= (a \Rightarrow b) \to \big((b \Rightarrow a) \to b\big) && \text{by (3.22)} \\
&= \big((a \Rightarrow b) \wedge (b \Rightarrow a)\big) \to b && \text{by (3.18)} \\
&= (a \Leftrightarrow b) \to b.
\end{aligned}$$

For (3.30), put $\alpha := a$ and $\beta := a \to b$. We have:

$$\begin{aligned}
a \to {\sim}(a \to b) &= \alpha \to {\sim} \beta \\
&= \alpha \to {\sim}(\alpha \wedge \beta) && \text{by (3.28)} \\
&= a \to {\sim}\big(a \wedge (a \to b)\big).
\end{aligned}$$

Put $\alpha := a \wedge {\sim} a$ and $\beta := a \wedge b$. We have:

$$\begin{aligned}
&a \to {\sim}\big(a \wedge (a \to b)\big) \\
&\qquad = a \to {\sim}\big(a \wedge ({\sim} a \vee b)\big) && \text{by (3.16)}
\end{aligned}$$

$$
\begin{aligned}
&= a \to {\sim}\big((a \wedge {\sim} a) \vee (a \wedge b)\big) && \text{by (3.19)}\\
&= a \to {\sim}(\alpha \vee \beta) && \\
&= a \to ({\sim}\alpha \wedge {\sim}\beta) && \text{by (3.12)}\\
&= a \to \big({\sim}(a \wedge {\sim} a) \wedge {\sim}(a \wedge b)\big). &&
\end{aligned}
$$

Put $\alpha := a$, $\beta := {\sim}(a \wedge {\sim} a)$ and $\gamma := {\sim}(a \wedge b)$. We have:

$$
\begin{aligned}
&a \to \big({\sim}(a \wedge {\sim} a) \wedge {\sim}(a \wedge b)\big) && \\
&\quad= \alpha \to (\beta \wedge \gamma) && \\
&\quad= (\alpha \to \beta) \wedge (\alpha \to \gamma) && \text{by (3.20)}\\
&\quad= \big(a \to {\sim}(a \wedge {\sim} a)\big) \wedge \big(a \to {\sim}(a \wedge b)\big) && \\
&\quad= (a \to a) \wedge \big(a \to {\sim}(a \wedge b)\big) && \text{by (3.27)}\\
&\quad= 1 \wedge \big(a \to {\sim}(a \wedge b)\big) && \text{by (3.13)}\\
&\quad= a \to {\sim}(a \wedge b) && \\
&\quad= a \to {\sim} b && \text{by (3.28).}
\end{aligned}
$$

For (3.31), put $\alpha := a \to b$, $\beta := a$ and $\gamma := c$. We have:

$$
\begin{aligned}
&(a \to b) \to (a \to c) && \\
&\quad= \alpha \to (\beta \to \gamma) && \\
&\quad= \beta \to (\alpha \to \gamma) && \text{by (3.17)}\\
&\quad= (\beta \wedge \alpha) \to \gamma && \text{by (3.18)}\\
&\quad= \big(a \wedge (a \to b)\big) \to c && \\
&\quad= \big(a \wedge {\sim}(a \wedge {\sim} b)\big) \to c && \text{by (3.26).}
\end{aligned}
$$

Put $\alpha := a$, $\beta := {\sim}(a \wedge {\sim} b)$ and $\gamma := c$. We have:

$$
\begin{aligned}
&\big(a \wedge {\sim}(a \wedge {\sim} b)\big) \to c && \\
&\quad= (\alpha \wedge \beta) \to \gamma && \\
&\quad= \alpha \to (\beta \to \gamma) && \text{by (3.18)}
\end{aligned}
$$

$$= a \to \big({\sim}(a \wedge {\sim} b) \to c\big).$$

Put $\alpha := a$, $\beta := {\sim} b$ and $\gamma := c$. We have:

$$\begin{aligned}
& a \to \big({\sim}(a \wedge {\sim} b) \to c\big) \\
& \quad = a \to \big({\sim}(\alpha \wedge \beta) \to \gamma\big) \\
& \quad = a \to \big(({\sim} \alpha \to \gamma) \wedge ({\sim} \beta \to \gamma)\big) && \text{by (3.25)} \\
& \quad = a \to \big(({\sim} a \to c) \wedge ({\sim}{\sim} b \to c)\big) \\
& \quad = a \to \big(({\sim} a \to c) \wedge (b \to c)\big) && \text{by (3.10).}
\end{aligned}$$

Put $\alpha := a$, $\beta := {\sim} a \to c$ and $\gamma := b \to c$. We have:

$$\begin{aligned}
& a \to \big(({\sim} a \to c) \wedge (b \to c)\big) \\
& \quad = \alpha \to (\beta \wedge \gamma) \\
& \quad = (\alpha \to \beta) \wedge (\alpha \to \gamma) && \text{by (3.20)} \\
& \quad = \big(a \to ({\sim} a \to c)\big) \wedge \big(a \to (b \to c)\big) \\
& \quad = 1 \wedge \big(a \to (b \to c)\big) && \text{by (3.23)} \\
& \quad = a \to (b \to c).
\end{aligned}$$

For (3.32), let $\alpha := a$, $\beta := a$ and $\gamma := (a \to b) \vee (a \to c)$. We have:

$$\begin{aligned}
& a \to \big((a \to b) \vee (a \to c)\big) \\
& \quad = 1 \wedge \big(a \to ((a \to b) \vee (a \to c))\big) \\
& \quad = (a \to a) \wedge \big(a \to ((a \to b) \vee (a \to c))\big) && \text{by (3.13)} \\
& \quad = (\alpha \to \beta) \wedge (\alpha \to \gamma) \\
& \quad = \alpha \to (\beta \wedge \gamma) && \text{by (3.20)} \\
& \quad = a \to \big(a \wedge ((a \to b) \vee (a \to c))\big).
\end{aligned}$$

Put $\alpha := a$, $\beta := a \to b$ and $\gamma := a \to c$. We have:

$$\begin{aligned} a \to \big(a \wedge ((a \to b) \vee (a \to c))\big) & \\ = a \to \big(\alpha \wedge (\beta \vee \gamma)\big) & \\ = a \to \big((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)\big) & \qquad \text{by (3.19)} \\ = a \to \big((a \wedge (a \to b)) \vee (a \wedge (a \to c))\big). & \end{aligned}$$

Put $\alpha := a$, $\beta := \sim a \vee b$ and $\gamma := \sim a \vee c$. We have:

$$\begin{aligned} a \to \big((a \wedge (a \to b)) \vee (a \wedge (a \to c))\big) & \\ = a \to \big((a \wedge (\sim a \vee b)) \vee (a \wedge (\sim a \vee c))\big) & \qquad \text{by (3.16)} \\ = a \to \big((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)\big) & \\ = a \to \big(\alpha \wedge (\beta \vee \gamma)\big) & \qquad \text{by (3.19)} \\ = a \to \big(a \wedge (\sim a \vee b \vee \sim a \vee c)\big). & \end{aligned}$$

Put $\alpha := a$, $\beta := b \vee c$. We have:

$$\begin{aligned} a \to \big(a \wedge (\sim a \vee b \vee \sim a \vee c)\big) & \\ = a \to \big(a \wedge (\sim a \vee (b \vee c))\big) & \\ = a \to \big(\alpha \wedge (\sim \alpha \vee \beta)\big) & \\ = a \to \big(\alpha \wedge (\alpha \to \beta)\big) & \qquad \text{by (3.16)} \\ = a \to \big(a \wedge (a \to (b \vee c))\big). & \end{aligned}$$

Put $\alpha := a$, $\beta := a$ and $\gamma := a \to (b \vee c)$. We have:

$$\begin{aligned} a \to \big(a \wedge (a \to (b \vee c))\big) & \\ = \alpha \to (\beta \wedge \gamma) & \\ = (\alpha \to \beta) \wedge (\alpha \to \gamma) & \qquad \text{by (3.20)} \\ = (a \to a) \wedge \big(a \to (a \to (b \vee c))\big) & \\ = 1 \wedge \big(a \to (a \to (b \vee c))\big) & \qquad \text{by (3.13)} \end{aligned}$$

$$
\begin{aligned}
&= a \to (a \to (b \vee c)) \\
&= a \to (b \vee c) && \text{by (3.24).}
\end{aligned}
$$

∎

Remark 3.1.32 (Added in proof). After obtaining the derivation of the identity (3.31) given in the proof of Lemma 3.1.31 we were made aware of the existence of Viglizzo [232, (1.11), p. 8], wherein a simpler derivation of (3.31) may be found. ∎

Theorem 3.1.33. *For any Nelson algebra* $\mathbf{A}$ *and* $a, b, c, d \in A$,

$$c \equiv d \left(\operatorname{mod} \Theta^{\mathbf{A}}(a, b)\right) \quad \textit{iff} \quad p^{\mathbf{A}}(a, b, c) = p^{\mathbf{A}}(a, b, d)$$

where $p(x, y, z) := (x \Leftrightarrow y) \to z$. *Thus the variety of Nelson algebras has EDPC. Moreover,* $p(x, y, z)$ *is a commutative TD term for* N*; it is not regular.*

Proof. To see the variety of Nelson algebras has EDPC, let $\mathbf{A}$ be a Nelson algebra and let $\Phi := \{\langle g, g' \rangle : (a \Leftrightarrow b) \to g = (a \Leftrightarrow b) \to g'\}$. Clearly Φ is an equivalence relation. Let $c, c', d, d' \in A$ and suppose $c \equiv c' \,(\operatorname{mod} \Phi)$ and $d \equiv d' \,(\operatorname{mod} \Phi)$. To see Φ is a congruence relation we show:

(i) $c \wedge d \equiv c' \wedge d' \,(\operatorname{mod} \Phi)$; (iii) $c \to d \equiv c' \to d' \,(\operatorname{mod} \Phi)$;
(ii) $c \vee d \equiv c' \vee d' \,(\operatorname{mod} \Phi)$; (iv) ${\sim} c \equiv {\sim} c' \,(\operatorname{mod} \Phi)$.

For (i), we have:

$$
\begin{aligned}
(a \Leftrightarrow b) \to (c \wedge d) &= ((a \Leftrightarrow b) \to c) \wedge ((a \Leftrightarrow b) \to d) && \text{by (3.20)} \\
&= ((a \Leftrightarrow b) \to c') \wedge ((a \Leftrightarrow b) \to d') \\
&= (a \Leftrightarrow b) \to (c' \wedge d') && \text{by (3.20).}
\end{aligned}
$$

So $c \wedge d \equiv c' \wedge d' \,(\operatorname{mod} \Phi)$. For (ii), we have:

$$
\begin{aligned}
(a \Leftrightarrow b) \to (c \vee d) &= (a \Leftrightarrow b) \to \\
&\qquad (((a \Leftrightarrow b) \to c) \vee ((a \Leftrightarrow b) \to d)) && \text{by (3.32)} \\
&= (a \Leftrightarrow b) \to
\end{aligned}
$$

$$\begin{aligned}&\big(((a \Leftrightarrow b) \to c') \vee ((a \Leftrightarrow b) \to d')\big)\\ &= (a \Leftrightarrow b) \to (c' \vee d') && \text{by (3.32).}\end{aligned}$$

So $c \vee d \equiv c' \vee d' \pmod{\Phi}$. For (iii), we have:

$$\begin{aligned}(a \Leftrightarrow b) \to (c \to d) &= \big((a \Leftrightarrow b) \to c\big) \to \big((a \Leftrightarrow b) \to d\big) && \text{by (3.31)}\\ &= \big((a \Leftrightarrow b) \to c'\big) \to \big((a \Leftrightarrow b) \to d'\big)\\ &= (a \Leftrightarrow b) \to (c' \to d') && \text{by (3.31).}\end{aligned}$$

So $c \to d \equiv c' \to d' \pmod{\Phi}$. For (iv), we have:

$$\begin{aligned}(a \Leftrightarrow b) \to {\sim} c &= (a \Leftrightarrow b) \to {\sim}\big((a \Leftrightarrow b) \to c\big) && \text{by (3.30)}\\ &= (a \Leftrightarrow b) \to {\sim}\big((a \Leftrightarrow b) \to c'\big)\\ &= (a \Leftrightarrow b) \to {\sim} c' && \text{by (3.30).}\end{aligned}$$

Thus ${\sim} c \equiv {\sim} c' \pmod{\Phi}$ and Φ is a congruence relation. Moreover $\langle a, b\rangle \in \Phi$ by (3.29), so $\Theta^{\mathbf{A}}(a, b) \subseteq \Phi$. Conversely, if $c \equiv d \pmod{\Phi}$ then $c = 1 \to c = (a \Leftrightarrow a) \to c \equiv_{\Theta^{\mathbf{A}}(a,b)} (a \Leftrightarrow b) \to c = (a \Leftrightarrow b) \to d \equiv_{\Theta^{\mathbf{A}}(a,b)} (a \Leftrightarrow a) \to d = 1 \to d = d$ since $\to$ and $\Leftrightarrow$ are respectively subtractive weak relative pseudocomplementation and Gödel equivalence terms for N. So $\Phi \subseteq \Theta^{\mathbf{A}}(a, b)$ and the terms $p(x, y, z), p(x, y, w)$ witness EDPC for N.

Let $a, b, c \in A$. To see $p(x, y, z)$ is a TD term, it is sufficient to note $p^{\mathbf{A}}(a, a, c) = (a \Leftrightarrow a) \to c = 1 \to c = c$, just because $\Leftrightarrow$ is a Gödel equivalence term for N and $\to$ is a subtractive weak relative pseudocomplementation. To see $p(x, y, z)$ is commutative, let $a, b, a', b', c \in A$. We have:

$$\begin{aligned}p^{\mathbf{A}}\big(a, b, p^{\mathbf{A}}(a', b', c)\big) &= (a \Leftrightarrow b) \to \big((a' \Leftrightarrow b') \to c\big)\\ &= (a' \Leftrightarrow b') \to \big((a \Leftrightarrow b) \to c\big) && \text{by (3.17)}\\ &= p^{\mathbf{A}}\big(a', b', p^{\mathbf{A}}(a, b, c)\big).\end{aligned}$$

Finally, an easy inspection of the unique (to within isomorphism) 3-element Nelson algebra establishes that $p(x, y, z)$ is not regular. ■

Remark 3.1.34. Because of (3.18), $p(x,y,z) := (x \Rightarrow y) \to \big((y \Rightarrow x) \to z\big)$ is also a commutative (but not regular) TD term for the variety of Nelson algebras; compare Blok and Pigozzi [34, Corollary 5.2(i)]. ∎

Recall from [34, Corollary 5.2] that dual Brouwerian semilattices possess two distinct TD terms: the commutative (but not regular) TD term $\big(z * (x * y)\big) * (y * x)$ and the commutative and regular TD term $z \vee \big((x * y) \vee (y * x)\big)$. Thus the existence of a commutative (but not regular) TD term for a variety V need not preclude the existence of a commutative and regular TD term for V.

Problem 3.1.35. Does the variety of Nelson algebras possess a commutative, regular TD term? ∎

Remark 3.1.36. In view of work due to Blok and Pigozzi [34, Section 5] an obvious candidate for a commutative, regular TD term for the variety of Nelson algebras is the term $p(x,y,z) := (x \Leftrightarrow y) \wedge z$. To see $p(x,y,z)$ is not a TD term for N, consider the following 6-element algebra $\mathbf{A}$:

$\wedge^{\mathbf{A}}$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	a	a	a	a
b	0	a	b	c	d	b
c	0	a	c	c	a	c
d	0	a	d	a	d	d
1	0	a	b	c	d	1

$\vee^{\mathbf{A}}$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	b	c	d	1
b	b	b	b	b	b	1
c	c	c	b	c	b	1
d	d	d	b	b	d	1
1	1	1	1	1	1	1

$\to^{\mathbf{A}}$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	1	1	1	1	1	1
b	a	a	1	c	d	1
c	d	d	1	1	d	1
d	c	c	1	c	1	1
1	0	a	b	c	d	1

$\sim^{\mathbf{A}}$	
0	1
a	b
b	a
c	d
d	c
1	0

An easy sequence of checks shows $\mathbf{A}$ is a subdirectly irreducible Nelson algebra and that the monolith of $\mathbf{A}$ is the congruence μ induced by the parti-

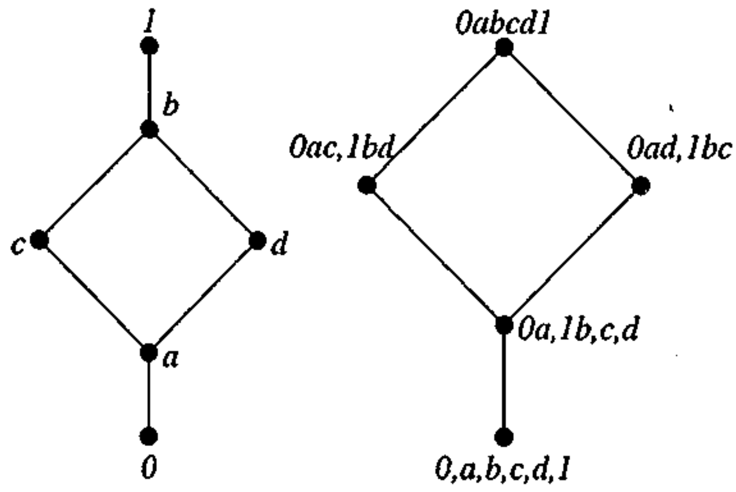

Figure 3.3. (a) The Nelson algebra **A** of Remark 3.1.36; (b) **Con A**.

tion $\{\{0,a\},\{1,b\},\{c\},\{d\}\}$: see Figure 3.3 (note that in Figure 3.3(b), the congruences of **A** are represented by their corresponding partitions (with all parentheses dropped)). Observe now that $0 \equiv a \left(\mathrm{mod}\ \mu(1,b)\right)$, but $(1 \Leftrightarrow b) \wedge 0 = b \wedge 0 = 0 \neq a = b \wedge a = (1 \Leftrightarrow b) \wedge a$. Thus $p(x,y,z)$ does not even witness EDPC for **A**, and so in particular cannot be a commutative, regular TD term for N. ∎

In [29, p. 361] Blok and Pigozzi assert without proper proof that the variety of Nelson algebras is not congruence permutable. This assertion is corrected in [34, p. 606], where Blok and Pigozzi (citing an unpublished result of Idziak) announce that the variety of Nelson algebras is congruence permutable. They do not provide a proof, and in particular do not give a Mal'cev term.

Lemma 3.1.37. *The variety of Nelson algebras satisfies the following identities:*

$$\sim(x \to y) \to x \approx \mathbf{1} \tag{3.33}$$

$$\sim(x \to y) \to z \approx x \to (\sim y \to z) \tag{3.34}$$

$$\mathbf{1} \Rightarrow x \approx x. \tag{3.35}$$

Proof. Let **A** be a Nelson algebra and let $a, b, c \in A$. For (3.33), observe that $\sim(a \to b) \preceq a \wedge \sim b$ by (N7). But $a \wedge \sim b \leq a$, which implies $a \wedge \sim b \preceq a$

by (N4). Thus $\sim(a \to b) \preceq a \wedge \sim b \preceq a$, whence $\sim(a \to b) \preceq a$ by transitivity. By (N2) we conclude $\sim(a \to b) \to a = 1$. For (3.34), put $\alpha := \sim(a \to b)$, $\beta := a$ and $\gamma := c$. We have:

$$\begin{aligned}
\sim(a \to b) \to c &= 1 \to \big(\sim(a \to b) \to c\big) \\
&= \big(\sim(a \to b) \to a\big) \to \big(\sim(a \to b) \to c\big) && \text{by (3.33)} \\
&= (\alpha \to \beta) \to (\alpha \to \gamma) \\
&= \alpha \to (\beta \to \gamma) && \text{by (3.31)} \\
&= \beta \to (\alpha \to \gamma) && \text{by (3.17)} \\
&= a \to \big(\sim(a \to b) \to c\big).
\end{aligned}$$

Put $\alpha := a$, $\beta := \sim(a \to b)$ and $\gamma := c$. We have:

$$\begin{aligned}
a \to (\sim(a \to b) \to c) &= \alpha \to (\beta \to \gamma) \\
&= (\alpha \to \beta) \to (\alpha \to \gamma) && \text{by (3.31)} \\
&= \big(a \to \sim(a \to b)\big) \to (a \to c) \\
&= (a \to \sim b) \to (a \to c) && \text{by (3.30)} \\
&= a \to (\sim b \to c) && \text{by (3.31).}
\end{aligned}$$

For (3.35), observe $1 \Rightarrow a = (1 \to a) \wedge (\sim a \to \sim 1) = a \wedge (\sim a \to \sim 1)$ by (3.14). Therefore to see $1 \Rightarrow a = a$ it is sufficient to show $a \leq \sim a \to \sim 1$, or equivalently (by (N4), (N2)) both $a \to (\sim a \to \sim 1) = 1$ and $\sim(\sim a \to \sim 1) \to \sim a = 1$. Now $a \to (\sim a \to \sim 1) = 1$ by (3.23). Put $\alpha := \sim a$, $\beta := \sim 1$ and $\gamma := \sim a$. Then $\sim(\sim a \to \sim 1) \to \sim a = \sim(\alpha \to \beta) \to \gamma = \alpha \to (\sim \beta \to \gamma) = \sim a \to (\sim\sim 1 \to \sim a) = \sim a \to (1 \to \sim a) = \sim a \to \sim a = 1$ by (3.34),(3.10) and (3.13). Thus $a \leq \sim a \to \sim 1$ as required. ∎

Lemma 3.1.38. *Let* $\mathbf{A}$ *be a Nelson algebra. The following inequalities are identically satisfied for any* $a, b \in A$*:*

$$a \Rightarrow b \leq a \to b \tag{3.36}$$

$$a \Rightarrow b \leq \sim b \to \sim a. \tag{3.37}$$

Thus the following equations are identically satisfied for any $a, b \in A$:

$$(a \Rightarrow b) \to (a \to b) = 1 \tag{3.38}$$
$$\sim(a \to b) \to \sim(a \Rightarrow b) = 1 \tag{3.39}$$

$$(a \Rightarrow b) \to \sim(b \to \sim a) = 1 \tag{3.40}$$
$$\sim(\sim b \to \sim a) \to \sim(a \Rightarrow b) = 1. \tag{3.41}$$

Proof. Let $\mathbf{A}$ be a Nelson algebra and let $a, b \in A$. We have $(a \Rightarrow b) \wedge (a \to b) = \big((a \to b) \wedge (\sim b \to \sim a)\big) \wedge (a \to b) = (a \to b) \wedge (\sim b \to \sim a) = a \Rightarrow b$, which is sufficient to establish (3.36). Also $(a \Rightarrow b) \wedge (\sim b \to \sim a) = \big((a \to b) \wedge (\sim b \to \sim a)\big) \wedge (\sim b \to \sim a) = (a \to b) \wedge (\sim b \to \sim a) = a \Rightarrow b$, which is sufficient to establish (3.37). Equations (3.38)–(3.41) now follow immediately by (N4), (N2). ∎

Lemma 3.1.39. *Let $\mathbf{A}$ be a Nelson algebra. The following inequalities are identically satisfied for any $a, b \in A$:*

$$a \preceq (a \Rightarrow b) \Rightarrow b \tag{3.42}$$
$$\sim\big((a \Rightarrow b) \Rightarrow b\big) \preceq \sim a. \tag{3.43}$$

Thus the following inequality is identically satisfied for any $a, b \in A$:

$$a \leq (a \Rightarrow b) \Rightarrow b. \tag{3.44}$$

Proof. Let $\mathbf{A}$ be a Nelson algebra and let $a, b \in A$. For (3.42) observe first that:

$$\begin{aligned} a &\to \big((a \Rightarrow b) \to b\big) \\ &= (a \Rightarrow b) \to (a \to b) && \text{by (3.17)} \\ &= 1 && \text{by (3.38).} \end{aligned} \tag{3.45}$$

Also, put $\alpha := a$, $\beta := b$ and $\gamma := \sim(a \Rightarrow b)$ and observe that:

$$\begin{aligned}
&a \to \big(\sim b \to \sim(a \Rightarrow b)\big) \\
&\qquad = \alpha \to (\sim \beta \to \gamma) \\
&\qquad = \sim(\alpha \to \beta) \to \gamma && \text{by (3.34)} \\
&\qquad = \sim(a \to b) \to \sim(a \Rightarrow b) \\
&\qquad = 1 && \text{by (3.39).} \quad (3.46)
\end{aligned}$$

Now put $\alpha := a$, $\beta := (a \Rightarrow b) \to b$ and $\gamma := \sim b \to \sim(a \Rightarrow b)$. We have:

$$\begin{aligned}
&a \to \big((a \Rightarrow b) \Rightarrow b\big) \\
&\qquad = a \to \big(((a \Rightarrow b) \to b) \wedge \\
&\qquad\qquad\qquad (\sim b \to \sim(a \Rightarrow b))\big) \\
&\qquad = \alpha \to (\beta \wedge \gamma) \\
&\qquad = (\alpha \to \beta) \wedge (\alpha \to \gamma) && \text{by (3.20)} \\
&\qquad = \big(a \to ((a \Rightarrow b) \to b)\big) \wedge \\
&\qquad\qquad\qquad \big(a \to (\sim b \to \sim(a \Rightarrow b))\big) \\
&\qquad = 1 \wedge 1 && \text{by (3.45),(3.46)} \\
&\qquad = 1
\end{aligned}$$

which (in view of (N2)) establishes (3.42). For (3.43), put $\alpha := a \Rightarrow b$, $\beta := b$ and $\gamma := \sim a$ and observe first that:

$$\begin{aligned}
&\sim\big((a \Rightarrow b) \to b\big) \to \sim a \\
&\qquad = \sim(\alpha \to \beta) \to \gamma \\
&\qquad = \alpha \to (\sim \beta \to \gamma) && \text{by (3.34)} \\
&\qquad = (a \Rightarrow b) \to (\sim b \to \sim a) \\
&\qquad = 1 && \text{by (3.40).} \quad (3.47)
\end{aligned}$$

Also, put $\alpha := \sim b$, $\beta := \sim(a \Rightarrow b)$ and $\gamma := \sim a$ and observe that:

$$
\begin{aligned}
&\big(\sim(\sim b \to \sim(a \Rightarrow b)) \to \sim a\big) \\
&\qquad = \sim(\alpha \to \beta) \to \gamma \\
&\qquad = \alpha \to (\sim \beta \to \gamma) && \text{by (3.34)} \\
&\qquad = \sim b \to \big(\sim\sim(a \Rightarrow b) \to \sim a\big) \\
&\qquad = \sim b \to \big((a \Rightarrow b) \to \sim a\big) && \text{by (3.10)} \\
&\qquad = (a \Rightarrow b) \to (\sim b \to \sim a) && \text{by (3.17)} \\
&\qquad = 1 && \text{by (3.40).} \qquad (3.48)
\end{aligned}
$$

Now put $\alpha := (a \Rightarrow b) \to b$, $\beta := \sim b \to \sim(a \Rightarrow b)$ and $\gamma := \sim a$. We have:

$$
\begin{aligned}
&\sim\big((a \Rightarrow b) \Rightarrow b\big) \to \sim a \\
&\qquad = \sim\big(((a \Rightarrow b) \to b) \wedge \\
&\qquad\qquad\qquad (\sim b \to \sim(a \Rightarrow b))\big) \to \sim a \\
&\qquad = \sim(\alpha \wedge \beta) \to \gamma \\
&\qquad = (\sim \alpha \to \gamma) \wedge (\sim \beta \to \gamma) && \text{by (3.25)} \\
&\qquad = \big(\sim((a \Rightarrow b) \to b) \to \sim a\big) \wedge \\
&\qquad\qquad\qquad \big(\sim(\sim b \to \sim(a \Rightarrow b)) \to \sim a\big) \\
&\qquad = 1 \wedge 1 && \text{by (3.47),(3.48)} \\
&\qquad = 1
\end{aligned}
$$

which (in view of (N2)) establishes (3.43). The remaining assertion of the lemma now follows immediately by (N4). ∎

Theorem 3.1.40. *(Idziak) The variety of Nelson algebras is congruence permutable. A Mal'cev term witnessing congruence permutability is:*

$$p(x, y, z) := \big((x \Rightarrow y) \Rightarrow z\big) \wedge \big((z \Rightarrow y) \Rightarrow x\big).$$

Proof. Let $\mathbf{A}$ be a Nelson algebra and $a, b \in A$. We have:

$$\begin{aligned} p^{\mathbf{A}}(a,a,b) &= \big((a \Rightarrow a) \Rightarrow b\big) \wedge \big((b \Rightarrow a) \Rightarrow a\big) \\ &= (1 \Rightarrow b) \wedge \big((b \Rightarrow a) \Rightarrow a\big) \\ &= b \wedge \big((b \Rightarrow a) \Rightarrow a\big) && \text{by (3.35)} \\ &= b && \text{by (3.44)} \end{aligned}$$

and:

$$\begin{aligned} p^{\mathbf{A}}(a,b,b) &= \big((a \Rightarrow b) \Rightarrow b\big) \wedge \big((b \Rightarrow b) \Rightarrow a\big) \\ &= \big((a \Rightarrow b) \Rightarrow b\big) \wedge (1 \Rightarrow a) \\ &= \big((a \Rightarrow b) \Rightarrow b\big) \wedge a && \text{by (3.35)} \\ &= a && \text{by (3.44).} \end{aligned}$$

∎

Corollary 3.1.41. *The variety of Nelson algebras is arithmetical.*

Proof. Congruence permutability is clear in view of the preceding theorem. Since the variety of Nelson algebras has EDPC (by Theorem 3.1.33), it is also congruence distributive (by Theorem 1.5.2(3)). Thus N is arithmetical as asserted. ∎

In [29, p. 361] Blok and Pigozzi erroneously assert that the variety of Nelson algebras does not have a QD term since it is not congruence permutable. The following theorem corrects this assertion.

Theorem 3.1.42. *The variety of Nelson algebras is a congruence permutable WBSO variety. A QD term for* N *is:*

$$q(x,y,z,w) := p\big(e(x,y,z), e(x,y,w), w\big)$$

where:

$$e(x,y,z) := (x \Leftrightarrow y) \to z$$

is the commutative TD term of Theorem 3.1.35 and:

$$p(x,y,z) := \big((x \Rightarrow y) \Rightarrow z\big) \wedge \big((z \Rightarrow y) \Rightarrow x\big)$$

is the Mal'cev term of Theorem 3.1.40.

Proof. The first assertion is clear from previous remarks and Theorem 3.1.40. The second assertion follows immediately from Theorem 3.1.33, Theorem 3.1.40 and Proposition 1.5.11(1). ∎

Corollary 3.1.43. *The class* N_{SS} *of all semisimple Nelson algebras, axiomatised relative to* N *by the identity:*

$$\big((x \to y) \to x\big) \to x \approx \mathbf{1},$$

is a discriminator variety. A discriminator term for N_{SS} *is given by* $q(x,y,z,x)$, *where* $q(x,y,z,w)$ *is the QD term of Theorem 3.1.42.*

Proof. By Viglizzo [232, Theorem 4.2] the class N_{SS} of all semisimple Nelson algebras is a variety, axiomatised relative to N by the identity:

$$\big((x \to y) \to x\big) \to x \approx \mathbf{1}.$$

Since N_{SS} is a semisimple congruence permutable WBSO variety, it is a discriminator variety with discriminator term $q(x,y,z,x)$, where $q(x,y,z,w)$ is the QD term of Theorem 3.1.42. ∎

By an *implicative subreduct* of a Nelson algebra $\langle A;\ \wedge,\vee,\to,\sim,0,1\rangle$ we mean a subalgebra of the reduct $\langle A;\ \to,1\rangle$. If V is a variety of Nelson algebras, we denote the class of implicative subreducts of V by $\mathbf{S}(\mathsf{V}^{\{\to,1\}})$. The following theorem is an easy modification of a result due to Blok and Pigozzi [34].

Theorem 3.1.44. *(cf. [34, Corollary 5.3]) For any variety* V *of Nelson algebras, the class* $\mathbf{S}(\mathsf{V}^{\{\to,1\}})$ *of implicative subreducts of* V *is a variety.*

Proof. We prove $\mathbf{S}(\mathsf{V}^{\{\to,1\}})$ is a variety by showing it is closed under $\mathbf{S}$, $\mathbf{P}$ and $\mathbf{H}$. It is closed under $\mathbf{S}$ by definition, and it is easy to see it is closed under $\mathbf{P}$. So we have only to show $\mathbf{HS}(\mathsf{V}^{\{\to,1\}}) \subseteq \mathbf{S}(\mathsf{V}^{\{\to,1\}})$.

Let $\mathbf{A} := \langle A;\ \rightarrow, 1\rangle \in \mathbf{S}(\mathsf{V}^{\{\rightarrow,1\}})$ and let $\mathbf{B} := \langle B;\ \wedge, \vee, \rightarrow, \sim, 0, 1\rangle \in \mathsf{V}$ with $\mathbf{A} \in \mathbf{S}\big(\langle B;\ \rightarrow, 1\rangle\big)$. Let Φ be a congruence on $\mathbf{A}$, and let $F := [1]_\Phi$. By Lemma 1.5.4 (see also Rasiowa [195, Theorem V§4.4, Theorem V§4.3]) we have that F is an implicative filter. That is, $1 \in F$ and F has the detachment property: $1 \in F$ trivially, and, if $c \equiv 1 \,(\mathrm{mod}\,\Phi)$ and $c \rightarrow d \equiv 1 \,(\mathrm{mod}\,\Phi)$, then:

$$d = 1 \rightarrow d \equiv_\Phi c \rightarrow d \equiv_\Phi 1.$$

Let G be the filter on $\mathbf{B}$ generated by F, and let $\Theta(G)$ be the congruence on $\mathbf{B}$ such that $G = [1]_{\Theta(G)}$; of course, $\Theta(G)$ exists and is unique, just because of Proposition 1.5.5. By Proposition 1.5.5 (see also Rasiowa [195, Theorem V§4.5]) $\Theta(G) = \big\{\langle a, b\rangle \in B \times B : a \Rightarrow b, b \Rightarrow a \in G\big\}$. Moreover, by (3.17) and Rasiowa [195, Theorem V§4.8] we have that $a \equiv b \,\big(\mathrm{mod}\,\Theta(G)\big)$ iff:

$$\begin{aligned}
&c_1 \rightarrow \big(c_2 \rightarrow (\cdots \rightarrow (c_k \rightarrow (a \Rightarrow b)))\cdots\big) = 1 \quad \text{and} \\
&c_1 \rightarrow \big(c_2 \rightarrow (\cdots \rightarrow (c_k \rightarrow (b \Rightarrow a)))\cdots\big) = 1
\end{aligned}$$

for some $c_1, \ldots, c_k \in F$. If $a, b \in A$, then $a \Rightarrow b, b \Rightarrow a \in F$ follows from the fact that F has the detachment property. Thus by Theorem 3.1.33, Remark 3.1.34 and the definition of F,

$$\begin{aligned}
a &= 1 \rightarrow (1 \rightarrow a) \\
&\equiv_\Phi (a \Rightarrow b) \rightarrow \big((b \Rightarrow a) \rightarrow a\big) \\
&= (a \Rightarrow b) \rightarrow \big((b \Rightarrow a) \rightarrow b\big) \\
&\equiv_\Phi 1 \rightarrow (1 \rightarrow b) \\
&= b.
\end{aligned}$$

We have shown $\Theta(G) \cap A^2 \subseteq \Phi$. Conversely, if $a, b \in A$ and $a \equiv b \,(\mathrm{mod}\,\Phi)$, then $a \Rightarrow b \equiv_\Phi 1 \equiv_\Phi b \Rightarrow a$ by Rasiowa [195, Theorem V§4.5]. Hence $a \Rightarrow b, b \Rightarrow a \in F \subseteq G$ and consequently $a \equiv b \,\big(\mathrm{mod}\,\Theta(G)\big)$. Thus $\Phi \subseteq \Theta(G) \cap A^2$. So $\Theta(G) \cap A^2 = \Phi$, and thus $\mathbf{A}/\Phi$ is isomorphic to a subalgebra of the implicative reduct of $\mathbf{B}/\Theta(G)$. Hence $\mathbf{A}/\Phi \in \mathbf{S}(\mathsf{V}^{\{\rightarrow,1\}})$, which completes the proof of the corollary. ∎

Problem 3.1.45. Axiomatise the variety $\mathsf{S}(\mathsf{N}^{\{\to,1\}})$. Is $\mathsf{S}(\mathsf{N}^{\{\to,1\}})$ finitely axiomatisable? ∎

Let V be a variety of Nelson algebras. By the previous theorem, the class $\mathsf{S}(\mathsf{V}^{\{\to,1\}})$ of implicative subreducts of V is a variety. Moreover, by Rasiowa [195, Theorem V§1.3] we have that $\mathsf{N} \models$ (1.45)–(1.48), so $\mathsf{S}(\mathsf{V}^{\{\to,1\}})$ is a variety of MINI-algebras (this observation also follows from Theorem 3.1.44 and Proposition 3.1.24, since $\to$ is a subtractive weak relative pseudocomplementation for N). Consider now the following 4-element Nelson algebra $\mathbf{A}$:

$\wedge^{\mathbf{A}}$	0	a	b	1
0	0	0	0	0
a	0	a	b	a
b	0	b	b	b
1	0	a	b	1

$\vee^{\mathbf{A}}$	0	a	b	1
0	0	a	b	1
a	a	a	a	1
b	b	a	b	1
1	1	1	1	1

$\to^{\mathbf{A}}$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	1	1	1	1
1	0	a	b	1

$\sim^{\mathbf{A}}$	
0	1
a	b
b	a
1	0

Since $\big((a \to 0) \to 0\big) \to 0 = (b \to 0) \to 0 = 1 \to 0 = 0 \neq 1$, we conclude that $\mathbf{A} \not\models \big((x \to y) \to x\big) \to x \approx \mathbf{1}$. Therefore in general $\mathsf{S}(\mathsf{V}^{\{\to,1\}}) \not\models \big((x \to y) \to x\big) \to x \approx \mathbf{1}$. From these remarks it clearly follows that the varieties of {commutative, positive implicative, implicative}-pre-BCK-algebras and the variety of implicative BCS-algebras do not exhaust the naturally occurring subvarieties of pre-BCK-algebras that may be of interest in universal algebra and algebraic logic.

3.2 Binary and Dual Binary Discriminator Varieties

Recall from Example 2.3.11 that on any binary discriminator algebra the binary discriminator is precisely implicative BCS difference, and thus that any member of a binary discriminator variety has a canonical implicative BCS-algebra polynomial reduct. This observation calls for a study of the role played by implicative BCS-algebras in binary discriminator varieties. More generally, this remark (in conjunction with the remarks of §1.7.20) calls for the study of bands in dual binary discriminator varieties.

Dual binary discriminator varieties are considered in §3.2.1. It is shown that the variety of left normal bands with zero coincides with the pure dual binary discriminator variety, namely the variety generated by the class of all dual binary discriminator algebras $\langle A;\ h, 0\rangle$, where h is the dual binary discriminator on A and 0 is a nullary operation. We also give a semigroup-theoretic characterisation of dual binary discriminator varieties in terms of left normal bands with zero.

In §3.2.6 binary discriminator varieties proper are studied. It is shown that the variety of implicative BCS-algebras coincides with the pure binary discriminator variety, namely the variety generated by the class of all binary discriminator algebras $\langle A;\ b, 0\rangle$, where b is the binary discriminator on A and 0 is a nullary operation. We prove that any member $\mathbf{A}$ of a binary discriminator variety V has an implicative BCS-algebra polynomial reduct whose iBCS-ideals coincide with the V-ideals of $\mathbf{A}$. We characterise binary discriminator varieties in ideal-theoretic terms: a pointed variety is a binary discriminator variety iff it is subtractive with EDPI and is generated by a class of ideal simple algebras. A characterisation of binary discriminator varieties in the spirit of Agliano and Ursini's characterisation of subtractive varieties with EDPI is also given. The results are illustrated with some examples.

In §3.2.22 attention is focussed on point regular binary discriminator varieties. We prove two results that together show a pointed variety is a point regular binary discriminator variety iff it is a 'pointed' fixedpoint discriminator variety. In the main result of the section, the 'pointed' fixedpoint discriminator varieties are characterised: a pointed variety is a 'pointed' fixedpoint discriminator variety iff it is ideal determined, semisimple and has EDPC. The results give a partial answer to a question of Blok and Pigozzi.

The relationship between binary and pointed ternary discriminator varieties is considered in §3.2.32. It is shown that a pointed variety is a pointed ternary discriminator variety iff it is a congruence permutable point regular binary discriminator variety. We also give an (almost) trivial syntactic criterion both necessary and sufficient for a point regular binary discriminator variety to be a pointed ternary discriminator variety.

3.2.1. Dual Binary Discriminator Varieties. In [58, Example (2), p. 242] Chajda, Halaš and Rosenberg observed that the simplest non-trivial example of a dual binary discriminator algebra is $\mathbf{2}$, the one-element semilattice with a zero adjoined. Since $\mathbf{2}$ is, to within isomorphism, the only subdirectly irreducible meet semilattice with zero, it follows from this example that the variety of meet semilattices with zero is a dual binary discriminator variety. This result may be regarded as a specialisation to meet semilattices with zero of the following theorem, in the statement of which the *pure dual binary discriminator variety* (in symbols, PdBD) denotes the variety generated by the class of all dual binary discriminator algebras $\langle A;\ h, 0\rangle$, where h is the dual binary discriminator function on A and 0 is a nullary operation.

Theorem 3.2.2. *The variety of left normal bands with zero coincides with the pure dual binary discriminator variety.*

Proof. Let K denote the class of all dual binary discriminator algebras $\langle A;\ h, 0\rangle$ where h is the dual binary discriminator function on A and 0 is a nullary operation. By definition of the dual binary discriminator and Example 1.3.20 every member of K is a left normal band with zero, so $\mathsf{K} \subseteq \mathsf{lnB_0}$, the variety of left normal bands with zero. Hence the variety $\mathbf{V}(\mathsf{K})$ generated by K, namely PdBD, is a subvariety of $\mathsf{lnB_0}$. Conversely, the three-element left normal band with zero $\mathbf{3}_L$ is a member of K; since an easy consequence of Corollary 1.3.19 shows $\mathbf{3}_L$ generates $\mathsf{lnB_0}$ as a variety we have that $\mathsf{lnB_0} = \mathbf{V}(\mathbf{3}_L) \subseteq \mathbf{V}(\mathsf{K}) = \mathsf{PdBD}$. ∎

Given a band with zero $\mathbf{A}$, we say $\mathbf{A}$ is *$\mathcal{D}$-simple* or *primitive* if the only $\mathcal{D}$-equivalence classes of $\mathbf{A}$ are $\{0\}$ and $A - \{0\}$. Notice that $\mathbf{A}$ is $\mathcal{D}$-simple iff $\mathbf{A}/\mathcal{D}$ is isomorphic to $\mathbf{2}$, the one element semilattice with a zero adjoined. By the proof of Theorem 3.2.2, any dual binary discriminator algebra (considered as a band with zero) is $\mathcal{D}$-simple, which observation suggests the following semigroup-theoretic characterisation of dual binary discriminator varieties (implicit in [58, Section 5]). In the statement of the theorem and in the sequel the dual binary discriminator [term] is denoted $\sqcap$; of course, this change in notation is legitimised by Theorem 3.2.2.

Theorem 3.2.3. *(cf. [58, Theorem 5.2]) For a variety* V *the following are equivalent:*

1. V *is a dual binary discriminator variety;*

2. *There exists a binary term* $x \sqcap y$ *of* V *such that for any* $\mathbf{A} \in \mathsf{V}$*, the polynomial reduct* $\langle A;\ \sqcap^{\mathbf{A}}, 0\rangle$ *is a left normal band with zero, and* any one *of the following conditions is satisfied:*

   (a) V *is generated by a class of algebras* K *such that for each* $\mathbf{A} \in \mathsf{K}$*, the left normal band with zero polynomial reduct* $\langle A;\ \sqcap^{\mathbf{A}}, 0\rangle$ *is* $\mathcal{D}_{\langle A;\, \sqcap^{\mathbf{A}}, 0\rangle}$*-simple;*

   (b) V *is generated by a class of algebras* K *such that for each* $\mathbf{A} \in \mathsf{K}$*, the maximal semilattice homomorphic image* $\langle A;\ \sqcap^{\mathbf{A}}, 0\rangle / \mathcal{D}_{\langle A;\, \sqcap^{\mathbf{A}}, 0\rangle}$ *of the left normal band with zero polynomial reduct* $\langle A;\ \sqcap^{\mathbf{A}}, 0\rangle$ *is isomorphic to* $\mathbf{2}$*, the one-element semilattice with a zero adjoined;*

Proof. Let V be a variety with $\mathbf{0}$. By previous remarks the equivalence (2)(a) $\Leftrightarrow$ (2)(b) is clear, so it only remains to prove the equivalence (1) $\Leftrightarrow$ (2)(a).

(1) $\Rightarrow$ (2)(a) Suppose V is a dual binary discriminator variety with dual binary discriminator term $x \sqcap y$ generated by a class K of dual binary discriminator algebras. By definition of the dual binary discriminator and Example 1.3.20, the polynomial reduct $\langle A;\ \sqcap^{\mathbf{A}}, 0\rangle$ of each $\mathbf{A} \in \mathsf{K}$ is a left normal band with zero whose only $\mathcal{D}_{\langle A;\, \sqcap^{\mathbf{A}}, 0\rangle}$-equivalence classes are $\{0\}$ and $A - \{0\}$. For each $\mathbf{A} \in \mathsf{K}$, therefore, $\langle A;\ \sqcap^{\mathbf{A}}, 0\rangle$ is primitive. Upon recalling that the class of left normal bands with zero is equationally definable, we may also infer the polynomial reduct $\langle B;\ \sqcap^{\mathbf{B}}, 0\rangle$ of any $\mathbf{B} \in \mathsf{V}$ is a left normal band with zero, just because the identities satisfied by V are precisely those satisfied by K.

