Forced Brakke flows

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DECLARATION

In accordance with Doctorate Regulation 17 of Monash University the following declaration is made:

I hereby declare that this thesis contains no material which has been accepted for the award of any other degree or diploma at any University or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or ~~written by another~~ person, except where due reference is made in the text of the thesis. [redacted]

David Graham

The past is what man should not have been. The present is what man ought not to be. The future is what artists are.

Oscar Wilde

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Introduction

Consider a family of smooth n-dimensional hypersurfaces $\mathcal{M} = \{\Gamma_t\}_{0 \le t < T}$ satisfying

$$\frac{\partial x}{\partial t} = \vec{H}_{\Gamma_t}(x), \quad x \in \Gamma_t \tag{MCF}$$

for all $t \geq 0$ where $\vec{H}_{\Gamma_t}$ is the mean curvature vector of Γ_t. Then we say that $\mathcal{M} = \{\Gamma_t\}_{0 \le t < T}$ is a *mean curvature flow*. The mean curvature flow of a hypersurface will decrease surface area in the most efficient way. In some physical phenomena such as the evolution of the interface between two liquid pure metals, the energy of the interface is proportional to the surface area of the interface. If we ignore additional energy arising from momentum, pressure, gravitation, etc., then the evolution of the interface can be accurately modelled by (MCF). Weak versions of (MCF) have been widely used to model crystal growth (see, for example, [ATW], [AW], [NP], and [WJE]) and have had a heavy focus on numerical techniques.

However, there are certain physical situations where the additional energy cannot be ignored, for example *magnetised* liquid crystals (called *ferronemetics*). In this case, the additional energy terms involving pressure and magnetic energy are important aspects to the physical model that should not be ignored [ZI]. These additional terms are important for explaining and understanding certain patterns in so-called *zebra rocks* that contain both magnetic and neutral clay (see figure 1).

Figure 1 (*Photo: Dr. E. Mattievich, converted to digital by Ms. S. Mattievich*)

Ferronemetics evolve according to (MCF) plus additional forces (see [ZI]). These additional forces could be L^p on the surface (see Appendix). Therefore we will consider the evolution of n-dimensional hypersurfaces $\mathcal{M} = \{\Gamma_t\}_{0 \le t < T}$ given by

$$\frac{\partial x}{\partial t} = \vec{H}_{\Gamma_t}(x) + \vec{g}(x,t), \quad x \in \Gamma_t \tag{fMCF}$$

for all $t \geq 0$. We will call this a *forced mean curvature flow*. Such flows have previously been studied in [DER] where g is constant.

As with (MCF), even some of the simplest examples will develop a singularity in finite time. Take for example a sphere of unit radius and $g(t)$ given by

$$g(t) = \begin{cases} 1-t & 0 \leq t \leq 1 \\ 0 & t > 1 \end{cases}.$$

Then the family $\mathcal{M} = \{S^2_{\sqrt{1-2t-t^2}}(0)\}_{0\leq t<-1+\sqrt{2}}$ is a forced mean curvature flow and the flow becomes singular at $t = -1+\sqrt{2}$.

We would like to define the forced mean curvature flow after the onset of singularities. To do this, we will use an approach similar to Brakke's geometric measure theoretic method [B].