(2)(a) $\Rightarrow$ (1) Suppose (2)(a) holds for V. Let K be a subclass of V satisfying the conditions of (2)(a) and let $\mathbf{A} \in \mathsf{K}$. Let $a, b \in A$. If $b = 0$, then $a \sqcap^{\mathbf{A}} b = 0$ since the polynomial reduct $\langle A;\ \sqcap^{\mathbf{A}}, 0\rangle$ is a left normal band with zero. So assume $b \neq 0$. If $a = 0$, then $a \sqcap^{\mathbf{A}} b = 0 = a$, also because the polynomial reduct $\langle A;\ \sqcap^{\mathbf{A}}, 0\rangle$ is a left normal band with zero. So assume $a \neq 0$. Then $a, b \in A - \{0\}$, whence $a \equiv b \,(\operatorname{mod} \mathcal{D}_{\langle A;\, \sqcap^{\mathbf{A}}, 0\rangle})$ since $\langle A;\ \sqcap^{\mathbf{A}}, 0\rangle$ is primitive. By the Clifford-McLean theorem for bands, $a \sqcap^{\mathbf{A}} b \sqcap^{\mathbf{A}} a = a$, whence $a \sqcap^{\mathbf{A}} b = a$ by left normality. We have shown that, for any $a, b \in A$, $a \sqcap^{\mathbf{A}} b = 0$ if $b = 0$ and a otherwise, so $\sqcap^{\mathbf{A}}$ is the dual binary discriminator on A. Thus K is a class of dual binary discriminator algebras; since K generates V as a variety we have that V is a dual binary discriminator variety. ■

Example 3.2.4. (*cf.* [58, Section 2, p. 241]) Let V be a pointed dual ternary discriminator variety (say with 0) with dual ternary discriminator term $d(x, y, z)$ generated by a class $\mathsf{K} \subseteq \mathsf{V}$ of dual ternary discriminator algebras. For any $\mathbf{A} \in \mathsf{K}$, let $x \sqcap y := d(0, y, x)$. By definition of the dual ternary discriminator,

$$\begin{aligned} a \sqcap^{\mathbf{A}} b &= d^{\mathbf{A}}(0, b, a) \\ &= \begin{cases} 0 & \text{if } b = 0 \\ a & \text{otherwise} \end{cases} \end{aligned}$$

whence $\mathbf{A}$ is a dual binary discriminator algebra. Hence K is a class of dual binary discriminator algebras and V is a dual binary discriminator variety. ■

Let $\mathbf{A}$ be a band with zero. For any $B \subseteq A$, let BAB [BA; AB] denote the set $\{bab : b \in B, a \in A\}$ [$\{ba;\ b \in B, a \in A\}$; $\{ab : a \in A, b \in B\}$]. A non-empty subset $I \subseteq A$ such that both $IA \subseteq A$ and $AI \subseteq A$ is an *ideal* of $\mathbf{A}$ (in the usual semigroup-theoretic sense) [111, pp. 4–5]. By remarks of Ursini [222, Remarks (b), pp. 211–212], the semigroup-theoretic ideals of $\mathbf{A}$ are precisely the ideals of $\mathbf{A}$ in the sense of §1.7.1; from remarks due to Petrich [180, Chapter I.2.3, pp. 4–5], it follows that $\mathbf{A}$ is ideal simple iff $B = BcB$ for all $c \in B$, where $B := A - \{0\}$. Suppose $\mathbf{A}$ is primitive. By assumption, $a \equiv b \,(\operatorname{mod} \mathcal{D})$ for any $a, b \in B$, whence $BcB = \{dcd : d \in B\} = \{d : d \in B\} = B$ for all $c \in B$. Hence $\mathbf{A}$ is ideal simple. Conversely, suppose $\mathbf{A}$ is ideal simple. By hypothesis, $BaB = B = BbB$ for all $a, b \in B$. But for any $a, b \in A$ we have that $a\mathcal{D}b$ iff $AaA = AbA$ by Howie [111, Section 2.4, p. 55]. It follows that the only $\mathcal{D}$-equivalence classes of $\mathbf{A}$ are $\{0\}$ and B $(:= A - \{0\})$, whence $\mathbf{A}$ is primitive. Thus $\mathbf{A}$ is ideal simple iff it is primitive. This observation, in conjunction with Theorem 3.2.3, gives rise to the following problem.

Problem 3.2.5. Give an ideal-theoretic characterisation of dual binary discriminator varieties. ∎

3.2.6. Binary Discriminator Varieties. To within isomorphism, there is just one two-element binary discriminator algebra, namely the two-element flat implicative BCK-algebra $\mathbf{C}_1$ of Example 1.6.18. Since $\mathbf{C}_1$ is, to within isomorphism, the only subdirectly irreducible implicative BCK-algebra (recall Theorem 1.6.19), the variety of implicative BCK-algebras is a binary discriminator variety. We take this observation, which does not seem to have been made by Chajda, Halaš and Rosenberg in [58], as the starting point for our study of binary discriminator varieties. To begin, denote by the *pure binary discriminator variety* (in symbols, PBD) the variety generated by the class of all binary discriminator algebras $\langle A;\ b, 0\rangle$, where b is the binary discriminator function on A and 0 is a nullary operation.

Theorem 3.2.7. *The variety of implicative BCS-algebras coincides with the pure binary discriminator variety.*

Proof. Let K denote the class of all binary discriminator algebras $\langle A;\ b, 0\rangle$ where b is the binary discriminator function on A and 0 is a nullary operation. By Example 2.3.8 every member of K is a flat implicative BCS-algebra, whence $\mathsf{K} \subseteq \mathsf{iBCS}$. It follows that the variety $\mathbf{V}(\mathsf{K})$ generated by K, namely PBD, is a subvariety of iBCS. Conversely, the three-element flat implicative BCS-algebra $\mathbf{B}_2$ is a member of K; since $\mathbf{B}_2$ generates iBCS as a variety (by Theorem 2.3.73) we have that $\mathsf{iBCS} = \mathbf{V}(\mathbf{B}_2) \subseteq \mathbf{V}(\mathsf{K}) = \mathsf{PBD}$. ∎

The following result is due to the author's Ph.D. supervisor. In the statement of the theorem and in the sequel we denote the binary discriminator [term] by $\backslash$; of course, this change in notation is justified by Theorem 3.2.7.

Theorem 3.2.8. *(Bignall) Any algebra* $\mathbf{A}$ *in a binary discriminator variety* V *has an implicative BCS-algebra polynomial reduct whose* iBCS*-ideals coincide with the* V*-ideals of* $\mathbf{A}$.

Proof. By Theorem 3.2.7, any algebra $\mathbf{A}$ in a binary discriminator variety V has a canonical implicative BCS-algebra polynomial reduct $\langle A;\ \backslash, 0\rangle$. Let $I \in \mathrm{I}_{\mathsf{V}}(\mathbf{A})$. Since V is subtractive, $I = [0]_\theta$ for some $\theta \in \mathrm{Con}\,\mathbf{A}$ by Proposition 1.7.5. Since θ is also a congruence on $\langle A;\ \backslash, 0\rangle$, we have that I is an iBCS-ideal of $\langle A;\ \backslash, 0\rangle$, just because the ideals of any implicative BCS-algebra are the 0-classes of its congruences. For the converse, let $\langle a\rangle_{\langle A;\ \backslash, 0\rangle}$ be a principal iBCS-ideal of the canonical implicative BCS-algebra polynomial reduct $\langle A;\ \backslash, 0\rangle$. Let ϑ_a be the relation defined on $A \times A$ by $b \equiv c\,(\mathrm{mod}\,\vartheta_a)$ iff $b\backslash a = c\backslash a$. By Lemma 2.3.45(1)(a)$'$,(2)(a) ϑ_a is a congruence on $\langle A;\ \backslash, 0\rangle$ with the property that $[0]_{\vartheta_a} = \langle a\rangle_{\langle A;\ \backslash, 0\rangle}$. Suppose f is an n-ary fundamental operation on $\mathbf{A}$ and that $b_i \equiv c_i\,(\mathrm{mod}\,\vartheta_a)$ for $i = 1, \ldots, n$. By Theorem 1.7.21,

$$\begin{aligned} f(b_1, \ldots, b_n)\backslash a &= f(b_1\backslash a, \ldots, b_n\backslash a)\backslash a \\ &= f(c_1\backslash a, \ldots, c_n\backslash a)\backslash a \\ &= f(c_1, \ldots, c_n)\backslash a \end{aligned}$$

so $f(b_1, \ldots, b_n) \equiv f(c_1, \ldots, c_n)\,(\mathrm{mod}\,\vartheta_a)$. Hence ϑ_a is a congruence on $\mathbf{A}$ and $\langle a\rangle_{\langle A;\ \backslash, 0\rangle} \in \mathrm{I}_{\mathsf{V}}(\mathbf{A})$. Let now J be an arbitrary iBCS-ideal of $\langle A;\ \backslash, 0\rangle$. Put:

$$\psi := \bigvee\{\vartheta_a \in \mathrm{Con}\,\langle A;\ \backslash, 0\rangle : a \in J\},$$

where the join is taken in the lattice of equivalence relations. Then ψ must be a congruence on both $\mathbf{A}$ and $\langle A;\ \backslash, 0\rangle$, just because each ϑ_a is. Since the 0-class of ψ is clearly J, we have that J is a V-ideal of $\mathbf{A}$ as required. ∎

In general, the converse of Theorem 3.2.8 fails to hold: see Remark 3.2.21 in the sequel.

Corollary 3.2.9. *Let* V *be a binary discriminator variety with binary discriminator term* $x\backslash y$ *and dual binary discriminator term* $x \sqcap y$*. The following statements hold:*

1. V *has EDPI witness* $x\backslash y$*;*

2. V *is ideal distributive;*

3. V *has the ideal extension property;*

4. $x \sqcap y$ *is a meet generator term for* V*;*

Further, if V *is with* $\{\mathbf{0}, \mathbf{1}\}$*, then:*

5. $x \sqcup y := \mathbf{1}\backslash\big((\mathbf{1}\backslash x) \sqcap (\mathbf{1}\backslash y)\big)$ *is a join generator term for* V.

Proof. Because of Theorem 3.2.8, Proposition 2.3.5 and Proposition 2.2.30, (1) follows from Theorem 2.2.20, while (2) and (3) follow from Proposition 2.1.24. For (4) and (5), notice that on any binary discriminator algebra, the binary discriminator coincides with implicative BCS difference, while the dual binary discriminator coincides with the implicative BCS meet. Because of these remarks and Theorem 3.2.8, Proposition 2.3.5 and Proposition 2.2.30, $x \sqcap y$ and $x \sqcup y$ are meet and join generator terms for V by Proposition 2.2.31 and Proposition 2.3.60(2) respectively. ∎

In introducing the binary discriminator as a generalisation of the ternary discriminator to varieties exhibiting congruence permutability only locally at $\mathbf{0}$, Chajda, Halaš and Rosenberg were primarily concerned with generalising a well-known result of Pixley [184, Theorem 3.1] to the effect that a (finite) algebra $\mathbf{A}$ is a ternary discriminator algebra iff $\mathbf{V}(\mathbf{A})$ is arithmetical and $\mathbf{A}$ is hereditarily simple. In particular, in [58] they proved:

Proposition 3.2.10. *[58, Corollary 2.2] If* $\mathbf{A}$ *is a binary discriminator algebra then* $\mathbf{V}(\mathbf{A})$ *is arithmetic at* $\mathbf{0}$ *and* $\mathbf{A}$ *is hereditarily ideal simple.*

Chadja, Halaš and Rosenberg were unable to establish a converse of Proposition 3.2.10, except in the restricted case of main ideal term algebras. A *main ideal term algebra* is an algebra $\mathbf{A}$ with $\mathbf{0}$ for which there exists a binary term function $\circ$ and a unary term function $'$ of $\mathbf{A}$ such that the following equations are identically satisfied for any $a, b \in A$:

$$(a \circ b'') \circ b' = 0 \quad \text{and} \quad a \circ 0' = a$$

and moreover:

$$\langle a \rangle_{\mathbf{A}} = \{b \circ a'' : b \in A\} \qquad \text{for every } a \in A - \{0\},$$

where a'' is shorthand notation for $(a')'$. By [58, Theorem 4.1] a main ideal term algebra $\mathbf{A}$ is a binary discriminator algebra iff $\mathbf{A}$ is ideal simple. In contradistinction to this result, the following theorem characterises binary discriminator algebras solely in ideal-theoretic terms.

Theorem 3.2.11. *(cf. [229, Proposition 6.13]) An algebra $\mathbf{A}$ is a binary discriminator algebra iff $\mathbf{A}$ is ideal simple and the variety $\mathsf{V}(\mathbf{A})$ generated by $\mathbf{A}$ is subtractive with EDPI.*

Proof. ($\Rightarrow$) Let $\mathbf{A}$ be a binary discriminator algebra. By Proposition 3.2.10, $\mathbf{A}$ is ideal simple and $\mathsf{V}(\mathbf{A})$ is subtractive. Since $\mathsf{V}(\mathbf{A})$ is a binary discriminator variety, by Corollary 3.2.9(1) we have that $\mathsf{V}(\mathbf{A})$ has EDPI.

($\Leftarrow$) Let $\mathbf{A}$ be ideal simple and suppose $\mathsf{V}(\mathbf{A})$ is subtractive with EDPI. By Theorem 1.7.9 there exists a term $x\backslash y$ of $\mathsf{V}(\mathbf{A})$ that witnesses *both* subtractivity and EDPI in the sense that $\mathsf{V}(\mathbf{A}) \models x\backslash x \approx \mathbf{0}, x\backslash \mathbf{0} \approx x$ and $a\backslash^{\mathbf{B}} b = 0$ iff $a \in \langle b \rangle_{\mathbf{B}}$ for any $\mathbf{B} \in \mathsf{V}(\mathbf{A})$ and $a, b \in B$. We will show $x\backslash y$ induces the binary discriminator on $\mathbf{A}$. Let $a, b \in A$. Suppose $b = 0$. By subtractivity $a\backslash^{\mathbf{A}} b = a\backslash^{\mathbf{A}} 0 = a$. Suppose $b \neq 0$. From $b \in \langle b \rangle_{\mathbf{A}}$ we have that $\langle b \rangle_{\mathbf{A}} \neq \{0\}$, which implies by ideal simplicity that $\langle b \rangle_{\mathbf{A}} = A$. Therefore $a \in \langle b \rangle_{\mathbf{A}}$, whence $a\backslash^{\mathbf{A}} b = 0$ by EDPI. Thus $\mathbf{A}$ is a binary discriminator algebra. ∎

Corollary 3.2.12. *A variety V is a binary discriminator variety iff it is subtractive with EDPI and is generated by a class of ideal simple algebras.*

In the following theorem, we give an alternative characterisation of binary discriminator varieties, in the spirit of Agliano and Ursini's characterisation of subtractive varieties with EDPI (Theorem 3.1.6); note that this characterisation of binary discriminator varieties is implicit in [58, Section 5].

Theorem 3.2.13. *(cf. Theorem 1.7.21) For a variety V the following are equivalent:*

1. *V is a binary discriminator variety;*

2. *There exists a binary term $x\backslash y$ of V such that:*

(a) *For any* $\mathbf{A} \in \mathsf{V}$, *the polynomial reduct* $\langle A;\ \backslash^{\mathbf{A}}, 0\rangle$ *is an implicative BCS-algebra;*

(b) *For any* $\mathbf{A} \in \mathsf{V}$ *and* $a \in A$, $\langle a\rangle_{\mathbf{A}} = \langle a\rangle_{\langle A;\ \backslash^{\mathbf{A}}, 0\rangle}$*;*

(c) V *is generated by a class* $\mathsf{K} \subseteq \mathsf{V}$ *whose members are ideal simple.*

Proof. (1) $\Rightarrow$ (2) Suppose V is a binary discriminator variety with binary discriminator term $x\backslash y$. For any $\mathbf{A} \in \mathsf{V}$ the polynomial reduct $\langle A;\ \backslash^{\mathbf{A}}, 0\rangle$ is an implicative BCS-algebra by Example 2.3.11, which establishes (2)(a). Moreover, for any $\mathbf{A} \in \mathsf{V}$ and $a \in A$ we have $\langle a\rangle_{\mathbf{A}} = \langle a\rangle_{\langle A;\ \backslash^{\mathbf{A}}, 0\rangle}$ as a particular case of Proposition 3.2.8; thus (2)(b) holds. And, since V is a binary discriminator variety, from Theorem 1.7.21 we have that V is generated by a class $\mathsf{K} \subseteq \mathsf{V}$ of ideal simple algebras, which establishes (2)(c).

(2) $\Rightarrow$ (1) Suppose V is a variety satisfying (2)(a)–(c). By (2)(a) and Theorem 2.1.3 we have that V is subtractive, while from (2)(b) and Theorem 2.2.20 we have that V has EDPI. Because of (2)(c), it follows that V is a subtractive variety with EDPI generated by a class $\mathsf{K} \subseteq \mathsf{V}$ of ideal simple algebras; the result now follows from Corollary 3.2.12. ∎

In the following series of examples, we list some binary discriminator varieties beyond those given by Chajda, Halaš and Rosenberg in [58].

Example 3.2.14. Let V be a fixedpoint discriminator variety with $\mathbf{0}$ generated by a class $\mathsf{K} \subseteq \mathsf{V}$ of fixedpoint discriminator algebras such that 0 realises the discriminating element on any $\mathbf{A} \in \mathsf{K}$. Let $f(x, y, z)$ be a fixedpoint discriminator term for V. By Example 2.3.13, V is a binary discriminator variety with binary discriminator term $f(\mathbf{0}, y, x)$, while K is a class of binary discriminator algebras generating V. ∎

Example 3.2.15. (*cf.* [58, Section 2, p. 241]) Let V be a pointed ternary discriminator variety (say with $\mathbf{0}$) with ternary discriminator term $t(x, y, z)$ generated by a class $\mathsf{K} \subseteq \mathsf{V}$ of ternary discriminator algebras. For any $\mathbf{A} \in \mathsf{K}$, let $x\backslash y := t(\mathbf{0}, y, x)$. By definition of the ternary discriminator,

$$a\backslash^{\mathbf{A}} b = t^{\mathbf{A}}(0, b, a)$$

$$= \begin{cases} a & \text{if } b = 0 \\ 0 & \text{otherwise} \end{cases}$$

whence the polynomial reduct $\langle A;\ \backslash^{\mathbf{A}}, 0\rangle$ is a flat implicative BCS-algebra. Hence K is a class of binary discriminator algebras and V is a binary discriminator variety. ∎

Example 3.2.16. By the proof of Proposition 3.1.8, the reduct $\langle A;\ \backslash, 0\rangle$ of any primitive skew Boolean algebra $\mathbf{A}$ is a flat implicative BCS-algebra. Hence $\mathbf{A}$ is a binary discriminator algebra, and the binary discriminator on A is standard difference. By Theorem 1.4.29, the class SBA of skew Boolean algebras is generated as a variety by any family of primitive skew Boolean algebras that contains the 3-element left and right handed primitive algebras $\mathbf{3}_L^p$ and $\mathbf{3}_R^p$. Hence SBA is a binary discriminator variety. ∎

Example 3.2.17. By Example 2.3.12 the class of pseudocomplemented semilattices is a binary discriminator variety (with binary discriminator term $x\backslash y := x \wedge y^*$), generated (as a binary discriminator variety) by the 3-element chain $\mathbf{3}$ (considered as a pseudocomplemented semilattice). More generally, PCSL is generated (as a binary discriminator variety) by any subclass of the family of bounded chains (considered as pseudocomplemented semilattices) that includes $\mathbf{3}$. Indeed, if $\mathbf{A}$ is a bounded chain, then $b^* = 0$ if $b \neq 0$ and 0 otherwise, whence $a\backslash b = a$ if $b = 0$ and a otherwise for any $a, b \in A$. Hence the canonical implicative BCS-algebra polynomial reduct $\langle A;\ \backslash, 0\rangle$ of $\mathbf{A}$ is flat, and $\mathbf{A}$ is a binary discriminator algebra. ∎

Example 3.2.18. Recall from Example 1.4.30 that a Stone algebra is a distributive lattice with pseudocomplementation satisfying $x^* \vee x^{**} \approx \mathbf{1}$. By Balbes and Dwinger [14, Example VIII§7.2] any bounded chain $\mathbf{A}$ (considered as a distributive lattice with pseudocomplementation) is a Stone algebra. By Example 2.3.12 and the remarks of §1.3.5, $\mathbf{A}$ has a canonical implicative BCS-algebra polynomial reduct $\langle A;\ \backslash, 0\rangle$, which must be flat by Example 3.2.17. Hence $\mathbf{A}$ is a binary discriminator algebra. Since the class of Stone algebras is generated (as a variety) by any subclass of the family of bounded chains that includes $\mathbf{3}$, the 3-element chain considered as a Stone algebra (see Balbes and

Dwinger [14, Theorem VIII§7.1] or Grätzer [100]), the class of Stone algebras is a binary discriminator variety. ∎

Remark 3.2.19 (Added in proof). The author's Ph.D. supervisor has pointed out that the class of Abelian Rickart semirings, studied by Cornish in [63], is also a binary discriminator variety. This observation generalises Example 3.2.18, since Stone algebras are a subvariety of the variety of Abelian Rickart semirings. ∎

The final example of this subsection shows that the variety DLPC of distributive lattices with pseudocomplementation is *not* a binary discriminator variety. This example is of interest since every distributive lattice with pseudocomplementation has a canonical implicative BCS-algebra polynomial reduct. Binary discriminator varieties therefore do *not* exhaust those classes of algebras in which implicative BCS-algebras arise naturally (recall the remarks following Problem 2.3.16 in the prequel).

Example 3.2.20. By Example 2.3.12 and the remarks of §1.3.5 any distributive lattice with pseudocomplementation $\mathbf{A}$ has a canonical implicative BCS-algebra polynomial reduct $\langle A;\ \backslash, 0\rangle$, where $a\backslash b := a \wedge b^*$ for any $a, b \in A$. However, DLPC is not a binary discriminator variety. To see this it is sufficient to show there is no subclass K of DLPC with a binary term $b(x, y)$ that realises the binary discriminator on each $\mathbf{A} \in$ K and for which DLPC $= \mathbf{V}$(K). Observe first by inspection of both the DLPC-free algebra on two free generators [217] and the subdirectly irreducible members of DLPC [101, Section 16] that the only binary term $b(x, y)$ of DLPC inducing implicative BCS difference is $x \wedge y^*$. Because of Example 2.3.61(2) and Lemma 2.3.63(2), this implies $\mathbf{A}$ is a binary discriminator algebra iff $\langle A;\ \backslash, 0\rangle$ is flat iff $\mathbf{A}$ is dense (that is, any element in $A - \{0\}$ is dense). By results due to Agliano and Ursini [9, Result, p. 256] and Grätzer [101, Exercise 3§14.3, p. 164] every dense distributive lattice with pseudocomplementation $\langle B;\ \wedge, \vee, ^*, 0\rangle$ arises from a bounded distributive lattice $\langle B;\ \wedge, \vee\rangle$ upon: (i) adjoining a new element 0 to B such that $0 \leq b$ for all $b \in B$; (ii) defining a pseudocomplementation operation on B by $b^* := 0$ if $b \neq 0$ and $b^* := 1$ otherwise, where 1 is the greatest element of $\mathbf{B}$; and (iii) distinguishing the operation * and the least element 0. (See also Balbes and

Dwinger [14, Example VIII§7.3].) Let $\mathbf{A}$ be a dense distributive lattice with pseudocomplementation. Because of the definition of * on A, it is easy to see that $\mathbf{A} \models x^* \vee x^{**} \approx \mathbf{1}$, whence $\mathbf{A}$ is a Stone algebra. Hence the class of all binary discriminator algebras of DLPC generates the variety of Stone algebras, not the variety of distributive lattices with pseudocomplementation, so DLPC is not a binary discriminator variety. ∎

Remark 3.2.21. Example 3.2.20 shows also that Condition (2)(c) of Theorem 3.2.13 is not artificial. Indeed, we have already observed in Example 3.2.20 that any distributive lattice with pseudocomplementation $\mathbf{A}$ has a canonical implicative BCS-algebra polynomial reduct $\langle A;\ \backslash, 0\rangle$, whence DLPC satisfies Condition (2)(a) of Theorem 3.2.13. Also, by Agliano and Ursini [9, Example 6.1, p. 256] the principal DLPC-ideal $\langle b\rangle_{\mathbf{A}}$ generated by $b \in A$ is $(b^{**}]$, whence $a \in \langle b\rangle_{\mathbf{A}}$ iff $a \leq b^{**}$ iff $a \wedge b^* = 0$ iff $a\backslash b = 0$ iff $a \in \langle b\rangle_{\langle A;\ \backslash, 0\rangle}$. Thus the principal DLPC-ideals of $\mathbf{A}$ coincide with the principal iBCS-ideals of $\langle A;\ \backslash, 0\rangle$, and DLPC satisfies Condition (2)(b) of Theorem 3.2.13. Because DLPC satisfies Conditions (2)(a)–(2)(b) of Theorem 3.2.13 but is not a binary discriminator variety, Condition (2)(c) cannot be omitted from the assertion of the theorem. ∎

3.2.22. Point Regular Binary Discriminator Varieties. Corollary 3.2.9 prompts us to investigate $\mathbf{0}$-regular binary discriminator varieties, since $\mathbf{0}$-regularity implies ideal determinacy for such varieties (by Proposition 1.7.3) and hence EDPC (by Corollary 3.2.9(1) and Proposition 1.7.10). To begin, recall from §1.5.9 that the fixedpoint discriminator on a set A is the ternary operation $f : A^3 \to A$ defined for any $a, b, c \in A$ by:

$$f(a,b,c) := \begin{cases} c & \text{if } a = b \\ 0 & \text{otherwise,} \end{cases}$$

where $0 \in A$ is the discriminating element of f. A *pointed fixedpoint discriminator algebra* is an algebra $\mathbf{A}$ with $\mathbf{0}$ for which there is a ternary term f of $\mathbf{A}$ that realises the fixedpoint discriminator on $\mathbf{A}$ such that $\mathbf{0}^{\mathbf{A}}$ is the discriminating element. A *pointed fixedpoint discriminator variety* is a variety V with $\mathbf{0}$ for which there is a subclass K of V such that $\mathsf{V} = \mathbf{V}(\mathsf{K})$ and a ternary term f of V

such that f realises the fixedpoint discriminator with discriminating element $\mathbf{0}^{\mathbf{A}}$ on each $\mathbf{A} \in \mathsf{K}$. In this case f is called a *pointed fixedpoint discriminator term* for V.

Remark 3.2.23. Pointed fixedpoint discriminator varieties should not be confused with fixedpoint discriminator varieties that happen to be pointed. In particular, while every pointed fixedpoint discriminator variety is a fixedpoint discriminator variety that is pointed, the converse does not hold. To see this, let $A := \langle A;\ \mathbf{0}^{\mathbf{A}} \rangle$ be a pointed set. Let $f : A^3 \to A$ be a fixedpoint discriminator on A with discriminating element $\mathbf{0}^{\mathbf{A}} \neq d \in A$. Let $\mathbf{A} := \langle A;\ f, \mathbf{0}^{\mathbf{A}} \rangle$. Then $\mathbf{V}(\mathbf{A})$ is a fixedpoint discriminator variety that is pointed, but it is not a pointed fixedpoint discriminator variety. See Blok and Pigozzi [34, p. 580]. ∎

Theorem 3.2.24. *Let* V *be a variety with* $\mathbf{0}$. *If* V *is a* $\mathbf{0}$*-regular binary discriminator variety then* V *is a pointed fixedpoint discriminator variety. In this case a pointed fixedpoint discriminator term for* V *is given by:*

$$f(x, y, z) := (\cdots(z\backslash d_1(x, y))\backslash \cdots)\backslash d_n(x, y)$$

where $x\backslash y$ *is a binary discriminator term for* V *and* $d_1(x, y), \ldots, d_n(x, y)$ *are binary terms witnessing the* $\mathbf{0}$*-regularity of* V *in the sense of Proposition 1.2.6.*

Proof. Let V be a variety with $\mathbf{0}$. Suppose V is a $\mathbf{0}$-regular binary discriminator variety with binary discriminator term $x\backslash y$ and that $d_1(x, y), \ldots, d_n(x, y)$ are binary terms witnessing the $\mathbf{0}$-regularity of V in the sense of Proposition 1.2.6. Let $\mathsf{K} \subseteq \mathsf{V}$ be a class of binary discriminator algebras generating V as a variety. We will show $f(x, y, z)$ induces the pointed fixedpoint discriminator on any member of K. So let $\mathbf{A} \in \mathsf{K}$ and $a, b, c \in A$. Suppose $a = b$. By Proposition 1.2.6 we have that $d_i(x, x) \approx \mathbf{0}$ for all $1 \leq i \leq n$, whence:

$$\begin{aligned}(\cdots(c\backslash^{\mathbf{A}} d_1^{\mathbf{A}}(a, b))\backslash^{\mathbf{A}} \cdots)\backslash^{\mathbf{A}} d_n^{\mathbf{A}}(a, b) &= (\cdots(c\backslash^{\mathbf{A}} 0)\backslash^{\mathbf{A}} \cdots)\backslash^{\mathbf{A}} 0 \\ &= c\end{aligned}$$

by repeated application of the identity $x\backslash\mathbf{0} \approx x$. Suppose instead that $a \neq b$. If $d_i^{\mathbf{A}}(a, b) = 0$ for all $1 \leq i \leq n$ then $a = b$ by Proposition 1.2.6, a contradiction. Thus $d_i^{\mathbf{A}}(a,b) \neq 0$ for some $1 \leq i \leq n$. By definition of the binary discriminator, this implies that the subexpression of $f^{\mathbf{A}}(a,b,c)$ of the form:

$$(\cdots(c\backslash^{\mathbf{A}} d_1^{\mathbf{A}}(a,b))\backslash^{\mathbf{A}}\cdots)\backslash^{\mathbf{A}} d_i^{\mathbf{A}}(a,b)$$

must equal 0. We therefore have that:

$$\begin{aligned}
&(\cdots(((\cdots(c\backslash^{\mathbf{A}} d_1^{\mathbf{A}}(a,b))\backslash^{\mathbf{A}}\cdots)\backslash^{\mathbf{A}} d_i^{\mathbf{A}}(a,b))\backslash^{\mathbf{A}} d_{i+1}^{\mathbf{A}}(a,b))\backslash^{\mathbf{A}}\cdots)\backslash^{\mathbf{A}} d_n^{\mathbf{A}}(a,b)\\
&= (\cdots(0\backslash^{\mathbf{A}} d_{i+1}^{\mathbf{A}}(a,b))\backslash^{\mathbf{A}}\cdots)\backslash^{\mathbf{A}} d_n^{\mathbf{A}}(a,b)\\
&= 0
\end{aligned}$$

by repeated application of the identity $\mathbf{0}\backslash x \approx \mathbf{0}$. Hence $f^{\mathbf{A}}(a,b,c) = c$ if $a = b$ and 0 otherwise, whence $f(x,y,z)$ induces the fixedpoint discriminator on any member of K. Since K generates V as a variety we have that V is a pointed fixedpoint discriminator variety and that $f(x,y,z)$ is a pointed fixedpoint discriminator term for V. ∎

Corollary 3.2.25. *Let* V *be a variety with* $\{\mathbf{0},\mathbf{1}\}$. *If* V *is a* $\mathbf{0}$*-regular binary discriminator variety then the following assertions hold:*

1. *The term* $f(x,y,z)$ *of Theorem 3.2.24 is a commutative TD term for* V *that is regular with respect to* $\mathbf{1}$*;*
2. V *is a variety of subtractive weak Boolean algebras with filter preserving operations. Weak meet, weak relative pseudocomplement and Gödel equivalence terms for* V *are defined respectively by:*

$$\begin{aligned}
x\cdot y &:= f(x,\mathbf{1},y)\\
x\to y &:= f\big(f(x,\mathbf{1},y), f(x,\mathbf{1},\mathbf{1}),\mathbf{1}\big)\\
x\Delta y &:= f(x,y,\mathbf{1}).
\end{aligned}$$

Proof. Immediate by Theorem 3.2.24 and Theorem 1.5.14. ∎

Remark 3.2.26. The hypothesis that V is double-pointed in Corollary 3.2.25 is essential if the conclusions of the corollary are to obtain, since the variety of implicative BCK-algebras is a $\mathbf{0}$-regular binary discriminator variety that does not possess either a TD term that is regular or a weak meet. See Blok and Pigozzi [34, p. 589]. ∎

Example 3.2.14 shows that a pointed fixedpoint discriminator variety is a binary discriminator variety. Given Theorem 3.2.24, it is therefore natural to ask if the converse of Theorem 3.2.24 also holds. Perhaps surprisingly, the answer to this question is 'yes'.

Theorem 3.2.27. *Let* V *be a variety with* $\mathbf{0}$. *If* V *is a pointed fixedpoint discriminator variety, then* V *is a* $\mathbf{0}$*-regular binary discriminator variety. In this case a binary discriminator term for* V *is given by:*

$$x \backslash y := f(\mathbf{0}, y, x)$$

where $f(x, y, z)$ *is a pointed fixedpoint discriminator term for* V, *while the binary terms:*

$$d_1(x, y) := x \backslash f(x, y, x) \quad \textit{and} \quad d_2(x, y) := y \backslash f(y, x, y)$$

witness $\mathbf{0}$*-regularity for* V *in the sense of Proposition 1.2.6.*

Proof. Let V be a variety with $\mathbf{0}$. Suppose V is a pointed fixedpoint discriminator variety with pointed fixedpoint discriminator term $f(x, y, z)$. Let $\mathsf{K} \subseteq \mathsf{V}$ be a class of fixedpoint discriminator algebras generating V as a variety. Put $x \backslash y := f(\mathbf{0}, y, x)$. For any $\mathbf{A} \in \mathsf{K}$ and $a, b \in A$, by Example 2.3.13 we have that:

$$\begin{aligned} a \backslash^{\mathbf{A}} b &= f^{\mathbf{A}}(0, b, a) \\ &= \begin{cases} a & \text{if } b = 0 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

whence $\langle A; \backslash, 0 \rangle$ is a flat implicative BCS-algebra. Thus $x \backslash y$ induces the binary discriminator on any member of K. Since K generates V as a variety, it follows that V is a binary discriminator variety. To see V is $\mathbf{0}$-regular, put $x/y := x\backslash f(x, y, x)$. For any $\mathbf{A} \in \mathsf{K}$ and $a, b \in A$, by definition of the fixedpoint discriminator we have that:

$$\begin{aligned} a/b &= a\backslash^{\mathbf{A}} f^{\mathbf{A}}(a, b, a) \\ &= f^{\mathbf{A}}\big(0, f^{\mathbf{A}}(a, b, a), a\big) \\ &= \begin{cases} a & \text{if } a \neq b \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

whence $\langle A;\ /, 0\rangle$ is (by Example 1.6.18) a flat implicative BCK-algebra. Because the identities satisfied by V are precisely those satisfied by K, it follows that any algebra $\mathbf{B} \in \mathsf{V}$ has an implicative BCK-algebra polynomial reduct $\langle B;\ /^{\mathbf{B}}, 0\rangle$. Since any algebra with a point regular polynomial reduct must itself be point regular, we conclude from the $\mathbf{0}$-regularity of $\langle B;\ /^{\mathbf{B}}, 0\rangle$ (recall Theorem 1.6.17 and Theorem 1.6.10(3)) that $\mathbf{B}$ is $\mathbf{0}$-regular. Thus V is $\mathbf{0}$-regular and the binary terms $d_1(x, y) := x/y$, $d_2(x, y) := y/x$ witness $\mathbf{0}$-regularity for V in the sense of Proposition 1.2.6. ∎

Remark 3.2.28 (Added in proof). An *implicative BCSK-algebra* is an algebra $\langle A;\ /, \backslash, 0\rangle$ of type $\langle 2, 2, 0\rangle$ such that: (i) the reduct $\langle A;\ /, 0\rangle$ is an implicative BCK-algebra; (ii) the reduct $\langle A;\ \backslash, 0\rangle$ is an implicative BCS-algebra; and (iii) the implicative BCK partial order $\leq^{\langle A;\,/,0\rangle}$ and the implicative BCS partial order $\leq^{\langle A;\,\backslash,0\rangle}$ coincide. By an unpublished result of the author, an algebra $\mathbf{A} := \langle A;\ /, \backslash, 0\rangle$ of type $\langle 2, 2, 0\rangle$ is an implicative BCSK algebra iff $\mathbf{A} \models$ (1.35)–(1.38), $\mathbf{A} \models$ (2.47)–(2.50) and $\mathbf{A}$ satisfies the identities $(x\backslash y)/x \approx \mathbf{0}$ and $x\backslash\big(x\backslash(x/y)\big) \approx x/y$, whence the class iBCSK of implicative BCSK-algebras is a variety.

The variety iBCSK arises in the first instance from algebraic logic. Let $\mathcal{L} := \{\rightarrow, \Rightarrow\}$ be a language of type $\langle 2, 2\rangle$. *BCSK logic* is the deductive system $\mathbb{BCSK} := \langle \mathcal{L}, \vdash_{\mathbb{BCSK}}\rangle$ defined by the following axioms and inference rule:

$$p \Rightarrow (q \Rightarrow p) \tag{B1}$$

$$\big(p \Rightarrow (q \Rightarrow r)\big) \Rightarrow \big((p \Rightarrow q) \Rightarrow (p \Rightarrow r)\big) \tag{B2}$$

$$\big((p \Rightarrow q) \Rightarrow p\big) \Rightarrow p \tag{B3}$$

$$p \Rightarrow (q \to p) \tag{B4}$$
$$\bigl(p \to (q \to r)\bigr) \Rightarrow \bigl((p \to q) \to (p \to r)\bigr) \tag{B5}$$
$$\bigl(p \to (q \to r)\bigr) \Rightarrow \bigl(q \to (p \to r)\bigr) \tag{B6}$$
$$\bigl((p \to q) \to p\bigr) \Rightarrow p \tag{B7}$$

$$\bigl((p \Rightarrow q) \to q\bigr) \Rightarrow \bigl((q \Rightarrow p) \to p\bigr) \tag{B8}$$
$$(p \Rightarrow q) \to (p \to q) \tag{B9}$$

$$p, p \to q \vdash_{\mathbf{BCSK}} q. \tag{BCSK-MP}$$

BCSK logic was introduced by the author in [211] and has been extensively investigated by Humberstone in [112], where connections with modal logic (including the Lewis system $\mathbb{S}5$) are established. Unpublished results of the author show $\mathbb{BCSK}$ is algebraisable (with equivalence formulas $\{p \Rightarrow q, q \Rightarrow p\}$ and defining equation $p \approx p \Rightarrow p$) and that its equivalent algebraic semantics is termwise definitionally equivalent to iBCSK.

Call an implicative BCSK-algebra *flat* if its underlying poset is flat. The proof of Theorem 3.2.27 shows any pointed fixedpoint discriminator algebra $\langle A; f, 0\rangle$ has a flat implicative BCSK-algebra polynomial reduct $\langle A; /, \backslash, 0\rangle$, where:

$$a\backslash b := f(0, b, a) \quad \text{and} \quad a/b := a\backslash f(a, b, a)$$

for any $a, b \in A$. Conversely, unpublished results due to the author and the author's Ph.D. supervisor show that an implicative BCSK-algebra is subdirectly irreducible iff it is flat (for some details, see Humberstone [112, Section 1, Appendix B]), whence the class iBCSK is a pointed fixedpoint discriminator variety with pointed fixedpoint discriminator term:

$$f(x, y, z) := \bigl(z\backslash(x/y)\bigr)\backslash(y/x).$$

Let FPD_0 denote the *pure pointed fixedpoint discriminator variety*, namely the variety of type $\langle 3, 0\rangle$ generated by the class of all pointed fixedpoint discriminator algebras $\langle A; f, 0\rangle$, where 0 is a nullary operation and f is the fixedpoint discriminator on A with discriminating element 0. In view of the preceding discussion, it is easy to see that FPD_0 is termwise definitionally equivalent to iBCSK. Moreover, because the congruence structure of any algebra in a fixedpoint discriminator variety is (by Lemma 1.5.10) completely determined by the fixedpoint discriminator term, any algebra $\mathbf{A}$ in a pointed fixedpoint discriminator variety must have an implicative BCSK-algebra polynomial reduct whose congruences coincide with those of $\mathbf{A}$. These remarks extend and contrast with Blok and Pigozzi [34, Section 3]: see in particular [34, Corollary 3.6]. ∎

In their study [34] of varieties with equationally definable principal congruences, Blok and Pigozzi posed the following problem [34, Problem 7.3]: *Does there exist a purely algebraic characterisation of fixedpoint discriminator varieties similar to the one for ternary discriminator varieties given in Theorem 1.5.13(4)?* For *pointed* fixedpoint discriminator varieties, the following theorem provides an affirmative answer to this question.

Theorem 3.2.29. *For a variety* V *with* $\mathbf{0}$*, the following are equivalent:*

1. V *is a pointed fixedpoint discriminator variety;*

2. V *is a* $\mathbf{0}$*-regular binary discriminator variety;*

3. V *is congruence* $\mathbf{0}$*-permutable,* $\mathbf{0}$*-regular, semisimple with EDPC;*

4. V *is ideal determined, semisimple with EDPC.*

Proof. Let V be a variety with $\mathbf{0}$. The equivalence (1) $\Leftrightarrow$ (2) follows from Theorem 3.2.24 and Theorem 3.2.27, while the equivalence (3) $\Leftrightarrow$ (4) is clear from Proposition 1.7.3. Thus it only remains to demonstrate the equivalence (2) $\Leftrightarrow$ (3).

(2) $\Rightarrow$ (3) Suppose V is a $\mathbf{0}$-regular binary discriminator variety. Then V is $\mathbf{0}$-regular and subtractive by hypothesis. Also, V is a pointed fixedpoint discrim-

inator variety by Theorem 3.2.24, and so is semisimple (by Theorem 1.5.12) and has EDPC (by the remarks of §1.5.9).

(3) $\Rightarrow$ (2) Suppose V is congruence **0**-permutable, **0**-regular and semisimple with EDPC. Since V is subtractive, by Proposition 1.7.10 we have that V has EDPI. Since V is semisimple with EDPC, from Theorem 1.5.2(4) we have that V is generated as a variety by a class K of simple algebras. By normality of ideals we have that K is ideal simple, and so V is a subtractive variety with EDPI that is generated by a class K of ideal simple algebras. From Corollary 3.2.12 it follows that V is a binary discriminator variety; since V is **0**-regular by hypothesis, V is a **0**-regular binary discriminator variety. ∎

Although pointed fixedpoint discriminator varieties do not encompass even those fixedpoint discriminator varieties that are pointed (by Remark 3.2.23), the hypotheses of Theorem 3.2.29 are nonetheless satisfied by most fixedpoint discriminator varieties arising naturally as 'quasivarieties of logic'. In particular, varieties of k-potent Wajsberg algebras and their implicational subreducts (including Boolean algebras and implicative BCK-algebras) are pointed fixedpoint discriminator varieties. See [34, Corollary 3.6].

Example 3.2.30. By Example 2.3.14(1) each variety $\mathsf{ce_n BCK}$, $n \in \omega$, is a fixedpoint discriminator variety, with fixedpoint discriminator term $f(x, y, z) := \big(z \dot{-} (x \dot{-} y)^n\big) \dot{-} (y \dot{-} x)^n$. Although each $\mathsf{ce_n BCK}$ is pointed, the proof of Example 2.3.14(1) does not show each $\mathsf{ce_n BCK}$ is a pointed fixedpoint discriminator variety: recall Remark 2.3.15. In contrast, we may immediately conclude from Theorem 3.2.29 that each $\mathsf{ce_n BCK}$, $n \in \omega$, is a pointed fixedpoint discriminator variety, just because each $\mathsf{ce_n BCK}$ is ideal determined (by Theorem 1.6.10(3)) and semisimple with EDPC (by Example 2.3.14(1)).

Via Theorem 3.2.24, the proof of Theorem 3.2.29 yields a fixedpoint discriminator term for each $\mathsf{ce_n BCK}$, $n \in \omega$, namely $f'(x, y, z) := \big(z \backslash (x \dot{-} y)\big) \backslash (y \dot{-} x)$, where $x \backslash y$ is a binary discriminator term for $\mathsf{ce_n BCK}$. For each $n \in \omega$, let $x \backslash y := x \dot{-} y^n$. By [68, Lemma 1.1(ii),(iii)] $\mathsf{ce_n BCK} \models x \backslash x \approx \mathbf{0}$ and $\mathsf{ce_n BCK} \models x \backslash 0 \approx x$, whence $\backslash$ witnesses subtractivity for $\mathsf{ce_n BCK}$. Also, $\backslash$ witnesses EDPI for $\mathsf{ce_n BCK}$, since it is implicit in the proof of Blok and Raftery [39, Theorem 4.2] that $a \in \langle b \rangle_{\mathbf{A}}$ iff $a \backslash^{\mathbf{A}} b = 0$ for any $\mathbf{A} \in \mathsf{ce}_n\mathsf{BCK}$. Hence $\backslash$ is a binary discriminator term for each $\mathsf{ce}_n\mathsf{BCK}$, $n \in \omega$, and $f'(x,y,z) = \big(z\backslash(x \dot{-} y)\big)\backslash(y \dot{-} x) = \big(z \dot{-} (x \dot{-} y)^n\big) \dot{-} (y \dot{-} x)^n = f(x,y,z)$. ∎

Let K be a quasivariety and $\mathbf{A} \in \mathsf{K}$. Recall that a strong ideal of $\mathbf{A}$ in the sense of Blok and Raftery [40] is an $\mathbb{S}(\mathsf{K},\tau)$-filter of $\mathbf{A}$, where $\mathbb{S}(\mathsf{K},\tau)$ is a certain deductive system 'extracted' from the quasi-equational theory of K by means of a translation τ [40, Section 5]. Under inclusion, the set $\mathrm{SId}_{\mathsf{K},\tau}$ of all strong ideals of $\mathbf{A}$ forms an algebraic lattice $\mathbf{SId}_{\mathsf{K},\tau}$. In general, the natural map $\tau^{\mathbf{A}}/- : \mathrm{Con}_{\mathsf{K}}\,\mathbf{A} \to \mathrm{SId}_{\mathsf{K},\tau}\,\mathbf{A}$ sending a K-congruence to its associated strong ideal is neither injective nor surjective. We say K is *strongly ideal determined* if, for any $\mathbf{A} \in \mathsf{K}$, the map $\tau^{\mathbf{A}}/- : \mathbf{Con}_{\mathsf{K}}\,\mathbf{A} \to \mathbf{SId}_{\mathsf{K},\tau}\,\mathbf{A}$ is a lattice isomorphism; for details, see [40, Section 5, Theorem 5.2]. Theorem 3.2.29 and preceding remarks invite the following problem:

Problem 3.2.31. Let V be a variety. Is V a fixedpoint discriminator variety iff V is strongly ideal determined, semisimple with EDPC? ∎

3.2.32. Binary Discriminator Varieties and Pointed Ternary Discriminator Varieties. Theorem 3.2.29 shows a variety with $\mathbf{0}$ is a pointed fixedpoint discriminator variety iff it is a $\mathbf{0}$-regular binary discriminator variety. Since the theory of the fixedpoint discriminator closely parallels that of the ternary discriminator [34, p. 548], this observation calls for a study of the relationship between binary and pointed ternary discriminator varieties. The following theorem is an obvious consequence of Theorem 3.2.29 and the results of §1.5.9.

Theorem 3.2.33. *For a variety V with $\mathbf{0}$ the following are equivalent:*

1. *V is a ternary discriminator variety;*
2. *V is a congruence permutable $\mathbf{0}$-regular binary discriminator variety.*

In particular, if $x\backslash y$ is a binary discriminator term for V, $p(x,y,z)$ is a Mal'cev term for V, and $d_1(x,y),\ldots,d_n(x,y)$ are binary terms of V witnessing the $\mathbf{0}$-regularity of V in the sense of Proposition 1.2.6, then:

$$t(x,y,z) := p\big(f(x,y,z), f(x,y,x), x\big)$$

is a ternary discriminator term for V*, where* $f(x, y, z)$ *is the pointed fixedpoint discriminator term of Theorem 3.2.24.*

Proof. Let V be a variety with **0**.

(1) $\Rightarrow$ (2) Suppose V is a ternary discriminator variety. By Theorem 1.5.13 we have that V is a congruence permutable pointed fixedpoint discriminator variety, so V is a congruence permutable **0**-regular binary discriminator variety by Theorem 3.2.27.

(2) $\Rightarrow$ (1) Suppose V is a congruence permutable **0**-regular binary discriminator variety. Then V is a pointed fixedpoint discriminator variety by Theorem 3.2.24. Let $f(x, y, z)$ be the pointed fixedpoint discriminator term for V of Theorem 3.2.24 and let $p(x, y, z)$ be a Mal'cev term for V. Since $f(x, y, z)$ is a TD term for V (by Theorem 1.5.12), from Lemma 1.5.11(1) we have that $q(x, y, z, w) := p\big(f(x, y, z), f(x, y, w), w\big)$ is a QD term for V. Since V has permuting congruences $q(x, y, z, x)$ is a ternary discriminator term for V; clearly $q(x, y, z, x) = t(x, y, z)$, completing the proof. ∎

By Corollary 2.2.6 and the remarks of §3.2.6 the variety of implicative BCK-algebras is a **0**-regular binary discriminator variety that is not congruence permutable. Thus the hypothesis of congruence permutability cannot be omitted from the statement of Theorem 3.2.33. The following example shows that the assumption of **0**-regularity also cannot be dropped.

Example 3.2.34. Let $A := \{0, 1, 2\}$ be a set and let $t : A^3 \to A$ be the ternary discriminator on A. Let $\mathbf{A} := \langle A;\ p, 0\rangle$ be the algebra with distinguished element 0 and ternary operation $p : A^3 \to A$ defined by:

$$p(a, b, c) := \begin{cases} a & \text{if } a = 1,\ b = 2 \text{ and } c = 0 \\ b & \text{if } a = 2,\ b = 1 \text{ and } c = 0 \\ t(a, b, c) & \text{otherwise} \end{cases}$$

for any $a, b, c \in A$. Let $a \backslash b := p(0, b, a)$ for any $a, b \in A$. Clearly $a \backslash b$ is the binary discriminator on A, so $\mathbf{A}$ is a binary discriminator algebra. Moreover, one easily checks that $\mathbf{A} \models p(x, x, y) \approx y$ and $\mathbf{A} \models p(x, y, y) \approx x$; thus

$\mathsf{V}(\mathbf{A})$, the variety generated by $\mathbf{A}$, is a congruence permutable binary discriminator variety. However, $\mathsf{V}(\mathbf{A})$ is not point regular, and in particular is not $\mathbf{0}$-regular. Indeed, one easily shows (by inspection of the $\mathsf{V}(\mathbf{A})$-free algebra on one free generator) that the only constant term of $\mathsf{V}(\mathbf{A})$ is $\mathbf{0}$. But the partition $\{\{0\},\{1,2\}\}$ induces a non-trivial (in fact, the only non-trivial) congruence on $\mathbf{A}$, whence $\mathbf{A}$ itself is not $\mathbf{0}$-regular. Thus $\mathsf{V}(\mathbf{A})$ is a congruence permutable binary discriminator variety that is not point-regular (and in particular not $\mathbf{0}$-regular), and hence is not a ternary discriminator variety. ∎

In general, it is a non-trivial task to construct a Mal'cev term for a congruence permutable variety: witness for example Theorem 3.1.40. A simpler syntactic criterion both necessary and sufficient for a $\mathbf{0}$-regular binary discriminator variety to be a ternary discriminator variety is therefore desirable. In the following proposition we give just such a syntactic criterion. But first, let A be a set and let $0 \in A$ be fixed but arbitrary. Recall from multiple-valued switching theory [172, Chapter 3] that a binary function $+ : A^2 \to A$ is called a *sum-like* operation (*with respect to* 0) if $a + 0 = a = 0 + a$ for any $a \in A$. Let K be a class of algebras with $\mathbf{0}$. A binary term $x + y$ of K is said to be *sum-like* (*with respect to* $\mathbf{0}$) if the canonical interpretation of $x + y$ on any $\mathbf{A} \in \mathsf{K}$ is a sum-like operation with respect to $\mathbf{0}^{\mathbf{A}}$. See also Werner [237, Theorem 1.3].

Proposition 3.2.35. *(cf. [39, Theorem 6.1(i)]) For a variety* V *with* $\mathbf{0}$ *the following are equivalent:*

1. V *is a ternary discriminator variety;*
2. V *is a* $\mathbf{0}$*-regular binary discriminator variety with a binary term* $x + y$ *that is sum-like (with respect to* $\mathbf{0}$*).*

In particular, if $x \backslash y$ *and* $x + y$ *are respectively a binary discriminator term and a sum-like term for* V*, then:*

$$t(x, y, z) := f(x, y, z) + \big(x \backslash f(x, y, x)\big)$$

is a ternary discriminator term for V*, where* $f(x, y, z)$ *is the pointed fixedpoint discriminator term of Theorem 3.2.24.*

Proof. Let V be a variety with $\mathbf{0}$.

(1) $\Rightarrow$ (2) Suppose V is a ternary discriminator variety with ternary discriminator term $t(x, y, z)$. By Theorem 3.2.33 we have that V is a $\mathbf{0}$-regular binary discriminator variety. Put $x + y := t(y, 0, x)$. Because of Theorem 1.4.39, we have that $x +^{\mathbf{A}} y$ is the left handed skew lattice join on any $\mathbf{A} \in \mathsf{V}$, whence V has a sum-like term.

(2) $\Rightarrow$ (1) Suppose V is a $\mathbf{0}$-regular regular binary discriminator variety with binary discriminator term $x \backslash y$. By Theorem 3.2.24 we have that V is a pointed fixedpoint discriminator variety. Let $\mathsf{K} \subseteq \mathsf{V}$ be a class of fixedpoint discriminator algebras generating V as a variety. Let $x + y$ be a sum-like term for V and put $t(x, y, z) := f(x, y, z) + (x \backslash f(x, y, x))$, where $f(x, y, z)$ is the pointed fixedpoint discriminator term of Theorem 3.2.24. By definition of the fixedpoint discriminator, for any $\mathbf{A} \in \mathsf{K}$ and $a, b, c \in A$ we have:

$$f^{\mathbf{A}}(a, b, c) = \begin{cases} c & \text{if } a = b \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad a \backslash^{\mathbf{A}} f^{\mathbf{A}}(a, b, a) \; = \begin{cases} a & \text{if } a \neq b \\ 0 & \text{otherwise.} \end{cases}$$

Since $+$ is a sum-like term, it follows that $t^{\mathbf{A}}(a, b, c) = a$ if $a \neq b$ and c otherwise. Thus $t(x, y, z)$ induces the ternary discriminator on any member of K; since K generates V as a variety we have that V is a ternary discriminator variety with ternary discriminator term $t(x, y, z)$. ∎

Remark 3.2.36. Let V be a dual binary discriminator variety with dual binary discriminator term $x \wedge y$. In [19, Section 4.10] Bignall and Leech essentially assert that a necessary and sufficient condition for V to be a ternary discriminator variety is the existence of a binary term $x \oplus y$ that is both a sum-like term and a Gödel equivalence term for V; in this case a ternary discriminator term for V is given by:

$$t(x, y, z) := \big(x \wedge (x \oplus y)\big) \oplus \big(z \oplus (z \wedge (x \oplus y))\big).$$

∎

3.3 Pre-BCK Quasilattices and BCK Paralattices

Collectively, the results of §1.4.32, §1.4.37, §3.2.6 and §3.2.22 intimate that pre-BCK-algebras structurally enriched with (locally Boolean) band operations arise naturally in pointed discriminator, pointed fixedpoint discriminator and pointed discriminator varieties, which remark calls for a generalisation of Idziak's theory of BCK-[semi]lattices to the non-commutative case. Lemma 1.6.24, which shows any BCK-[semi]lattice may be viewed as the conjunction of a BCK-algebra and a [semi]lattice such that the underlying BCK-algebra partial ordering and the [semi]lattice partial ordering either dualise or coincide, suggests that an appropriate generalisation of BCK-[semi]lattices is to algebras consisting of a pre-BCK-algebra reduct and a band or non-commutative lattice reduct such that the underlying pre-BCK-algebra ordering either coincides with or dualises an ordering on the band or non-commutative lattice reduct. Because there exist two fundamental orderings $\preceq_{\mathcal{D}}$ and $\leq_{\mathcal{H}}$ on any band, Idziak's theory of BCK-[semi]lattices bifurcates when generalised to the non-commutative case. On the one hand, the theory of BCK-[semi]lattices generalises to a theory of pre-BCK bands and pre-BCK quasilattices $\mathsf{PQ}_{\mathcal{C}}$, $\{\dot{-}, 0\} \subseteq \mathcal{C} \subseteq \{\wedge, \vee, \dot{-}, 0\}$; and on the other, to a theory of BCK bands and BCK paralattices $\mathsf{BP}_{\mathcal{C}}$, $\{\dot{-}, 0\} \subseteq \mathcal{C} \subseteq \{\wedge, \vee, \dot{-}, 0\}$. Within the context of the families $\mathsf{PQ}_{\mathcal{C}}$, $\mathsf{PQ}_{\mathcal{C}}$, it is natural to focus attention (ignoring issues of type) on certain varieties $\mathsf{IQ}_{\mathcal{C}'}$, $\{\wedge, \backslash, 0\} \subseteq \mathcal{C}' \subseteq \{\wedge, \vee, \backslash, 0\}$, of implicative pre-BCK bands and implicative BCS quasilattices and certain varieties $\mathsf{IP}_{\mathcal{C}''}$, $\{\wedge, /, 0\} \subseteq \mathcal{C}'' \subseteq \{\wedge, \vee, /, 0\}$, of implicative BCK bands and implicative BCK paralattices respectively, since these classes exhibit 'locally Boolean' behaviour. Throughout this section, our study of all the various families $\mathsf{PQ}_{\mathcal{C}}$, $\mathsf{IQ}_{\mathcal{C}'}$, $\mathsf{BP}_{\mathcal{C}}$ and $\mathsf{IP}_{\mathcal{C}''}$ is informed by both Idziak's theory of BCK-[semi]lattices and Laslo and Leech's recent study of paralattices and quasilattices [145].

In §3.3.2 we study the family of classes $\mathsf{PQ}_{\mathcal{C}}$, $\{\dot{-}, 0\} \subseteq \mathcal{C} \subseteq \{\wedge, \vee, \dot{-}, 0\}$. Members of the classes $\mathsf{PQ}_{\mathcal{C}}$ consist of pre-BCK-algebras (possibly) structurally enriched with band operations $\wedge$ or $\vee$ such that the natural band quasiorder $\preceq_{\mathcal{D}}$

coincides with or dualises the underlying pre-BCK-algebra quasiorder $\preceq^{\langle \dot{-},0\rangle}$. Let $\wedge \in \mathcal{C}$ or $\vee \in \mathcal{C}$. For each choice of $\mathcal{C}$, it is shown that the class $\mathsf{PQ}_{\mathcal{C}}$ is a variety. For each $\mathsf{PQ}_{\mathcal{C}}$, we present analogues of the Clifford-McLean theorem for bands or quasilattices; some sufficient conditions for each $\mathsf{PQ}_{\mathcal{C}}$ to be regular (in a sense suitably modified from §1.3.15) are also presented.

In §3.3.13 we study the family of classes $\mathsf{IQ}_{\mathcal{C}'}$, $\{\backslash, 0\} \subseteq \mathcal{C}' \subseteq \{\wedge, \vee, \backslash, 0\}$. Members of the classes $\mathsf{IQ}_{\mathcal{C}'}$ consist of implicative BCS-algebras (possibly) structurally enriched with band operations $\wedge$ and $\vee$ such that the natural band partial order $\leq_{\mathcal{H}}$ and quasiorder $\preceq_{\mathcal{D}}$ respectively coincide with or dualise the underlying implicative BCS-algebra partial order $\leq^{\langle \backslash,0\rangle}$ and quasiorder $\preceq^{\langle \backslash,0\rangle}$. For each choice of $\mathcal{C}'$, $\wedge \in \mathcal{C}'$ or $\vee \in \mathcal{C}'$, it is shown that the class $\mathsf{IQ}_{\mathcal{C}'}$ is a variety. Let $\wedge \in \mathcal{C}'$. For each $\mathcal{C}'$ and any $\mathbf{A} \in \mathsf{IQ}_{\mathcal{C}'}$, it is shown that the band with zero reduct $\langle A;\ \wedge, 0\rangle$ is locally Boolean (in the sense of §1.3.15). In one of the two main results of the section, the skew Boolean algebras are characterised amongst the members of $\mathsf{IQ}_{\mathcal{C}'}$, $\{\wedge, \vee\} \subseteq \mathcal{C}'$. Ignoring issues of similarity type, we also show that Idziak's variety of BCK-lattices is the splitting variety associated with the variety of left handed skew Boolean algebras in a certain large subvariety of $\mathsf{PQ}_{\mathcal{C}}$, $\{\wedge, \vee\} \subseteq \mathcal{C}$.

In §3.3.27 we study the family of classes $\mathsf{BP}_{\mathcal{C}}$, $\{\dot{-}, 0\} \subseteq \mathcal{C} \subseteq \{\wedge, \vee, \dot{-}, 0\}$. Members of the classes $\mathsf{BP}_{\mathcal{C}}$ consist of BCK-algebras (possibly) structurally enriched with band operations $\wedge$ or $\vee$ such that the natural band partial order $\leq_{\mathcal{H}}$ coincides with or dualises the underlying BCK-algebra partial order $\leq^{\langle \dot{-},0\rangle}$. Let $\wedge \in \mathcal{C}$ or $\vee \in \mathcal{C}$. For each choice of $\mathcal{C}$, we show that the class $\mathsf{BP}_{\mathcal{C}}$ is a variety. It is shown that no non-trivial analogue of the Clifford-McLean theorem exists for each $\mathsf{BP}_{\mathcal{C}}$ and hence that each $\mathsf{BP}_{\mathcal{C}}$ is only trivially regular (in a sense suitably modified from §1.3.15). We also prove that each $\mathsf{BP}_{\mathcal{C}}$ is ideal determined, congruence distributive and (when $\vee \in \mathcal{C}$) congruence permutable.

In §3.3.43 we study the family of classes $\mathsf{IP}_{\mathcal{C}''}$, $\{/, 0\} \subseteq \mathcal{C}'' \subseteq \{\wedge, \vee, /, 0\}$. Members of the classes $\mathsf{IP}_{\mathcal{C}''}$ consist of implicative BCK-algebras (possibly) structurally enriched with band operations $\wedge$ and $\vee$ such that the natural band partial order $\leq_{\mathcal{H}}$ coincides with or dualises the underlying implicative

BCK-algebra partial order $\leq^{\langle /,0 \rangle}$. For each choice of C'', $\wedge \in C''$ or $\vee \in C''$, it is shown that the class $\mathsf{IP}_{C''}$ is a variety. Let $\wedge \in C''$. For each C'' and any $\mathbf{A} \in \mathsf{IP}_{C''}$, it is shown that the band with zero reduct $\langle A;\ \wedge, 0\rangle$ is locally Boolean (in the sense of §1.3.15). In the other main result of the section, the skew Boolean $\cap$-algebras are characterised (to within termwise definitional equivalence) amongst the members of $\mathsf{IP}_{C''}$, $\{\wedge, \vee\} \subseteq C''$. We also present a simple equational axiomatisation of the variety of skew Boolean $\cap$-algebras.

The final two subsections of this section are devoted to the further exploration of the theory of skew Boolean $\cap$-algebras, and hence, by extension, the theory of pointed discriminator varieties. In §3.3.59 the theory of skew Boolean $\cap$-algebras as presented in §1.4.32, §1.4.37 is extended to double-pointed skew Boolean $\cap$-algebras. It is shown that the class of double-pointed skew Boolean $\cap$-algebras is a variety, and the subdirectly irreducible double-pointed skew Boolean $\cap$-algebras are characterised to within isomorphism. It is also shown that the variety of double-pointed left handed skew Boolean $\cap$-algebras coincides with the pure double-pointed discriminator variety, namely the variety of type $\langle 3, 0, 0\rangle$ generated by the class of all double-pointed discriminator algebras. In consequence, we infer that any algebra $\mathbf{A}$ in a double-pointed discriminator variety has a double-pointed left handed skew Boolean $\cap$-algebra polynomial reduct whose congruences coincide with those of $\mathbf{A}$.

In §3.3.69 we present an axiomatisation of a certain deductive system $\mathbb{SBPC}$. We show $\mathbb{SBPC}$ is definitionally equivalent to the assertional logic of the variety of double-pointed left handed skew Boolean $\cap$-algebras, and hence infer that $\mathbb{SBPC}$ is definitionally equivalent to the assertional logic of the pure double-pointed discriminator variety. It is also show that, in principle, there exists an axiomatisation of $\mathbb{SBPC}$ such that (MP) is the only (proper) rule of inference.

Remark 3.3.1. We impose two fundamental restrictions on the scope of our study of the varieties $\mathsf{BP}_{\mathcal{C}}$ (and hence, by extension, on our study of the varieties $\mathsf{PQ}_{\mathcal{C}}$) in this section. First, our study of the varieties $\mathsf{BP}_{\mathcal{C}}$ does *not* extend to a study of those members of $\mathsf{BP}_{\mathcal{C}}$ for which the BCK-algebra reduct has condition (S) (recall the definition of condition (S) from §2.1.1). Although BCK-[semi]lattices with condition (S) were considered by Idziak in his original paper [116] on BCK-[semi]lattices, a study of those members of the varieties $\mathsf{BP}_{\mathcal{C}}$ for which the BCK-algebra reduct has condition (S) contravenes (at least in spirit) the restrictions of Remark 1.1.2, because BCK-algebras with condition (S) are precisely the $\langle \dot{-}, 0\rangle$-*reducts* of pocrims (by results due to Iséki [121, 123]). Second, our study of the varieties $\mathsf{BP}_{\mathcal{C}}$ does *not* extend to a study of the assertional logics $\mathbb{S}(\mathsf{BP}_{\mathcal{C}}, \mathbf{0})$ (with the obvious exception of the assertional logic of the variety of double-pointed left handed skew Boolean $\cap$-algebras), because the deductive systems $\mathbb{S}(\mathsf{BP}_{\mathcal{C}}, \mathbf{0})$ are, by remarks due to Restall [199], not in general amenable to standard logical analysis (for example, in the sense of [198]): *cf.* Remark 3.3.80 and the remarks of §4.2.27 in the sequel. By extension, the preceding restrictions apply *mutatis mutandis* to our study of the varieties $\mathsf{PQ}_{\mathcal{C}}$, to better enable the uniform development of the theory of the varieties $\mathsf{BP}_{\mathcal{C}}$ and $\mathsf{PQ}_{\mathcal{C}}$. ∎

3.3.2. Pre-BCK Bands and Pre-BCK Quasilattices. A *lower pre-BCK-band* is an algebra $\langle A;\ \wedge, \dot{-}, 0\rangle$ of type $\langle 2,2,0\rangle$ such that: (i) the reduct $\langle A;\ \wedge, 0\rangle$ is a band with zero; (ii) the reduct $\langle A;\ \dot{-}, 0\rangle$ is a pre-BCK-algebra; and (iii) the natural band quasiorder $\preceq_{\mathcal{D}}$ coincides with the pre-BCK quasiorder $\preceq^{\langle A;\ \dot{-}, 0\rangle}$. Clearly a lower pre-BCK band $\mathbf{A}$ is a lower BCK-semilattice iff either $\mathbf{A} \models$ (2.5) or $\mathbf{A} \models x \wedge y \approx y \wedge x$.

Theorem 3.3.3. *An algebra $\langle A;\ \wedge, \dot{-}, 0\rangle$ of type $\langle 2,2,0\rangle$ is a lower pre-BCK-band iff the following identities are satisfied:*

$$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z \tag{3.49}$$

$$x \wedge x \approx x \tag{3.50}$$

$$((x \dot{-} y) \dot{-} (x \dot{-} z)) \dot{-} (z \dot{-} y) \approx \mathbf{0} \tag{3.51}$$

$$x \dot{-} \mathbf{0} \approx x \tag{3.52}$$

$$\mathbf{0} \dot{-} x \approx \mathbf{0} \tag{3.53}$$

$$(x \wedge y \wedge x) \dot{-} y \approx 0 \tag{3.54}$$

$$\big(x \dot{-} (x \dot{-} y)\big) \wedge y \wedge \big(x \dot{-} (x \dot{-} y)\big) \approx x \dot{-} (x \dot{-} y). \tag{3.55}$$

Thus the class of lower pre-BCK bands is a variety.

Proof. Let $\mathbf{A} := \langle A;\ \wedge, \dot{-}, 0\rangle$ be an algebra of type $\langle 2,2,0\rangle$ satisfying (3.49)–(3.53). To prove the theorem it is sufficient to show:

(i) $\mathbf{A} \models$ (3.54)–(3.55) iff $\preceq^{\langle A;\ \dot{-},0\rangle}$ and $\preceq_{\mathcal{D}}$ coincide;
(ii) $\mathbf{A} \models$ (3.54)–(3.55) implies $\langle A;\ \wedge, 0\rangle$ is a band with zero.

For (i), let $a, b \in A$ and suppose $\mathbf{A} \models$ (3.54)–(3.55). Suppose $a \preceq^{\langle A;\ \dot{-},0\rangle} b$. Then $a = a \dot{-} 0$ (by (3.52)) $= a \dot{-} (a \dot{-} b) = \big(a \dot{-} (a \dot{-} b)\big) \wedge b \wedge \big(a \dot{-} (a \dot{-} b)\big)$ (by (3.55)) $= (a \dot{-} 0) \wedge b \wedge (a \dot{-} 0) = a \wedge b \wedge a$ (by (3.52)). Hence $a \preceq_{\mathcal{D}} b$. For the opposite implication, assume $a \preceq_{\mathcal{D}} b$. Then $a \wedge b \wedge a = a$, whence $a \dot{-} b = (a \wedge b \wedge a) \dot{-} b = 0$ by (3.54). Hence $a \preceq^{\langle A;\ \dot{-},0\rangle} b$ and thus $a \preceq_{\mathcal{D}} b$ iff $a \preceq^{\langle A;\ \dot{-},0\rangle} b$. Conversely, suppose the quasiorders $\preceq_{\mathcal{D}}$ and $\preceq^{\langle A;\ \dot{-},0\rangle}$ coincide. From $a \dot{-} (a \dot{-} b) \preceq^{\langle A;\ \dot{-},0\rangle} b$ we have $a \dot{-} (a \dot{-} b) \preceq_{\mathcal{D}} b$, which implies $\big(a \dot{-} (a \dot{-} b)\big) \wedge b \wedge \big(a \dot{-} (a \dot{-} b)\big) = a \dot{-} (a \dot{-} b)$. Also, from $a \wedge b \wedge a \preceq_{\mathcal{D}} b$ we have $a \wedge b \wedge a \preceq^{\langle A;\ \dot{-},0\rangle} b$ whence $(a \wedge b \wedge a) \dot{-} b = 0$. Thus $\mathbf{A} \models$ (3.54)–(3.55).

For (ii), suppose $\mathbf{A} \models$ (3.54)–(3.55). We have $b \wedge 0 \wedge b = (b \wedge 0 \wedge b) \dot{-} 0 = 0$ for any $b \in A$ by (3.52) and (3.54). Thus $0 = (a \wedge 0) \wedge 0 \wedge (a \wedge 0) = (a \wedge 0) \wedge (a \wedge 0) = a \wedge 0$ for any $a \in A$. But then $0 = a \wedge 0 \wedge a = 0 \wedge a$, so the reduct $\langle A;\ \wedge, 0\rangle$ is a band with zero. ∎

An *upper pre-BCK-band* is an algebra $\langle A;\ \vee, \dot{-}, 0\rangle$ of type $\langle 2,2,0\rangle$ such that: (i) the reduct $\langle A;\ \vee, 0\rangle$ is a band with identity; (ii) the reduct $\langle A;\ \dot{-}, 0\rangle$ is a pre-BCK-algebra; and (iii) the natural band quasiorder $\preceq_{\mathcal{D}}$ dualises the pre-BCK quasiorder $\preceq^{\langle A;\ \dot{-},0\rangle}$ in the sense that $a \preceq_{\mathcal{D}} b$ iff $b \preceq^{\langle A;\ \dot{-},0\rangle} a$. Clearly an upper pre-BCK band $\mathbf{A}$ is an upper BCK-semilattice iff either $\mathbf{A} \models$ (2.5) or $\mathbf{A} \models x \vee y \approx y \vee x$. The proof of the following theorem is similar to the proof of Theorem 3.3.3 and is omitted.

Theorem 3.3.4. *An algebra* $\langle A;\ \wedge, \dot{-}, 0\rangle$ *of type* $\langle 2,2,0\rangle$ *is an upper pre-BCK-band iff the following identities are satisfied:*

$$x \vee (y \vee z) \approx (x \vee y) \vee z \tag{3.56}$$

$$x \vee x \approx x \tag{3.57}$$

$$x \vee \mathbf{0} \approx x \tag{3.58}$$

$$\mathbf{0} \vee x \approx x \tag{3.59}$$

$$\big((x \dot{-} y) \dot{-} (x \dot{-} z)\big) \dot{-} (z \dot{-} y) \approx \mathbf{0} \tag{3.60}$$

$$x \dot{-} \mathbf{0} \approx x \tag{3.61}$$

$$\mathbf{0} \dot{-} x \approx \mathbf{0} \tag{3.62}$$

$$x \vee \big(y \dot{-} (y \dot{-} x)\big) \vee x \approx x \tag{3.63}$$

$$x \dot{-} (y \vee x \vee y) \approx \mathbf{0}. \tag{3.64}$$

Thus the class of upper pre-BCK-bands is a variety.

Remark 3.3.5. The identities (3.58)–(3.59) cannot be omitted from the axiomatisation of the variety of upper pre-BCK bands given in Theorem 3.3.4. To see this, let $A := \{0,1,2\}$, let $\mathbf{A} := \langle A;\ \vee^{\mathbf{A}}, \dot{-}^{\mathbf{A}}, 0^{\mathbf{A}}\rangle$ be the algebra with distinguished element 0 and whose binary operations $\vee^{\mathbf{A}}$ and $\dot{-}^{\mathbf{A}}$ are determined by the following operation tables:

$\vee^{\mathbf{A}}$	0	1	2
0	0	1	2
1	1	1	2
2	1	1	2

$\dot{-}^{\mathbf{A}}$	0	1	2
0	0	0	0
1	1	0	0
2	2	0	0

and let $\mathbf{A}' := \langle A;\ \vee^{\mathbf{A}'}, \dot{-}^{\mathbf{A}'}, \mathbf{0}^{\mathbf{A}'}\rangle$ be the algebra of type $\langle 2,2,0\rangle$ obtained from $\mathbf{A}$ upon defining $\mathbf{0}^{\mathbf{A}'} := \mathbf{0}^{\mathbf{A}}$, $a \dot{-}^{\mathbf{A}'} b := a \dot{-}^{\mathbf{A}} b$ and $a \vee^{\mathbf{A}'} b := b \vee^{\mathbf{A}} a$ for any $a, b \in A$. An easy sequence of checks shows $\mathbf{A} \models$ (3.56)–(3.57), (3.59)–(3.64). However $\mathbf{A} \not\models$ (3.58) since $2 \vee 0 = 1 \neq 2$. From these remarks and the definition of $\mathbf{A}'$ it follows at once that $\mathbf{A}' \models$ (3.56)–(3.58), (3.60)–(3.64) but $\mathbf{A} \not\models$ (3.59). ∎

A *quasilattice with zero* is a quasilattice $\langle A;\ \wedge, \vee\rangle$ for which there exists $0 \in A$ (the *zero* of $\langle A;\ \wedge, \vee\rangle$) such that both $0 \preceq_{\mathcal{H}}^{\langle A;\,\wedge\rangle} a$ and $a \preceq_{\mathcal{H}}^{\langle A;\,\vee\rangle} 0$ for all $a \in A$; by abuse of language and notation we often identify a quasilattice with zero $\mathbf{A} := \langle A;\ \wedge, \vee\rangle$ with the algebra $\langle A;\ \wedge, \vee, 0\rangle$ obtained from $\mathbf{A}$ upon augmenting the language of $\mathbf{A}$ with a new nullary operation symbol $\mathbf{0}$ whose canonical interpretation on $\langle A;\ \wedge, \vee, 0\rangle$ is the zero element $0 \in A$. A *pre-BCK quasilattice* is an algebra $\langle A;\ \wedge, \vee, \dot{-}, 0\rangle$ of type $\langle 2, 2, 2, 0\rangle$ such that: (i) the reduct $\langle A;\ \wedge, \vee, 0\rangle$ is a quasilattice with zero; (ii) the reduct $\langle A;\ \dot{-}, 0\rangle$ is a pre-BCK-algebra; and (iii) the natural quasilattice quasiorder $\preceq_{\mathcal{D}}$ coincides with the pre-BCK quasiorder $\preceq^{\langle A;\ \dot{-}, 0\rangle}$. Clearly a pre-BCK quasilattice $\mathbf{A}$ is a BCK-lattice iff either $\mathbf{A} \models$ (2.5), $\mathbf{A} \models x \wedge y \approx y \wedge x$ or $\mathbf{A} \models x \vee y \approx y \vee x$. From Theorem 3.3.3 and Theorem 3.3.4 the following result is clear.

Theorem 3.3.6. *An algebra* $\mathbf{A} := \langle A;\ \wedge, \vee, \dot{-}, 0\rangle$ *of type* $\langle 2, 2, 2, 0\rangle$ *is a pre-BCK quasilattice iff the reduct* $\langle A;\ \wedge, \dot{-}, 0\rangle$ *is a lower pre-BCK band and the reduct* $\langle A;\ \vee, \dot{-}, 0\rangle$ *is an upper pre-BCK band. Hence* $\mathbf{A}$ *is a pre-BCK quasilattice iff* $\mathbf{A} \models$ (3.49)–(3.64). *Therefore the class of pre-BCK quasilattices is a variety.*

Let $\mathcal{C}$ denote an arbitrary subset of the language $\{\wedge, \vee, \dot{-}, \mathbf{0}\}$ of pre-BCK quasilattices that contains both $\dot{-}$ and $\mathbf{0}$. Let:

- $\mathsf{PQ}_{\mathcal{C}}$ denote the variety of pre-BCK-algebras when $\mathcal{C} = \{\dot{-}, \mathbf{0}\}$;
- $\mathsf{PQ}_{\mathcal{C}}$ denote the variety of lower pre-BCK bands when $\mathcal{C} = \{\wedge, \dot{-}, \mathbf{0}\}$;
- $\mathsf{PQ}_{\mathcal{C}}$ denote the variety of upper pre-BCK bands when $\mathcal{C} = \{\vee, \dot{-}, \mathbf{0}\}$;
- $\mathsf{PQ}_{\mathcal{C}}$ denote the variety of pre-BCK quasilattices when $\mathcal{C} = \{\wedge, \vee, \dot{-}, \mathbf{0}\}$.

Notice that for each $\mathcal{C}$, $\mathsf{PQ}_{\mathcal{C}}$ is the class of algebras with language $\mathcal{C}$ axiomatised by those identities among (3.49)–(3.64) that use only operation symbols from $\mathcal{C}$; of course, this observation is dependent upon the axiomatisation of the variety of pre-BCK quasilattices given in Theorem 3.3.6.

Remark 3.3.7. Let $\mathbf{A}$ be an algebra of type $\mathcal{L}$. For $\mathcal{L}' \subseteq \mathcal{L}$, by an $\mathcal{L}'$*-subreduct* of $\mathbf{A}$ we mean a subalgebra of the reduct $\mathbf{A}|_{\mathcal{L}'} := \langle A;\ f^{\mathbf{A}}\rangle_{f\in\mathcal{L}'}$. For a class K of algebras, let $\mathcal{L}'$-K denote the class of all $\mathcal{L}'$-subreducts of members of K. For each $\mathcal{C}$, $\mathsf{PQ}_{\mathcal{C}}$ should not be confused with $\mathcal{C}$-PQ. In particular, in contrast to each $\mathsf{PQ}_{\mathcal{C}}$, we do not know in general if each $\mathcal{C}$-PQ is even a quasivariety. ∎

Throughout the remainder of this subsection we assume $\wedge \in \mathcal{C}$ or $\vee \in \mathcal{C}$. Recall from §1.4.3 that quasilattices satisfy the following modified form of the Clifford-McLean theorem: every quasilattice is a lattice of its maximal rectangular subalgebras. Because any member of any $\mathsf{PQ}_{\mathcal{C}}$ possesses a coherent $\mathcal{D}$-quasiordering by definition, it is natural to anticipate that the Clifford-McLean theorem for quasilattices extends to the varieties $\mathsf{PQ}_{\mathcal{C}}$.

Theorem 3.3.8 (Clifford-McLean Theorem for $\mathsf{PQ}_{\mathcal{C}}$). *Let $\mathbf{A} \in \mathsf{PQ}_{\mathcal{C}}$. For any $a, b \in A$ the following are equivalent:*

1. $a\mathcal{D}b$;

2. $a \Xi b$.

Thus the following assertions hold:

1. *If $\wedge \in \mathcal{C}$ then $\mathcal{D}$-equivalence is a congruence relation on $\langle A;\ \wedge, \dot{-}\, 0\rangle$. The $\mathcal{D}$-equivalence classes are the maximal rectangular subalgebras of $\langle A;\ \wedge, 0\rangle$, while the quotient algebra $\langle A;\ \wedge, \dot{-}, 0\rangle/\mathcal{D}$ is the maximal lower BCK-semilattice homomorphic image of $\langle A;\ \wedge, \dot{-}, 0\rangle$;*

2. *If $\vee \in \mathcal{C}$ then $\mathcal{D}$-equivalence is a congruence relation on $\langle A;\ \vee, \dot{-}, 0\rangle$. The $\mathcal{D}$-equivalence classes are the maximal rectangular subalgebras of $\langle A;\ \vee, 0\rangle$, while the quotient algebra $\langle A;\ \vee, \dot{-}, 0\rangle/\mathcal{D}$ is the maximal upper BCK-semilattice homomorphic image of $\langle A;\ \vee, \dot{-}, 0\rangle$;*

3. *If $\{\wedge, \vee\} \subseteq \mathcal{C}$ then $\mathcal{D}$-equivalence is a congruence relation on $\mathbf{A}$. The $\mathcal{D}$-equivalence classes are the maximal rectangular subalgebras of $\langle A;\ \wedge, \vee, 0\rangle$, while the quotient algebra $\mathbf{A}/\mathcal{D}$ is the maximal BCK-lattice homomorphic image of $\mathbf{A}$.*

Proof. Let $\mathbf{A} \in \mathsf{PQ}_{\mathcal{C}}$. For the first assertion of the theorem, just note $a\mathcal{D}b$ iff $a \preceq_{\mathcal{D}} b, b \preceq_{\mathcal{D}} a$ iff $a \preceq^{\langle A;\ \dot{-},0\rangle} b, b \preceq^{\langle A;\ \dot{-},0\rangle} a$ iff $a \Xi b$. We prove the remaining assertions of the theorem only in the case that $\{\wedge, \vee\} \subseteq \mathcal{C}$; the proofs in the other cases are not essentially different and are omitted. So suppose $\{\wedge, \vee\} \subseteq \mathcal{C}$. Since Ξ is a congruence on $\langle A;\ \dot{-}, 0\rangle$ and $\mathcal{D}$ is a congruence on $\langle A;\ \wedge, \vee, 0\rangle$, it follows from the first assertion of the theorem that $\mathcal{D}$ is a congruence on $\mathbf{A}$. By the Clifford-McLean theorem for quasilattices and Theorem 2.1.14, we deduce that $\mathbf{A}/\mathcal{D}$ is the maximal BCK-lattice homomorphic image of $\mathbf{A}$ and that the $\mathcal{D}$-equivalence classes are the maximal rectangular subalgebras of $\langle A;\ \wedge, \vee, 0\rangle$. This establishes the theorem in the case that $\{\wedge, \vee\} \subseteq \mathcal{C}$, so the proof is complete. ∎

Let $\mathbf{A}$ be a non-commutative lattice. In general, the four Green's equivalences $\mathcal{L}_{\langle A;\ \wedge\rangle}$, $\mathcal{R}_{\langle A;\ \wedge\rangle}$, $\mathcal{L}_{\langle A;\ \vee\rangle}$ and $\mathcal{R}_{\langle A;\ \vee\rangle}$ on A need not be full congruences on $\mathbf{A}$, even if $\mathbf{A}$ is a quasilattice (compare this situation to that of skew lattices—recall Theorem 1.4.11). $\mathbf{A}$ is said to be *regular* if all of $\mathcal{L}_{\langle A;\ \wedge\rangle}$, $\mathcal{R}_{\langle A;\ \wedge\rangle}$, $\mathcal{L}_{\langle A;\ \vee\rangle}$ and $\mathcal{R}_{\langle A;\ \vee\rangle}$ are congruences on $\mathbf{A}$; notice this definition is consistent with both §1.3.11 and Theorem 1.4.11. Necessary and sufficient conditions for the regularity of non-commutative lattices have been studied extensively by Laslo and Leech in [145, Section 4]. In more detail, recall from non-commutative lattice theory that a non-commutative lattice is *one-sided* if any one of the following pairs of identities is satisfied [145, Section 4]:

$$x \wedge y \wedge x \approx x \wedge y \quad \text{and} \quad x \vee y \vee x \approx x \vee y \tag{l, l}$$

$$x \wedge y \wedge x \approx x \wedge y \quad \text{and} \quad x \vee y \vee x \approx y \vee x \tag{l, r}$$

$$x \wedge y \wedge x \approx y \wedge x \quad \text{and} \quad x \vee y \vee x \approx x \vee y \tag{r, l}$$

$$x \wedge y \wedge x \approx y \wedge x \quad \text{and} \quad x \vee y \vee x \approx y \vee x. \tag{r, r}$$

A non-commutative lattice is *two-sided* if it is not one-sided. By Laslo and Leech [145, Theorem 20] one-sided quasilattices are regular. In turn, the class of all regular quasilattices is a variety [145, Theorem 19] that is generated by the class of all one-sided quasilattices [145, Theorem 20].

Remark 3.3.9. One-sided non-commutative lattices were introduced by Laslo and Leech in [145, Section 4] under the name *flat* non-commutative lattices, in conformance with standard semigroup terminology. From the perspective of algebraic logic, however, the use of the adjective 'flat' to describe one-sided non-commutative lattices is unfortunate, for it conflicts with the established meaning of the term 'flat' as employed in domain theory. In particular, it conflicts with the term 'flat' as used in this thesis to describe algebras that are flat posets (and hence flat domains) with respect to some underlying partial ordering. We adopt alternative terminology here for this reason. ∎

Let $\mathbf{A} \in \mathsf{PQ}_{\mathcal{C}}$. By analogy with non-commutative lattice theory, for $\wedge \in \mathcal{C}$ [$\vee \in \mathcal{C}$; $\{\wedge, \vee\} \subseteq \mathcal{C}$] we say $\langle A;\ \wedge, \dot{-}, 0\rangle$ [$\langle A;\ \vee, \dot{-}, 0\rangle$; $\langle A;\ \wedge, \vee, \dot{-}, 0\rangle$] is *regular* if $\mathcal{L}_{\langle A;\, \wedge\rangle}$ and $\mathcal{R}_{\langle A;\, \wedge\rangle}$ [$\mathcal{L}_{\langle A;\, \vee\rangle}$, $\mathcal{R}_{\langle A;\, \vee\rangle}$; $\mathcal{L}_{\langle A;\, \wedge\rangle}$, $\mathcal{R}_{\langle A;\, \wedge\rangle}$, $\mathcal{L}_{\langle A;\, \vee\rangle}$, $\mathcal{R}_{\langle A;\, \vee\rangle}$] are congruences on $\langle A;\ \wedge, \dot{-}, 0\rangle$ [$\langle A;\ \vee, \dot{-}, 0\rangle$; $\langle A;\ \wedge, \vee, \dot{-}, 0\rangle$]. For $\mathcal{C} = \{\wedge, \dot{-}, \mathbf{0}\}$ [$\mathcal{C} = \{\vee, \dot{-}, \mathbf{0}\}$] we say $\mathbf{A}$ is *one-sided* if its band reduct $\langle A;\ \wedge\rangle$ [$\langle A;\ \vee\rangle$] is either left regular or right regular (recall §1.3.11). We say $\mathbf{A}$ is *two-sided* if it is not one-sided. For $\mathcal{C} = \{\wedge, \vee, \dot{-}, \mathbf{0}\}$ we say $\mathbf{A}$ is *one-sided* if its quasilattice reduct $\langle A;\ \wedge, \vee\rangle$ is one-sided; $\mathbf{A}$ is *two-sided* if it is not one-sided.

Proposition 3.3.10. *Let $\mathbf{A} \in \mathsf{PQ}_{\mathcal{C}}$. If $\mathbf{A}$ is one-sided then $\mathbf{A}$ is regular.*

Proof. We prove the proposition only in the case that $\{\wedge, \vee\} \subseteq \mathcal{C}$; the proofs in the other cases are not essentially different and are omitted. Let $\mathbf{A} \in \mathsf{PQ}_{\mathcal{C}}$. By the Clifford-McLean theorem for $\mathsf{PQ}_{\mathcal{C}}$, the $\mathcal{D}_{\mathbf{A}}$-classes of $\mathbf{A}$ form maximal rectangular bands in both $\langle A;\ \wedge\rangle$ and $\langle A;\ \vee\rangle$. Since the identities (l, l)–(r, r) are respectively equivalent to asserting that:

$a \wedge b = a$ [$a \vee b = a$] in each $\wedge$-rectangular [$\vee$-rectangular] class;
$a \wedge b = b$ [$a \vee b = a$] in each $\wedge$-rectangular [$\vee$-rectangular] class;
$a \wedge b = a$ [$a \vee b = b$] in each $\wedge$-rectangular [$\vee$-rectangular] class;
$a \wedge b = b$ [$a \vee b = b$] in each $\wedge$-rectangular [$\vee$-rectangular] class

it follows that each of $\mathcal{L}_{\langle A;\, \wedge\rangle}$, $\mathcal{R}_{\langle A;\, \wedge\rangle}$, $\mathcal{L}_{\langle A;\, \vee\rangle}$ and $\mathcal{R}_{\langle A;\, \vee\rangle}$ must be either $\mathcal{D}_{\mathbf{A}}$ or $\omega_{\mathbf{A}}$. Hence all of $\mathcal{L}_{\langle A;\, \wedge\rangle}$, $\mathcal{R}_{\langle A;\, \wedge\rangle}$, $\mathcal{L}_{\langle A;\, \vee\rangle}$ and $\mathcal{R}_{\langle A;\, \vee\rangle}$ are congruences on $\mathbf{A}$, so $\mathbf{A}$ is regular. ∎

Let $\mathbf{A} \in \mathsf{PQ}_{\mathcal{C}}$. In general (that is, when $\mathbf{A}$ is two-sided), the equivalences $\mathcal{L}_{\langle A;\, \wedge\rangle}$, $\mathcal{R}_{\langle A;\, \wedge\rangle}$, $\mathcal{L}_{\langle A;\, \vee\rangle}$ and $\mathcal{R}_{\langle A;\, \vee\rangle}$ (if they exist) need not be full congruences on **A**. This is so even if **A** is a pre-BCK quasilattice whose quasilattice with zero reduct $\langle A;\ \wedge, \vee, 0\rangle$ is a skew lattice. To see this, consider the following 5-element algebra **A**:

$\wedge^{\mathbf{A}}$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	b	c	d
b	0	b	b	b	b
c	0	c	c	c	c
d	0	a	b	c	d

$\vee^{\mathbf{A}}$	0	a	b	c	d
0	0	a	b	c	d
a	a	a	a	a	a
b	b	a	b	c	d
c	c	a	b	c	d
d	d	d	d	d	d

$\dot{-}^{\mathbf{A}}$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	d	0
b	b	0	0	0	0
c	c	0	0	0	0
d	d	0	a	a	0

An easy sequence of checks establishes that **A** is a pre-BCK quasilattice for which the quasilattice with zero reduct $\langle A;\ \wedge, \vee, 0\rangle$ is a skew lattice with zero. Moreover, $a \equiv a \,(\mathrm{mod}\, \mathcal{L}_{\langle A;\, \wedge,\vee\rangle})$ and $b \equiv c \,(\mathrm{mod}\, \mathcal{L}_{\langle A;\, \wedge,\vee\rangle})$ (since $\mathcal{L}_{\langle A;\, \wedge\rangle} = \mathcal{R}_{\langle A;\, \vee\rangle}$ and $\mathcal{R}_{\langle A;\, \wedge\rangle} = \mathcal{L}_{\langle A;\, \vee\rangle}$—recall §1.4.5). However, $(a \dot{-} b) \wedge (a \dot{-} c) = a \wedge d = d \neq a = a \dot{-} b$, so $\mathcal{L}_{\langle A;\, \wedge,\vee\rangle}$ is not a congruence on **A**.

For each $\mathcal{C}$, the following proposition provides a sufficient condition for $\mathbf{A} \in \mathsf{PQ}_{\mathcal{C}}$ to be regular. Theorem 3.3.21 below shows the hypotheses of the proposition are not artificial; indeed, among natural syntactic conditions on members of $\mathsf{PQ}_{\mathcal{C}}$ implying regularity, the assumptions of the proposition are the most general known to us.

Proposition 3.3.11. *Let $\mathbf{A} \in \mathsf{PQ}_{\mathcal{C}}$. The following assertions hold:*

1. *If $\wedge \in \mathcal{C}$, $\langle A;\ \wedge\rangle$ is regular and* **A** *satisfies:*

$$(x \dot{-} z) \wedge (y \dot{-} z) \approx (x \wedge y) \dot{-} z \tag{3.65}$$

$$(z \dot{-} x) \wedge (z \dot{-} y) \approx ((z \dot{-} (x \wedge y)) \dot{-} (x \dot{-} y)) \dot{-} (y \dot{-} x) \tag{3.66}$$

   then $\langle A;\ \wedge, \dot{-}, 0\rangle$ is regular. That is, $\mathcal{L}_{\langle A;\, \wedge\rangle}$ and $\mathcal{R}_{\langle A;\, \wedge\rangle}$ are congruences on $\langle A;\ \wedge, \dot{-}, 0\rangle$;

2. *If $\vee \in \mathcal{C}$, $\langle A;\ \vee\rangle$ is regular and* **A** *satisfies:*

$$(x \dot{-} z) \vee (y \dot{-} z) \approx (x \vee y) \dot{-} z \tag{3.67}$$

$$(z \dot{-} x) \vee (z \dot{-} y) \approx z \dot{-} (((x \vee y) \dot{-} (x \dot{-} y)) \dot{-} (y \dot{-} x)) \tag{3.68}$$

then $\langle A;\ \vee, \dot{-}, 0\rangle$ *is regular. That is,* $\mathcal{L}_{\langle A;\, \vee\rangle}$ *and* $\mathcal{R}_{\langle A;\, \vee\rangle}$ *are congruences on* $\langle A;\ \vee, \dot{-}, 0\rangle$*;*

3. *If* $\{\wedge, \vee\} \subseteq \mathcal{C}$*,* $\langle A;\ \wedge, \vee, 0\rangle$ *is regular (in particular, if* $\langle A;\ \wedge, \vee, 0\rangle$ *is a skew lattice with zero) and* $\mathbf{A}$ *satisfies* (3.65)–(3.68)*, then* $\mathbf{A}$ *is regular. That is,* $\mathcal{L}_{\langle A;\, \wedge\rangle}$*,* $\mathcal{R}_{\langle A;\, \wedge\rangle}$*,* $\mathcal{L}_{\langle A;\, \vee\rangle}$ *and* $\mathcal{R}_{\langle A;\, \vee\rangle}$ *are congruences on* $\mathbf{A}$*.*

Proof. Let $\mathbf{A} \in \mathsf{PQ}_{\mathcal{C}}$ and let $a, b, c \in A$. For (1), suppose $\wedge \in \mathcal{C}$, $\langle A;\ \wedge\rangle$ is regular and that $\mathbf{A} \models$ (3.65), (3.66). We show only that $\mathcal{L}_{\langle A;\, \wedge\rangle}$ is a congruence on $\langle A;\ \wedge, \dot{-}, 0\rangle$; the proof that $\mathcal{R}_{\langle A;\, \wedge\rangle}$ is a congruence on $\langle A;\ \wedge, \dot{-}, 0\rangle$ is similar and is omitted. Since $\langle A;\ \wedge\rangle$ is regular, to see $\mathcal{L}_{\langle A;\, \wedge\rangle}$ is a congruence on $\langle A;\ \wedge, \dot{-}, 0\rangle$ it is sufficient to show:

(i) $a \equiv b \,(\mathrm{mod}\, \mathcal{L}_{\langle A;\, \wedge\rangle})$ implies $a \dot{-} c \equiv b \dot{-} c \,(\mathrm{mod}\, \mathcal{L}_{\langle A;\, \wedge\rangle})$;
(ii) $a \equiv b \,(\mathrm{mod}\, \mathcal{L}_{\langle A;\, \wedge\rangle})$ implies $c \dot{-} a \equiv c \dot{-} b \,(\mathrm{mod}\, \mathcal{L}_{\langle A;\, \wedge\rangle})$

since (i) and (ii) together guarantee the substitution property for the $\dot{-}$ operation. So suppose $a \equiv b \,(\mathrm{mod}\, \mathcal{L}_{\langle A;\, \wedge\rangle})$. For (i), we have $(a \dot{-} c) \wedge (b \dot{-} c) = (a \wedge b) \dot{-} c = a \dot{-} c$ by (3.65); likewise $(b \dot{-} c) \wedge (a \dot{-} c) = b \dot{-} c$. For (ii), notice $a \equiv b \,(\mathrm{mod}\, \mathcal{L}_{\langle A;\, \wedge\rangle})$ implies $a \equiv b \,(\mathrm{mod}\, \mathcal{D})$ and hence $a \equiv b \,(\mathrm{mod}\, \Xi)$. By (3.66),

$$\begin{aligned}
(c \dot{-} a) \wedge (c \dot{-} b) &= ((c \dot{-} (a \wedge b)) \dot{-} (a \dot{-} b)) \dot{-} (b \dot{-} a) \\
&= ((c \dot{-} (a \wedge b)) \dot{-} 0) \dot{-} 0 \\
&= c \dot{-} (a \wedge b) && \text{by (3.52)} \\
&= c \dot{-} a.
\end{aligned}$$

Similarly $(c \dot{-} b) \wedge (c \dot{-} a) = c \dot{-} b$. Thus $\mathcal{L}_{\langle A;\, \wedge\rangle}$ has the substitution property for the $\dot{-}$ operation and $\mathcal{L}_{\langle A;\, \wedge\rangle}$ is a congruence on $\langle A;\ \wedge, \dot{-}, 0\rangle$.

For (2), suppose $\vee \in \mathcal{C}$, $\langle A;\ \vee\rangle$ is regular and that $\mathbf{A} \models$ (3.67), (3.68). We show only that $\mathcal{L}_{\langle A;\, \vee\rangle}$ is a congruence on $\langle A;\ \vee, \dot{-}, 0\rangle$; the proof that $\mathcal{R}_{\langle A;\, \vee\rangle}$ is a congruence on $\langle A;\ \vee, \dot{-}, 0\rangle$ is analogous and is omitted. Since $\langle A;\ \vee\rangle$ is regular, to see $\mathcal{L}_{\langle A;\, \vee\rangle}$ is a congruence on $\langle A;\ \vee, \dot{-}, 0\rangle$ it is sufficient to show:

(i) $a \equiv b \,(\mathrm{mod}\, \mathcal{L}_{\langle A;\,\vee\rangle})$ implies $a \dot{-} c \equiv b \dot{-} c \,(\mathrm{mod}\, \mathcal{L}_{\langle A;\,\vee\rangle})$;
(ii) $a \equiv b \,(\mathrm{mod}\, \mathcal{L}_{\langle A;\,\vee\rangle})$ implies $c \dot{-} a \equiv c \dot{-} b \,(\mathrm{mod}\, \mathcal{L}_{\langle A;\,\vee\rangle})$

since (i) and (ii) together guarantee the substitution property for the $\dot{-}$ operation. So suppose $a \equiv b \,(\mathrm{mod}\, \mathcal{L}_{\langle A;\,\vee\rangle})$. For (i), we have $(a \dot{-} c) \vee (b \dot{-} c) = (a \vee b) \dot{-} c = a \dot{-} c$ by (3.67); $(b \dot{-} c) \vee (a \dot{-} c) = b \dot{-} c$ likewise. For (ii), notice $a \equiv b \,(\mathrm{mod}\, \mathcal{L}_{\langle A;\,\vee\rangle})$ implies $a \equiv b \,(\mathrm{mod}\, \mathcal{D})$ and hence $a \equiv b \,(\mathrm{mod}\, \Xi)$. By (3.68),

$$\begin{aligned}
(c \dot{-} a) \vee (c \dot{-} b) &= c \dot{-} (((a \vee b) \dot{-} (a \dot{-} b)) \dot{-} (b \dot{-} a)) \\
&= c \dot{-} (((a \vee b) \dot{-} 0) \dot{-} 0) \\
&= c \dot{-} (a \vee b) && \text{by (3.52)} \\
&= c \dot{-} a.
\end{aligned}$$

Similarly $(c \dot{-} b) \vee (c \dot{-} a) = c \dot{-} b$. Thus $\mathcal{L}_{\langle A;\,\vee\rangle}$ has the substitution property for the $\dot{-}$ operation and $\mathcal{L}_{\langle A;\,\vee\rangle}$ is a congruence on $\langle A;\ \vee, \dot{-}, 0\rangle$.

Item (3) now follows as a trivial consequence of (1), (2) and the regularity of $\langle A;\ \wedge, \vee, 0\rangle$. ∎

Problem 3.3.12. For each $\mathcal{C}$, let $\mathsf{K}_{\mathcal{C}}$ denote the class of all members of $\mathsf{PQ}_{\mathcal{C}}$ that are regular. Is $\mathsf{K}_{\mathcal{C}}$ equationally definable? ∎

3.3.13. Implicative BCS Bands and Implicative BCS Quasilattices. By the results of §1.4.32, §1.4.37, §3.2.6 and §3.2.22, algebras arising in binary discriminator, pointed fixedpoint discriminator and pointed ternary discriminator varieties all support an underlying 'locally Boolean' structure (in the sense of either §1.3.15 or §1.4.24), which observation motivates the study of those members of $\mathsf{PQ}_{\mathcal{C}}$ for which every (appropriately defined) principal subalgebra is a Boolean lattice. Because of the results of §2.3.19, these remarks lead naturally to a consideration of those members of $\mathsf{PQ}_{\mathcal{C}}$ for which the pre-BCK-algebra reduct is an implicative BCS-algebra.

Proposition 3.3.14. *Let* $\mathbf{A} := \langle A;\ \wedge, \backslash, 0\rangle$ *be an algebra of type* $\langle 2, 2, 0\rangle$ *such that the reduct* $\langle A;\ \wedge, 0\rangle$ *is a band with zero and the reduct* $\langle A;\ \backslash, 0\rangle$ *is an implicative BCS-algebra. The following are equivalent:*

1. *For any* $a, b \in A$,

$$a \leq_{\mathcal{H}} b \textit{ iff } a \leq^{\langle A;\,\backslash,0\rangle} b \quad \textit{and} \quad a \preceq_{\mathcal{D}} b \textit{ iff } a \preceq^{\langle A;\,\backslash,0\rangle} b;$$

2. $\mathbf{A}$ *satisfies the identity:*

$$x \wedge y \wedge x \approx x \sqcap y. \tag{3.69}$$

Proof. Let $\mathbf{A} := \langle A;\ \wedge, \backslash, 0\rangle$ be an algebra of type $\langle 2,2,0\rangle$ such that the reduct $\langle A;\ \wedge, 0\rangle$ is a band with zero and the reduct $\langle A;\ \backslash, 0\rangle$ is an implicative BCS-algebra.

(1) $\Rightarrow$ (2) To prove the implication, we first observe that for any $a, b \in A$,

(i) $a \wedge b \wedge a \equiv a \sqcap b \,(\operatorname{mod} \Xi)$;
(ii) $a\backslash\big(a\backslash(a \wedge b \wedge a)\big) = a \wedge b \wedge a$.

For (i), $a \sqcap b$ is a greatest lower bound of $\{a, b\}$ with respect to $\preceq^{\langle A;\,\backslash,0\rangle}$ by Proposition 2.3.5, Proposition 2.2.30 and Proposition 2.2.11. Also, $a \wedge b \wedge a$ is a greatest lower bound of $\{a, b\}$ with respect to $\preceq^{\langle A;\,\backslash,0\rangle}$ since it is a greatest lower bound of $\{a, b\}$ with respect to $\preceq_{\mathcal{D}}$. Therefore $a \wedge b \wedge a \equiv a \sqcap b \,(\operatorname{mod} \Xi)$ by Lemma 1.2.3(2).

For (ii), just note that $a \wedge b \wedge a \leq_{\mathcal{H}} a$ implies $a \wedge b \wedge a \leq^{\langle A;\,\backslash,0\rangle} a$, whence $a\backslash\big(a\backslash(a \wedge b \wedge a)\big) = a \wedge b \wedge a$.

To complete the proof of the implication, let $a, b \in A$ and observe that $a \wedge b \wedge a, a \sqcap b \in (a]_{\langle A;\,\backslash,0\rangle}$. Because of Proposition 2.3.31, this implies the equivalence $a \wedge b \wedge a \equiv a \sqcap b \,(\operatorname{mod} \Xi)$ of (i) collapses in $(a]_{\langle A;\,\backslash,0\rangle}$ to the equality $a \wedge b \wedge a = a \sqcap b$. Hence $a\backslash b = (a \sqcap b)^{*}_{(a]_{\langle A;\,\backslash,0\rangle}} = (a \wedge b \wedge a)^{*}_{(a]_{\langle A;\,\backslash,0\rangle}} = \big(a \sqcap (a \wedge b \wedge a)\big)^{*}_{(a]_{\langle A;\,\backslash,0\rangle}}$ (by (ii)) $= a\backslash(a \wedge b \wedge a)$. But then $a \wedge b \wedge a = a\backslash\big(a\backslash(a \wedge b \wedge a)\big)$ (by (ii)) $= a\backslash(a\backslash b)$ as required.

(2) $\Rightarrow$ (1) To see $a \leq^{\langle A;\,\backslash,0\rangle} b$ iff $a \leq_{\mathcal{H}} b$ for any $a, b \in A$, suppose $a \leq^{\langle A;\,\backslash,0\rangle} b$. Then $b \sqcap a = a$, so $a \wedge b = (b \sqcap a) \wedge b = (b \wedge a \wedge b) \wedge b = b \wedge a \wedge b = b \sqcap a = a$ by (3.69) applied twice. Similarly, $b \wedge a = a$. Therefore $a \leq_{\mathcal{H}} b$. For

the converse, suppose $a \leq_{\mathcal{H}} b$. Then $a \wedge b \wedge a = a$, so $a = a \wedge b \wedge a = a \sqcap b$ by (3.69). Similarly, $a = b \sqcap a$. Therefore $a \leq^{\langle A;\, \backslash, 0\rangle} b$.

To see $a \preceq^{\langle A;\, \backslash, 0\rangle} b$ iff $a \preceq_{\mathcal{D}} b$ for any $a, b \in A$, suppose $a \preceq^{\langle A;\, \backslash, 0\rangle} b$. Then $a \backslash b = 0$, so $a = a \backslash 0 = a \backslash (a \backslash b) = a \wedge b \wedge a$ by (3.69). Hence $a \preceq_{\mathcal{D}} b$. For the converse, suppose $a \preceq_{\mathcal{D}} b$. Then $a \wedge b \wedge a = a$ and so $a = a \wedge b \wedge a = a \sqcap b$ by (3.69). Therefore $a \backslash b = 0$ by Lemma 2.1.42(1). Hence $a \preceq^{\langle A;\, \backslash, 0\rangle} b$. ■

A *lower implicative BCS band* is an algebra $\langle A;\ \wedge, \backslash, 0\rangle$ of type $\langle 2, 2, 0\rangle$ such that: (i) the reduct $\langle A;\ \wedge, 0\rangle$ is a band with zero; (ii) the reduct $\langle A;\ \backslash, 0\rangle$ is an implicative BCS-algebra; (iii) the natural band quasiorder $\preceq_{\mathcal{D}}$ coincides with the implicative BCS-algebra quasiorder $\preceq^{\langle A;\, \backslash, 0\rangle}$; and (iv) the natural band partial order $\leq_{\mathcal{H}}$ coincides with the implicative BCS-algebra partial order $\leq^{\langle A;\, \backslash, 0\rangle}$. From Proposition 3.3.14 the following result is clear.

Theorem 3.3.15. *An algebra* $\mathbf{A} := \langle A;\ \wedge, \backslash, 0\rangle$ *of type* $\langle 2, 2, 0\rangle$ *is a lower implicative BCS band iff the reduct* $\langle A;\ \wedge, 0\rangle$ *is a band with zero, the reduct* $\langle A;\ \backslash, 0\rangle$ *is an implicative BCS-algebra, and* $\mathbf{A} \models (3.69)$. *Thus the class of lower implicative BCS bands is a variety.*

An *upper implicative BCS band* is an algebra $\langle A;\ \wedge, \backslash, 0\rangle$ of type $\langle 2, 2, 0\rangle$ such that: (i) the reduct $\langle A;\ \vee, 0\rangle$ is a band with zero; (ii) the reduct $\langle A;\ \backslash, 0\rangle$ is an implicative BCS-algebra; (iii) the natural band quasiorder $\preceq_{\mathcal{D}}$ dualises the implicative BCS-algebra quasiorder $\preceq^{\langle A;\, \backslash, 0\rangle}$ in the sense that $a \preceq_{\mathcal{D}} b$ iff $b \preceq^{\langle A;\, \dot{-}, 0\rangle} a$ for any $a, b \in A$; and (iv) the natural band partial order $\leq_{\mathcal{H}}$ dualises the implicative BCS-algebra partial order $\leq^{\langle A;\, \backslash, 0\rangle}$ in the sense that $a \preceq_{\mathcal{H}} b$ iff $b \leq^{\langle A;\, \dot{-}, 0\rangle} a$ for any $a, b \in A$. Although we know of no elegant characterisation of upper implicative BCS bands analogous to that of Theorem 3.3.15 in general, we nonetheless have the following result, the proof of which is omitted.

Theorem 3.3.16. *An algebra* $\mathbf{A} := \langle A;\ \vee, \backslash, 0\rangle$ *of type* $\langle 2, 2, 0\rangle$ *is an upper implicative BCS band iff the reduct* $\langle A;\ \vee, 0\rangle$ *is a band with zero, the reduct* $\langle A;\ \backslash, 0\rangle$ *is an implicative BCS-algebra, and* $\mathbf{A}$ *satisfies the following identities:*

$$x \vee \big(y \backslash (y \backslash x)\big) \vee x \approx x \tag{3.70}$$

$$x\backslash(y \vee x \vee y) \approx \mathbf{0} \tag{3.71}$$

$$x\backslash\big(x\backslash x \vee y \vee x)\big) \approx x \tag{3.72}$$

$$(x \vee y \vee x)\backslash\big((x \vee y \vee x)\backslash x\big) \approx x \tag{3.73}$$

$$x \vee (x\backslash y) \approx x \tag{3.74}$$

$$(x\backslash y) \vee x \approx x. \tag{3.75}$$

Thus the class of upper implicative BCS bands is a variety.

Recall from [145, Section 1] that a *fine quasilattice* is a non-commutative lattice that is simultaneously both a quasilattice and a paralattice. By the remarks of §1.4.3 and §1.4.4, the class of all fine quasilattices is a variety. Fine quasilattices naturally generalise skew lattices, inasmuch as several important structural results for skew lattices extend to fine quasilattices: see Laslo and Leech [145, Section 5, pp. 28–29].

An *implicative BCS quasilattice* is an algebra of type $\langle 2, 2, 2, 0\rangle$ such that: (i) the reduct $\langle A;\ \wedge, \vee, 0\rangle$ is a fine quasilattice with zero; (ii) the reduct $\langle A;\ \backslash, 0\rangle$ is an implicative BCS-algebra; (iii) the natural quasilattice quasiorder $\preceq_{\mathcal{D}}$ coincides with the implicative BCS-algebra quasiorder $\preceq^{\langle A;\, \backslash, 0\rangle}$; and (iv) the natural quasilattice partial order $\leq_{\mathcal{H}}$ coincides with the implicative BCS-algebra partial order $\leq^{\langle A;\, \backslash, 0\rangle}$. From Proposition 3.3.14 the following result is clear.

Theorem 3.3.17. *An algebra* $\mathbf{A} := \langle A;\ \wedge, \vee, \backslash, 0\rangle$ *of type* $\langle 2, 2, 0\rangle$ *is an implicative BCS quasilattice iff the reduct* $\langle A;\ \wedge, \vee, 0\rangle$ *is a fine quasilattice with zero, the reduct* $\langle A;\ \backslash, 0\rangle$ *is an implicative BCS-algebra, and* $\mathbf{A} \models$ (3.69). *Thus the class of implicative BCS quasilattices is a variety.*

For consistency with the prequel, let C' denote an arbitrary subset of the language $\{\wedge, \vee, \backslash, \mathbf{0}\}$ of implicative BCS quasilattices that contains all of $\wedge, \backslash$ and $\mathbf{0}$. Ignoring issues of type, let:

- $\mathsf{PQ}_{C'}$ denote the variety of pre-BCK-algebras when $C' = \{\backslash, \mathbf{0}\}$;
- $\mathsf{PQ}_{C'}$ denote the variety of lower pre-BCK bands when $C' = \{\wedge, \backslash, \mathbf{0}\}$;

- $\mathsf{PQ}_{C'}$ denote the variety of upper pre-BCK bands when $C' = \{\wedge, \setminus, 0\}$;
- $\mathsf{PQ}_{C'}$ denote the variety of pre-BCK quasilattices when $C' = \{\wedge, \vee, \setminus, 0\}$.

Also, let:

- $\mathsf{IQ}_{C'}$ denote the variety of implicative BCS-algebras for $C' = \{\setminus, 0\}$;
- $\mathsf{IQ}_{C'}$ denote the variety of lower implicative BCS bands for $C' = \{\wedge, \setminus, 0\}$;
- $\mathsf{IQ}_{C'}$ denote the variety of upper implicative BCS bands for $C' = \{\vee, \setminus, 0\}$;
- $\mathsf{IQ}_{C'}$ denote the variety of implicative BCS quasilattices for $C' = \{\wedge, \vee, \setminus, 0\}$.

Given the above notation, clearly $\mathsf{IQ}_{C'} \subseteq \mathsf{PQ}_{C'}$ for any fixed choice of C'.

In non-commutative lattice theory, there exists a fundamental connection between principal subalgebras of normal bands in semigroup theory and the study of 'locally Boolean' structures: see Leech [150, Section 0.10]. For the varieties $\mathsf{IQ}_{C'}$, $\wedge \in C'$, however, it is the underlying implicative BCS-algebra principal subalgebra structure that is decisive. To see this, let $\mathbf{A} \in \mathsf{PQ}_{C'}$, $\wedge \in C'$, be such that the reduct $\langle A;\ \setminus, 0\rangle$ is an implicative BCS-algebra. Notice that in this case $\mathbf{A}$ has both a band with zero reduct $\langle A;\ \wedge, 0\rangle$ and a left normal band with zero polynomial reduct $\langle A;\ \sqcap, 0\rangle$, where $\langle A;\ \sqcap, 0\rangle$ is determined by $\langle A;\ \setminus, 0\rangle$ as per Corollary 2.3.22(1). From this observation it follows that every $a \in A$ generates both a principal subalgebra $(\mathbf{a}]_{\langle A;\ \wedge,0\rangle}$ of the band with zero reduct $\langle A;\ \wedge, 0\rangle$ (recall Lemma 1.3.13) and a principal subalgebra $(\mathbf{a}]_{\langle A;\ \setminus,0\rangle}$ of the left normal band with zero polynomial reduct $\langle A;\ \sqcap, 0\rangle$ (recall Proposition 2.3.31), whence we have the following proposition.

Proposition 3.3.18. *Let $\wedge \in C$ and let $\mathbf{A} \in \mathsf{PQ}_{C'}$ be such that the reduct $\langle A;\ \setminus, 0\rangle$ of $\mathbf{A}$ is an implicative BCS-algebra. Then $\mathbf{A} \in \mathsf{IQ}_{C'}$ iff the principal subalgebras $(\mathbf{a}]_{\langle A;\ \wedge,0\rangle}$ and $(\mathbf{a}]_{\langle A;\ \setminus,0\rangle}$ coincide for each $a \in A$.*

Proof. Let $\wedge \in C$ and let $\mathbf{A} \in \mathsf{PQ}_{C'}$ be such that the reduct $\langle A;\ \setminus, 0\rangle$ of $\mathbf{A}$ is an implicative BCS-algebra. Throughout the proof, we denote by $(a]_{\langle A;\ \wedge,0\rangle}$ and $(a]_{\langle A;\ \setminus,0\rangle}$ the respective universes of the principal subalgebras $(\mathbf{a}]_{\langle A;\ \wedge,0\rangle}$

and $(\mathbf{a}]_{\langle A;\ \dot{-},0\rangle}$. To simplify notation, we may also write simply $(a]_{\mathbf{A}}$ for both $(a]_{\langle A;\,\wedge,0\rangle}$ and $(a]_{\langle A;\ \dot{-},0\rangle}$ when these sets coincide.

($\Rightarrow$) Suppose $\mathbf{A} \in \mathsf{IQ}_{\mathcal{C}'}$. To see the principal subalgebras $(\mathbf{a}]_{\langle A;\,\wedge,0\rangle}$ and $(\mathbf{a}]_{\langle A;\,\backslash,0\rangle}$ coincide for every $a \in A$, it is sufficient to show:

(i) For any $a \in A$, $(a]_{\langle A;\,\wedge,0\rangle} = (a]_{\langle A;\,\backslash,0\rangle}$;
(ii) If $b, c \in (a]_{\mathbf{A}}$ then $b \wedge c = b \sqcap c$.

For (i), simply observe that for any $a \in A$, $(a]_{\langle A;\,\wedge,0\rangle} = \{b : b \leq_{\mathcal{H}} a\} = \{b : b \leq^{\langle A;\,\backslash,0\rangle} a\} = (a]_{\langle A;\,\backslash,0\rangle}$ because $\leq_{\mathcal{H}}$ and $\leq^{\langle A;\,\backslash,0\rangle}$ coincide.

For (ii), by (i) we have that $(a]_{\langle A;\,\wedge,0\rangle}$ and $(a]_{\langle A;\ \dot{-},0\rangle}$ coincide, so the reference to $(a]_{\mathbf{A}}$ makes sense. Let $b, c \in (a]_{\mathbf{A}}$. From $b \wedge c \equiv b \wedge c \wedge b \pmod{\mathcal{D}}$ we have that $b \wedge c \equiv b \wedge c \wedge b \pmod{\Xi}$. Also, $b \wedge c \wedge b \equiv b \sqcap c \pmod{\Xi}$ by the proof of Theorem 3.3.14. Hence $b \wedge c \equiv b \sqcap c \pmod{\Xi}$. Now for any $a \in A$ we have that the restriction of Ξ to $(a]_{\langle A;\,\backslash,0\rangle}$ is the identity congruence on $(\mathbf{a}]_{\langle A;\,\backslash,0\rangle}$ (by Proposition 2.3.31), whence the equivalence $b \wedge c \equiv b \sqcap c \pmod{\Xi}$ collapses to the equality $b \wedge c = b \sqcap c$ as desired.

($\Leftarrow$) Suppose the principal subalgebras $(\mathbf{a}]_{\langle A;\,\wedge,0\rangle}$ and $(\mathbf{a}]_{\langle A;\,\backslash,0\rangle}$ coincide for each $a \in A$. Since $\mathbf{A} \in \mathsf{PQ}_{\mathcal{C}'}$, the quasiorders $\preceq_{\mathcal{D}}$ and $\preceq^{\langle A;\,\backslash,0\rangle}$ coincide by definition, so to establish the assertion we need only show that the partial orders $\leq_{\mathcal{H}}$ and $\leq^{\langle A;\,\backslash,0\rangle}$ coincide also. For this, just note that for any $b, c \in A$, $b \leq_{\mathcal{H}} c$ iff $(b]_{\langle A;\,\wedge,0\rangle} \subseteq (c]_{\langle A;\,\wedge,0\rangle}$ iff $(b]_{\langle A;\,\backslash,0\rangle} \subseteq (c]_{\langle A;\,\backslash,0\rangle}$ iff $b \leq^{\langle A;\,\backslash,0\rangle} c$. ∎

Because of the proposition, for any $\mathbf{A} \in \mathsf{IQ}_{\mathcal{C}'}$, $\wedge \in \mathcal{C}$, we may unambiguously denote by $(\mathbf{a}]_{\mathbf{A}}$ the principal subalgebra generated by $a \in A$; we follow this convention in the sequel.

Corollary 3.3.19. *For any $\mathbf{A} \in \mathsf{IQ}_{\mathcal{C}'}$, the principal subalgebra $(\mathbf{a}]_{\mathbf{A}}$ generated by $a \in A$ is a Boolean lattice. Consequently the band with zero reduct $\langle A;\ \wedge, 0\rangle$ is normal.*

Proof. Let $\mathbf{A} \in \mathsf{IQ}_{\mathcal{C}'}$. For every $a \in A$, the principal subalgebra $(\mathbf{a}]_{\mathbf{A}}$ generated by a is a Boolean lattice, because of Proposition 3.3.18 and Proposition 2.3.31. Hence $\langle A\cdot\ \wedge, 0\rangle$ is normal (by Lemma 1.3.16), and the proof is complete. ∎

Given the preceding corollary, throughout the remainder of this subsection we assume $\wedge \in C'$.

Corollary 3.3.20. *Let* $\vee \in C'$ *and let* $\mathbf{A} \in \mathsf{IQ}_{C'}$. *If the quasilattice with zero reduct* $\langle A;\ \wedge, \vee, 0\rangle$ *is a skew lattice with zero, then it is distributive local.*

Proof. Let $\vee \in C'$ and let $\mathbf{A} \in \mathsf{IQ}_{C'}$ be such that the quasilattice with zero reduct $\langle A;\ \wedge, \vee, 0\rangle$ is a skew lattice with zero. It is clear that $\langle A;\ \wedge, \vee, 0\rangle$ is local. Since for each $a \in A$, the sublattice $(\boldsymbol{a}]_{\mathbf{A}}$ is distributive, from Proposition 1.4.22 we deduce that $\langle A;\ \wedge, \vee, 0\rangle$ is also distributive. Thus $\langle A;\ \wedge, \vee, 0\rangle$ is distributive local. ∎

An *implicative BCS skew lattice* is an algebra $\langle A;\ \wedge, \vee, \backslash, 0\rangle$ of type $\langle 2, 2, 2, 0\rangle$ such that: (i) the reduct $\langle A;\ \wedge, \vee, 0\rangle$ is a skew lattice with zero; (ii) the reduct $\langle A;\ \backslash, 0\rangle$ is an implicative BCS-algebra; (iii) the natural skew lattice quasiorder $\preceq_{\mathcal{D}}$ coincides with the implicative BCS-algebra quasiorder $\preceq^{\langle A;\ \backslash, 0\rangle}$; and (iv) the natural skew lattice partial order $\leq_{\mathcal{K}}$ coincides with the implicative BCS-algebra partial order $\leq^{\langle A;\ \backslash, 0\rangle}$. Clearly the class of implicative BCS skew lattices is a subvariety of the variety of implicative BCS quasilattices.

Corollary 3.3.19 and Corollary 3.3.20 direct attention towards those members of $\mathsf{IQ}_{C'}$, $\vee \in C'$, that are implicative BCS skew lattices, inasmuch as these algebras preserve several important structural properties of skew Boolean algebras. In particular, if $\langle A;\ \wedge, \vee, \backslash, 0\rangle$ is an implicative BCS skew lattice, then: (i) the skew lattice with zero reduct $\langle A;\ \wedge, \vee, 0\rangle$ is distributive local; (ii) the reduct $\langle A;\ \backslash, 0\rangle$ is an implicative BCS-algebra; and (iii) for every $a \in A$, the principal subalgebra $(\boldsymbol{a}]_{\mathbf{A}}$ generated by a is a Boolean sublattice. The precise relationship between implicative BCS skew lattices and skew Boolean algebras is clarified in the following theorem, a first-order proof of (a slightly less general form of) which may be found in [210, Section 4.2].

Theorem 3.3.21. *An algebra* $\mathbf{A} := \langle A;\ \wedge, \vee, \backslash, 0\rangle$ *of type* $\langle 2, 2, 2, 0\rangle$ *is a skew Boolean algebra iff the following conditions are satisfied:*

1. *The reduct* $\langle A;\ \wedge, \vee, 0\rangle$ *is a join symmetric skew lattice with zero;*
2. *The reduct* $\langle A;\ \backslash, 0\rangle$ *is an implicative BCS algebra;*

3. **A** *satisfies the identity:*

$$x \wedge y \wedge x \approx x \sqcap y. \tag{3.69}$$

Given assertions (1)–(3) *above, implicative BCS difference coincides with standard difference.*

Proof. Let $\mathbf{A} := \langle A;\ \wedge, \vee, \backslash, 0\rangle$ be an algebra of type $\langle 2,2,2,0\rangle$.

($\Rightarrow$) Suppose **A** is a skew Boolean algebra. Then the reduct $\langle A;\ \wedge, \vee, 0\rangle$ is a locally Boolean skew lattice (in the sense of §1.4.24) by definition and so is join symmetric. Also, the reduct $\langle A;\ \backslash, 0\rangle$ is an implicative BCS-algebra by the proof of Proposition 3.1.8. Moreover, $\mathbf{A} \models (3.69)$ since $\mathbf{2}^p, \mathbf{3}^p_L, \mathbf{3}^p_R \models (3.69)$. Hence **A** is an implicative BCS skew lattice.

($\Leftarrow$) Suppose **A** satisfies Conditions (1)–(3) of the theorem. By (3) and Proposition 3.3.14, $\mathbf{A} \in \mathsf{IQ}_{\mathcal{C}'}$. By Corollary 3.3.20, therefore, the skew lattice with zero reduct $\langle A;\ \wedge, \vee, 0\rangle$ is distributive local. By (1) and Lemma 1.4.17 it follows that $\langle A;\ \wedge, \vee, 0\rangle$ is symmetric. Since for every $a \in A$, the principal subalgebras $(\mathbf{a}]_{\langle A;\ \wedge, 0\rangle}$ and $(\mathbf{a}]_{\langle A;\ \wedge, \vee, 0\rangle}$ must coincide, by Corollary 3.3.19 we have that $\langle A;\ \wedge, \vee, 0\rangle$ is locally Boolean. To complete the proof it remains to show $a\backslash b$ is the standard difference of $a, b \in A$. For this, just note that $a\backslash b$ is the complement of $a \sqcap b$ in $(\mathbf{a}]_{\langle A;\ \backslash, 0\rangle}$ by (2) and Corollary 2.3.33, and hence that $a\backslash b$ is the complement of $a \wedge b \wedge a$ in $(\mathbf{a}]_{\langle A;\ \wedge, 0\rangle}$ by (3) and Proposition 3.3.18. Since $(\mathbf{a}]_{\langle A;\ \wedge, 0\rangle}$ and $(\mathbf{a}]_{\langle A;\ \wedge, \vee, 0\rangle}$ must coincide, $a\backslash b$ is the complement of $a \wedge b \wedge a$ in $(\mathbf{a}]_{\langle A;\ \wedge, \vee, 0\rangle}$. Hence $a\backslash b$ is the standard difference of $a, b \in A$, and **A** is a skew Boolean algebra. ∎

Corollary 3.3.22. *An algebra* $\mathbf{A} := \langle A;\ \wedge, \vee, \backslash, 0\rangle$ *of type* $\langle 2,2,2,0\rangle$ *is a left handed skew Boolean algebra iff the following conditions are satisfied:*

1. *The reduct* $\langle A;\ \wedge, \vee, 0\rangle$ *is a join symmetric skew lattice with zero;*

2. *The reduct* $\langle A;\ \backslash, 0\rangle$ *is an implicative BCS algebra;*

3. *The skew lattice meet* $\wedge$ *coincides with the implicative BCS meet* $\sqcap$.

Given assertions (1)–(3) *above, implicative BCS difference coincides with standard difference.*

Remark 3.3.23. The condition of join symmetry cannot be omitted from the assertion of Theorem 3.3.21 and its corollary since the variety of left handed skew Boolean algebras is properly contained within the class (in fact, variety) of all implicative BCS skew lattices for which the skew lattice reduct is left handed. To see this, consider the algebra $\mathbf{A} := \langle A;\ \wedge, \vee, \backslash, 0\rangle$ of type $\langle 2,2,2,0\rangle$ with universe $A := \{0, a, b, c, d, e, f\}$ and derived binary operation $\sqcap$ defined by $i \sqcap j := i\backslash(i\backslash j)$ for any $i, j \in A$ determined by the following operation tables:

$\wedge^{\mathbf{A}}$	0	a	b	c	d	e	f
0	0	0	0	0	0	0	0
a	0	a	0	a	a	0	a
b	0	0	b	b	b	b	0
c	0	a	e	c	c	e	a
d	0	f	b	d	d	b	f
e	0	0	e	e	e	e	0
f	0	f	0	f	f	0	f

$\vee^{\mathbf{A}}$	0	a	b	c	d	e	f
0	0	a	b	c	d	e	f
a	a	a	d	c	d	c	f
b	b	c	b	c	d	e	d
c	c	c	d	c	d	c	d
d	d	c	d	c	d	c	d
e	e	c	b	c	d	e	d
f	f	a	d	c	d	c	f

$\backslash^{\mathbf{A}}$	0	a	b	c	d	e	f
0	0	0	0	0	0	0	0
a	a	0	a	0	0	a	0
b	b	b	0	0	0	0	b
c	c	e	a	0	0	a	e
d	d	b	f	0	0	f	b
e	e	e	0	0	0	0	e
f	f	0	f	0	0	f	0

$\sqcap^{\mathbf{A}}$	0	a	b	c	d	e	f
0	0	0	0	0	0	0	0
a	0	a	0	a	a	0	a
b	0	0	b	b	b	b	0
c	0	a	e	c	c	e	a
d	0	f	b	d	d	b	f
e	0	0	e	e	e	e	0
f	0	f	0	f	f	0	f

An easy sequence of checks shows: (i) the reduct $\langle A;\ \wedge, \vee, 0\rangle$ is a left handed skew lattice with zero; (ii) the reduct $\langle A;\ \backslash, 0\rangle$ is an implicative BCS-algebra (with implicative BCS meet $\sqcap$); and (iii) $i \leq_{\mathcal{H}} j$ iff $i \leq^{\langle A;\ \backslash, 0\rangle} j$ and $i \preceq_{\mathcal{D}} j$ iff $i \preceq^{\langle A;\ \backslash, 0\rangle} j$ for any $i, j \in A$. Hence $\mathbf{A}$ is an implicative BCS skew lattice for which the skew lattice reduct is left handed. However, $\mathbf{A}$ is not a left handed skew Boolean algebra, since the skew lattice with zero reduct $\langle A;\ \wedge, \vee, 0\rangle$ is not locally Boolean (in the sense of §1.4.24). In particular, $\langle A;\ \wedge, \vee, 0\rangle$ is not symmetric, since it is not join symmetric: $(a \vee b) \vee (a \wedge b \wedge a) = d \vee 0 = d$ but $(a \wedge b \wedge a) \vee (b \vee a) = 0 \vee c = c$. ∎

Denote by lrPQ the variety of (l, r)-*pre-BCK quasilattices*, namely the subvariety of $\mathsf{PQ}_{C'}$, $C' = \{\wedge, \vee, \backslash, \mathbf{0}\}$, satisfying the identities (l, r); also, recall the definition of Idziak's variety BL of BCK-lattices from §1.6.23. Perhaps surprisingly, there exists an intimate connection between the variety of left handed skew Boolean algebras and the variety of BCK-lattices (in the context of the variety lrPQ). In more detail, the variety of left handed skew Boolean algebras is the natural conjugate of BL in $\mathbf{\Lambda}^V(\mathsf{lrPQ})$ in the same way that BCK is the natural conjugate of the quasivariety $\mathbf{Q}(\mathbf{B}_2)$ in $\mathbf{\Lambda}^V(\mathsf{PBCK})$ (recall Proposition 2.2.5). This is shown by the following proposition, in the statement of which (and throughout the remainder of this subsection), the type of Idziak's variety BL is understood to be $\{\wedge, \vee, \backslash, \mathbf{0}\}$.

Proposition 3.3.24. *The pair* $\langle \mathbf{V}(\mathbf{3}_L^p), \mathsf{BL}\rangle$ *[equivalently* $\langle \mathsf{lhSBA}, \mathsf{BL}\rangle$*] splits the lattice of varieties* $\mathbf{\Lambda}^V(\mathsf{lrPQ})$ *of* (l, r)*-pre-BCK quasilattices. Thus for every variety* $\mathsf{K} \subseteq \mathsf{lrPQ}$*, either* $\mathsf{K} \subseteq \mathsf{BL}$ *or* $\mathbf{V}(\mathbf{3}_L^p) \subseteq \mathsf{K}$ *[equivalently* $\mathsf{lhSBA} \subseteq \mathsf{K}$*] (and not both).*

Proof. Suppose $\mathsf{K} \in \Lambda^V(\mathsf{lrPQ})$ and $\mathsf{K} \not\subseteq \mathsf{BL}$. By hypothesis, K does not satisfy (2.5), so there is an algebra $\mathbf{A} \in \mathsf{K}$ and there are elements $a, b \in A$ such that $a\backslash b = 0 = b\backslash a$ but $a \neq b$. Therefore $a \equiv b \pmod{\Xi}$ and so $a \equiv b \pmod{\mathcal{D}}$. Thus $a \wedge b \wedge a = a$ and $b \wedge a \wedge b = b$; because $\mathbf{A} \models$ (l, r) we infer $a \wedge b = a$ and $b \wedge a = b$. Similarly we deduce that $a \vee b = b$ and $b \vee a = a$. Because the reduct $\langle A;\ \wedge, \vee, 0\rangle$ is a quasilattice with zero, we conclude that $\{0, a, b\}$ is the universe of a subalgebra of $\mathbf{A}$ isomorphic to $\mathbf{3}_L^p$. Hence $\mathbf{3}_L^p \in \mathsf{K}$ and, by Theorem 1.4.29, $\mathsf{lhSBA} = \mathbf{V}(\mathbf{3}_L^p) \subseteq \mathsf{K}$. ∎

Remark 3.3.25. BL is also the splitting variety associated with the 3-element right handed skew Boolean algebra $\mathbf{3}_R^p$ in the variety of (r, l)-*pre-BCK quasilattices*, namely the subvariety of $\mathsf{PQ}_{C'}$, $C' = \{\wedge, \vee, \backslash, \mathbf{0}\}$, satisfying the identities (r, l). ∎

Corollary 3.3.26. *For a variety* V *of* (l, r)*-pre-BCK quasilattices the following statements hold:*

1. V *is* **0***-regular iff* V $\subseteq$ BL*;*

2. V *is congruence distributive iff* V $\subseteq$ BL*;*

3. V *is congruence n-permutable for some $n \geq 2$ iff* V $\subseteq$ BL. *In particular, if* V $\subseteq$ BL *then* V *is congruence permutable.*

Proof. Let V be a variety of (l, r)-pre-BCK quasilattices. Suppose V $\subseteq$ BL. Put $d_1(x, y) := x\backslash y$ and $d_2(x, y) := y\backslash x$. Then $d_1(x, y)$ and $d_2(x, y)$ are binary terms of V satisfying the identities and quasi-identities of Proposition 1.2.6, so V is **0**-regular. Because V is subtractive (witness $x\backslash y$), from Proposition 1.7.3 we infer that V is ideal determined. Since V $\models$ (1.3), (1.4), (1.7), V is ideal distributive by the remarks of §1.7.4, so V is congruence distributive by ideal determinacy. Moreover, $p(x, y, z) := \big(x\backslash(y\backslash z)\big) \vee \big(z\backslash(y\backslash x)\big)$ is a Mal'cev term for V by Idziak [116, Theorem 2], so V is congruence permutable. For the converse, suppose V $\not\subseteq$ BL. Then lhSBA $= \mathbf{V}(\mathbf{3}_L^p) \subseteq$ V by Proposition 3.3.24. Since the variety of left handed skew Boolean algebras is not **0**-regular, V is not **0**-regular. Moreover, because the variety of left handed skew Boolean algebras satisfies no non-trivial congruence identities (by Lemma 4.8 of Cornish [65]) V does not satisfy any non-trivial congruence identity. In particular, V is not congruence n-permutable for any $n \geq 2$. ∎

Corollary 3.3.26 in conjunction with the theory of BCK-lattices shows that several important properties of BCK-lattices do not extend to pre-BCK quasilattices. On the other hand, by Theorem 3.3.8 the varieties $\mathsf{PQ}_\mathcal{C}$ satisfy a modified form of the Clifford-McLean theorem, while by Proposition 3.3.10, one-sided members of any $\mathsf{PQ}_\mathcal{C}$ are regular. Thus fundamental properties of bands and quasilattices are preserved by pre-BCK quasilattices. Collectively, these observations suggest that, for pre-BCK bands and pre-BCK quasilattices, the behaviour of these algebras is more closely aligned to that of bands and quasilattices than to that of BCK-semilattices and BCK-lattices. We return to this point in the following subsection.

3.3.27. BCK Bands and BCK Paralattices. Recall from Lemma 1.6.2 and Proposition 1.6.4 that for any BCK-algebra $\mathbf{A}$, there exist distinct partial orders $\leq_n$, $n = 0, 1, 2$ on A, where for any $a, b \in A$,

$$a \leq_0 b \quad \text{iff} \quad ab = 0$$
$$a \leq_1 b \quad \text{iff} \quad b \cap a = a$$
$$a \leq_2 b \quad \text{iff} \quad (b \cap a)(ba) = a.$$

A *lower* $\leq_n$*-BCK band*, $n = 0, 1, 2$, is an algebra $\langle A;\ \wedge, \dot{-}, 0\rangle$ of type $\langle 2, 2, 0\rangle$ such that: (i) the reduct $\langle A;\ \wedge, 0\rangle$ is a band with zero; (ii) the reduct $\langle A;\ \dot{-}, 0\rangle$ is a BCK-algebra; and (iii) the natural band partial order $\leq_{\mathcal{H}}$ coincides with the BCK partial order $\leq_n^{\langle A;\ \dot{-}, 0\rangle}$.

Theorem 3.3.28. *An algebra* $\langle A;\ \wedge, \dot{-}, 0\rangle$ *of type* $\langle 2, 2, 0\rangle$ *is a lower* $\leq_0$*-BCK band iff the following identities are satisfied:*

$$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z \tag{3.76}$$
$$x \wedge x \approx x \tag{3.77}$$

$$((x \dot{-} y) \dot{-} (x \dot{-} z)) \dot{-} (z \dot{-} y) \approx 0 \tag{3.78}$$
$$x \dot{-} \mathbf{0} \approx x \tag{3.79}$$
$$\mathbf{0} \dot{-} x \approx \mathbf{0} \tag{3.80}$$

$$\big(x \dot{-} (x \dot{-} y)\big) \wedge y \approx x \dot{-} (x \dot{-} y) \tag{3.81}$$
$$y \wedge \big(x \dot{-} (x \dot{-} y)\big) \approx x \dot{-} (x \dot{-} y) \tag{3.82}$$
$$(x \wedge y \wedge x) \dot{-} x \approx \mathbf{0}. \tag{3.83}$$

Thus the class of lower $\leq_0$*-BCK bands is a variety.*

Proof. Let $\mathbf{A} = \langle A;\ \wedge, \dot{-}, 0\rangle$ be an algebra of type $\langle 2, 2, 0\rangle$ satisfying (3.76)–(3.80) and let $a, b \in A$. To prove the theorem it is sufficient to show:

(i) $\mathbf{A} \models$ (3.81)–(3.83) implies $\langle A;\ \dot{-}, 0\rangle$ is a BCK-algebra;

(ii) $\mathbf{A} \models (3.81)$–$(3.83)$ iff $\leq_0^{\langle A;\, \dot{-},0\rangle}$ and $\leq_{\mathcal{H}}$ coincide;

(iii) $\mathbf{A} \models (3.81)$–$(3.83)$ implies $\langle A;\ \wedge, 0\rangle$ is a band with zero.

For (i), let $a, b \in A$ and suppose $\mathbf{A} \models (3.81)$–$(3.83)$. To see $\langle A;\ \dot{-}, 0\rangle$ is a BCK-algebra it is sufficient to show $a \dot{-} b = 0 = b \dot{-} a$ implies $a = b$, just because $\langle A;\ \dot{-}, 0\rangle$ is a pre-BCK-algebra by (3.78)–(3.80). So let $a \dot{-} b = 0 = b \dot{-} a$. Then $a \wedge b = (a \dot{-} 0) \wedge b$ (by (3.79)) $= \big(a \dot{-} (a \dot{-} b)\big) \wedge b = a \dot{-} (a \dot{-} b)$ (by (3.81)) $= a \dot{-} 0 = a$ (by (3.79)). Also $a \wedge b = a \wedge (b \dot{-} 0)$ (by (3.79)) $= a \wedge \big(b \dot{-} (b \dot{-} a)\big) = b \dot{-} (b \dot{-} a)$ (by (3.82)) $= b \dot{-} 0 = b$ (by (3.79)). Hence $a = a \wedge b = b$.

For (ii), suppose $\mathbf{A} \models (3.81)$–$(3.83)$. By (i), $\langle A;\ \dot{-}, 0\rangle$ is a BCK-algebra, so the reference to $\leq_0^{\langle A;\, \dot{-},0\rangle}$ makes sense. Assume $a \leq_0^{\langle A;\, \dot{-},0\rangle} b$. Then $a = a \dot{-} 0$ (by (3.79)) $= a \dot{-} (a \dot{-} b) = \big(a \dot{-} (a \dot{-} b)\big) \wedge b$ (by (3.81)) $=$ $(a \dot{-} 0) \wedge b$ (by (3.79)) $= a \wedge b$. Also $b \wedge a = b \wedge (a \dot{-} 0)$ (by (3.79)) $= b \wedge \big(a \dot{-} (a \dot{-} b)\big) = a \dot{-} (a \dot{-} b)$ (by (3.82)) $= a \dot{-} 0 = a$ (by (3.79)). Hence $a \leq_0^{\langle A;\, \dot{-},0\rangle} b$ implies $a \leq_{\mathcal{H}} b$. For the opposite implication assume $a \leq_{\mathcal{H}} b$. Then $a \wedge b = a = b \wedge a$, so $b \wedge a \wedge b = a$ and $a \dot{-} b =$ $(b \wedge a \wedge b) \dot{-} b = 0$ by (3.83). Thus $a \leq_0^{\langle A;\, \dot{-},0\rangle} b$ and so $a \leq_{\mathcal{H}}$ iff $a \leq_0^{\langle A;\, \dot{-},0\rangle} b$. Conversely, suppose the partial orders $\leq_{\mathcal{H}}$ and $\leq_0^{\langle A;\, \dot{-},0\rangle}$ coincide. From $\big(a \dot{-} (a \dot{-} b)\big) \leq_0^{\langle A;\, \dot{-},0\rangle} b$ we have $\big(a \dot{-} (a \dot{-} b)\big) \leq_{\mathcal{H}} b$, which implies $\big(a \dot{-} (a \dot{-} b)\big) \wedge b = a \dot{-} (a \dot{-} b) = b \wedge \big(a \dot{-} (a \dot{-} b)\big)$. Moreover, from $a \wedge b \wedge a \leq_{\mathcal{H}} a$ we have $a \wedge b \wedge a \leq_0^{\langle A;\, \dot{-},0\rangle} a$ and thus $(a \wedge b \wedge a) \dot{-} a = 0$. Hence $\mathbf{A} \models (3.81)$–$(3.83)$.

For (iii), suppose $\mathbf{A} \models (3.81)$–$(3.83)$. Then $\langle A;\ \dot{-}, 0\rangle$ is a BCK-algebra by (i), so $0 \leq_0^{\langle A;\, \dot{-},0\rangle} a$ for all $a \in A$ by Lemma 1.6.2. By (ii), $0 \leq_{\mathcal{H}} a$ for all $a \in A$, so $\langle A;\ \wedge, 0\rangle$ is a band with zero. ∎

For a lower $\leq_1$-BCK band $\mathbf{A}$ and $a, b \in A$, an argument similar to the proof of Theorem 3.3.28 shows that $a \leq_1^{\langle A;\, \dot{-},0\rangle} b$ implies $a \leq_{\mathcal{H}} b$ iff $\mathbf{A}$ satisfies the identities:

$$(x \dot{-} y) \wedge x \approx x \dot{-} y \tag{3.84}$$

$$x \wedge (x \dot{-} y) \approx x \dot{-} y. \tag{3.85}$$

Conversely, $a \leq_{\mathcal{H}} b$ implies $a \leq_1^{\langle A;\ \dot{-},0\rangle} b$ iff **A** satisfies the identity:

$$x \dot{-} \big(x \dot{-} (x \wedge y \wedge x)\big) \approx x \wedge y \wedge x. \tag{3.86}$$

(For a lower $\leq_2$-BCK band, we know of no corresponding equational characterisation of the coincidence of the partial orders $\leq_2^{\langle A;\ \dot{-},0\rangle}$ and $\leq_{\mathcal{H}}$.) Nonetheless, the identities (3.76)–(3.80), (3.84)–(3.86) are not sufficient to ensure satisfaction of the quasi-identity (1.5). To see this, consider $\langle \wedge, \backslash, \mathbf{0}\rangle$-$\mathbf{3}_L^p$, the $\langle \wedge, \backslash, \mathbf{0}\rangle$-reduct of the 3-element left handed primitive skew Boolean algebra $\mathbf{3}_L^p$. An easy sequence of checks (ignoring issues of type) shows $\langle \wedge, \backslash, \mathbf{0}\rangle$-$\mathbf{3}_L^p \models$ (3.76)–(3.80), (3.84)–(3.86) but that $\mathbf{A} \not\models$ (1.5).

Problem 3.3.29. For $n = 1, 2$, is the class of lower $\leq_n$-BCK bands equationally definable? ∎

An *upper* $\leq_n$*-BCK band,* $n = 0, 1, 2$, is an algebra $\langle A;\ \vee, \dot{-}, 0\rangle$ of type $\langle 2, 2, 0\rangle$ such that: (i) the reduct $\langle A;\ \vee, 0\rangle$ is a band with identity; (ii) the reduct $\langle A;\ \dot{-}, 0\rangle$ is a BCK-algebra; and (iii) the natural band partial order $\leq_{\mathcal{H}}$ dualises the BCK partial order $\leq_n^{\langle A;\ \dot{-},0\rangle}$ in the sense that $a \leq_{\mathcal{H}} b$ iff $b \leq_n^{\langle A;\ \dot{-},0\rangle} a$ for any $a, b \in A$. The proof of the following theorem is similar to the proof of Theorem 3.3.28 and is omitted.

Theorem 3.3.30. *An algebra* $\langle A;\ \vee, \dot{-}, 0\rangle$ *of type* $\langle 2, 2, 0\rangle$ *is an upper* $\leq_0$*-BCK band iff the following identities are satisfied:*

$$x \vee (y \vee z) \approx (x \vee y) \vee z \tag{3.87}$$

$$x \vee x \approx x \tag{3.88}$$

$$\big((x \dot{-} y) \dot{-} (x \dot{-} z)\big) \dot{-} (z \dot{-} y) \approx \mathbf{0} \tag{3.89}$$

$$x \dot{-} \mathbf{0} \approx x \tag{3.90}$$

$$\mathbf{0} \dot{-} x \approx \mathbf{0} \tag{3.91}$$

$$\big(x \dot{-} (x \dot{-} y)\big) \vee y \approx y \tag{3.92}$$

$$y \vee (x \dot{-} (x \dot{-} y)) \approx y \tag{3.93}$$

$$x \dot{-} (x \vee y \vee x) \approx 0. \tag{3.94}$$

Thus the class of upper $\leq_0$-BCK bands is a variety.

For $n = 1, 2$, remarks concerning lower $\leq_n$-BCK bands apply *mutatis mutandis* to upper $\leq_n$-BCK bands. In particular, for an upper $\leq_1$-BCK band $\mathbf{A}$ and $a, b \in A$, $a \leq_n^{\langle A;\ \dot{-},0\rangle} b$ implies $b \leq_{\mathcal{H}} a$ iff $\mathbf{A}$ satisfies the identities:

$$(x \dot{-} y) \vee x \approx x \tag{3.95}$$

$$x \vee (x \dot{-} y) \approx x. \tag{3.96}$$

Conversely, $a \leq_{\mathcal{H}} b$ implies $b \leq_n^{\langle A;\ \dot{-},0\rangle} a$ iff $\mathbf{A}$ satisfies the identity:

$$(x \vee y \vee x) \dot{-} ((x \vee y \vee x) \dot{-} x) \approx x. \tag{3.97}$$

(For an upper $\leq_2$-BCK band, we know of no corresponding equational characterisation of the dualisation of the partial orders $\leq_2^{\langle A;\ \dot{-},0\rangle}$ and $\leq_{\mathcal{H}}$.) However, the identities (3.87)–(3.91), (3.95)–(3.97) are not sufficient to ensure satisfaction of the quasi-identity (2.5). To see this, consider $\langle \vee, \backslash, \mathbf{0}\rangle$-$\mathbf{3}_R^p$, the $\langle \vee, \backslash, \mathbf{0}\rangle$-reduct of the 3-element right handed primitive skew Boolean algebra $\mathbf{3}_R^p$. An easy sequence of checks (ignoring issues of type) shows that $\langle \vee, \backslash, \mathbf{0}\rangle$-$\mathbf{3}_R^p \models$ (3.87)–(3.91), (3.95)–(3.97) but that $\mathbf{A} \not\models$ (1.5).

Problem 3.3.31. For $n = 1, 2$, is the class of upper $\leq_n$-BCK bands equationally definable? ∎

A *paralattice with zero* is a paralattice $\langle A;\ \wedge, \vee\rangle$ for which there exists $0 \in A$ (the *zero* of $\langle A;\ \wedge, \vee\rangle$) such that 0 is the least element under the natural paralattice partial order. As is usual, by abuse of language and notation we often identify a paralattice with zero $\mathbf{A} := \langle A;\ \wedge, \vee\rangle$ with the algebra $\langle A;\ \wedge, \vee, 0\rangle$ obtained from $\mathbf{A}$ upon enriching the language of $\mathbf{A}$ with a new nullary operation symbol $\mathbf{0}$ whose canonical interpretation on $\langle A;\ \wedge, \vee, 0\rangle$ is the zero element $0 \in A$. A *$\leq_0$-BCK paralattice* is an algebra $\langle A;\ \wedge, \vee, \dot{-}, 0\rangle$ of type $\langle 2, 2, 2, 0\rangle$ such that: (i) the reduct $\langle A;\ \wedge, \vee, 0\rangle$ is a paralattice with zero; (ii) the reduct $\langle A;\ \dot{-}, 0\rangle$ is a BCK-algebra; and (iii) the natural paralattice partial order $\leq_{\mathcal{H}}$ coincides with the BCK partial order $\leq_0^{\langle A;\ \dot{-}, 0\rangle}$. From Theorem 3.3.28 and Theorem 3.3.30 the following result is clear.

Theorem 3.3.32. *An algebra* $\mathbf{A} := \langle A;\ \wedge, \vee, \dot{-}, 0\rangle$ *of type* $\langle 2, 2, 2, 0\rangle$ *is a* $\leq_0$*-BCK paralattice iff the reduct* $\langle A;\ \wedge, \dot{-}, 0\rangle$ *is a lower* $\leq_0$*-BCK band and the reduct* $\langle A;\ \vee, \dot{-}, 0\rangle$ *is an upper* $\leq_0$*-BCK band. Thus* $\mathbf{A}$ *is a* $\leq_0$*-BCK paralattice iff* $\mathbf{A} \models (3.76) - -(3.83), (3.87) - -(3.94)$. *Therefore the class of* $\leq_0$*-BCK paralattices is a variety.*

Let $\mathcal{C}$ denote an arbitrary subset of the language $\{\wedge, \vee, \dot{-}, \mathbf{0}\}$ of $\leq_0$-BCK paralattices that contains both $\dot{-}$ and $\mathbf{0}$. Let:

- $\mathsf{BP}_{\mathcal{C}}$ denote the quasivariety of BCK-algebras when $\mathcal{C} = \{\dot{-}, \mathbf{0}\}$;
- $\mathsf{BP}_{\mathcal{C}}$ denote the variety of lower $\leq_0$-BCK bands when $\mathcal{C} = \{\wedge, \dot{-}, \mathbf{0}\}$;
- $\mathsf{BP}_{\mathcal{C}}$ denote the variety of upper $\leq_0$-BCK bands when $\mathcal{C} = \{\vee, \dot{-}, \mathbf{0}\}$;
- $\mathsf{BP}_{\mathcal{C}}$ denote the variety of $\leq_0$-BCK paralattices when $\mathcal{C} = \{\wedge, \vee, \dot{-}, \mathbf{0}\}$.

Notice that for each $\mathcal{C}$ distinct from $\{\dot{-}, \mathbf{0}\}$, $\mathsf{BP}_{\mathcal{C}}$ is the class of algebras with language $\mathcal{C}$ axiomatised by those identities among (3.76)–(3.83), (3.87)–(3.94) that use only operation symbols from $\mathcal{C}$; of course, this observation is dependent upon the axiomatisation of the variety of $\leq_0$-BCK paralattices given in Theorem 3.3.32.

Remark 3.3.33. As per pre-BCK quasilattices, $\mathsf{BP}_{\mathcal{C}}$ should not be confused (for each $\mathcal{C}$) with $\mathcal{C}$-BP, the class of all $\mathcal{C}$-subreducts of the variety of $\leq_0$-BCK paralattices. In particular, in contrast to each $\mathsf{BP}_{\mathcal{C}}$, we do not know in general if each $\mathcal{C}$-BP is even a quasivariety. ∎

Throughout the remainder of this subsection assume $\wedge \in \mathcal{C}$ or $\vee \in \mathcal{C}$. By remarks due to Laslo and Leech [145, Section 5, p. 23], the Green's equivalences $\mathcal{D}_{\langle A;\ \wedge\rangle}$ and $\mathcal{D}_{\langle A;\ \vee\rangle}$ on a paralattice $\mathbf{A}$ are not typically congruences, whence paralattices do not in general possess a coherent Clifford-McLean structure. In fact, by Laslo and Leech [145, Theorem 26], a paralattice $\mathbf{A}$ supports a coherent Clifford-McLean structure iff it is simultaneously a quasilattice, in which case

$\mathcal{D}_{\langle A;\, \wedge \rangle} = \mathcal{D}_{\langle A;\, \vee \rangle}$. Concerning the varieties BP_C, the following proposition shows that the presence of BCK difference causes members of BP_C to have a coherent Clifford-McLean structure only in the trivial (that is, commutative) case.

Proposition 3.3.34. *For any* $\mathbf{A} \in \mathsf{BP}_C$ *the following assertions hold:*

1. *If* $\wedge \in C$ *then* $\mathcal{D}_{\langle A;\, \wedge \rangle}$ *is a congruence on* $\langle A;\, \wedge, \dot{-}, 0 \rangle$ *iff* $\mathcal{D}_{\langle A;\, \wedge \rangle} = \omega_{\mathbf{A}}$;
2. *If* $\vee \in C$ *then* $\mathcal{D}_{\langle A;\, \vee \rangle}$ *is a congruence on* $\langle A;\, \vee, \dot{-}, 0 \rangle$ *iff* $\mathcal{D}_{\langle A;\, \vee \rangle} = \omega_{\mathbf{A}}$;
3. *If* $\{\wedge, \vee\} \subseteq C$ *then* $\mathcal{D}_{\langle A;\, \wedge \rangle}$ *and* $\mathcal{D}_{\langle A;\, \vee \rangle}$ *are congruences on* $\mathbf{A}$ *iff* $\mathcal{D}_{\langle A;\, \wedge \rangle} = \omega_{\mathbf{A}} = \mathcal{D}_{\langle A;\, \vee \rangle}$.

Proof. We prove the proposition only for the case where $\{\wedge, \vee\} \subseteq C$; the proofs in the remaining cases do not differ significantly and are omitted. Let $\{\wedge, \vee\} \subseteq C$ and let $\mathbf{A} \in \mathsf{BP}_C$. Suppose both $\mathcal{D}_{\langle A;\, \wedge \rangle}$ and $\mathcal{D}_{\langle A;\, \vee \rangle}$ are congruences on $\mathbf{A}$. Then in particular $\mathcal{D}_{\langle A;\, \wedge \rangle}$ and $\mathcal{D}_{\langle A;\, \vee \rangle}$ are congruences on the paralattice with zero reduct $\langle A;\, \wedge, \vee, 0 \rangle$, so $\mathcal{D}_{\langle A;\, \wedge \rangle} = \mathcal{D}_{\langle A;\, \vee \rangle}$ by previous remarks. In view of this last, to simplify notation we can and will write $\mathcal{D}$ for $\mathcal{D}_{\langle A;\, \wedge \rangle} = \mathcal{D}_{\langle A;\, \vee \rangle}$ throughout the remainder of the proof. Let $a, b \in A$. To complete the proof it is sufficient to show $a \equiv b \pmod{\mathcal{D}}$ implies $a = b$. So suppose $a \equiv b \pmod{\mathcal{D}}$. From $a \wedge b \wedge a \equiv a \wedge b \wedge a \pmod{\mathcal{D}}$ and $a \equiv b \pmod{\mathcal{D}}$ we have that $(a \wedge b \wedge a) \dot{-} a \equiv (a \wedge b \wedge a) \dot{-} b \pmod{\mathcal{D}}$, since $\mathcal{D}$ has the substitution property for the $\dot{-}$ operation. Because $(a \wedge b \wedge a) \dot{-} a = 0$ by (3.83) we have that $(a \wedge b \wedge a) \dot{-} b \equiv 0 \pmod{\mathcal{D}}$, whence $(a \wedge b \wedge a) \dot{-} b = 0$ by Lemma 1.4.13. But then $a \dot{-} b = 0$ since $a \equiv b \pmod{\mathcal{D}}$. An analogous argument shows $b \dot{-} a = 0$; since $\langle A;\, \dot{-}, 0 \rangle$ is a BCK-algebra we have that $a = b$. Thus $\mathcal{D} = \omega_{\mathbf{A}}$. Since the converse holds trivially, the proposition is proved. ∎

Let $\mathbf{A} \in \mathsf{BP}_C$. Because $\mathcal{L}_{\langle A;\, \wedge \rangle}, \mathcal{R}_{\langle A;\, \wedge \rangle} \subseteq \mathcal{D}_{\langle A;\, \wedge \rangle}$ and $\mathcal{L}_{\langle A;\, \vee \rangle}, \mathcal{R}_{\langle A;\, \vee \rangle} \subseteq \mathcal{D}_{\langle A;\, \vee \rangle}$ (when these equivalences exist), an easy modification of the proof of Proposition 3.3.34 yields:

Proposition 3.3.35. *For any* $\mathbf{A} \in \mathsf{BP}_C$ *the following assertions hold:*

1. *If* $\wedge \in \mathcal{C}$ *then* $\mathcal{L}_{\langle A;\, \wedge\rangle}$ *and* $\mathcal{R}_{\langle A;\, \wedge\rangle}$ *are congruences on* $\langle A;\ \wedge, \dot{-}, 0\rangle$ *iff* $\mathcal{L}_{\langle A;\, \wedge\rangle} = \omega_{\mathbf{A}} = \mathcal{R}_{\langle A;\, \wedge\rangle}$;

2. *If* $\vee \in \mathcal{C}$ *then* $\mathcal{L}_{\langle A;\, \vee\rangle}$ *and* $\mathcal{R}_{\langle A;\, \vee\rangle}$ *are congruences on* $\langle A;\ \vee, \dot{-}, 0\rangle$ *iff* $\mathcal{L}_{\langle A;\, \vee\rangle} = \omega_{\mathbf{A}} = \mathcal{R}_{\langle A;\, \vee\rangle}$;

3. *If* $\{\wedge, \vee\} \subseteq \mathcal{C}$ *then* $\mathcal{L}_{\langle A;\, \wedge\rangle}$, $\mathcal{R}_{\langle A;\, \wedge\rangle}$, $\mathcal{L}_{\langle A;\, \vee\rangle}$ *and* $\mathcal{R}_{\langle A;\, \vee\rangle}$ *are all congruences on* $\mathbf{A}$ *iff* $\mathcal{L}_{\langle A;\, \wedge\rangle} = \omega_{\mathbf{A}} = \mathcal{R}_{\langle A;\, \wedge\rangle}$ *and* $\mathcal{L}_{\langle A;\, \vee\rangle} = \omega_{\mathbf{A}} = \mathcal{R}_{\langle A;\, \vee\rangle}$.

Let $\mathbf{A} \in \mathsf{BP}_{\mathcal{C}}$. For $\wedge \in \mathcal{C}$ $[\vee \in \mathcal{C};\ \{\wedge, \vee\} \subseteq \mathcal{C}]$ $\langle A;\ \wedge, \dot{-}, 0\rangle$ $[\langle A;\ \vee, \dot{-}, 0\rangle;$ $\langle A;\ \wedge, \vee, \dot{-}, 0\rangle]$ is *regular* if $\mathcal{L}_{\langle A;\, \wedge\rangle}$ and $\mathcal{R}_{\langle A;\, \wedge\rangle}$ $[\mathcal{L}_{\langle A;\, \vee\rangle}, \mathcal{R}_{\langle A;\, \vee\rangle};\ \mathcal{L}_{\langle A;\, \wedge\rangle}, \mathcal{R}_{\langle A;\, \wedge\rangle}, \mathcal{L}_{\langle A;\, \vee\rangle}, \mathcal{R}_{\langle A;\, \vee\rangle}]$ are congruences on $\langle A;\ \wedge, \dot{-}, 0\rangle$ $[\langle A;\ \vee, \dot{-}, 0\rangle;\ \langle A;\ \wedge, \vee, \dot{-}, 0\rangle]$. Because of Proposition 3.3.35, an algebra $\mathbf{A} \in \mathsf{BP}_{\mathcal{C}}$ is regular only in the trivial (commutative) case.

Corollary 3.3.36. *For* $\mathbf{A} \in \mathsf{BP}_{\mathcal{C}}$ *the following assertions hold:*

1. *If* $\wedge \in \mathcal{C}$ *then* $\langle A;\ \wedge, \dot{-}, 0\rangle$ *is regular iff* $\mathcal{D}_{\langle A;\, \wedge\rangle} = \omega_{\mathbf{A}}$;

2. *If* $\vee \in \mathcal{C}$ *then* $\langle A;\ \vee, \dot{-}, 0\rangle$ *is regular iff* $\mathcal{D}_{\langle A;\, \vee\rangle} = \omega_{\mathbf{A}}$;

3. *If* $\{\wedge, \vee\} \subseteq \mathcal{C}$ *then* $\mathbf{A}$ *is regular iff* $\mathcal{D}_{\langle A;\, \wedge\rangle} = \omega_{\mathbf{A}} = \mathcal{D}_{\langle A;\, \vee\rangle}$.

Proof. We prove the corollary only for the case that $\{\wedge, \vee\} \subseteq \mathcal{C}$; the proofs in the other cases are not significantly different and are omitted. So let $\mathbf{A} \in \mathsf{BP}_{\mathcal{C}}$ where $\{\wedge, \vee\} \subseteq \mathcal{C}$. Suppose $\mathbf{A}$ is regular. Then $\mathcal{L}_{\langle A;\, \wedge\rangle}$ and $\mathcal{R}_{\langle A;\, \wedge\rangle}$ are congruences on $\mathbf{A}$, so $\mathcal{L}_{\langle A;\, \wedge\rangle} = \omega_{\mathbf{A}} = \mathcal{R}_{\langle A;\, \wedge\rangle}$ by Proposition 3.3.35(3). Since $\mathcal{D}_{\langle A;\, \wedge\rangle} = \mathcal{L}_{\langle A;\, \wedge\rangle} \circ \mathcal{R}_{\langle A;\, \wedge\rangle}$ (by Howie [111, p. 46]) we have that $\mathcal{D}_{\langle A;\, \wedge\rangle} = \omega_{\mathbf{A}}$; an analogous argument shows $\mathcal{D}_{\langle A;\, \vee\rangle} = \omega_{\mathbf{A}}$. For the converse, suppose $\mathcal{D}_{\langle A;\, \wedge\rangle} = \omega_{\mathbf{A}} = \mathcal{D}_{\langle A;\, \vee\rangle}$. Since both $\mathcal{L}_{\langle A;\, \wedge\rangle} \subseteq \mathcal{D}_{\langle A;\, \wedge\rangle}$ and $\mathcal{R}_{\langle A;\, \wedge\rangle} \subseteq \mathcal{D}_{\langle A;\, \wedge\rangle}$ we must have both $\mathcal{L}_{\langle A;\, \wedge\rangle} = \omega_{\mathbf{A}}$ and $\mathcal{R}_{\langle A;\, \wedge\rangle} = \omega_{\mathbf{A}}$. Similar reasoning shows $\mathcal{L}_{\langle A;\, \vee\rangle} = \omega_{\mathbf{A}}$ and $\mathcal{R}_{\langle A;\, \vee\rangle} = \omega_{\mathbf{A}}$. Thus all of $\mathcal{L}_{\langle A;\, \wedge\rangle}$, $\mathcal{R}_{\langle A;\, \wedge\rangle}$, $\mathcal{L}_{\langle A;\, \vee\rangle}$ and $\mathcal{R}_{\langle A;\, \vee\rangle}$ are congruences on $\mathbf{A}$, so $\mathbf{A}$ is regular. ∎

Recall from the remarks concluding §3.3.2 that the behaviour of pre-BCK bands and pre-BCK quasilattices more closely resembles that of bands and quasilattices than that of BCK-semilattices and BCK-lattices. In contrast, Proposition 3.3.34 and Corollary 3.3.36 suggest that the behaviour of $\leq_0$-BCK bands and $\leq_0$-BCK paralattices is more closely aligned to that of BCK-semilattices and BCK-lattices than to that of bands and paralattices. Further support for this last contention is provided in the remaining results of this subsection, which show that properties of 0-regularity, congruence distributivity and congruence permutability enjoyed by BCK-semilattices and BCK-lattices are preserved upon passing to $\leq_0$-BCK bands and $\leq_0$-BCK paralattices.

Proposition 3.3.37. *(cf. [116, Theorem 1]) Let* $\mathbf{A} \in \mathsf{BP}_{\mathcal{C}}$. *For any* $a, b \in A$, *the following are equivalent:*

1. $a = b$;
2. $a \dot{-} b = 0 = b \dot{-} a$.

If $\vee \in \mathcal{C}$, *then either of* (1) *or* (2) *is equivalent to:*

3. $(a \dot{-} b) \vee (b \dot{-} a) = 0$.

Thus any variety $\mathsf{BP}_{\mathcal{C}}$ *is* **0**-*regular and hence ideal determined. That is, for any* $\mathbf{A} \in \mathsf{BP}_{\mathcal{C}}$, *the map* $\theta \mapsto [0]_\theta$ $(\theta \in \operatorname{Con} \mathbf{A})$ *is a lattice isomorphism from* $\mathbf{Con\,A}$ *into* $\mathbf{I}(\mathbf{A})$.

Proof. For the first assertion of the proposition, the only non-trivial implication to prove is (3) $\Rightarrow$ (2). So let $\vee \in \mathcal{C}$ and let $\mathbf{A} \in \mathsf{BP}_{\mathcal{C}}$. Let $a, b \in A$ and suppose $(a \dot{-} b) \vee (b \dot{-} a) = 0$. We have $a \dot{-} b = (a \dot{-} b) \vee 0 = (a \dot{-} b) \vee (a \dot{-} b) \vee (b \dot{-} a) = (a \dot{-} b) \vee (b \dot{-} a) = 0$, just because the reduct $\langle A;\ \vee, 0\rangle$ is a band with identity. Similarly, $b \dot{-} a = 0 \vee (b \dot{-} a) = (a \dot{-} b) \vee (b \dot{-} a) \vee (b \dot{-} a) = (a \dot{-} b) \vee (b \dot{-} a) = 0$. Thus $a \dot{-} b = 0 = b \dot{-} a$ as required.

From the first assertion of the proposition and Proposition 1.2.6 it follows that any $\mathsf{BP}_{\mathcal{C}}$ is **0**-regular and hence (by Proposition 1.7.3) ideal determined. Thus for any $\mathbf{A} \in \mathsf{BP}_{\mathcal{C}}$ the map $\theta \mapsto [0]_\theta$ $(\theta \in \operatorname{Con} \mathbf{A})$ is a lattice isomorphism from $\mathbf{Con\,A}$ into $\mathbf{I}(\mathbf{A})$. ∎

Remark 3.3.38. Because any $\mathsf{BP}_{\mathcal{C}}$ is ideal determined, it has a finite basis of ideal terms (in the sense of [57]): see Chajda and Halas [57] or Ursini [221]. Nonetheless, in general we know of no simple description of the $\mathsf{BP}_{\mathcal{C}}$-ideals for any variety $\mathsf{BP}_{\mathcal{C}}$. In particular, for each choice of $\mathcal{C}$, there exists $\mathbf{A} \in \mathsf{BP}_{\mathcal{C}}$ such that $\mathrm{I}_{\mathsf{BP}_{\mathcal{C}}}(\mathbf{A}) \not\subseteq \mathrm{I}_{\mathsf{BCK}}(\langle A;\ \dot{-}, 0\rangle)$ (by an easy modification of a result due to Idziak [116, Lemma 2(ii)], for each choice of $\mathcal{C}$ the converse $\mathrm{I}_{\mathsf{BCK}}(\langle A;\ \dot{-}, 0\rangle) \subseteq \mathrm{I}_{\mathsf{BP}_{\mathcal{C}}}(\mathbf{A})$ does obtain for any $\mathbf{A} \in \mathsf{BP}_{\mathcal{C}}$). Indeed, examples due to Idziak [116, p. 979] show there exists $\mathbf{A} \in \mathsf{BP}_{\mathcal{C}}$ such that the inclusion $\mathrm{I}_{\mathsf{BP}_{\mathcal{C}}}(\mathbf{A}) \subseteq \mathrm{I}_{\mathsf{BCK}}\big(\langle A;\ \dot{-}, 0\rangle\big)$ is strict when $\vee \in \mathcal{C}$. To see the inclusion $\mathrm{I}_{\mathsf{BP}_{\mathcal{C}}}(\mathbf{A}) \subseteq \mathrm{I}_{\mathsf{BCK}}\big(\langle A;\ \dot{-}, 0\rangle\big)$ is strict for $\mathcal{C} = \{\wedge, \dot{-}, 0\}$, consider the following 4-element lower $\leq_0$-BCK band $\mathbf{A}$:

$\wedge^{\mathbf{A}}$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	c	c

$\dot{-}^{\mathbf{A}}$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	a	0	a
c	c	c	c	0

An easy sequence of checks shows that $\mathbf{A}$ is simple but that $\{0, a, b\}$ is a BCK-ideal of the BCK-algebra reduct $\langle A;\ \dot{-}, 0\rangle$. Hence $\mathrm{I}_{\mathsf{BP}_{\mathcal{C}}}(\mathbf{A}) \not\subseteq \mathrm{I}_{\mathsf{BCK}}\big(\langle A;\ \dot{-}, 0\rangle\big)$.
∎

Theorem 3.3.39. *For each $\mathcal{C}$, the variety $\mathsf{BP}_{\mathcal{C}}$ is congruence distributive.*

Proof. Because $\mathsf{BP}_{\mathcal{C}} \models (1.3), (1.4), (1.7)$, from the remarks of §1.7.4 we have that $\mathsf{BP}_{\mathcal{C}}$ is ideal distributive. Since $\mathsf{BP}_{\mathcal{C}}$ is ideal determined (by Proposition 3.3.37) we conclude that $\mathsf{BP}_{\mathcal{C}}$ is congruence distributive. ∎

Remark 3.3.40. It is easily verified that if $\vee \in \mathcal{C}$ then the term $M(x, y, z) := \big(x \dot{-} (x \dot{-} z)\big) \vee \big(z \dot{-} (z \dot{-} y)\big) \vee \big(y \dot{-} (y \dot{-} x)\big)$ is a majority term for $\mathsf{BP}_{\mathcal{C}}$ (see also Idziak [116, p. 841, Theorem 2]). In contrast, the varieties $\mathsf{BP}_{\mathcal{C}}$ have no majority term when $\vee \notin \mathcal{C}$: see Idziak [116, Remark, p. 842]. ∎

Proposition 3.3.41. *Concerning the varieties $\mathsf{BP}_{\mathcal{C}}$, the following assertions hold:*

1. *(cf. [187, Theorem]) If $\wedge \in \mathcal{C}$ then $\mathsf{BP}_{\mathcal{C}}$ is congruence 4-permutable;*

2. (cf. [116, p. 841, Theorem 2]) If $\vee \in \mathcal{C}$ *then* $\mathsf{BP}_{\mathcal{C}}$ *is congruence permutable, with Mal'cev term:*

$$p(x, y, z) := \big(x \mathbin{\dot{-}} (y \mathbin{\dot{-}} z)\big) \vee \big(z \mathbin{\dot{-}} (y \mathbin{\dot{-}} x)\big).$$

Proof. For (1), suppose $\wedge \in \mathcal{C}$. Let:

$$\begin{aligned} p_1(x, y, z) &:= x \mathbin{\dot{-}} (y \mathbin{\dot{-}} z) \\ p_2(x, y, z) &:= \big(x \mathbin{\dot{-}} (x \mathbin{\dot{-}} y)\big) \wedge \big(z \mathbin{\dot{-}} (z \mathbin{\dot{-}} y)\big) \\ p_3(x, y, z) &:= z \mathbin{\dot{-}} (y \mathbin{\dot{-}} x). \end{aligned}$$

By (1.7), (3.81), (3.82), (3.76) and (1.3), the variety $\mathsf{BP}_{\wedge\mathcal{C}}$ satisfies the identities:

$$\begin{aligned} p_1(x, y, y) &\approx x \\ p_i(x, x, y) &\approx p_{i+1}(x, y, y), \qquad\qquad i = 1, 2 \\ p_3(x, x, y) &\approx y \end{aligned}$$

and so is congruence 4-permutable by Hagemann and Mitschke [106, Theorem 2].

For (2), suppose $\vee \in \mathcal{C}$ and let $p(x, y, z)$ be as in the statement of the proposition. Because of (3.92) and (3.93), it is easily verified that $\mathsf{BP}_{\mathcal{C}}$ satisfies the identities $p(x, x, y) \approx y$ and $p(x, y, y) \approx x$. Hence $p(x, y, z)$ is a Mal'cev term for V and $\mathsf{BP}_{\mathcal{C}}$ is congruence permutable. ∎

Remark 3.3.42. The statement of Proposition 3.3.41(1) cannot be strengthened to the assertion that if $\wedge \in \mathcal{C}$ then $\mathsf{BP}_{\mathcal{C}}$ is congruence 3-permutable, in view of a result of Raftery [187, Theorem] showing that the variety of lower BCK-semilattices is not congruence 3-permutable. ∎

3.3.43. Implicative BCK Bands and Implicative BCK Paralattices. By the results of §1.4.32, §1.4.37, §3.2.6 and §3.2.22, algebras arising in binary discriminator, pointed fixedpoint discriminator and pointed ternary discriminator varieties all support an underlying 'locally Boolean' structure (in the sense of either §1.3.15 or §1.4.24). As in §3.3.13, this observation motivates the study of those members of $\mathsf{BP}_{\mathcal{C}}$ for which every (appropriately defined) principal subalgebra is a Boolean lattice. To begin, recall the definition of the BCK meet $\sqcap$ from §1.6.11. For a BCK-algebra $\mathbf{A}$ and any $a \in A$, let $(\boldsymbol{a}] := \langle a \sqcap A;\ \sqcap^{\mathbf{A}|_{a\sqcap A}} \rangle$. By analogy with semigroup theory we call $(\boldsymbol{a}]$ the *principal subalgebra generated by* a, even though the polynomial reduct $\langle A;\ \sqcap \rangle$ is not a semigroup in general.

Remark 3.3.44. For a BCK-algebra $\mathbf{A}$, the principal subalgebra $(\boldsymbol{a}]$ generated by $a \in A$ should not be confused with the BCK-subalgebra $\langle (a]_0;\ \dot{-}^{\mathbf{A}|_{(a]}}, 0 \rangle$, where $(a]_0 := \{b : b \leq_0 a\}$, and which is also denoted $(\boldsymbol{a}]$. In the sequel, it will always be clear from context whether $(\boldsymbol{a}]$ denotes the BCK-subalgebra $\langle (a]_0;\ \dot{-}^{\mathbf{A}|_{(a]}}, 0 \rangle$ or the principal subalgebra $\langle a \sqcap A;\ \sqcap^{\mathbf{A}|_{a\sqcap A}} \rangle$ generated by a.

For a BCK-algebra $\mathbf{A}$, the principal subalgebra $(\boldsymbol{a}]$ generated by $a \in A$ may be alternatively defined as $\langle (a]_1;\ \sqcap^{\mathbf{A}|_{a\sqcap A}} \rangle$, where $(a]_1 := \{b : b \leq_1 a\}$, because $\{b : b \leq_1 a\} = a \sqcap A$ by Proposition 1.6.4(1). From this observation it follows that if $\mathbf{A}$ is commutative then $(\boldsymbol{a}] = \langle (a]_0;\ \sqcap^{\mathbf{A}|_{(a]}} \rangle$, where $(a]_0 := \{b : b \leq_0 a\}$, since in this case $\leq_0 = \leq_1$ by Lemma 1.6.12. In other words, if $\mathbf{A}$ is commutative, then $(\boldsymbol{a}]$ is precisely the principal subalgebra generated by a of the semilattice polynomial reduct $\langle A;\ \sqcap \rangle$. It is this observation that motivates our description of the algebra $\langle a \sqcap A;\ \sqcap^{\mathbf{A}|_{a\sqcap A}} \rangle$ above as the principal subalgebra generated by a. ∎

As per §3.3.13, our study of varieties $\mathsf{BP}_{\mathcal{C}}$ supporting a 'locally Boolean' structure centres on the interplay of the underlying band with zero principal subalgebra structure and the underlying BCK-algebra principal subalgebra structure. In more detail, let $\wedge \in \mathcal{C}$ and let $\mathbf{A} \in \mathsf{BP}_{\mathcal{C}}$. Then $\mathbf{A}$ has both a band with zero reduct $\langle A;\ \wedge, 0 \rangle$ and a BCK-algebra reduct $\langle A;\ \sqcap, 0 \rangle$. Hence every $a \in A$ generates both a principal subalgebra $(\boldsymbol{a}]_{\langle A;\ \wedge, 0\rangle}$ of the band with zero reduct $\langle A;\ \wedge, 0 \rangle$ (recall Lemma 1.3.13) and a principal subalgebra $(\boldsymbol{a}]_{\langle A;\ \dot{-}, 0\rangle}$ of the BCK-algebra reduct $\langle A;\ \sqcap, 0 \rangle$. Although the principal subalgebra $(\boldsymbol{a}]_{\langle A;\ \dot{-}, 0\rangle}$ generated by a of the BCK-algebra reduct $\langle A;\ \dot{-}, 0 \rangle$ is not a principal subalgebra in the usual semigroup theoretic sense, the following two results show nonetheless that for the varieties $\mathsf{BP}_{\mathcal{C}}$, $\wedge \in \mathcal{C}$, the behaviour of the underlying BCK-algebra principal subalgebra structure is decisive (*cf.* Proposition 3.3.18).

Proposition 3.3.45. *Let* $\wedge \in \mathcal{C}$ *and let* $\mathbf{A} \in \mathsf{BP}_{\mathcal{C}}$. *For every* $a \in A$, *the principal subalgebras* $(\boldsymbol{a}]_{\langle A;\, \wedge,0\rangle}$ *and* $(\boldsymbol{a}]_{\langle A;\, \dot{-},0\rangle}$ *coincide iff the reduct* $\langle A;\ \wedge, 0\rangle$ *is a normal band with zero and the reduct* $\langle A;\ \dot{-}, 0\rangle$ *is a commutative BCK-algebra.*

Proof. Let $\wedge \in \mathcal{C}$ and let $\mathbf{A} \in \mathsf{BP}_{\mathcal{C}}$. Throughout the proof, we denote by $(a]_{\langle A;\, \wedge,0\rangle}$ and $(a]_{\langle A;\, \dot{-},0\rangle}$ the respective universes of the principal subalgebras $(\boldsymbol{a}]_{\langle A;\, \wedge,0\rangle}$ and $(\boldsymbol{a}]_{\langle A;\, \dot{-},0\rangle}$. To simplify notation, we may also write simply $(a]_{\mathbf{A}}$ for both $(a]_{\langle A;\, \wedge,0\rangle}$ and $(a]_{\langle A;\, \dot{-},0\rangle}$ when these sets coincide.

($\Rightarrow$) Suppose that for every $a \in A$, the principal subalgebras $(\boldsymbol{a}]_{\langle A;\, \wedge,0\rangle}$ and $(\boldsymbol{a}]_{\langle A;\, \dot{-},0\rangle}$ coincide. To see the reduct $\langle A;\ \dot{-}, 0\rangle$ is a commutative BCK-algebra, it is sufficient by Lemma 1.6.12 to show the partial orders $\leq_0^{\langle A;\, \dot{-},0\rangle}$ and $\leq_1^{\langle A;\, \dot{-},0\rangle}$ coincide. For this, observe that for any $a, b \in A$,

$$
\begin{array}{rlll}
b \leq_1^{\langle A;\, \dot{-},0\rangle} a & \text{iff} & b \in a \cap A & \text{by Proposition 1.6.4(1)} \\
& \text{iff} & b \in (a]_{\langle A;\, \dot{-},0\rangle} & \\
& \text{iff} & b \in (a]_{\langle A;\, \wedge,0\rangle} & \text{by hypothesis} \\
& \text{iff} & b \leq_{\mathcal{H}} a & \text{by Lemma 1.3.13(1)} \\
& \text{iff} & b \leq_0^{\langle A;\, \dot{-},0\rangle} a & \text{since } \mathbf{A} \in \mathsf{BP}_{\mathcal{C}}.
\end{array}
$$

Hence $\leq_0^{\langle A;\, \dot{-},0\rangle}$ and $\leq_1^{\langle A;\, \dot{-},0\rangle}$ coincide, and $\langle A;\ \dot{-}, 0\rangle$ is commutative.

It remains to show the band with zero reduct $\langle A;\ \wedge, 0\rangle$ is normal. To this end, recall from §1.6.11 that since the BCK-algebra reduct $\langle A;\ \dot{-}, 0\rangle$ is commutative, it has a distributive nearlattice polynomial reduct $\langle A;\ \sqcap\rangle$. For any $a \in A$, let $(\boldsymbol{a}]_{\langle A;\, \sqcap\rangle}$ denote the principal subalgebra of $\langle A;\ \sqcap\rangle$ generated by a. Because $\langle A;\ \sqcap\rangle$ is a distributive nearlattice, every principle subalgebra $(\boldsymbol{a}]_{\langle A;\, \sqcap\rangle}$ is a distributive sublattice by the remarks of §1.6.11. Because $(\boldsymbol{a}]_{\langle A;\, \sqcap\rangle}$ coincides with $(\boldsymbol{a}]_{\langle A;\, \dot{-},0\rangle}$ (by commutativity of $\langle A;\ \dot{-}, 0\rangle$ and Remark 3.3.44), and $(\boldsymbol{a}]_{\langle A;\, \dot{-},0\rangle}$ coincides with $(\boldsymbol{a}]_{\langle A;\, \wedge,0\rangle}$ (by hypothesis), we have that $(\boldsymbol{a}]_{\langle A;\, \wedge,0\rangle}$ is a distributive sublattice. Hence for every $a \in A$, the principal subalgebra

$(\boldsymbol{a}]_{\langle A;\, \wedge, 0\rangle}$ generated by a is a sublattice. By Lemma 1.3.16 the band with zero reduct $\langle A;\ \wedge, 0\rangle$ is normal, and the proof is complete.

($\Leftarrow$) Suppose the reducts $\langle A;\ \wedge, 0\rangle$ and $\langle A;\ \dot{-}, 0\rangle$ are respectively a normal band with zero and a commutative BCK-algebra. To see the principal subalgebras $(\boldsymbol{a}]_{\langle A;\, \wedge,0\rangle}$ and $(\boldsymbol{a}]_{\langle A;\ \dot{-},0\rangle}$ coincide for every $a \in A$ it is sufficient to show:

(i) For any $a \in A$, $(a]_{\langle A;\, \wedge,0\rangle} = (a]_{\langle A;\ \dot{-},0\rangle}$;
(ii) If $b, c \in (a]_{\mathbf{A}}$ then $b \wedge c = b \cap c$.

For (i), just observe that for any $a \in A$, $(a]_{\langle A;\, \wedge,0\rangle} = \{b : b \leq_{\mathcal{H}} a\} = \{b : b \leq_0^{\langle A;\ \dot{-},0\rangle} a\} = \{b : b \leq_1^{\langle A;\ \dot{-},0\rangle} a\} = (a]_{\langle A;\ \dot{-},0\rangle}$ by commutativity of $\langle A;\ \dot{-}, 0\rangle$ and the coincidence of the partial orders $\leq_{\mathcal{H}}$ and $\leq_0^{\langle A;\, \backslash,0\rangle}$.

For (ii), by (i) we have that $(a]_{\langle A;\, \wedge,0\rangle}$ and $(a]_{\langle A;\ \dot{-},0\rangle}$ coincide, so the reference to $(a]_{\mathbf{A}}$ makes sense. Let $b, c \in (a]_{\mathbf{A}}$. Since $\langle A;\ \wedge, 0\rangle$ is normal, $b \wedge c$ is the greatest lower bound of $\{b, c\}$ with respect to the restriction of $\leq_{\mathcal{H}}$ to $(a]_{\langle A;\, \wedge,0\rangle}$. Also, $b \cap c$ is the greatest lower bound of $\{b, c\}$ with respect to the restriction of $\leq_0^{\langle A;\ \dot{-},0\rangle}$ to $(a]_{\langle A;\ \dot{-},0\rangle}$, because $\langle A;\ \dot{-}, 0\rangle$ is commutative. Since $(a]_{\langle A;\, \wedge,0\rangle} = (a]_{\langle A;\ \dot{-},0\rangle}$ (by (i)) and $\leq_{\mathcal{H}} = \leq_0^{\langle A;\ \dot{-},0\rangle}$ (by our assumptions on $\mathbf{A}$), we have that $b \wedge c = b \cap c$ as desired. ∎

Remark 3.3.46. Let $\mathbf{A}$ be a BCK-algebra and let $a \in A$. Clearly, the definition of the principal subalgebra $(\boldsymbol{a}]$ generated by a plays a crucial role in the preceding proof. In particular, if $(\boldsymbol{a}]$ is instead defined as the algebra $\langle A \cap a;\ \cap^{\mathbf{A}|_{A\cap a}}\rangle$, then the argument of the proof is not sufficient to establish Proposition 3.3.45, since in this case $A \cap a = \{b : b \leq_0 a\}$. ∎

Let $\mathbf{A} \in \mathsf{BP}_{\mathcal{C}}$, $\wedge \in \mathcal{C}$, be such that the reduct $\langle A;\ \wedge, 0\rangle$ is a normal band with zero and the reduct $\langle A;\ \dot{-}, 0\rangle$ is a commutative BCK-algebra. In view of Proposition 3.3.45, we can unambiguously write $(\boldsymbol{a}]_{\mathbf{A}}$ for the principal subalgebra generated by a; we adopt this practice henceforth.

Proposition 3.3.47. *Let $\wedge \in \mathcal{C}$ and let $\mathbf{A} \in \mathsf{BP}_{\mathcal{C}}$ be such that for every $a \in A$, the principal subalgebras $(\boldsymbol{a}]_{\langle A;\, \wedge,0\rangle}$ and $(\boldsymbol{a}]_{\langle A;\ \dot{-},0\rangle}$ coincide. Then for every $a \in A$, the principal subalgebra $(\boldsymbol{a}]_{\mathbf{A}}$ is a Boolean sublattice iff the reduct $\langle A;\ \dot{-}, 0\rangle$ is an implicative BCK-algebra.*

Proof. Let $\wedge \in C$ and let $\mathbf{A} \in \mathsf{BP}_C$ be such that for every $a \in A$, the principal subalgebras $(a]_{\langle A;\ \wedge,0\rangle}$ and $(a]_{\langle A;\ \dot{-},0\rangle}$ coincide.

($\Rightarrow$) Suppose that, for every $a \in A$, the principal subalgebra $(a]_{\mathbf{A}}$ is a Boolean sublattice. By our assumptions on $\mathbf{A}$ and Proposition 3.3.45, the band with zero reduct $\langle A;\ \wedge, 0\rangle$ is normal and the BCK-algebra reduct $\langle A;\ \dot{-}, 0\rangle$ is commutative. Since $\langle A;\ \dot{-}, 0\rangle$ is commutative, $\mathbf{A}$ possesses a meet semilattice polynomial reduct $\langle A;\ \cap\rangle$. For any $a \in A$, let $(a]_{\langle A;\, \cap\rangle}$ denote the principal subalgebra of $\langle A;\ \cap\rangle$ generated by a. Because $b \leq^{\langle A;\, \cap\rangle} c$ iff $b \leq^{\langle A;\ \dot{-},0\rangle} c$ for any $b, c \in A$ and the band with zero reduct $\langle A;\ \wedge, 0\rangle$ is normal, an argument similar to the proof of Proposition 3.3.45 shows that for every $a \in A$, $(a]_{\langle A;\ \wedge,0\rangle}$ and $(a]_{\langle A;\, \cap\rangle}$ coincide. Hence the principal subalgebra $(a]_{\langle A;\, \cap\rangle}$ is a Boolean lattice for every $a \in A$, and so $\langle A;\ \cap\rangle$ is semi-Boolean. From Theorem 1.6.21 it follows that $\langle A;\ \cap\rangle$ has an induced implicative BCK difference operation $/$, where $b/c := (b \cap c)'_{(b]_{\langle A;\, \cap\rangle}}$ for any $b, c \in A$. But for any $b, c \in A$,

$$
\begin{aligned}
b/c &= (b \cap c)'_{(b]_{\langle A;\, \cap\rangle}} \\
&= (b \dot{-} (b \dot{-} c))'_{(b]_{\langle A;\ \dot{-},0\rangle}} \\
&= b \dot{-} \big(b \dot{-} (b \dot{-} c)\big) \\
&= b \dot{-} c && \text{by (1.31),}
\end{aligned}
$$

so $b/c = b \dot{-} c$. Hence the reduct $\langle A;\ \dot{-}, 0\rangle$ is an implicative BCK-algebra.

($\Leftarrow$) Suppose the reduct $\langle A;\ \dot{-}, 0\rangle$ is an implicative BCK-algebra. Then for every $a \in A$, the principal subalgebra $(a]_{\mathbf{A}}$ is a Boolean lattice by our assumptions on $\mathbf{A}$ and Corollary 1.6.22. ∎

Unless otherwise specified, throughout the remainder of this section we assume $\wedge \in C''$. Given this convention, Proposition 3.3.47 leads to the study in the sequel of those members of BP_C for which the band with zero reduct $\langle A;\ \wedge, 0\rangle$ is normal and the BCK-algebra reduct $\langle A;\ \dot{-}, 0\rangle$ is implicative.

A *lower implicative* $\leq_0$*-BCK normal band* is an algebra $\langle A;\ \wedge, /, 0\rangle$ of type $\langle 2, 2, 0\rangle$ such that: (i) the reduct $\langle A;\ \wedge, 0\rangle$ is a normal band with zero; (ii) the reduct $\langle A;\ /, 0\rangle$ is an implicative BCK-algebra; and (iii) the natural band partial order $\leq_{\mathcal{H}}$ coincides with the BCK partial order $\leq_0^{\langle A;\ /,0\rangle}$.

Theorem 3.3.48. *An algebra* $\mathbf{A} := \langle A;\ \wedge, /, 0\rangle$ *of type* $\langle 2, 2, 0\rangle$ *is a lower implicative* $\leq_0$*-BCK normal band iff the reduct* $\langle A;\ \wedge, 0\rangle$ *is a normal band with zero, the reduct* $\langle A;\ /, 0\rangle$ *is an implicative BCK-algebra, and* $\mathbf{A} \models (3.83), (3.84), (3.85)$. *Thus the class of lower implicative* $\leq_0$*-BCK normal bands is a variety.*

Proof. Let $\mathbf{A} := \langle A;\ \wedge, /, 0\rangle$ be an algebra of type $\langle 2, 2, 0\rangle$ such that the reduct $\langle A;\ \wedge, 0\rangle$ is a normal band with zero and the reduct $\langle A;\ /, 0\rangle$ is an implicative BCK-algebra. Let $a, b \in A$. To prove the theorem it is sufficient to show:

(i) $(a \wedge b \wedge a) \dot{-} a = 0$ iff $a \leq_{\mathcal{H}} b$ implies $a \leq_0^{\langle A;\ /,0\rangle} b$;
(ii) $(a \dot{-} b) \wedge a = a \dot{-} b = a \wedge (a \dot{-} b)$ iff $a \leq_0^{\langle A;\ /,0\rangle} b$ implies $a \leq_{\mathcal{H}} b$.

The proof of (i) is implicit in the proof of Theorem 3.3.28. For (ii), the remarks immediately following Theorem 3.3.28 imply $(a \dot{-} b) \wedge a = a \dot{-} b = a \wedge (a \dot{-} b)$ iff $a \leq_1^{\langle A;\ /,0\rangle} b$ implies $a \leq_{\mathcal{H}} b$. Since the BCK partial orders $\leq_0^{\langle A;\ /,0\rangle}$ and $\leq_1^{\langle A;\ /,0\rangle}$ coincide (by the proof of Proposition 3.3.45), the result follows. ∎

By analogy with the theory of skew lattices, call a paralattice $\langle A;\ \wedge, \vee\rangle$ *local* if its band reduct $\langle A;\ \wedge\rangle$ is normal. An *implicative* $\leq_0$*-BCK local paralattice* is an algebra $\langle A;\ \wedge, \vee, /, 0\rangle$ of type $\langle 2, 2, 2, 0\rangle$ such that: (i) the reduct $\langle A;\ \wedge, \vee, 0\rangle$ is a local paralattice with zero; (ii) the reduct $\langle A;\ /, 0\rangle$ is an implicative BCK-algebra; and (iii) the natural band partial order $\leq_{\mathcal{H}}$ coincides with the BCK partial order $\leq_0^{\langle A;\ /,0\rangle}$. The proof of the following result may be established by an argument similar to the proof of Theorem 3.3.48 and hence is omitted.

Theorem 3.3.49. *An algebra* $\mathbf{A} := \langle A;\ \wedge, /, 0\rangle$ *of type* $\langle 2, 2, 0\rangle$ *is an implicative* $\leq_0$*-BCK local paralattice iff the reduct* $\langle A;\ \wedge, \vee, 0\rangle$ *is a local paralattice with zero, the reduct* $\langle A;\ /, 0\rangle$ *is an implicative BCK-algebra, and* $\mathbf{A} \models (3.83), (3.95), (3.96)$. *Thus the class of implicative* $\leq_0$*-BCK local paralattices is a variety.*

For the sake of notational consistency with the prequel, throughout the remainder of this subsection let C'' denote an arbitrary subset of the language

$\{\wedge, \vee, /, 0\}$ of implicative $\leq_0$-BCK local paralattices that contains either $/$ and 0 or all of $\wedge, /$ and 0. Ignoring issues of type, let:

- $\mathsf{BP}_{C''}$ denote the quasivariety of BCK-algebras when $C'' = \{/, 0\}$;
- $\mathsf{BP}_{C''}$ denote the variety of lower $\leq_0$-BCK bands when $C'' = \{\wedge, /, 0\}$;
- $\mathsf{BP}_{C'}$ denote the variety of $\leq_0$-BCK paralattices when $C'' = \{\wedge, \vee, /, 0\}$.

Also, let:

- $\mathsf{IP}_{C''}$ denote the variety of implicative BCK-algebras when $C'' = \{/, 0\}$;
- $\mathsf{IP}_{C''}$ denote the variety of normal lower implicative $\leq_0$-BCK bands when $C'' = \{\wedge, /, 0\}$;
- $\mathsf{IP}_{C''}$ denote the variety of local implicative $\leq_0$-BCK paralattices when $C'' = \{\wedge, \vee, /, 0\}$.

Given the above notation, clearly $\mathsf{IP}_{C''} \subseteq \mathsf{BP}_{C''}$ for any fixed choice of C''.

Remark 3.3.50. An *upper implicative* $\leq_0$*-BCK band* is an algebra $\langle A;\ \vee, /, 0\rangle$ of type $\langle 2, 2, 0\rangle$ such that: (i) the reduct $\langle A;\ \vee, 0\rangle$ is a band with identity; (ii) the reduct $\langle A;\ /, 0\rangle$ is an implicative BCK-algebra; and (iii) the natural band partial order $\leq_{\mathcal{H}}$ dualises the BCK partial order $\leq_0^{\langle A;\, /, 0\rangle}$ in the sense that $a \leq_{\mathcal{H}} b$ iff $b \leq_0^{\langle A;\, /, 0\rangle} a$ for any $a, b \in A$. Clearly the class of upper implicative $\leq_0$-BCK bands is a variety.

By normality, it is clear that no non-trivial (that is, non-commutative) normal subvariety of the variety of upper implicative $\leq_0$-BCK bands exists. Hence there exists no non-trivial variety of upper implicative $\leq_0$-BCK normal bands that stands in relation to the variety of upper implicative BCS bands as the varieties $\mathsf{IP}_{C'}$, $\{\wedge, \backslash, 0\} \subseteq C'$, stand in relation to the varieties $\mathsf{IP}_{C''}$, $\{\wedge, /, 0\} \subseteq C''$. For consistency and the sake of parity with the theory of the varieties $\mathsf{PQ}_{C'}$, $\mathsf{IQ}_{C'}$, we let:

- $\mathsf{BP}_{C''}$ denote the variety of upper $\leq_0$-BCK bands when $C'' = \{\vee, /, 0\}$;
- $\mathsf{IP}_{C''}$ denote the variety of upper implicative $\leq_0$-BCK bands when $C'' = \{\vee, /, 0\}$,

even though our interest in this thesis lies exclusively in the varieties $\mathsf{IP}_{\mathcal{C}''}$, $\wedge \in \mathcal{C}''$. That the preceding definitions are coherent in the context of a unified theory of the varieties $\mathsf{BP}_{\mathcal{C}''}$, $\{/, 0\} \subseteq \mathcal{C}'' \subseteq \{\wedge, \vee, /, 0\}$, follows from Theorem 3.3.53 and the remarks of §4.2.27 in the sequel. ∎

Proposition 3.3.51. *Let* $\mathbf{A} \in \mathsf{IP}_{\mathcal{C}''}$. *Then the polynomial reduct* $\langle A;\ \backslash, 0\rangle$ *is an implicative BCS-algebra and the polynomial reduct* $\langle A;\ \wedge, \backslash, 0\rangle$ *is a lower implicative BCS band, where in both cases* $a\backslash b := a/(a \wedge b \wedge a)$ *for any* $a, b \in A$. *Consequently, if* $\vee \in \mathcal{C}''$ *then the polynomial reduct* $\langle A;\ \wedge, \vee, \backslash, 0\rangle$ *is an implicative BCS quasilattice. In particular, if* $\vee \in \mathcal{C}''$ *and the paralattice with zero reduct* $\langle A;\ \wedge, \vee, 0\rangle$ *is a skew lattice with zero, then the polynomial reduct* $\langle A;\ \wedge, \vee, \backslash, 0\rangle$ *is an implicative BCS skew lattice.*

Proof. Let $\mathbf{A} \in \mathsf{IP}_{\mathcal{C}''}$. For the first assertion, by Theorem 1.6.21, Proposition 3.3.47 and the definition of $\backslash$ we have that $a\backslash b = \big(a \sqcap (a \wedge b \wedge a)\big)'_{(a]_{\mathbf{A}}}$ for any $a, b \in A$. But,

$$\begin{aligned}
a \sqcap (a \wedge b \wedge a) &= (a \wedge b \wedge a) \sqcap a && \text{by commutativity of } \sqcap\\
&= (a \wedge b \wedge a)/\big((a \wedge b \wedge a)/a\big)\\
&= (a \wedge b \wedge a)/0 && \text{by (3.83)}\\
&= a \wedge b \wedge a, && (3.98)
\end{aligned}$$

so $a\backslash b = (a \wedge b \wedge a)'_{(a]_{\mathbf{A}}}$. From Remark 2.3.37 it follows that the polynomial reduct $\langle A;\ \backslash, 0\rangle$ is an implicative BCS-algebra. To prove the remaining assertions of the proposition it is sufficient to show the induced implicative BCS-algebra partial order $\leq^{\langle A;\ \backslash, 0\rangle}$ and quasiorder $\preceq^{\langle A;\ \backslash, 0\rangle}$ coincide with the natural band partial order $\leq_{\mathcal{H}}$ and quasiorder $\preceq_{\mathcal{D}}$ respectively. Because of Proposition 3.3.14, this reduces to establishing that $a\backslash(a\backslash b) = a \wedge b \wedge a$ for any $a, b \in A$. So let $a, b \in A$. We have:

$$\begin{aligned}
a\backslash(a\backslash b) &= a/\big(a \wedge (a/(a \wedge b \wedge a)) \wedge a\big)\\
&= a/\big(a/(a \wedge b \wedge a)\big) && \text{by (3.84), (3.85)}\\
&= a \sqcap (a \wedge b \wedge a)\\
&= a \wedge b \wedge a && \text{by (3.98).}
\end{aligned}$$

■

Proposition 3.3.52. *Let* $\mathcal{V} \in \mathcal{C}''$ *and let* $\mathbf{A} \in \mathsf{IP}_{\mathcal{C}''}$. *If the paralattice with zero reduct* $\langle A;\ \wedge, \vee, 0\rangle$ *is a skew lattice with zero, then the following assertions hold:*

1. *The skew lattice with zero reduct* $\langle A;\ \wedge, \vee, 0\rangle$ *is distributive local;*

2. *The polynomial reduct* $\langle A;\ \wedge, \vee, \cap\rangle$ *is a skew lattice with intersections, where* $a \cap b := a/(a/b)$ *for any* $a, b \in A$.

Proof. Let $\mathcal{V} \in \mathcal{C}''$ and let $\mathbf{A} \in \mathsf{IP}_{\mathcal{C}''}$ be such that the paralattice with zero reduct $\langle A;\ \wedge, \vee, 0\rangle$ is a skew lattice with zero.

For (1), we have that $\langle A;\ \wedge, \vee, 0\rangle$ is a local skew lattice by hypothesis. Since for each $a \in A$, the sublattice $(\boldsymbol{a}]_{\mathbf{A}}$ is distributive (by the proof of Proposition 3.3.45), from Proposition 1.4.22 we deduce that $\langle A;\ \wedge, \vee, 0\rangle$ is distributive.

For (2), the reduct $\langle A;\ \wedge, \vee\rangle$ is a skew lattice, while the reduct $\langle A;\ \cap\rangle$ is a meet semilattice. Since $a \cap b = a$ iff $a/b = 0$ for any $a, b \in A$, the underlying partial order $\leq^{A;\,\cap}$ on the semilattice polynomial reduct $\langle A;\ \cap\rangle$ coincides with the underlying BCK partial order $\leq_0^{\langle A;\ \dot{-},0\rangle}$ on $\langle A;\ \dot{-}, 0\rangle$. Since $\leq_0^{\langle A;\ \dot{-}0\rangle}$ and $\leq_{\mathcal{H}}$ coincide by hypothesis, $\leq^{\langle A;\,\cap\rangle}$ and $\leq_{\mathcal{H}}$ must coincide also. Therefore the polynomial reduct $\langle A;\ \wedge, \vee, \cap\rangle$ is a skew lattice with intersections. ■

An *implicative* $\leq_0$*-BCK local skew lattice* is an algebra $\langle A;\ \wedge, \vee, \backslash, 0\rangle$ of type $\langle 2, 2, 2, 0\rangle$ such that: (i) the reduct $\langle A;\ \wedge, \vee, 0\rangle$ is a local skew lattice with zero; (ii) the reduct $\langle A;\ \backslash, 0\rangle$ is an implicative BCK-algebra; (iii) the natural skew lattice partial order $\leq_{\mathcal{H}}$ coincides with the implicative BCK-algebra partial order $\leq_0^{\langle A;\,/,0\rangle}$. Clearly the class of implicative $\leq_0$-BCK local skew lattices is a subvariety of the variety of implicative $\leq_0$-BCK local paralattices.

Proposition 3.3.52 and Proposition 3.3.51 direct attention towards those members of $\mathsf{IP}_{\mathcal{C}''}$, $\mathcal{V} \in \mathcal{C}''$, that are implicative $\leq_0$-BCK local skew lattices, inasmuch as these algebras enjoy several important structural properties of skew Boolean $\cap$-algebras. In particular, if $\langle A;\ \wedge, \vee, /, 0\rangle$ is an implicative $\leq_0$-BCK local skew lattice, then: (i) the skew lattice with zero reduct $\langle \wedge, \vee, 0\rangle$ is distributive local; (ii) the polynomial reduct $\langle A;\ \backslash, 0\rangle$ is an implicative BCS-algebra; (iii) the polynomial reduct $\langle A;\ \cap\rangle$ is a meet semilattice; and (iv) for every $a \in A$, the principal subalgebra $(\mathbf{a}]$ generated by a is a Boolean sublattice. The precise relationship between implicative $\leq_0$-BCK local skew lattices and skew Boolean $\cap$-algebras is clarified in the following theorem, a version of which is asserted with second-order proof in Bignall and Leech [19, Section 4]. For the sake of completeness, we provide a direct proof here.

Theorem 3.3.53. *(cf. [19, Theorem 4.2]) A skew Boolean $\cap$-algebra is term equivalent to an algebra $\langle A;\ \wedge, \vee, /, 0\rangle$ of type $\langle 2, 2, 2, 0\rangle$ where:*

1. *The reduct $\langle A;\ \wedge, \vee, 0\rangle$ is a join symmetric local skew lattice with zero;*
2. *The reduct $\langle A;\ /, 0\rangle$ is an implicative BCK-algebra;*
3. *The natural skew lattice partial order $\leq_{\mathcal{H}}$ and the BCK partial order $\leq_0^{\langle A;\ /, 0\rangle}$ coincide.*

In particular, given such an algebra $\langle A;\ \wedge, \vee, /, 0\rangle$, standard difference $\backslash$ and the intersection operation $\cap$ are respectively defined on A by:

$$a \backslash b := a/(a \wedge b \wedge a) \quad \text{and} \quad a \cap b := a/(a/b)$$

for any $a, b \in A$. Conversely, given a skew Boolean $\cap$-algebra $\langle A;\ \wedge, \vee, \backslash, \cap, 0\rangle$ and $a, b \in A$, implicative BCK difference $/$ is defined on A by:

$$a/b := a \backslash (a \cap b).$$

Thus the variety of skew Boolean $\cap$-algebras is termwise definitionally equivalent to the variety of implicative $\leq_0$-BCK join symmetric local skew lattices.

Proof. Let $\mathbf{A} := \langle A;\ \wedge, \vee, /, 0\rangle$ be an algebra of type $\langle 2, 2, 2, 0\rangle$ satisfying Conditions (1)–(3) of the theorem. Then $\mathbf{A} \in \mathsf{IP}_{C''}$, $C'' = \{\wedge, \vee, /, 0\}$. To see the derived algebra $\langle A;\ \wedge, \vee, \backslash, \cap, 0\rangle$ (where $a \backslash b := a/(a \wedge b \wedge a)$ and $a \cap b := a/(a/b)$ for any $a, b \in A$) is a skew Boolean $\cap$-algebra it is sufficient to show:

(i) The polynomial reduct $\langle A;\ \wedge, \vee, \backslash, 0\rangle$ is a skew Boolean algebra;
(ii) The polynomial reduct $\langle A;\ \cap\rangle$ is a meet semilattice;

(iii) The induced semilattice partial order $\leq^{\langle A;\, \cap\rangle}$ coincides with the natural skew lattice partial order $\leq_{\mathcal{H}}$.

For (i), because $\mathbf{A} \in \mathsf{IP}_{\mathcal{C}''}$, $\vee \in \mathcal{C}''$, from Proposition 3.3.51 we have that the polynomial reduct $\langle A;\ \wedge, \vee, \backslash, 0\rangle$ is an implicative BCS skew lattice. Since the skew lattice with zero reduct $\langle A;\ \wedge, \vee, 0\rangle$ is join symmetric, from Theorem 3.3.21 it follows that the polynomial reduct $\langle A;\ \wedge, \vee, \backslash, 0\rangle$ is a skew Boolean algebra.

For (ii), because $\mathbf{A} \in \mathsf{IP}_{\mathcal{C}''}$, $\vee \in \mathcal{C}''$, from Proposition 3.3.52(2) we have that polynomial reduct $\langle A;\ \wedge, \vee, \cap\rangle$ is a skew lattice with intersections.

For (iii), we have $a \leq^{\langle A;\, \cap\rangle} b$ iff $a \leq_0^{\langle A;\, /, 0\rangle} b$ for any $a, b \in A$. Since $\mathbf{A} \in \mathsf{IP}''_{\mathcal{C}}$, we infer that $a \leq^{\langle A;\, \cap\rangle} b$ iff $a \leq_{\mathcal{H}} b$ for any $a, b \in A$. Hence the induced semilattice partial order $\leq^{\langle A;\, \cap\rangle}$ coincides with the natural skew lattice partial order $\leq_{\mathcal{H}}$.

For the converse, let $\mathbf{A} := \langle A;\ \wedge, \vee, \backslash, \cap, 0\rangle$ be a skew Boolean $\cap$-algebra. We verify Conditions (1)–(3) above are satisfied.

For (1), the reduct $\langle A;\ \wedge, \vee, 0\rangle$ is a distributive symmetric local skew lattice with zero by definition, and so in particular is a join symmetric local skew lattice with zero.

For (2), we have that the reduct $\langle A;\ \cap\rangle$ is a meet semilattice by definition. For any $a \in A$, let $(a]_{\langle A;\, \cap\rangle}$ denote the principal subalgebra of $\langle A;\ \cap\rangle$ generated by a. Because of Proposition 1.4.34 and the locality of the skew lattice with zero reduct $\langle A;\ \wedge, \vee, 0\rangle$, an argument similar to the proof of Proposition 3.3.45 shows that for every $a \in A$, the principal subalgebra $(a]_{\langle A;\, \wedge, \vee, 0\rangle}$ coincides with $(a]_{\langle A;\, \cap\rangle}$. Without loss of generality, therefore, throughout the remainder of the proof we can and will write $(a]_{\mathbf{A}}$ for both $(a]_{\langle A;\, \wedge, \vee, 0\rangle}$ and $(a]_{\langle A;\, \cap\rangle}$. Since $\langle A;\ \wedge, \vee, 0\rangle$ is locally Boolean (in the sense of §1.4.24), $(a]_{\mathbf{A}}$ is a Boolean lattice for every $a \in A$. Hence $\langle A;\ \cap\rangle$ is semi-Boolean. From Theorem 1.6.21 it follows that $\langle A;\ \cap\rangle$ possesses an induced implicative BCK difference operation $/$, where $a/b := (a \cap b)'_{(a]_{\mathbf{A}}}$ for any $a, b \in A$. To see $a/b = a\backslash(a \cap b)$, note:

(i) $a\backslash(a \cap b)$ is the complement of $a \wedge (a \cap b) \wedge a$ in $(a]_{\mathbf{A}}$ by definition of standard difference;

(ii) $a \wedge (a \cap b) \wedge a = a \cap b$ by Proposition 1.4.34.

Because of (i) and (ii), we have that $a\backslash(a \cap b)$ is the complement of $a \cap b$ in $(a]_{\mathbf{A}}$; that is to say $a\backslash(a \cap b) = (a \cap b)'_{(a]_{\mathbf{A}}}$. Hence the induced implicative BCK difference $/$ is term definable, and $\mathbf{A}$ has an implicative BCK-algebra polynomial reduct $\langle A;\ /, 0\rangle$, where $a/b := a\backslash(a \cap b)$ for any $a, b \in A$.

For (3), it is sufficient to show the BCK partial ordering $\leq_0^{\langle A;\,/,0\rangle}$ coincides with the semilattice partial ordering $\leq^{\langle A;\,\cap\rangle}$. So suppose $a \leq_0^{\langle A;\,/,0\rangle} b$. Then $0 = a/b = a\backslash(a \cap b)$, so $0 = \big(a \wedge (a \cap b) \wedge a\big)^*_{(a]_{\mathbf{A}}} = (a \cap b)^*_{(a]_{\mathbf{A}}}$, which implies $a \cap b = a$. Thus $a \leq^{\langle A;\,\cap\rangle} b$. On the other hand, from $a \leq^{\langle A;\,\cap\rangle} b$ we have $a \cap b = a$, whence $a/b = a\backslash(a \cap b) = \big(a \wedge (a \cap b) \wedge a\big)^*_{(a]_{\mathbf{A}}} = (a \cap b)^*_{(a]_{\mathbf{A}}} = a^*_{(a]_{\mathbf{A}}} = 0$. Thus $a \leq_0^{\langle A;\,/,0\rangle} b$. ∎

Remark 3.3.54. The condition of join symmetry cannot be omitted from the assertion of Theorem 3.3.53 since the variety of left handed skew Boolean $\cap$-algebras is properly contained within the class (in fact, variety) of all implicative $\leq_0$-BCK local skew lattices for which the skew lattice reduct is left handed. To see this, consider the algebra $\mathbf{A} := \langle A;\ \wedge, \vee, /, 0\rangle$ of type $\langle 2, 2, 0\rangle$ whose reduct $\langle A;\ \wedge, \vee, 0\rangle$ is the skew lattice with zero reduct of the implicative BCS skew lattice of Remark 3.3.23, and whose operation $/^{\mathbf{A}}$ is determined by the following operation table:

$/^{\mathbf{A}}$	0	a	b	c	d	e	f
0	0	0	0	0	0	0	0
a	a	0	a	0	a	a	a
b	b	b	0	b	0	b	b
c	c	e	c	0	c	a	c
d	d	d	f	d	0	d	b
e	e	e	e	0	e	0	e
f	f	f	f	f	0	f	0

An easy sequence of checks shows that: (i) the skew lattice with zero reduct $\langle A;\ \wedge, \vee, 0\rangle$ is local; (ii) the reduct $\langle A;\ /, 0\rangle$ is an implicative BCK-algebra;

and (iii) $i \leq_{\mathcal{H}} j$ iff $i \leq^{\langle A;\, /, 0\rangle} j$ for any $i, j \in A$. Hence $\mathbf{A}$ is an implicative $\leq_0$-BCK local skew lattice for which the skew lattice reduct is left handed. However, A is not a left handed skew Boolean $\cap$-algebra, since the skew lattice with zero reduct $\langle A;\ \wedge, \vee, 0\rangle$ is not locally Boolean (in the sense of §1.4.24). In particular, $\langle A;\ \wedge, \vee, 0\rangle$ is not symmetric, since it is not join symmetric: $(a \vee b) \vee (a \wedge b \wedge a) = d \vee 0 = d$ but $(a \wedge b \wedge a) \vee (b \vee a) = 0 \vee c = c$. ∎

Remark 3.3.55 (Added in proof). A *lower implicative BCK left normal band* is an algebra $\langle A;\ /, \sqcap, 0\rangle$ of type $\langle 2, 2, 0\rangle$ such that: (i) the reduct $\langle A;\ /, 0\rangle$ is an implicative BCK-algebra; (ii) the reduct $\langle A;\ \sqcap, 0\rangle$ is a left normal band with zero; and (iii) the implicative BCK partial order $\leq_0^{\langle A;\, /, 0\rangle}$ and the natural band partial order $\leq_{\mathcal{H}}$ coincide. By an unpublished result of the author, an algebra $\mathbf{A} := \langle A;\ /, \sqcap, 0\rangle$ of type $\langle 2, 2, 0\rangle$ is a lower implicative BCK left normal band iff $\mathbf{A} \models$ (1.35)–(1.38), $\mathbf{A} \models (x \sqcap y) \sqcap z \approx x \sqcap (y \sqcap z), x \sqcap x \approx x, (x \sqcap y) \sqcap z \approx (x \sqcap z) \sqcap y$, and $\mathbf{A} \models (x \sqcap y)/x \approx \mathbf{0}, x \sqcap (x/y) \approx x/y$, whence the class $\mathsf{iBCKlnB}$ of lower implicative BCK left normal bands is a variety.

Recall the definition of an implicative BCSK-algebra from Remark 3.2.28. Because of Corollary 2.3.22(1), any implicative BCSK-algebra $\langle A;\ /, \backslash, 0\rangle$ has a lower implicative BCK left normal band polynomial reduct $\langle A;\ /, \sqcap, 0\rangle$, where $a \sqcap b := a\backslash(a\backslash b)$ for any $a, b \in A$. Conversely, an unpublished result of the author shows that any lower implicative BCK left normal band $\langle A;\ /, \sqcap, 0\rangle$ has an implicative BCSK-algebra polynomial reduct $\langle A;\ /, \backslash, 0\rangle$, where $a\backslash b := a/(a \sqcap b)$ for any $a, b \in A$. Hence the variety of lower implicative BCK left normal bands is termwise definitionally equivalent to the variety of implicative BCSK-algebras (compare this result to that of Theorem 3.3.53).

Call a lower implicative BCK left normal band *flat* if its underlying poset is flat. The proof of Theorem 3.2.27, in conjunction with preceding remarks, implies that any pointed fixedpoint discriminator algebra $\langle A;\ f, 0\rangle$ has a flat lower implicative BCK left normal band polynomial reduct $\langle A;\ /, \sqcap, 0\rangle$, where:

$$a/b := f\big(0, f(a, b, a), a\big) \quad \text{and} \quad a \sqcap b := f\big(0, f(0, b, a), a\big)$$

for any $a, b \in A$. Conversely, the description of the subdirectly irreducible implicative BCSK-algebras given in Remark 3.2.28 implies that a lower implicative BCK left normal band is subdirectly irreducible iff it is flat, whence the class $\mathsf{iBCKlnB}$ is a pointed fixedpoint discriminator variety, with pointed fixedpoint discriminator term:

$$f(x, y, z) := \big(z\backslash(z \sqcap (x/y))\big) \sqcap \big(z\backslash(z \sqcap (y/x))\big).$$

Recall from Remark 3.2.28 that $\mathsf{FPD_0}$ denotes the pure pointed fixedpoint discriminator variety. In view of the preceding discussion, it is easy to see that $\mathsf{FPD_0}$ is termwise definitionally equivalent to $\mathsf{iBCKlnB}$. Since the congruence structure of any algebra in a fixedpoint discriminator variety is (by Lemma 1.5.10) completely determined by the fixedpoint discriminator term, any algebra $\mathbf{A}$ in a pointed fixedpoint discriminator variety must have a lower implicative BCK left normal band polynomial reduct whose congruences coincide with those of $\mathbf{A}$ (compare this result to that of Corollary 1.4.40).

Ignoring issues of similarity type, it is clear from the conditions (i), (ii) and (iii) above defining lower implicative BCK left normal bands that $\mathsf{iBCKlnB}$ is a subvariety of the variety of lower implicative $\leq_0$-BCK normal bands. This observation, in conjunction with the above remarks and the results of §3.2.6 and §1.4.37, implies that the study of the classes $\mathsf{BP}_{\mathcal{C}}$, $\{\dot{-}, 0\} \subseteq \mathcal{C} \subseteq \{\wedge, \vee, \dot{-}, 0\}$ (and hence, by extension—recall Proposition 3.3.51—the classes $\mathsf{PQ}_{\mathcal{C}}$, $\{\dot{-}, 0\} \subseteq \mathcal{C} \subseteq \{\wedge, \vee, \dot{-}, 0\}$) encompasses, to within termwise definitional equivalence, the study of the pure binary discriminator, pure pointed fixedpoint discriminator and pure pointed ternary discriminator varieties. Consequently, the study of pre-BCK quasilattices, BCK paralattices and related structures provides a unifying framework for the study of several important classes of 'generalised Boolean structures' arising naturally in universal algebra and algebraic logic. ∎

Corollary 3.3.56. *An algebra $\langle A;\ \wedge, \vee, /, 0\rangle$ of type $\langle 2, 2, 0\rangle$ is a generalised Boolean algebra iff the following conditions hold:*

1. *The reduct $\langle A;\ \wedge, \vee, 0\rangle$ is a lattice with zero;*

2. *The reduct* $\langle A;\ /, 0\rangle$ *is an implicative BCK-algebra;*

3. *The BCK meet* $x \cap y$ *and the lattice meet* $x \wedge y$ *coincide.*

Henceforth, by abuse of language and notation we will always understand by the term 'skew Boolean $\cap$-algebra' an algebra $\langle A;\ \wedge, \vee, /, 0\rangle$ of type $\langle 2, 2, 2, 0\rangle$ satisfying conditions (1)–(3) of Theorem 3.3.53. Conversely, by abuse of language and notation an algebra $\langle A;\ \wedge, \vee, /, 0\rangle$ of type $\langle 2, 2, 2, 0\rangle$ satisfying the defining conditions of Theorem 3.3.53 will always be called a 'skew Boolean $\cap$-algebra'. See also Bignall and Leech [19, Section 4]. Given these conventions, we have the following result, a first-order proof of which may be found in [210, Section 5.2].

Theorem 3.3.57. *An algebra* $\langle A;\ \wedge, \vee, /, 0\rangle$ *of type* $\langle 2, 2, 2, 0\rangle$ *is a skew Boolean* $\cap$*-algebra iff the following identities are satisfied:*

$$(x \vee y) \vee z \approx x \vee (y \vee z) \tag{3.99}$$
$$(x \wedge y) \wedge z \approx x \wedge (y \wedge z) \tag{3.100}$$

$$x \wedge (x \vee y) \approx x \tag{3.101}$$
$$(y \wedge x) \vee x \approx x \tag{3.102}$$

$$x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z) \tag{3.103}$$
$$(x \vee y) \wedge z \approx (x \wedge z) \vee (y \wedge z) \tag{3.104}$$

$$x/(x/y) \approx y/(y/x) \tag{3.105}$$
$$(x/y)/z \approx (x/z)/y \tag{3.106}$$
$$x/(y/x) \approx x \tag{3.107}$$

$$(x \wedge y \wedge x)/x \approx \mathbf{0} \tag{3.108}$$
$$x \vee (x/y) \approx x \tag{3.109}$$
$$(x/y) \vee x \approx x. \tag{3.110}$$

Thus the class of skew Boolean $\cap$-algebras is a variety.

Proof. Let $\mathbf{A} := \langle A;\ \wedge, \vee, /, 0\rangle$ be an algebra of type $\langle 2,2,2,0\rangle$.

($\Rightarrow$) Suppose $\mathbf{A}$ is a skew Boolean $\cap$-algebra. Then $\mathbf{A} \models$ (3.99)–(3.102), (3.105)–(3.107) by the defining conditions of Theorem 3.3.53. Moreover, because of Theorem 3.3.53, $\mathbf{A} \in \mathsf{IP}_{C''}$, $C'' = \{\wedge, \vee, /, \mathbf{0}\}$, so $\mathbf{A} \models$ (3.108)–(3.110) by Theorem 3.3.49. Since $\mathbf{A} \in \mathsf{IP}_{C''}$, the skew lattice with zero reduct $\langle A;\ \wedge, \vee, 0\rangle$ is distributive local by Proposition 3.3.52, and so is symmetric by Lemma 1.4.17 and hypothesis. Therefore $\mathbf{A} \models$ (3.103)–(3.104) by Proposition 1.4.22, and the proof is complete.

($\Leftarrow$) Suppose $\mathbf{A} \models$ (3.99)–(3.110). To see $\mathbf{A}$ is a skew Boolean $\cap$-algebra it is sufficient to show the defining conditions of Theorem 3.3.53 are satisfied, *viz.*:

(i) The reduct $\langle A;\ \wedge, \vee, 0\rangle$ is a join symmetric local skew lattice with zero;
(ii) The reduct $\langle A;\ /, 0\rangle$ is an implicative BCK-algebra;
(iii) The partial orders $\leq_{\mathcal{H}}$ and $\leq_0^{\langle A;\ /, 0\rangle}$ coincide.

For (i), we first show $\langle A;\ \wedge, \vee, 0\rangle$ is a skew lattice with zero. So let $a, b \in A$. By (3.101) and (3.109) $a = a \wedge \big(a \vee (a/b)\big) = a \wedge a$, whence $\wedge$ is idempotent. By (3.101), idempotence of $\wedge$ and (3.103) we have $a = a \wedge (a \vee b) = (a \wedge a) \vee (a \wedge b) = a \vee (a \wedge b)$. Similarly $(b \vee a) \wedge a = (b \wedge a) \vee (a \wedge a) = (b \wedge a) \vee a = a$ by (3.104), idempotence of $\wedge$ and (3.102). By absorption, $a = a \vee \big(a \wedge (a \vee b)\big) = a \vee a$ and thus $\langle A;\ \wedge, \vee\rangle$ is a skew lattice. Also, $0 = a/(a \wedge a \wedge a) = a/a$ by (3.108) and idempotence of $\wedge$, whence $a = a \vee (a/a) = a \vee 0$ by (3.109) and $a = (a/a) \vee a = 0 \vee a$ by (3.110). Thus $\langle A;\ \wedge, \vee, 0\rangle$ is a skew lattice with zero, and the identities (3.103)–(3.104) in conjunction with Proposition 1.4.22 now ensure $\langle A;\ \wedge, \vee, 0\rangle$ is distributive symmetric local. Hence $\langle A;\ \wedge, \vee, 0\rangle$ is join symmetric local.

For (ii), we have already observed $a/a = 0$ for all $a \in A$, so $\mathbf{A} \models x/x \approx \mathbf{0}$. From this remark, the identities (3.105)–(3.107), and Theorem 1.6.17 it follows that the reduct $\langle A;\ /, 0\rangle$ is an implicative BCK-algebra.

For (iii), just notice that $\langle A;\ \wedge, \vee, /, 0\rangle$ is a BCK paralattice by (i), Lemma 1.4.8, (ii), (3.108)–(3.110) and Theorem 3.3.49. Therefore the natural skew lattice partial order $\leq_{\mathcal{H}}$ and the BCK partial order $\leq_0^{\langle A;\,/,0\rangle}$ coincide. ∎

Corollary 3.3.58. *An algebra $\langle A;\ \wedge, \vee, /, 0\rangle$ of type $\langle 2,2,2,0\rangle$ is a left handed skew Boolean $\cap$-algebra iff the identities of Theorem 3.3.57 are satisfied, where* (3.108) *is replaced by the identity:*

$$(x \wedge y)/x \approx \mathbf{0}. \tag{3.108$'$}$$

Proof. Let $\mathbf{A} := \langle A;\ \wedge, \vee, /, 0\rangle$ be an algebra of type $\langle 2,2,2,0\rangle$ such that $\mathbf{A} \models$ (3.99)–(3.107), (3.108′), (3.109)–(3.110) and let $a, b \in A$. By (3.103), we have that $a \wedge (b \vee a) = (a \wedge b) \vee (a \wedge a) = (a \wedge b) \vee a$. But by (3.108′) and (3.92), $(a \wedge b) \vee a = \big((a \wedge b)/0\big) \vee a = \big((a \wedge b)/((a \wedge b)/a)\big) \vee a = a$. Hence $a \wedge (b \vee a) = a = (a \wedge b) \vee a$, and $\mathbf{A}$ is left handed by the remarks of §1.4.14. Since the converse is clear, the corollary is proved. ∎

3.3.59. Double-Pointed Skew Boolean $\cap$-algebras. Let $\mathbf{A}$ be a skew Boolean $\cap$-algebra with maximal class M. By analogy with the theory of pre-BCK-algebras, an algebra $\mathbf{A}^1 := \langle A;\ \wedge, \vee, /, 0, 1\rangle$ obtained from $\mathbf{A}$ upon adjoining to the language of $\mathbf{A}$ a new nullary operation symbol $\mathbf{1}$ whose canonical interpretation on $\mathbf{A}^1$ is a fixed $1 \in M$ is called a *quasi-bounded skew Boolean $\cap$-algebra*. Clearly the class of quasi-bounded skew Boolean $\cap$-algebras is a variety, axiomatised relative to the variety of skew Boolean $\cap$-algebras by either of the identities:

$$x \backslash \mathbf{1} \approx \mathbf{0} \quad \text{or} \quad x \wedge \mathbf{1} \wedge x \approx x.$$

Remark 3.3.60. As per pre-BCK-algebras, in passing from a given skew Boolean $\cap$-algebra $\mathbf{A}$ with maximal class M to a quasi-bounded skew Boolean $\cap$-algebra $\mathbf{A}^1$ there is typically no natural choice of maximal element $1 \in M$. In general, it seems plausible that distinct choices of maximal element could give rise to non-isomorphic quasi-bounded skew Boolean $\cap$-algebras, although this possibility cannot occur in the *completely reducible* case (that is, when $\mathbf{A}$ is isomorphic to a direct product of primitive algebras). ∎

A version of the following result occurs in Blok and Pigozzi [34, Section 1].

Proposition 3.3.61. *(cf. [34, Theorem 1.1]) An algebra $\langle A;\ \wedge,\vee,/,0,1\rangle$ of type $\langle 2,2,2,0,0\rangle$ is a double-pointed skew Boolean $\cap$-algebra iff it is a quasi-bounded skew Boolean $\cap$-algebra. Thus the class SBIA^+ of double-pointed skew Boolean $\cap$-algebras is a variety, axiomatised relative to the variety of skew Boolean $\cap$-algebras by either of the identities $x\backslash\mathbf{1}\approx\mathbf{0}$ or $x\wedge\mathbf{1}\wedge x\approx x$.*

Proof. Because the ideals of any skew Boolean $\cap$-algebra coincide with the ideals of its implicative BCS-algebra polynomial reduct, the first assertion of the proposition follows from Lemma 2.2.27. The second assertion of the proposition now follows from previous remarks. ∎

Let $\mathbf{A}$ be a primitive skew Boolean $\cap$-algebra (recall Example 1.4.35). Since $\mathbf{A}$ is primitive, it has a maximal class M, which consists of all non-zero elements of A. Thus $\mathbf{A}$ induces a double-pointed *primitive* skew Boolean $\cap$-algebra $\mathbf{A}^+$. Because of Remark 3.3.60, $\mathbf{A}^+$ is unique to within isomorphism.

Theorem 3.3.62. *The following assertions hold in the variety of double-pointed skew Boolean $\cap$-algebras:*

1. *The double-pointed primitive skew Boolean $\cap$-algebras are the non-trivial simple algebras;*

2. *The double-pointed primitive skew Boolean $\cap$-algebras are the subdirectly irreducible algebras;*

3. *Every non-trivial double-pointed skew Boolean $\cap$-algebra is a subdirect product of primitive algebras.*

Proof. For Item (1) [Item (2)] let $\mathbf{A}^+$ be a non-trivial simple [subdirectly irreducible] double-pointed skew Boolean $\cap$-algebra. Its skew Boolean $\cap$-algebra reduct $\mathbf{A}:=\langle A;\ \wedge,\vee,/,0\rangle$ must also be non-trivial and simple [subdirectly irreducible], since $\mathbf{A}$ and $\mathbf{A}^+$ have the same congruences. By Theorem 1.4.36 we deduce that $\mathbf{A}$ is primitive, so $\mathbf{A}^+$ is primitive. Item (3) now follows from Birkhoff's subdirect representation theorem [55, Theorem II§8.6]. ∎

Recall from Theorem 1.4.38 that $t(x,y,z):=\big(z\backslash(x\Delta y)\big)\vee(x/y)$ is a ternary discriminator term for SBIA, where $x\Delta y\approx(x/y)\vee(y/x)$ (since $x\backslash(x\cap y)\approx x/y$). Because of Theorem 3.3.62, we may infer the following result.

Corollary 3.3.63. *The class of double-pointed skew Boolean $\cap$-algebras is a discriminator variety, with discriminator term given by:*

$$t(x,y,z) := \big(z\backslash(x\Delta y)\big) \vee (x/y).$$

Let V be a pointed discriminator variety (say with $\mathbf{0}$) with discriminator term $t(x,y,z)$. By the remarks of §1.5.9, V is a pointed fixedpoint discriminator variety, with pointed fixedpoint discriminator term $f(x,y,z) := t\big(t(x,y,z), t(x,y,\mathbf{0}), \mathbf{0}\big)$. For [double-pointed] skew Boolean $\cap$-algebras we have the following simplification.

Corollary 3.3.64. *The class of [double-pointed] skew Boolean $\cap$-algebras is a fixedpoint discriminator variety, with fixedpoint discriminator term given by $f(x,y,z) := z\backslash(x\Delta y)$.*

Proof. The result is established by an easy inspection of the subdirectly irreducible [double-pointed] skew Boolean $\cap$-algebras. ∎

Remark 3.3.65. Because of Remark 3.2.28, $f(x,y,z) := \big(z\backslash(x/y)\big)\backslash(y/x)$ is also a pointed fixedpoint discriminator term for both SBIA and SBIA^+. ∎

By the *pure double-pointed discriminator variety* PD_0^+ we mean the double-pointed discriminator variety of type $\langle 3,0,0\rangle$ generated by the class of all double-pointed discriminator algebras $\langle A;\ t,0,1\rangle$ where t is the discriminator and 0 and 1 are residually distinct nullary operations, while by the variety of *double-pointed left handed skew Boolean $\cap$-algebras* lhSBIA^+ we mean the variety of all left handed skew Boolean $\cap$-algebras that are double-pointed. From Theorem 1.4.39, Proposition 3.3.61 and [34, Theorem 1.1] we may infer the following result.

Theorem 3.3.66. *The variety PD_0^+ is termwise definitionally equivalent to the variety of double-pointed left handed skew Boolean $\cap$-algebras. In particular, given $\langle A;\ t,0,1\rangle \in \mathsf{PD}_0^+$, double-pointed left handed skew Boolean $\cap$-operations $\wedge$, $\vee$ and / are defined on A by:*

$$a \wedge b := t\big(b, t(b,0,a), a\big)$$

$$a \vee b := t(b, 0, a)$$
$$a/b := t(a, b, 0)$$

for any $a, b \in A$*. Conversely, given a double-pointed left handed skew Boolean* $\sqcap$*-algebra* $\langle A;\ \wedge, \vee, /, 0, 1\rangle$ *and* $a, b, c \in A$*, the operation* $t(a, b, c) := \big(c/(c \wedge b)\big) \vee (c \wedge a) \vee (a/b)$ *yields an algebra* $\langle A;\ t, 0, 1\rangle$ *in* PD_0^+*.*

Clearly PD_0^+ is also termwise definitionally equivalent to the variety of double-pointed right handed skew Boolean $\sqcap$-algebras.

Corollary 3.3.67. *Any algebra* $\mathbf{A}$ *in a double-pointed discriminator variety has a double-pointed left handed skew Boolean* $\sqcap$*-algebra polynomial reduct whose congruences coincide with those of* $\mathbf{A}$*.*

Proof. The result follows immediately from Theorem 3.3.66 and Lemma 1.5.10. ∎

Remark 3.3.68. A *pseudo-interior algebra* is an algebra $\langle A;\ \cdot, \rightarrow, {}^\circ, 1\rangle$ of type $\langle 2, 2, 1, 0\rangle$ that is essentially a hybrid of an interior algebra and a residuated partially ordered monoid (for a precise definition, see [35, Definition 2.6]). A *pseudo-interior algebra with compatible operations* is an algebra $\langle A;\ \cdot, \rightarrow, {}^\circ, 1, F_i\rangle_{i \in I}$ where $\langle A;\ \cdot, \rightarrow, {}^\circ, 1\rangle$ is a pseudo-interior algebra and the additional operations $\langle F_i\rangle_{i \in I}$ are such that every congruence on $\mathbf{A}$ has the substitution property with respect to each F_i [34, Corollary 2.17]. Pseudo-interior algebras with compatible operations were introduced by Blok and Pigozzi in [35] as the algebraic counterpart of a certain assertional logic inherent in any variety with a commutative, regular TD term; for details, see [34, Theorem 4.1]. By [35, Corollary 4.3], a double-pointed variety V is a ternary discriminator variety iff it is termwise definitionally equivalent to a congruence permutable, semisimple variety of pseudo-interior algebras with compatible operations; the hypothesis that V is double-pointed is essential. The description of double-pointed discriminator varieties and the associated assertional logics afforded by this result should be compared and contrasted with Corollary 3.3.67 and the developments of §3.3.69 in the sequel. ∎

3.3.69. The Assertional Logic of the Variety lhSBIA^+. Recall from §1.8.9 that for any quasivariety K with $\mathbf{1}$, the inherent assertional logic $\mathbb{S}(\mathsf{K}, \mathbf{1})$ of K may be defined by specifying that, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathbb{S}(\mathsf{K},\mathbf{1})} \varphi$ iff $\{\psi \approx \mathbf{1} : \psi \in \Gamma\} \models_{\mathsf{K}} \varphi \approx \mathbf{1}$. It follows from this observation and Theorem 3.3.66 that the assertional logic $\mathbb{S}(\mathsf{lhSBIA}^+, \mathbf{0})$ of the variety of double-pointed left handed skew Boolean $\sqcap$-algebras is definitionally equivalent to the assertional logic $\mathbb{S}(\mathsf{PD}_0^+, \mathbf{0})$ of the pure double-pointed discriminator variety. This remark calls for a study of $\mathbb{S}(\mathsf{lhSBIA}^+, \mathbf{0})$; in this subsection we provide a framework for such a study by axiomatising this deductive system.

Throughout this subsection, we work with a fixed language $\mathcal{L} := \langle \wedge, \vee, \Rightarrow, \mathbf{0}, \mathbf{1} \rangle$ of type $\langle 2, 2, 2, 0, 0 \rangle$, with fixed abbreviations:

$$
\begin{aligned}
p \Leftrightarrow q &:= (p \Rightarrow q) \wedge (q \Rightarrow p) \\
p \rightarrow q &:= (p \vee q) \Rightarrow q \\
p \leftrightarrow q &:= (p \rightarrow q) \wedge (q \rightarrow p) \\
\neg\, p &:= p \rightarrow \mathbf{0}
\end{aligned}
$$

for propositional variables p, q, and with a fixed and defining collection $Ax \cup Ir$ of axioms and inferences rules, *viz.*:

$$\big((p \wedge q) \wedge r\big) \Leftrightarrow \big(p \wedge (q \wedge r)\big) \tag{S1}$$
$$\big((p \vee q) \vee r\big) \Leftrightarrow \big(p \vee (q \vee r)\big) \tag{S2}$$

$$\big(p \wedge (p \vee q)\big) \Leftrightarrow p \tag{S3}$$
$$\big((q \wedge p) \vee p\big) \Leftrightarrow p \tag{S4}$$

$$\big((p \vee q) \wedge (p \vee r)\big) \Leftrightarrow \big(p \vee (q \wedge r)\big) \tag{S5}$$
$$\big((p \vee r) \wedge (q \vee r)\big) \Leftrightarrow \big((p \wedge q) \vee r\big) \tag{S6}$$

$$\big((p \Rightarrow q) \Rightarrow p\big) \Leftrightarrow p \tag{S7}$$
$$\big((q \Rightarrow p) \Rightarrow p\big) \Leftrightarrow \big((p \Rightarrow q) \Rightarrow q\big) \tag{S8}$$

$$\big(p \Rightarrow (q \Rightarrow r)\big) \Leftrightarrow \big(q \Rightarrow (p \Rightarrow r)\big) \tag{S9}$$

$$\big(p \Rightarrow (q \vee p)\big) \Leftrightarrow \mathbf{1} \tag{S10}$$

$$\big(p \wedge (q \Rightarrow p)\big) \Leftrightarrow p \tag{S11}$$

$$\big((q \Rightarrow p) \wedge p\big) \Leftrightarrow p \tag{S12}$$

$$(\mathbf{0} \vee p) \Leftrightarrow p \tag{S13}$$

$$\frac{}{p \Leftrightarrow p}\ (\mathrm{R}) \qquad \frac{p \Leftrightarrow q}{q \Leftrightarrow p}\ (\mathrm{S}) \qquad \frac{p \Leftrightarrow q, \quad q \Leftrightarrow r}{p \Leftrightarrow r}\ (\mathrm{T})$$

$$\frac{p \Leftrightarrow q, \quad r \Leftrightarrow s}{(p \wedge r) \Leftrightarrow (q \wedge s)}\ (\text{CP-}\wedge) \qquad \frac{p \Leftrightarrow q, \quad r \Leftrightarrow s}{(p \vee r) \Leftrightarrow (q \vee s)}\ (\text{CP-}\vee)$$

$$\frac{p \Leftrightarrow q, \quad r \Leftrightarrow s}{(p \Rightarrow r) \Leftrightarrow (q \Rightarrow s)}\ (\text{CP-}\Rightarrow)$$

$$\frac{p \Leftrightarrow \mathbf{1}}{p}\ (\mathbf{1}\text{-I}) \qquad \frac{p}{p \Leftrightarrow \mathbf{1}}\ (\mathbf{1}\text{-E})$$

The *skew Boolean propositional calculus* is the deductive system $\mathbb{SBPC}$ over the language $\mathcal{L}$ determined by the axiomatisation $Ax \cup Ir$. We denote by $\vdash_{\mathbb{SBPC}}$ the consequence relation of $\mathbb{SBPC}$. The skew Boolean propositional calculus was introduced implicitly by the author's Ph.D. supervisor in [18] and explicitly by the author's Ph.D. supervisor and the author in [20]; applications of the skew Boolean propositional calculus to theoretical computer science have been considered by the author and the author's Ph.D. supervisor in [21, 22].

Theorem 3.3.70. $\mathbb{SBPC}$ *is algebraisable with equivalence formula* $p \Leftrightarrow q$ *and defining equation* $p \approx \mathbf{1}$.

Proof. Immediate by Theorem 1.8.2 and the description of $Ax \cup Ir$. ∎

By Theorem 1.8.3, the equivalent algebraic semantics of $\mathbb{SBPC}$ is the quasivariety K over the language $\mathcal{L}$ axiomatised by the following collection of identities and quasi-identities:

$$\big((x \wedge y) \wedge z\big) \Leftrightarrow \big(x \wedge (y \wedge z)\big) \approx \mathbf{1} \tag{3.111}$$
$$\big((x \vee y) \vee z\big) \Leftrightarrow \big(x \vee (y \vee z)\big) \approx \mathbf{1} \tag{3.112}$$

$$\big(x \wedge (x \vee y)\big) \Leftrightarrow x \approx \mathbf{1} \tag{3.113}$$
$$\big((y \wedge x) \vee x\big) \Leftrightarrow x \approx \mathbf{1} \tag{3.114}$$

$$\big((x \vee y) \wedge (x \vee z)\big) \Leftrightarrow \big(x \vee (y \wedge z)\big) \approx \mathbf{1} \tag{3.115}$$
$$\big((x \vee z) \wedge (y \vee z)\big) \Leftrightarrow \big((x \wedge y) \vee z\big) \approx \mathbf{1} \tag{3.116}$$

$$\big((x \Rightarrow y) \Rightarrow x\big) \Leftrightarrow x \approx \mathbf{1} \tag{3.117}$$
$$\big((y \Rightarrow x) \Rightarrow x\big) \Leftrightarrow \big((x \Rightarrow y) \Rightarrow y\big) \approx \mathbf{1} \tag{3.118}$$
$$\big(x \Rightarrow (y \Rightarrow z)\big) \Leftrightarrow \big(y \Rightarrow (x \Rightarrow z)\big) \approx \mathbf{1} \tag{3.119}$$
$$\big(x \Rightarrow (y \vee x)\big) \Leftrightarrow \mathbf{1} \approx \mathbf{1} \tag{3.120}$$

$$\big(x \wedge (y \Rightarrow x)\big) \Leftrightarrow x \approx \mathbf{1} \tag{3.121}$$
$$\big((y \Rightarrow x) \wedge x\big) \Leftrightarrow x \approx \mathbf{1} \tag{3.122}$$

$$(\mathbf{0} \vee x) \Leftrightarrow x \approx \mathbf{1} \tag{3.123}$$

$$x \Leftrightarrow x \approx \mathbf{1} \tag{3.124}$$
$$y \Leftrightarrow x \approx \mathbf{1} \supset x \Leftrightarrow y \approx \mathbf{1} \tag{3.125}$$
$$x \Leftrightarrow y \approx \mathbf{1} \,\&\, y \Leftrightarrow z \approx \mathbf{1} \supset x \Leftrightarrow z \approx \mathbf{1} \tag{3.126}$$

$$x \Leftrightarrow y \approx \mathbf{1} \,\&\, z \Leftrightarrow w \approx \mathbf{1} \supset x \wedge z \Leftrightarrow y \wedge w \approx \mathbf{1} \tag{3.127}$$
$$x \Leftrightarrow y \approx \mathbf{1} \,\&\, z \Leftrightarrow w \approx \mathbf{1} \supset x \vee z \Leftrightarrow y \vee w \approx \mathbf{1} \tag{3.128}$$
$$x \Leftrightarrow y \approx \mathbf{1} \,\&\, z \Leftrightarrow w \approx \mathbf{1} \supset x \Rightarrow z \Leftrightarrow y \Rightarrow w \approx \mathbf{1} \tag{3.129}$$

$$x \approx \mathbf{1} \supset x \Leftrightarrow \mathbf{1} \approx \mathbf{1} \tag{3.130}$$
$$x \Leftrightarrow \mathbf{1} \approx \mathbf{1} \supset x \approx \mathbf{1} \tag{3.131}$$

$$x \Leftrightarrow y \approx \mathbf{1} \supset x \approx y \tag{3.132}$$

where $x \Leftrightarrow y$ abbreviates $(x \Rightarrow y) \wedge (y \Rightarrow x)$ for individual variables x, y. To see the quasivariety K has a familiar description, let lhSBIA^{+D} denote the variety with language $\mathcal{L}$ defined by the following set of identities:

$$(x \wedge y) \wedge z \approx x \wedge (y \wedge z) \tag{3.133}$$
$$(x \vee y) \vee z \approx x \vee (y \vee z) \tag{3.134}$$

$$x \wedge (x \vee y) \approx x \tag{3.135}$$
$$(y \wedge x) \vee x \approx x \tag{3.136}$$

$$(x \vee y) \wedge (x \vee z) \approx x \vee (y \wedge z) \tag{3.137}$$
$$(x \vee z) \wedge (y \vee z) \approx (x \wedge y) \vee z \tag{3.138}$$

$$(x \Rightarrow y) \Rightarrow x \approx x \tag{3.139}$$
$$(y \Rightarrow x) \Rightarrow x \approx (x \Rightarrow y) \Rightarrow y \tag{3.140}$$
$$x \Rightarrow (y \Rightarrow z) \approx y \Rightarrow (x \Rightarrow z) \tag{3.141}$$
$$x \Rightarrow (y \vee x) \approx \mathbf{1} \tag{3.142}$$

$$x \wedge (y \Rightarrow x) \approx x \tag{3.143}$$
$$(y \Rightarrow x) \wedge x \approx x \tag{3.144}$$

$$\mathbf{0} \vee x \approx x. \tag{3.145}$$

Let $\mathbf{T}_{\mathcal{L}}(\mathbf{X})$ denote the term algebra of type $\mathcal{L}$ over $\mathbf{X}$ and let $\mathbf{T}_{\mathcal{L}^D}(\mathbf{X})$ denote the term algebra of type $\mathcal{L}^D$ over $\mathbf{X}$, where $\mathcal{L}^D$ is the language of skew Boolean $\cap$-

algebras. Consider the maps $\xi : \mathrm{T}_{\mathcal{L}}(\mathrm{X}) \to \mathrm{T}_{\mathcal{L}^D}(\mathrm{X})$ and $\eta : \mathrm{T}_{\mathcal{L}^D}(\mathrm{X}) \to \mathrm{T}_{\mathcal{L}}(\mathrm{X})$ defined respectively by:

$$
\begin{aligned}
\xi(\mathbf{1}) &:= \mathbf{0} & & \\
\xi(\mathbf{0}) &:= \mathbf{1} & & \\
\xi(x) &:= x & & x \in \mathrm{X} \\
\xi(p \wedge q) &:= \xi(q) \vee \xi(p) & & p, q \in \mathrm{T}_{\mathcal{L}^D}(\mathrm{X}) \\
\xi(p \vee q) &:= \xi(q) \wedge \xi(p) & & p, q \in \mathrm{T}_{\mathcal{L}^D}(\mathrm{X}) \\
\xi(p \Rightarrow q) &:= \xi(q)/\xi(p) & & p, q \in \mathrm{T}_{\mathcal{L}^D}(\mathrm{X})
\end{aligned}
$$

and:

$$
\begin{aligned}
\eta(\mathbf{0}) &:= \mathbf{1} & & \\
\eta(\mathbf{1}) &:= \mathbf{0} & & \\
\eta(x) &:= x & & x \in \mathrm{X} \\
\eta(r \wedge s) &:= \eta(s) \vee \eta(r) & & r, s \in \mathrm{T}_{\mathcal{L}}(\mathrm{X}) \\
\eta(r \vee s) &:= \eta(s) \wedge \eta(r) & & r, s \in \mathrm{T}_{\mathcal{L}}(\mathrm{X}) \\
\eta(r/s) &:= \eta(s) \Rightarrow \eta(r) & & r, s \in \mathrm{T}_{\mathcal{L}}(\mathrm{X}).
\end{aligned}
$$

(The maps ξ and η so defined should not be confused with the similar maps of §2.1.33 or §3.1.1 in the prequel.) Because of Corollary 3.3.58 and the definition of lhSBIA^{+D}, the proof of the following lemma is trivial and so is omitted.

Lemma 3.3.71. *For $p, q \in \mathrm{T}_{\mathcal{L}}(\mathrm{X})$ and $r, s \in \mathrm{T}_{\mathcal{L}^D}(\mathrm{X})$ the following assertions hold:*

1. *If $\mathsf{lhSBIA}^{+D} \models p \approx q$ then $\mathsf{lhSBIA}^{+} \models \xi(p) \approx \xi(q)$;*

2. *If $\mathsf{lhSBIA}^{+} \models r \approx s$ then $\mathsf{lhSBIA}^{+D} \models \eta(r) \approx \eta(s)$.*

Moreover, $\eta \circ \xi = \omega_{\mathrm{T}_{\mathcal{L}}(\mathrm{X})}$ and $\xi \circ \eta = \omega_{\mathrm{T}_{\mathcal{L}^D}(\mathrm{X})}$.

By Lemma 3.3.71, the variety lhSBIA^{+D} is termwise definitionally equivalent to (in fact, is dually isomorphic to) the variety of double-pointed left handed skew Boolean $\cap$-algebras. In other words, lhSBIA^{+D} is precisely the variety of *dual double-pointed left handed skew Boolean* $\cap$*-algebras.*

Proposition 3.3.72. *The quasivariety* K *coincides with the variety* lhSBIA^{+D} *of dual double-pointed left handed skew Boolean* $\cap$*-algebras. Hence* $\mathbb{SBPC}$ *is strongly algebraisable.*

Proof. (Sketch) To prove the proposition, it is sufficient to show the quasivariety K and the variety lhSBIA^{+D} coincide. To establish the inclusion $\mathsf{K} \subseteq \mathsf{lhSBIA}^{+D}$, note first that:

$$\mathsf{lhSBIA}^{+D} \models x \approx y \quad \text{iff} \quad \mathsf{lhSBIA}^{+D} \models x \Leftrightarrow y \approx \mathbf{1} \tag{3.146}$$

by Lemma 3.3.71 and Proposition 3.3.37(3). To complete the proof we show:

(i) $\mathsf{lhSBIA}^{+D} \models$ (3.111)–(3.123);
(ii) $\mathsf{lhSBIA}^{+D} \models$ (3.124)–(3.132).

For (i), consider a defining identity of K, say (3.111). Then $\mathsf{K} \models$ (3.111), *viz.*:

$$\mathsf{K} \models \big((x \wedge y) \wedge z\big) \Leftrightarrow \big(x \wedge (y \wedge z)\big) \approx \mathbf{1}.$$

Now by (3.133), we have that:

$$\mathsf{lhSBIA}^{+D} \models (x \wedge y) \wedge z \approx x \wedge (y \wedge z),$$

so by (3.146), we have that:

$$\mathsf{lhSBIA}^{+D} \models \big((x \wedge y) \wedge z\big) \Leftrightarrow \big(x \wedge (y \wedge z)\big) \approx \mathbf{1},$$

whence $\mathsf{lhSBIA}^{+D} \models$ (3.111). Because of the axiomatisation of lhSBIA^{+D} by (3.133)–(3.145), a suitable modification of the preceding argument now shows that $\mathsf{lhSBIA}^{+D} \models$ (3.112)–(3.123) for each of the remaining identities (3.112)–(3.123) defining K.

For (ii), consider a defining quasi-identity of K, say (3.129). Then $\mathsf{K} \models$ (3.129);

that is to say:

$$\mathsf{K} \models x \Leftrightarrow y \approx \mathbf{1} \mathbin{\&} z \Leftrightarrow w \approx \mathbf{1} \supset x \Rightarrow z \Leftrightarrow y \Rightarrow w \approx \mathbf{1}.$$

By properties of $\approx$, we have that:

$$\mathsf{lhSBIA}^{+D} \models x \approx y \mathbin{\&} z \approx w \supset x \Rightarrow z \approx y \Rightarrow w$$

vacuously, so by (3.146) we have that:

$$\mathsf{lhSBIA}^{+D} \models x \Leftrightarrow y \approx \mathbf{1} \mathbin{\&} z \Leftrightarrow w \approx \mathbf{1} \supset x \Rightarrow z \Leftrightarrow y \Rightarrow w \approx \mathbf{1}.$$

Hence $\mathsf{lhSBIA}^{+D} \models$ (3.129). Because of properties of $\approx$, a suitable modification of the preceding argument now shows $\mathsf{lhSBIA}^{+D} \models$ (3.125)–(3.128), (3.130)–(3.132) for each of the remaining quasi-identities (3.125)–(3.128) and (3.130)–(3.132) axiomatising K.

By (i) and (ii), $\mathsf{lhSBIA}^{+D} \models$ (3.111)–(3.132), so $\mathsf{K} \subseteq \mathsf{lhSBIA}^{+D}$ as desired.

To establish the inclusion $\mathsf{lhSBIA}^{+D} \subseteq \mathsf{K}$, note first that:

$$\mathsf{K} \models x \approx y \quad \text{iff} \quad \mathsf{K} \models x \Leftrightarrow y \approx \mathbf{1}, \tag{3.147}$$

because of (3.124) and (3.132). To complete the proof, consider a defining identity of lhSBIA^{+D}, say (3.133). Then $\mathsf{lhSBIA}^{+D} \models$ (3.133); that is to say:

$$\mathsf{lhSBIA}^{+D} \models (x \wedge y) \wedge z \approx x \wedge (y \wedge z).$$

Now by (3.111), we have that:

$$\mathsf{K} \models \big((x \wedge y) \wedge z\big) \Leftrightarrow \big(x \wedge (y \wedge z)\big) \approx \mathbf{1},$$

so by (3.147) we have that:

$$\mathsf{K} \models (x \wedge y) \wedge z \approx x \wedge (y \wedge z).$$

Thus $\mathsf{K} \models$ (3.133). Because of the axiomatisation of K by (3.111)–(3.132), a suitable modification of the preceding argument now shows $\mathsf{K} \models$ (3.134)–(3.145) for each of the remaining identities (3.134)–(3.145) defining lhSBIA^{+D}. Hence $\mathsf{lhSBIA}^{+D} \subseteq \mathsf{K}$. ∎

Remark 3.3.73 (Added in proof). It is implicit in Blok and Pigozzi [36, Sections 3.4.5–3.4.7] and the constructive proof of Andréka, Kurucz, Németi and Sain [13, Theorem 3.2.3] that any axiomatisation of a K-$\mathbf{1}$-regular quasivariety K with $\mathbf{1}$ can be translated into an axiomatisation of the inherent assertional logic $\mathbb{S}(\mathsf{K}, \mathbf{1})$ of K. In more detail, let K be a K-$\mathbf{1}$-regular quasivariety (for some constant term $\mathbf{1}$) and let $\Delta := \{\Delta_1, \ldots, \Delta_m\}$ be a set of binary terms witnessing the K-$\mathbf{1}$-regularity of K in the sense of Czelakowski and Pigozzi [78, Theorem 2.3]. Let $\mathbb{S}_{\mathsf{K}}$ be the 2-dimensional deductive system associated with K in the sense of Blok and Pigozzi [36, Section 3.4.7] and let:

$$\tau(p) := \{\langle p, \mathbf{1}\rangle\} \quad \text{and} \quad \rho(\langle p, q\rangle) := \Delta(p, q)$$

be (1, 2)-translations and (2, 1)-translations in the sense of Blok and Pigozzi [36, Section 6.1]. Then the image $\rho[Id \cup QId]$ of any axiomatisation $Id \cup QId$ of K under ρ yields a deductive system $\mathbb{S}$. Adjoin to $\mathbb{S}$ the further rule:

$$\Delta_1(p, \mathbf{1}), \ldots, \Delta_m(p, \mathbf{1}) \vdash_{\mathbb{S}} p \tag{3.148}$$

and denote the resulting deductive system by $\mathbb{S}'$. Because of (3.148) and the description of $\rho[Id \cup QId]$, τ is an interpretation of $\mathbb{S}'$ in $\mathbb{S}_{\mathsf{K}}$ and ρ is an interpretation of $\mathbb{S}_{\mathsf{K}}$ in $\mathbb{S}'$, so $\mathbb{S}'$ and $\mathbb{S}_{\mathsf{K}}$ are equivalent: see for instance Blok and Pigozzi [36, Section 6.1]. Therefore $\mathbb{S}'$ is algebraisable with equivalence formulas $\Delta_1, \ldots, \Delta_m$ and defining equation $p \approx \mathbf{1}$. Let K' be the equivalent quasivariety semantics of $\mathbb{S}'$. Then $\mathbb{S}'$ is $\mathbb{S}(\mathsf{K}', \mathbf{1})$. Moreover, because both K and K' satisfy the quasi-identities:

$$x \approx y \supset \Delta_1(x, y) \approx \mathbf{1} \;\&\; \ldots \;\&\; \Delta_m(x, y) \approx \mathbf{1}$$

and:

$$\Delta_1(x,y) \approx \mathbf{1} \,\&\, \ldots \,\&\, \Delta_m(x,y) \approx \mathbf{1} \supset x \approx y$$

one sees easily that K and K' coincide. Hence $\mathbb{S}' = \mathbb{S}(\mathsf{K}', \mathbf{1}) = \mathbb{S}(\mathsf{K}, \mathbf{1})$. Theorem 3.3.70 and its counterpart Proposition 3.3.72 may be seen as a particular instance of this general argument, due in the form above to Professor James Raftery and the author [189]. ∎

Theorem 3.3.74. $\mathbb{SBPC}$ *coincides with* $\mathbb{S}(\mathsf{lhSBIA}^{+D}, \mathbf{1})$, *the assertional logic of the variety of double-pointed left handed skew Boolean* $\cap$*-algebras.*

Proof. By Theorem 3.3.70 and Proposition 3.3.72, $\mathbb{SBPC}$ is algebraisable with equivalent algebraic semantics lhSBIA^{+D} and defining equation $p \approx \mathbf{1}$. But by Theorem 1.8.15, $\mathbb{S}(\mathsf{lhSBIA}^{+D}, \mathbf{1})$ is algebraisable with the same equivalent algebraic semantics and defining equation, so by Lemma 1.8.8, $\mathbb{SBPC}$ and $\mathbb{S}(\mathsf{lhSBIA}^{+D}, \mathbf{1})$ coincide. ∎

In the sequel we continue to denote $\mathbb{S}(\mathsf{lhSBIA}^{+D}, \mathbf{1})$ by $\mathbb{SBPC}$. Upon recalling from §3.3.59 that PD_0^+ denotes the pure double-pointed discriminator variety, the following theorem may now be inferred from the preceding result and Theorem 3.3.66.

Corollary 3.3.75. $\mathbb{SBPC}$ *is definitionally equivalent to* $\mathbb{S}(\mathsf{PD}_0^+, \mathbf{0})$, *the assertional logic of the pure double-pointed discriminator variety.*

Although $\mathbb{SBPC}$ is a deductive system in the sense of §1.2.9, its presentation $Ax \cup Ir$ is too closely related to the underlying axiomatisation of its equivalent algebraic semantics to be a Hilbert-style axiomatisation in any familiar sense (for example, in the sense of Sundholm [213, Section 1]). In particular, $Ax \cup Ir$ is not an axiomatisation of $\mathbb{SBPC}$ for which (MP) is the only (proper) rule of proof. We claim that there exists just such an axiomatisation of $\mathbb{SBPC}$ (in principle). Our observation to this effect results as an easy consequence of Theorem 3.3.77 below, which characterises the logical connectives of $\mathbb{SBPC}$. But first, the following useful technical lemma, a version of which occurs in Bulman-Fleming and Werner [50]. See also Burris [53].

Lemma 3.3.76. *(cf. [50, Lemma 1.3]; cf. [53, Proposition 2.3.10]) For any (double-pointed) skew Boolean ⊓-algebra* $\mathbf{A}$ *and any* $a, b \in A$*, the following assertions hold:*

1. $\Theta^{\mathbf{A}}(a \wedge b \wedge a, 0) = \Theta^{\mathbf{A}}(a, 0) \wedge^{\mathbf{Con\,A}} \Theta^{\mathbf{A}}(b, 0)$;
2. $\Theta^{\mathbf{A}}(a \vee b \vee a, 0) = \Theta^{\mathbf{A}}(a, 0) \vee^{\mathbf{Con\,A}} \Theta^{\mathbf{A}}(b, 0)$;
3. $\Theta^{\mathbf{A}}(a \backslash b, 0) = \Theta^{\mathbf{A}}(a, 0) *^{\mathbf{Cp\,A}} \Theta^{\mathbf{A}}(b, 0)$;
4. $\Theta^{\mathbf{A}}(a \Delta b, 0) = \Theta^{\mathbf{A}}(a, b)$.

Proof. Let $\mathbf{A}$ be a (double-pointed) skew Boolean ⊓-algebra and $a, b \in A$.

For (1), from Proposition 2.2.31 we have that $\langle a \sqcap b \rangle_{\mathbf{A}} = \langle a \rangle_{\mathbf{A}} \cap \langle b \rangle_{\mathbf{A}}$, just because the ideals of $\mathbf{A}$ coincide with the ideals of its canonical implicative BCS-algebra polynomial reduct $\langle A; \backslash, 0 \rangle$. Since $a \wedge b \wedge a = a \sqcap b$ (by (3.69)), we have that $\langle a \wedge b \wedge a \rangle_{\mathbf{A}} = \langle a \rangle_{\mathbf{A}} \cap \langle b \rangle_{\mathbf{A}}$. By normality of ideals, $[0]_{\Theta^{\mathbf{A}}(a \wedge b \wedge a, 0)} = [0]_{\Theta^{\mathbf{A}}(a,0)} \cap [0]_{\Theta^{\mathbf{A}}(b,0)}$, so by ideal determinacy, $\Theta^{\mathbf{A}}(a \wedge b \wedge a, 0) = \Theta^{\mathbf{A}}(a, 0) \cap \Theta^{\mathbf{A}}(b, 0)$.

For (2), we first observe that $x \vee y \vee x$ is a join generator term for SBA. Indeed, let:

$$\begin{aligned} x \sqcup y &:= x \vee y \vee x \\ r(x, y, z) &:= z \backslash (y \backslash x) \\ t(x, y, z) &:= y \wedge \big(z \backslash (x \backslash y)\big) \wedge y. \end{aligned}$$

By an easy inspection of the subdirectly irreducible skew Boolean algebras, we see that SBA satisfies the identities:

$$\begin{aligned} r(x, y, \mathbf{0}) &\approx \mathbf{0} & t(x, y, \mathbf{0}) &\approx \mathbf{0} \\ r(x, y, x \sqcup y) &\approx x & t(x, y, x \sqcup y) &\approx y \\ \mathbf{0} \sqcup \mathbf{0} &\approx \mathbf{0} \end{aligned}$$

so $x \vee y \vee x$ is a join generator term for SBA by Proposition 1.7.13. Because the ideals of any skew Boolean ⊓-algebra coincide with the ideals of its canonical skew Boolean algebra polynomial reduct, $x \vee y \vee x$ is also a join generator term for SBIA. Hence $\langle a \vee b \vee a\rangle_{\mathbf{A}} = \langle a\rangle_{\mathbf{A}} \vee^{\mathbf{I(A)}} \langle b\rangle_{\mathbf{A}}$. By normality of ideals, $[0]_{\Theta^{\mathbf{A}}(a\vee b\vee a,0)} = [0]_{\Theta^{\mathbf{A}}(a,0)} \vee^{\mathbf{I(A)}} [0]_{\Theta^{\mathbf{A}}(b,0)}$, so by ideal determinacy, $\Theta^{\mathbf{A}}(a \vee b \vee a, 0) = \Theta^{\mathbf{A}}(a,0) \vee^{\mathbf{Con\,A}} \Theta^{\mathbf{A}}(b,0)$.

For (3), from Theorem 2.2.20(5) we have that $\langle a\backslash b\rangle_{\mathbf{A}} = \langle a\rangle_{\mathbf{A}} * \langle b\rangle_{\mathbf{A}}$ (where $*$ denotes dual relative pseudocomplementation in the join semilattice $\langle \mathrm{CI}(\mathbf{A});\ \vee, \langle 0\rangle_{\mathbf{A}}\rangle$ of compact ideals of $\mathbf{A}$), just because the ideals of $\mathbf{A}$ coincide with the ideals of its canonical implicative BCS-algebra polynomial reduct $\langle A;\ \backslash, 0\rangle$. By normality of ideals, $[0]_{\Theta^{\mathbf{A}}(a\backslash b,0)} = [0]_{\Theta^{\mathbf{A}}(a,0)} *^{\mathbf{CI(A)}} [0]_{\Theta^{\mathbf{A}}(b,0)}$, so by subtractivity and Proposition 1.7.10, $\Theta^{\mathbf{A}}(a\backslash b, 0) = \Theta^{\mathbf{A}}(a,0) *^{\mathbf{Cp\,A}} \Theta^{\mathbf{A}}(b,0)$.

For (4), recall $a\Delta b = (a/b) \vee (b/a)$ for any skew Boolean $\cap$-algebra $\mathbf{A}$ and $a, b \in A$. Let $\theta \in \mathrm{Con}\,\mathbf{A}$. From Proposition 3.3.37, we have that $a\Delta b = 0$ iff $a = b$. Applying this to the quotient algebra $\mathbf{A}/\theta$, we infer that $a\Delta b \equiv 0 \,(\mathrm{mod}\,\theta)$ iff $a \equiv b \,(\mathrm{mod}\,\theta)$, which implies that $\Theta^{\mathbf{A}}(a\Delta b, 0) = \Theta^{\mathbf{A}}(a, b)$. ∎

Theorem 3.3.77. IhSBIA$^+$ *is a WBSO$^\#$ variety with:*

1. *Weak join $x \wedge y \wedge x$;*
2. *Weak meet $x \vee y \vee x$;*
3. *Subtractive weak relative pseudocomplementation $x\backslash y$;*
4. *Gödel equivalence term $x\Delta y$.*

Hence the following assertions hold concerning the logical connectives of $\mathbb{SBPC}$*:*

1'. *$\wedge$ is a conjunction;*

2'. *$\vee$ is a disjunction;*

3'. *$\rightarrow$ is a conditional;*

4'. *$\Leftrightarrow$ is a G-identity;*

5'. *$\leftrightarrow$ is a biconditional;*

6'. *$\neg$ is a weak negation.*

Proof. The theorem follows easily from Lemma 3.3.76, Theorem 3.3.74 and Proposition 3.1.25. ∎

Corollary 3.3.78. *For any $\mathcal{L}$-formulas φ, ψ, χ, the following formulas are theorems of* $\mathbb{SBPC}$*:*

$$\varphi \to (\psi \to \varphi)$$
$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$
$$(\varphi \wedge \psi) \to \varphi$$
$$(\varphi \wedge \psi) \to \psi$$
$$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to (\varphi \wedge \psi)))$$
$$\varphi \to (\varphi \vee \psi)$$
$$\psi \to (\varphi \vee \psi)$$
$$(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi)).$$

Proof. The result follows from Theorem 3.3.77 and Wójcicki [238, Theorem 2.4.7]. ∎

Corollary 3.3.79 (Deduction-Detachment Theorem for $\mathbb{SBPC}$). *For all $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,*

$$\Gamma, \varphi \vdash_{\mathbb{SBPC}} \psi \quad \textit{iff} \quad \Gamma \vdash_{\mathbb{SBPC}} \varphi \to \psi.$$

Remark 3.3.80. By Corollary 3.3.79, $\varphi(\vec{p}) \vdash_{\mathbb{SBPC}} \psi(\vec{p})$ iff $\vdash_{\mathbb{SBPC}} \varphi(\vec{p}) \to \psi(\vec{p})$ iff $\mathsf{IhSBIA}^{+D} \models s(\vec{x}) \to t(\vec{x}) \approx 1$ iff $s^{\mathbf{A}}(\vec{a}) \preceq_D t^{\mathbf{A}}(\vec{a})$ for all $\mathbf{A} \in \mathsf{IhSBIA}^{+D}$ and $\vec{a} \in A$, where $s(\vec{x}), t(\vec{x})$ are $\mathcal{L}$-terms in the individual variables $\vec{x}$ identified respectively with $\mathcal{L}$-formulas $\varphi(\vec{p}), \psi(\vec{p})$ in the propositional variables $\vec{p}$. Hence the consequence relation $\vdash_{\mathbb{SBPC}}$ of $\mathbb{SBPC}$ induces only a quasiordering on its underlying algebraic models. In general, therefore, it is not the case that $\varphi(\vec{p}) \dashv\vdash_{\mathbb{SBPC}} \psi(\vec{p})$ iff $\mathsf{IhSBIA}^{+D} \models s(\vec{x}) \approx t(\vec{x})$. Because of Font and Jansana [91, Proposition 2.43], this shows that $\mathbb{SBPC}$ is not *congruential* in the sense of Rautenberg [197]. ∎

Apropos the claim prefixing Lemma 3.3.76, because $\mathbb{SBPC}$ has the DDT (witness $x \to y$), any axiomatisation of $\mathbb{SBPC}$ (including $Ax \cup Ir$) may be converted into one in which (MP) (for $\rightarrow$), *viz.*:

$$p, p \rightarrow q \vdash_{\mathbb{SBPC}} q$$

is the only (proper) rule of inference. To see this, apply the DDT from left to right repeatedly on all inference rules of a given axiomatisation of $\mathbb{SBPC}$ until they are in the form of axioms. Then the axiomatisation obtained by adjoining (MP) (for $\rightarrow$) as a (proper) rule of inference to the resulting collection of axioms is clearly an axiomatisation of $\mathbb{SBPC}$. (More generally, we remark that if K is a quasivariety with $\mathbf{1}$ such that the assertional logic $\mathbb{S}(\mathsf{K}, \mathbf{1})$ has a DDT with deduction-detachment set $\Sigma := \{\zeta_i(p, q) : i = 1, \ldots, n\}$, then any axiomatisation of $\mathbb{S}(\mathsf{K}, \mathbf{1})$ may be converted into one in which:

$$\varphi, \{\Sigma_i(\varphi, \psi) : i = 1, \ldots, n\} \vdash_{\mathbb{S}} \psi$$

is the only (proper) rule of inference [189].) Hence there exists (in principle) an axiomatisation of $\mathbb{SBPC}$ for which (MP) (for $\rightarrow$) is the only (proper) rule of inference.

By the preceding observation, a direct demonstration that (MP) (for $\rightarrow$) is a derived rule of proof for $\mathbb{SBPC}$ is of some independent interest. In the final result of this subsection, we provide just such a direct proof of the detachment property (for $\rightarrow$).

Lemma 3.3.31. *For any $\mathcal{L}$-formula φ, the following formula is a theorem of* $\mathbb{SBPC}$*:*

$$(\mathbf{1} \Rightarrow \varphi) \Leftrightarrow \varphi \tag{3.149}$$

Proof.

(1)	$(\varphi \Rightarrow (\varphi \vee \varphi)) \Leftrightarrow \mathbf{1}$	(S10), $p := \varphi$, $q := \varphi$
(2)	$\varphi \Leftrightarrow \varphi$	(R)
(3)	$((\varphi \Rightarrow (\varphi \vee \varphi)) \Rightarrow \varphi) \Leftrightarrow (\mathbf{1} \Rightarrow \varphi)$	(1), (2), (CP-$\Rightarrow$)
(4)	$(\mathbf{1} \Rightarrow \varphi) \Leftrightarrow ((\varphi \Rightarrow (\varphi \vee \varphi)) \Rightarrow \varphi)$	(3), (S)

(5)	$((\varphi \Rightarrow (\varphi \vee \varphi)) \Rightarrow \varphi) \Leftrightarrow \varphi$	(S7), $p := \varphi$, $q := \varphi \vee \varphi$
(6)	$(\mathbf{1} \Rightarrow \varphi) \Leftrightarrow \varphi$	(4), (5), (T)

∎

Proposition 3.3.82 (Modus Ponens for $\Rightarrow$). *For all $\mathcal{L}$-formulas φ, ψ,*

$$\varphi, \varphi \Rightarrow \psi \vdash_{\mathbb{SBPC}} \psi.$$

Proof.

(1)	φ	Assumption
(2)	$\varphi \Leftrightarrow \mathbf{1}$	(1), (**1**-I)
(3)	$\psi \Leftrightarrow \psi$	(R)
(4)	$(\varphi \Rightarrow \psi) \Leftrightarrow (\mathbf{1} \Rightarrow \psi)$	(2), (3), (CP-$\Rightarrow$)
(5)	$(\mathbf{1} \Rightarrow \psi) \Leftrightarrow \psi$	by (3.149)
(6)	$(\varphi \Rightarrow \psi) \Leftrightarrow \psi$	(4), (5), (T)
(7)	$\psi \Leftrightarrow (\varphi \Rightarrow \psi)$	(6), (S)
(8)	$\varphi \Rightarrow \psi$	Assumption
(9)	$(\varphi \Rightarrow \psi) \Leftrightarrow \mathbf{1}$	(8), (**1**-I)
(10)	$\psi \Leftrightarrow \mathbf{1}$	(7), (9), (T)
(11)	ψ	(10), (**1**-E).

∎

Lemma 3.3.83. *For any $\mathcal{L}$-formula φ, the following formulas are theorems of* $\mathbb{SBPC}$*:*

$$(\varphi \wedge \varphi) \Leftrightarrow \varphi \tag{3.150}$$

$$(\varphi \vee \varphi) \Leftrightarrow \varphi \tag{3.151}$$

$$\varphi \Rightarrow \varphi. \tag{3.152}$$

Proof. Let φ, ψ be $\mathcal{L}$-formulas. For (3.150), we have the following derivation:

(1)	$((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Leftrightarrow \varphi$	(S7), $p := \varphi$, $q := \psi$
(2)	$\varphi \Leftrightarrow \varphi$	(R)
(3)	$(((\varphi \Rightarrow \psi) \Rightarrow \varphi) \wedge \varphi) \Leftrightarrow (\varphi \wedge \varphi)$	(1), (2), (CP-$\wedge$)
(4)	$(\varphi \wedge \varphi) \Leftrightarrow (((\varphi \Rightarrow \psi) \Rightarrow \varphi) \wedge \varphi)$	(3), (S)
(5)	$(((\varphi \Rightarrow \psi) \Rightarrow \varphi) \wedge \varphi) \Leftrightarrow \varphi$	(S12), $q := \varphi \Rightarrow \psi$, $p := \varphi$
(6)	$(\varphi \wedge \varphi) \Leftrightarrow \varphi$	(4), (5), (T).

For (3.151), we have the following derivation:

(1)	$(\varphi \wedge \varphi) \Leftrightarrow \varphi$	by (3.150)
(2)	$\varphi \Leftrightarrow \varphi$	(R)
(3)	$((\varphi \wedge \varphi) \vee \varphi) \Leftrightarrow (\varphi \vee \varphi)$	(1), (2), (CP-$\vee$)
(4)	$(\varphi \vee \varphi) \Leftrightarrow ((\varphi \wedge \varphi) \vee \varphi)$	(3), (S)
(5)	$((\varphi \wedge \varphi) \vee \varphi) \Leftrightarrow \varphi$	(S4), $q := \varphi$, $p := \varphi$
(6)	$(\varphi \vee \varphi) \Leftrightarrow \varphi$	(4), (5), (T).

For (3.152), we have the following derivation:

(1)	$\varphi \Leftrightarrow \varphi$	(R)
(2)	$(\varphi \vee \varphi) \Leftrightarrow \varphi$	by (3.151)
(3)	$(\varphi \Rightarrow (\varphi \vee \varphi)) \Leftrightarrow (\varphi \Rightarrow \varphi)$	(1), (2), (CP-$\Rightarrow$)
(4)	$(\varphi \Rightarrow \varphi) \Leftrightarrow (\varphi \Rightarrow (\varphi \vee \varphi))$	(3), (S)
(5)	$(\varphi \Rightarrow (\varphi \vee \varphi)) \Leftrightarrow \mathbf{1}$	(S10), $p := \varphi$, $q := \varphi$
(6)	$(\varphi \Rightarrow \varphi) \Leftrightarrow \mathbf{1}$	(4), (5), (T)
(7)	$\varphi \Rightarrow \varphi$	(6), ($\mathbf{1}$-E).

∎

Lemma 3.3.84. *For any $\mathcal{L}$-formulas φ, ψ, the following formulas are theorems* *of* SBPC*:*

$$(\mathbf{1} \wedge \varphi) \Leftrightarrow \varphi \tag{3.153}$$

$$(\varphi \vee (\varphi \wedge \psi)) \Leftrightarrow \varphi \tag{3.154}$$

$$\mathbf{1} \vee \varphi. \tag{3.155}$$

Proof. Let φ, ψ, χ be $\mathcal{L}$-formulas. For (3.153) we have the following derivation:

(1)	$\varphi \Rightarrow \varphi$	by (3.152)
(2)	$(\varphi \Rightarrow \varphi) \Leftrightarrow \mathbf{1}$	(1), (**1**-I)
(3)	$\varphi \Leftrightarrow \varphi$	(R)
(4)	$((\varphi \Rightarrow \varphi) \wedge \varphi) \Leftrightarrow (\mathbf{1} \wedge \varphi)$	(2), (3), (CP-$\wedge$)
(5)	$(\mathbf{1} \wedge \varphi) \Leftrightarrow ((\varphi \Rightarrow \varphi) \wedge \varphi)$	(4), (S)
(6)	$((\varphi \Rightarrow \varphi) \wedge \varphi) \Leftrightarrow \varphi$	(S12), $q := \varphi$, $p := \varphi$
(7)	$(\mathbf{1} \wedge \varphi) \Leftrightarrow \varphi$	(5), (6), (T).

For (3.154) we have the following derivation:

(1)	$(\varphi \vee \varphi) \Leftrightarrow \varphi$	by (3.151)
(2)	$(\varphi \vee \psi) \Leftrightarrow (\varphi \vee \psi)$	(R)
(3)	$((\varphi \vee \varphi) \wedge (\varphi \vee \psi)) \Leftrightarrow (\varphi \wedge (\varphi \vee \psi))$	(1), (2), (CP-$\wedge$)
(4)	$(\varphi \wedge (\varphi \vee \psi)) \Leftrightarrow ((\varphi \vee \varphi) \wedge (\varphi \vee \psi))$	(3), (S)
(5)	$((\varphi \vee \varphi) \wedge (\varphi \vee \psi)) \Leftrightarrow (\varphi \vee (\varphi \wedge \psi))$	(S5), $p := \varphi$, $q := \varphi$, $r := \psi$
(6)	$(\varphi \wedge (\varphi \vee \psi)) \Leftrightarrow (\varphi \vee (\varphi \wedge \psi))$	(4), (5), (T)
(7)	$(\varphi \vee (\varphi \wedge \psi)) \Leftrightarrow (\varphi \wedge (\varphi \vee \psi))$	(6), (S)
(8)	$(\varphi \wedge (\varphi \vee \psi)) \Leftrightarrow \varphi$	(S3), $p := \varphi$, $q := \varphi$
(9)	$(\varphi \vee (\varphi \wedge \psi)) \Leftrightarrow \varphi$	(7), (8), (T).

For (3.155) we have the following derivation:

(1)	$\mathbf{1} \Leftrightarrow \mathbf{1}$	(R)

(2)	$(\mathbf{1} \wedge \varphi) \Leftrightarrow \varphi$	by (3.153)
(3)	$\big(\mathbf{1} \vee (\mathbf{1} \wedge \varphi)\big) \Leftrightarrow (\mathbf{1} \vee \varphi)$	(2), (3), (CP-$\vee$)
(4)	$(\mathbf{1} \vee \varphi) \Leftrightarrow \big(\mathbf{1} \vee (\mathbf{1} \wedge \varphi)\big)$	(3), (S)
(5)	$\big(\mathbf{1} \vee (\mathbf{1} \wedge \varphi)\big) \Leftrightarrow \mathbf{1}$	by (3.154)
(6)	$(\mathbf{1} \vee \varphi) \Leftrightarrow \mathbf{1}$	(4), (5), (T)
(7)	$\mathbf{1} \vee \varphi$	(6), (1-E).

∎

Proposition 3.3.85 (Modus Ponens for $\to$). *For all $\mathcal{L}$-formulas φ, ψ,*

$$\varphi, \varphi \to \psi \vdash_{\mathbb{SBPC}} \psi.$$

Proof. By the definition of $\to$ and Proposition 3.3.82, to prove the proposition we need only show $\varphi \vdash_{\mathbb{SBPC}} \varphi \vee \psi$ for all $\mathcal{L}$-formulas φ, ψ. For this we have:

(1)	φ	Assumption
(2)	$\varphi \Leftrightarrow \mathbf{1}$	(1), (1-I)
(3)	$\psi \Leftrightarrow \psi$	(R)
(4)	$(\varphi \vee \psi) \Leftrightarrow (\mathbf{1} \vee \psi)$	(2), (3), (CP-$\vee$)
(5)	$\mathbf{1} \vee \psi$	by (3.155)
(6)	$(\mathbf{1} \vee \psi) \Leftrightarrow \mathbf{1}$	(5), (1-I)
(7)	$(\varphi \vee \psi) \Leftrightarrow \mathbf{1}$	(4), (6), (T)
(8)	$\varphi \vee \psi$	(7), (1-E).

∎

The results of this subsection notwithstanding, we have been unable to obtain a Hilbert-style axiomatisation of $\mathbb{SBPC}$ in which (MP) (for $\to$) is the only (proper) rule of inference that is 'aesthetically pleasing' in the sense that its axioms have a familiar description. In particular, we have been unable to provide such an axiomatisation of $\mathbb{SBPC}$ whose axioms are based on the theorems (or some variants of the theorems) of Corollary 3.3.78. Hence we conclude this subsection with the following problem.

Problem 3.3.86. Give an 'aesthetically pleasing' Hilbert-style axiomatisation of $\mathbb{SBPC}$ for which the only rule of inference is (MP) (for $\rightarrow$): $p, p \rightarrow q \vdash_{\mathbb{SBPC}} q$.
■

Chapter 4

Conclusion

4.1 Summary

Motivated by work of Blok and Raftery [38, Section 4] and Agliano and Ursini [11, Example 3.7, Corollary 3.8], this thesis witnessed the introduction of pre-BCK-algebras as a generalisation of BCK-algebras to the subtractive but not BCK-0-regular case. In particular, this dissertation oversaw the investigation of the elementary theory of the variety of pre-BCK-algebras and some of its subvarieties, and the application of this theory to the study of some varieties arising naturally in universal algebra and algebraic logic.

Chapter 1 provided a structured account of the theory relevant to the study of pre-BCK-algebras, including: Laslo and Leech's theory of quasilattices, paralattices and skew lattices; Blok and Pigozzi's hierarchy of varieties with EDPC; the theory of BCK-algebras and BCK-lattices due to Iséki, Idziak and others; Agliano and Ursini's theory of ideals and subtractive varieties; and the theory of algebraisable and assertional logics due to Blok, Pigozzi, Raftery and others. The main new results concerned distributivity in skew lattices. By the results of §1.4.18, the equivalence of the middle distributive identities (1.19)–(1.20) for lattices extends to symmetric skew lattices, but not to skew lattices. The results provide support for Leech's contention (initially prompted by the remarks of §1.4.15) that '...symmetric skew lattices are the really *nice* skew lattices' [152, p. 17].

In Chapter 2 the elementary theory of the variety of pre-BCK-algebras and some of its subvarieties was investigated. In Section 2.1 pre-BCK-algebras proper were considered. The results obtained show that a significant fragment of the first-order theory of BCK-algebras extends to pre-BCK-algebras; in particular, the ideal theory of BCK-algebras carries over to pre-BCK-algebras almost in its entirety. In Section 2.2 varieties of pre-BCK-algebras were studied. It was shown that the correspondence between the theory of BCK-algebras and the theory of pre-BCK-algebras exhibited in Section 2.1 extends to subvarieties of BCK-algebras, inasmuch as with any variety V of BCK-algebras there may be associated a variety $\mathsf{V}_\flat$ of pre-BCK-algebras ('the natural pre-BCK-algebraic counterpart of V') such that $\mathsf{V}_\flat$ enjoys many of the same (first-order) properties as V. In Section 2.3 the variety of implicative BCS-algebras was investigated. Although the variety of implicative BCS-algebras fails to enjoy many of the properties typically associated with a 'tractable' class of algebras, the results of Section 2.3 nonetheless provide for these algebras a fairly complete elementary theory closely resembling that of implicative BCK-algebras. Collectively, the results of Chapter 2 indicate that pre-BCK-algebras enjoy a coherent elementary theory that largely parallels the theory of BCK-algebras. This suggests that pre-BCK-algebras are an appropriate generalisation of BCK-algebras to the subtractive but not BCK-0-regular case.

In Chapter 3 the theory of pre-BCK-algebras was applied to the study of some varieties arising naturally in universal algebra and algebraic logic. In Section 3.1 positive implicative pre-BCK-algebras in subtractive varieties with EDPI were considered. The results obtained show in particular that the study of ideals in subtractive varieties with EDPI reduces to the study of ideals in positive implicative pre-BCK-algebras. In Section 3.2 connections between implicative BCS-algebras and binary discriminator varieties were investigated. It was shown that the variety of implicative BCS-algebras provides a convenient framework for the study of binary discriminator varieties, and this observation was exploited in clarifying relationships between binary discriminator, pointed fixedpoint discriminator and pointed ternary discriminator varieties. In Section 3.3 varieties $\mathsf{PQ}_\mathcal{C}, \mathsf{BP}_\mathcal{C}$ of pre-BCK/BCK-algebras structurally enriched with band [double band] operations were studied as a generalisation of Idziak's varieties of lower/upper BCK-semilattices [BCK-lattices]. It was shown that the theory of the varieties $\mathsf{PQ}_{\mathcal{C}}$ and $\mathsf{BP}_{\mathcal{C}}$ encompasses the theory of skew Boolean algebras/skew Boolean $\cap$-algebras, and hence (to within termwise definitional equivalence) subsumes the theory of pointed discriminator varieties. Further, the results hint that the theory of the varieties $\mathsf{BP}_{\mathcal{C}}$ accommodates the theory of pointed fixedpoint discriminator varieties (to within termwise definitional equivalence). Collectively, the results of Chapter 3 indicate that (structurally enriched) pre-BCK-algebras (distinct from BCK-algebras) arise naturally in and simplify the study of several classes of varieties found in universal algebra and algebraic logic. This suggests that classes of (structurally enriched) pre-BCK-algebras may provide a unifying framework simplifying the study of several hitherto unrelated areas of universal algebraic logic.

As a generalisation of BCK-algebras to the subtractive but not BCK-0-regular case, the work of this thesis thus attests that pre-BCK-algebras are of interest both in their own right and in their application to the study of varieties arising naturally in universal algebra and algebraic logic. Pre-BCK-algebras are of interest in their own right, inasmuch as they are a natural and coherent generalisation of BCK-algebras to the subtractive but not BCK-0-regular case; and their applications to universal algebraic logic are of interest, insofar as such algebras provide a unifying framework simplifying the study of several important classes of varieties occurring naturally in universal algebra and algebraic logic.

4.2 Future Work

The remarks of Section 4.1 clearly call for a further study of pre-BCK-algebras and of their application to universal algebra and algebraic logic. In what follows we present a brief selection of problems outlining some possibilities for future research.

4.2.1. Pre-BCK-Algebras. By the remarks of §1.1.1, residuated structures play a central role in the algebraic study of logical systems. The residuated structures most commonly encountered in algebraic logic are (left) residuated partially ordered groupoids, or structures of the form $\langle A;\ \oplus;\ \leq\rangle$ where: (i) $\leq$ is a partial order on A; (ii) $\oplus$ is a binary operation on A isotone in each of its positions; and (iii) the equivalence $a \leq c \oplus b$ iff $a \dot{-} b \leq c$ is satisfied for any $a, b, c \in A$. Pocrims and polrims provide natural examples of (left) residuated partially ordered groupoids. On the other hand, members of pointed discriminator varieties, WBSO$^{\#}$ varieties and double-pointed binary discriminator varieties provide important examples of algebras that do not in general enjoy the underlying structure of a residuated partially ordered groupoid. Instead, such algebras have the underlying structure of a (left) residuated quasiordered groupoid, where a *(left) residuated quasiordered groupoid* is a structure of the form $\langle A;\ \oplus;\ \preceq\rangle$ such that: (i) $\preceq$ is a quasiorder on A; (ii) $\oplus$ is a binary operation on A that is isotone in each of its positions; and (iii) the equivalence $a \preceq c \oplus b$ iff $a \dot{-} b \preceq c$ is satisfied for any $a, b, c \in A$. Indeed, let V be a pointed discriminator variety, WBSO$^{\#}$ variety or double-pointed binary discriminator variety and let $\sqcup$ be a join generator term for V. For any algebra $\mathbf{A}$ in V, the structure $\langle A;\ \sqcup^{\mathbf{A}};\ \preceq\rangle$ is a (left) residuated quasiordered groupoid, where $\preceq$ denotes the underlying quasiorder of the canonical MINI-algebra polynomial reduct of $\mathbf{A}$ (which must exist as V has EDPI). Inasmuch as the results of Section 2.1 lend to the conjecture that the class of residuated quasiordered groupoids may admit a coherent elementary theory closely resembling that of the class of residuated ordered groupoids, the preceding remarks call attention to the following problem.

Problem 4.2.2. Investigate the class of residuated quasiordered groupoids.
■

Considered as residuated partially ordered groupoids, pocrims (or more generally polrims) are of particular interest in algebraic logic because they are amenable to purely algebraic investigation, since any such structure $\langle A;\ \oplus, 0;\ \leq\rangle$ satisfies $a \leq b$ iff $a \dot{-} b = 0$ for any $a, b \in A$, whence the partial order $\leq$ may be completely recovered from the residuation operation $\dot{-}$. These remarks suggest that attention be focussed on residuated quasiordered groupoids that (in some sense) naturally generalise pocrims, and in fact at least one such class of residuated quasiordered groupoids has already been considered in the literature. Call an algebra $\mathbf{A} := \langle A;\ \oplus, \dot{-}, 0\rangle$ of type $\langle 2,2,0\rangle$ a *pre-pocrim* if: (i) the reduct $\langle A;\ \oplus, 0\rangle$ is a commutative monoid; (ii) the reduct $\langle A;\ \dot{-}, 0\rangle$ is a pre-BCK-algebra; and (iii) $\mathbf{A}$ satisfies the following identities:

$$\big((x \dot{-} y) \dot{-} z\big) \dot{-} \big(x \dot{-} (y \oplus z)\big) \approx \mathbf{0}$$
$$\big(x \dot{-} (y \oplus z)\big) \dot{-} \big((x \dot{-} y) \dot{-} z\big) \approx \mathbf{0}.$$

Clearly any pre-pocrim $\langle A;\ \oplus, \dot{-}, 0\rangle$ has the underlying structure of a residuated quasiordered groupoid $\langle A;\ \oplus;\ \preceq\rangle$, where $\preceq$ denotes the underlying quasiordering of the pre-BCK-algebra reduct $\langle A;\ \dot{-}, 0\rangle$. Pre-pocrims were introduced by Higgs in [109] in connection with his example showing that the class of all pocrims is not a variety. By [109, pp. 72–73] pre-pocrims are known to preserve several important properties of pre-BCK-algebras: in particular, for any pre-pocrim $\mathbf{A}$, the relation Ξ of Theorem 2.1.14 is a congruence on $\mathbf{A}$ such that $\mathbf{A}/\Xi$ is the maximal pocrim homomorphic image of $\mathbf{A}$. Nonetheless, it is unclear if pre-pocrims (considered as residuated quasiordered groupoids) are the most appropriate generalisation of pocrims (understood as residuated partially ordered groupoids). For let $\mathbf{A}$ be a pseudocomplemented semilattice. By previous remarks, $\mathbf{A}$ has the underlying structure of a residuated quasiordered groupoid $\langle A;\ \sqcup^{\mathbf{A}};\ \preceq\rangle$, where $\sqcup$ is the join generator term of Proposition 2.3.60(2) and $\preceq$ denotes the underlying quasiorder of the canonical implicative BCS-algebra polynomial reduct $\langle A;\ \backslash, 0\rangle$ of $\mathbf{A}$. However, the polynomial reduct $\langle A;\ \sqcup^{\mathbf{A}}, \backslash, 0\rangle$ is not a pre-pocrim, since $\mathbf{A} \not\models x \sqcup \mathbf{0} \approx x$.

Problem 4.2.3. Identify an appropriate generalisation of pocrims (considered as residuated partially ordered groupoids) to residuated quasiordered groupoids. Does the quasivariety of pocrims admit a generalisation to a class of residuated quasiordered groupoids analogous to that of BCK-algebras to pre-BCK-algebras? ■

Recall from §1.1.1 that the quasivariety of BCK-algebras is precisely the class of all $\langle \dot{-}, 0\rangle$-subreducts of pocrims.

Problem 4.2.4. Is an algebra $\langle A;\ \dot{-}, 0\rangle$ of type $\langle 2,0\rangle$ a pre-BCK-algebra iff it is a $\langle \dot{-}, 0\rangle$-subreduct of a residuated quasiordered groupoid in the sense of

Problem 4.2.3? ■

Residuated quasiordered groupoids may not be the only generalisation of pre-BCK-algebras of interest in algebraic logic. Indeed, the results of Section 2.1, in conjunction with work due to van Alten [229], suggest that the varietal closure $\mathsf{H}(\mathsf{LR})$ of the quasivariety LR of left residuation algebras is of interest in algebraic logic as a naturally occurring generalisation of the variety of pre-BCK-algebras (recall Proposition 2.1.11). In the statement of the following problem, an algebra $\mathbf{A} := \langle A; \dot{-}, 0\rangle$ of type $\langle 2, 0\rangle$ is called an $\mathsf{H}(\mathsf{LR})$-*algebra* if $\mathbf{A} \in \mathsf{H}(\mathsf{LR})$.

Problem 4.2.5. Investigate the variety $\mathsf{H}(\mathsf{LR})$. Do $\mathsf{H}(\mathsf{LR})$-algebras stand in relation to pre-BCK-algebras as left residuation algebras stand in relation to BCK-algebras? ■

4.2.6. Varieties of Pre-BCK-Algebras. In Section 2.2 the varieties of commutative, positive implicative and implicative pre-BCK-algebras were studied as the pre-BCK-algebraic counterparts of the varieties of commutative, positive implicative and implicative BCK-algebras respectively. In light of the results of Section 2.3, Section 3.1 and Section 3.2, it is natural to anticipate that the pre-BCK-algebraic counterparts of other naturally occurring varieties of BCK-algebras may themselves be of interest in universal algebra and algebraic logic. Hence we posit:

Problem 4.2.7. Investigate those varieties $\mathsf{V}_\flat$ of pre-BCK-algebras arising as the pre-BCK-algebraic counterpart of a naturally occurring variety V of BCK-algebras. In particular, investigate the natural pre-BCK-algebraic counterparts of the varieties $\mathsf{e_n BCK}$, $n \in \omega$, and the natural pre-BCK-algebraic counterpart of the variety HBCK [38, Theorem 3.15] of all residuation subreducts of hoops. ■

Apropos the preceding problem, the remarks of §1.5.1 and the results of Section 3.1 collectively indicate that varieties of positive implicative pre-BCK-algebras generalising the variety of positive implicative BCK-algebras may be of particular interest in universal algebraic logic. Let K denote the class of all positive implicative pre-BCK-algebras $\mathbf{A}$ with a left normal band with zero polynomial reduct $\langle A;\ \sqcap_1, 0\rangle$, where $a \sqcap_1 b := \big(a(ab)\big)(ba)$ for any $a, b \in A$, such that the underlying natural band partial order $\leq_{\mathcal{H}}^{\langle A;\, \sqcap_1, 0\rangle}$ respects positive implicative pre-BCK difference. Members of the class K more closely resemble positive implicative BCK-algebras than do positive implicative pre-BCK-algebras, inasmuch as any positive implicative BCK-algebra $\mathbf{A}$ has a semi-Boolean algebra polynomial reduct $\langle A;\ \cap_1\rangle$, where $a \cap_1 b := \big(a(ab)\big)(ba)$ for any $a, b \in A$ (recall the remarks prior to Problem 2.3.16). Thus the following problem would seem relevant:

Problem 4.2.8. Characterise the class K. Is it a variety? What role, if any, does K play in the theory of pre-BCK-algebras? In algebraic logic? ∎

4.2.9. Implicative BCS-Algebras. Recall from Theorem 2.3.29 that implicative BCS-algebras are precisely those implicative pre-BCK-algebras $\mathbf{A}$ for which the polynomial reduct $\langle A;\ \sqcap, 0\rangle$ is a left normal band with zero whose underlying natural band partial order respects implicative pre-BCK difference. This result warrants the study of bands in pre-BCK-algebras, and in particular the study of those pre-BCK-algebras $\mathbf{A}$ for which the polynomial reduct $\langle A;\ \sqcap, 0\rangle$ is a (normal) band with zero. As we know of no such studies in the existing literature concerning (normal) bands in BCK-algebras, the preceding remarks call particularly for a study of bands arising in BCK-algebras.

Problem 4.2.10. Investigate (normal) bands in pre-BCK-algebras. In particular, investigate (normal) bands in BCK-algebras. Is the class of all BCK-algebras $\langle A;\ \dot{-}, 0\rangle$ such that the polynomial reduct $\langle A;\ \cap\rangle$ is a (normal) band equationally definable? ∎

By Theorem 2.3.73, the 3-element flat implicative BCS-algebra $\mathbf{B}_2$ generates the class iBCS of implicative BCS-algebras as a variety. However, by Proposition 2.3.76, the quasivariety $\mathbf{Q}(\mathbf{B}_2)$ generated by $\mathbf{B}_2$ is not a variety, whence the inclusion $\mathbf{Q}(\mathbf{B}_2) \subseteq \mathbf{V}(\mathbf{B}_2)$ is strict. In consequence, it is natural to focus attention on the quasivariety $\mathbf{Q}(\mathbf{B}_2)$ as a specialisation of iBCS. In particular, it is natural to ask to what extent membership of the quasivariety $\mathbf{Q}(\mathbf{B}_2)$ is reflected in special properties of implicative BCS-algebras. In the statement of the following problem and in the sequel, an algebra $\mathbf{A} := \langle A;\ \dot{-}, 0\rangle$ of type $\langle 2, 0\rangle$ is called a $\mathbf{Q}(\mathbf{B}_2)$-*algebra* if $\mathbf{A} \in \mathbf{Q}(\mathbf{B}_2)$.

Problem 4.2.11. Characterise the $\mathbf{Q}(\mathbf{B}_2)$-algebras amongst the implicative BCS-algebras. ∎

The quasivariety $\mathbf{Q}(\mathbf{B}_2)$ also seems of intrinsic interest in its own right, in view of the pivotal role it plays in the theory of BCK-algebras (recall Proposition 2.2.5). Two of the most pressing problems concerning $\mathbf{Q}(\mathbf{B}_2)$ are the following:

Problem 4.2.12. Give an implicational characterisation of the quasivariety $\mathbf{Q}(\mathbf{B}_2)$. Is $\mathbf{Q}(\mathbf{B}_2)$ finitely axiomatisable? ∎

Problem 4.2.13. Characterise the $\mathbf{Q}(\mathbf{B}_2)$-subdirectly irreducible $\mathbf{Q}(\mathbf{B}_2)$-algebras. To within isomorphism, are $\mathbf{C}_1, \mathbf{B}_2$ the only $\mathbf{Q}(\mathbf{B}_2)$-subdirectly irreducible $\mathbf{Q}(\mathbf{B}_2)$-algebras? ∎

In Problem 2.3.69 we asked if every implicative BCS-algebra arises as a 'residuation subreduct' of a pseudocomplemented semilattice. It would also be of interest to possess an embedding theorem for $\mathbf{Q}(\mathbf{B}_2)$-algebras. Upon recalling that the class of $\langle \backslash, 0\rangle$-reducts of members of $\mathsf{SBA}_{\mathrm{SI}}$ is, to within isomorphism, precisely $\{\mathbf{C}_1, \mathbf{B}_2\}$, the following problem is suggested by Problem 4.2.13 and Kalman's proof of Theorem 1.6.20.

Problem 4.2.14. Is an algebra $\langle A; \backslash, 0\rangle$ of type $\langle 2, 0\rangle$ a $\mathbf{Q}(\mathbf{B}_2)$-algebra iff it is a $\langle \backslash, 0\rangle$-subreduct of a skew Boolean algebra? ∎

By Theorem 2.3.75, iBCS is a cover of iBCK in $\Lambda^V(\mathsf{PBCK})$, the lattice of varieties of pre-BCK-algebras. However, by Proposition 2.3.76 this result does not extend to $\Lambda^Q(\mathsf{PBCK})$, the lattice of quasivarieties of pre-BCK-algebras. Therefore we pose the following problem:

Problem 4.2.15. Is $\mathbf{Q}(\mathbf{B}_2)$ a (unique) cover of the atom iBCK in $\Lambda^Q(\mathsf{PBCK})$? ∎

4.2.16. Subtractive Varieties with EDPI. Corollary 3.1.7 shows that the study of the ideal theory of subtractive varieties with EDPI may be reduced to the study of the ideal theory of MINI-algebras (or equivalently, positive implicative pre-BCK-algebras) inasmuch as a variety V with $\mathbf{1}$ is subtractive with EDPI (witness $y \to x$) iff every $\mathbf{A} \in \mathsf{V}$ has a MINI-algebra polynomial reduct $\langle A; \to^{\mathbf{A}}, 1\rangle$ whose MINI-ideals coincide with the V-ideals of $\mathbf{A}$. Because any dual Brouwerian semilattice $\mathbf{A} := \langle A; \vee, *, 0\rangle$ is termwise definitionally equivalent to a generalised Boolean algebra iff $\mathbf{A} \models x * (y * x) \approx x$ (recall the remarks of §1.3.4), the preceding observation calls for a study of the role played by MINI-algebras satisfying $\big((x \to y) \to x\big) \to x \approx \mathbf{1}$ in the theory of subtractive varieties with EDPI; see also Blok and Pigozzi [30, Corollary 4.3]. In the statement of the problem below, a MINI-algebra $\mathbf{A}$ is said to be *classical* if $\mathbf{A} \models \big((x \to y) \to x\big) \to x \approx \mathbf{1}$; clearly any classical MINI-algebra is term equivalent to (in fact, is dually isomorphic to) an implicative pre-BCK-algebra by Theorem 3.1.4.

Problem 4.2.17. Investigate the role played by classical MINI-algebras in the theory of subtractive varieties with EDPI. If V is a subtractive variety with EDPI (witness $y \to x$), does it follow that the polynomial reduct $\langle A; \to^{\mathbf{A}}, 1\rangle$ is a classical MINI-algebra iff for any $\mathbf{A} \in \mathsf{V}$, the join semilattice $\big\langle \mathrm{CI}(\mathbf{A}); \vee, \langle 0\rangle_{\mathbf{A}}\big\rangle$ of compact ideals of $\mathbf{A}$ is dually relatively complemented? ∎

By the *pure* [*subtractive*] *WBSO variety* we mean the variety generated by the class of all algebras $\mathbf{A} := \langle A; \cdot, \to, \Delta, 1\rangle$ of type $\langle 2, 2, 2, 1\rangle$ where the join semilattice $\langle \mathrm{Cp}\,\mathbf{A}; \vee, \omega_{\mathbf{A}}\rangle$ of compact congruences is dually relatively pseudocomplemented and $\cdot, \to$ and Δ are weak meet, [subtractive] weak relative pseudocomplemention and Gödel equivalence terms respectively. The pure [subtractive] WBSO variety may be of interest in algebraic logic inasmuch as it may provide a convenient framework for the study of [subtractive] WBSO varieties, similar to the manner in which skew Boolean $\cap$-algebras provide a point of reference for the study of pointed ternary discriminator varieties (recall Theorem 1.4.39) and implicative BCS-algebras provide a point of reference for the study of binary discriminator varieties (recall Theorem 3.2.7).

Problem 4.2.18. Investigate the pure [subtractive] WBSO variety. ∎

In [34, p. 549] Blok and Pigozzi note that with some important exceptions, the varieties of traditional algebraic logic all have (commutative) TD terms.

Included among these classes are the varieties of dual Brouwerian semilattices and positive implicative BCK-algebras, which have the (commutative) TD term $z \dot{-} \big((x \dot{-} y) \dot{-} (y \dot{-} x)\big)$ (by [34, Corollary 5.2(ii)]), and the variety of Nelson algebras, which has the (commutative) TD term $\big((x \Rightarrow y) \to (y \Rightarrow x)\big) \to z$ (by Theorem 3.1.33). Upon observing that the (commutative) TD terms for these varieties are all of the form $\big((\cdots(d_1(x,y) \to d_2(x,y)) \to \cdots) \to d_n(x,y)\big) \to z$, where $d_1(x,y), \ldots, d_n(x,y)$ are binary terms witnessing point regularity in the sense of Proposition 1.2.6 and $\to$ is a subtractive weak relative pseudocomplementation, the following problem suggests itself.

Problem 4.2.19. Investigate the class of all varieties with a (commutative) TD of the form $\big((\cdots(d_1(x,y) \to d_2(x,y)) \to \cdots) \to d_n(x,y)\big) \to z$, where $d_1(x,y), \ldots, d_n(x,y)$ are binary terms witnessing point regularity in the sense of Proposition 1.2.6 and $\to$ is a (subtractive) weak relative pseudocomplementation. In particular, investigate the class of all varieties with a (commutative) TD term of the form $(x \Delta y) \to z$, where Δ and $\to$ are a Gödel equivalence term and a (subtractive) weak relative pseudocomplementation respectively. ∎

4.2.20. Binary Discriminator Varieties. By Theorem 3.2.29, a variety with $\mathbf{0}$ is a pointed fixedpoint discriminator variety iff it is a $\mathbf{0}$-regular binary discriminator variety. From this result it follows that the binary discriminator may be legitimately considered a generalisation of the pointed fixedpoint discriminator to the subtractive (but not ideal determined) case. Inasmuch as the theory of the fixedpoint discriminator closely parallels that of the ternary discriminator [34, p. 548; Section 3], the preceding observation gives rise to the following problem:

Problem 4.2.21. To what extent does the theory of the binary discriminator parallel the theory of the (pointed) fixedpoint discriminator? Of the ternary discriminator? ∎

Apropos the preceding problem, a suitable representation theorem would help clarify connections between binary discriminator, pointed fixedpoint discriminator and ternary discriminator varieties. In particular, the following problem suggests itself, just because binary discriminator varieties have a well developed theory of prime ideals (by Theorem 3.2.8 and the results of §2.2.28).

Problem 4.2.22. Is there a sheaf-theoretic representation (in the sense of Burris and Werner [56]) of binary discriminator varieties? ■

In Remark 3.2.19, it was noted that the class of Abelian Rickart semirings is a binary discriminator variety. This observation seems to be of some interest, since it implies several classes of rings arising in real and complex analysis are binary discriminator varieties.

Problem 4.2.23. Investigate the class of Abelian Rickart semirings (considered as a binary discriminator variety). ■

By Example 2.3.12, another important example of a naturally occurring binary discriminator variety is the class of pseudocomplemented semilattices. Call an algebra $\langle A;\ \wedge, \backslash, 0\rangle$ of type $\langle 2, 2, 0\rangle$ a *locally pseudocomplemented semilattice* if: (i) the reduct $\langle A;\ \wedge, 0\rangle$ is a meet semilattice with zero; and (ii) for all $a, b \in A$, the difference $a \backslash b$ is the pseudocomplement of b in the principal subalgebra $(a]$ generated by a. Locally pseudocomplemented semilattices are a natural generalisation of pseudocomplemented semilattices that stand in relation to such algebras as generalised Boolean algebras stand in relation to Boolean algebras. By an unpublished theorem of the author, the class of locally pseudocomplemented semilattices is a finitely based variety. This result, in conjunction with preceding remarks, invites the study of the following problem.

Problem 4.2.24. Is the class of locally pseudocomplemented semilattices a binary discriminator variety? ■

A further unpublished result of the author shows that, for any locally pseudocomplemented semilattice $\mathbf{A}$, the reduct $\langle A;\ \backslash, 0\rangle$ of $\mathbf{A}$ is an implicative BCS-algebra. This result prompts the following question, which may be regarded as a generalisation of Problem 2.3.69.

Problem 4.2.25. Is an algebra $\langle A;\ \backslash, 0\rangle$ of type $\langle 2, 0\rangle$ an implicative BCS-algebra iff it is a $\langle \backslash, 0\rangle$*-reduct* of a locally pseudocomplemented semilattice? ■

Locally pseudocomplemented semilattices were introduced by Grätzer in [101] in connection with the following problem, which we include for the sake of completeness.

Problem 4.2.26. [101, Problem 1§6.22, p. 67] Let $\mathbf{A}$ be a meet semilattice with 0 for which $(a]$ is pseudocomplemented for each $a \in A$; let $\mathrm{S}(a)$ denote the pseudocomplements in $(a]$. Characterise the family of Boolean algebras $\{\mathrm{S}(a) : a \in A\}$. ∎

4.2.27. Pre-BCK Quasilattices and BCK Paralattices. Recall that, when generalised to bands, Idziak's theory of BCK-[semi]lattices bifurcates (owing to the absence of commutativity). In one direction, Idziak's theory of BCK-[semi]lattices generalises to a theory of pre-BCK bands and pre-BCK quasilattices $\mathsf{PQ}_{\mathcal{C}}$; while in another, it generalises to a theory of BCK bands and BCK paralattices $\mathsf{BP}_{\mathcal{C}}$. As it stands, the study of these complementary theories is largely orthogonal, since for each choice of $\mathcal{C}$ the varieties $\mathsf{PQ}_{\mathcal{C}}$ and $\mathsf{BP}_{\mathcal{C}}$ are incomparable. On the other hand, the results of §3.3.2 and §3.3.27 show that the study of both families $\mathsf{PQ}_{\mathcal{C}}$, $\mathsf{BP}_{\mathcal{C}}$ is relevant to the study of pointed fixedpoint discriminator and pointed ternary discriminator varieties. Prompted by the desire to provide a unified framework for the study of binary discriminator, pointed fixedpoint discriminator and pointed ternary discriminator varieties, these remarks engender the following question:

Problem 4.2.28. Find a common generalisation of pre-BCK bands [pre-BCK quasilattices] and BCK bands [BCK paralattices]. ∎

In §3.3.2 and §3.3.27, interest was naturally centred on those varieties $\mathsf{PQ}_{\mathcal{C}}$ and $\mathsf{BP}_{\mathcal{C}}$ for which $\{\wedge, \dot{-}, 0\} \subseteq \mathcal{C} \subseteq \{\wedge, \vee, \dot{-}, 0\}$. However, the varieties $\mathsf{PQ}_{\mathcal{C}}$ and $\mathsf{BP}_{\mathcal{C}}$ may also be of interest for other choices of $\mathcal{C}$. In particular, those varieties $\mathsf{PQ}_{\mathcal{C}}$ and $\mathsf{BP}_{\mathcal{C}}$ for which $\{\vee, \dot{-}, 0\} \subseteq \mathcal{C}$ would seem to merit attention, in view of the following two observations:

1. The variety SBA of skew Boolean algebras is termwise definitionally equivalent to a subvariety of $\mathsf{PQ}_{\mathcal{C}'}$, $\mathcal{C}' = \{\vee, \backslash, 0\}$;

2. The variety SBIA of skew Boolean $\cap$-algebras is termwise definitionally equivalent to a subvariety of $\mathsf{BP}_{\mathcal{C}''}$, $\mathcal{C}'' = \{\vee, /, 0\}$.

To justify (1), it is sufficient to note $\mathsf{SBA} \models x \wedge y \approx (y \vee x)\backslash((x\backslash y) \vee (y\backslash x))$, whence any skew Boolean algebra $\langle A;\ \wedge, \vee, \backslash, 0\rangle$ is term equivalent to its own reduct $\langle A;\ \vee, \backslash, 0\rangle$. Similarly, to justify (2) it is sufficient to note $\mathsf{SBIA} \models x \wedge y \approx (y \vee x)/((x/y) \vee (y/x))$, whence any skew Boolean $\cap$-algebra $\langle A;\ \wedge, \vee, /, 0\rangle$ is term equivalent to its own reduct $\langle A;\ \vee, /, 0\rangle$.

Problem 4.2.29. Investigate those varieties $\mathsf{PQ}_{\mathcal{C}}, \mathsf{BP}_{\mathcal{C}}$ for which $\{\vee, \dot{-}, \mathbf{0}\} \subseteq \mathcal{C} \subseteq \{\wedge, \vee, \dot{-}, \mathbf{0}\}$. ■

For any variety $\mathsf{V} \subseteq \mathsf{PQ}_{\mathcal{C}}$, an easy modification of the proof of Theorem 2.1.36 shows the assertional logic $\mathbb{S}(\mathsf{V}, \mathbf{0})$ coincides with the assertional logic $\mathbb{S}(\mathsf{V}_{\epsilon}, \mathbf{0})$ of the variety of BCK-[semi]lattices V_{ϵ} axiomatised relative to V by the quasi-identity (2.5). Thus the assertional logic $\mathbb{S}(\mathsf{V}, \mathbf{0})$ has a familiar description. In contrast, for any variety $\mathsf{V} \subseteq \mathsf{BP}_{\mathcal{C}}$ we have that V is $\mathbf{0}$-regular witness $\{x \dot{-} y, y \dot{-} x\}$ (by Proposition 3.3.37), whence the equivalent algebraic semantics of the algebraisable assertional logic $\mathbb{S}(\mathsf{V}, \mathbf{0})$ is exactly V (by Theorem 1.8.15). By Remark 3.3.1, therefore, the following problem is apposite:

Problem 4.2.30. For varieties $\mathsf{V} \subseteq \mathsf{BP}_{\mathcal{C}}$, investigate the assertional logics $\mathbb{S}(\mathsf{V}, \mathbf{0})$. In particular, investigate the assertional logics $\mathbb{S}(\mathsf{BP}_{\mathcal{C}}, \mathbf{0})$ of the varieties $\mathsf{BP}_{\mathcal{C}}$. ■

In terms of gaining insight into the unfamiliar properties and behaviours of the deductive systems $\mathbb{S}(\mathsf{BP}_{\mathcal{C}}, \mathbf{0})$ (recall Remark 3.3.1), the problem of obtaining a Gentzen-style axiomatisation for each $\mathbb{S}(\mathsf{BP}_{\mathcal{C}}, \mathbf{0})$ would seem particularly relevant.

Problem 4.2.31. For each $\mathsf{BP}_{\mathcal{C}}$, give a Gentzen-style axiomatisation (if one exists) of the assertional logic $\mathbb{S}(\mathsf{BP}_{\mathcal{C}}, \mathbf{0})$. ■

Let K be a K-$\mathbf{0}$-regular quasivariety. In traditional algebraic logic, a standard approach for constructing a Gentzen-style axiomatisation of $\mathbb{S}(\mathsf{K}, \mathbf{0})$ lies in conservatively extending $\mathbb{S}$ [193, Definition 9.1] with a multiplicative conjunction or *fusion* [199]. In algebraic logic, fusion of premisses is closely related to the calculation of residuals [86, Section 2]. Inasmuch as BCK-algebras satisfying Iséki's condition (S) are precisely the residuation *reducts* of pocrims, the preceding remarks call (for each $\mathsf{BP}_{\mathcal{C}}$) for the study of the class of all members of $\mathsf{BP}_{\mathcal{C}}$ for which the BCK-algebra reduct has condition (S).

Problem 4.2.32. For each $\mathsf{BP}_{\mathcal{C}}$, investigate the class of all members of $\mathsf{BP}_{\mathcal{C}}$ for which the BCK-algebra reduct enjoys condition (S). In particular, study the class of all BCK skew lattices for which the BCK-algebra reduct has condition (S). ■

By the remarks of §1.1.1, the study of residuated structures in universal algebra was initiated by the papers of Krull [143] and Ward and Dilworth [235] on residuated lattices. Recently, the algebraic logic community has shown renewed interest in residuated lattices and their associated logics [43, 42, 127]. In view of these remarks and the initial motivations of this thesis, the following problem (which is related to Problem 4.2.2) would seem pertinent.

Problem 4.2.33. Can the theory of residuated lattices be usefully extended to skew lattices? ■

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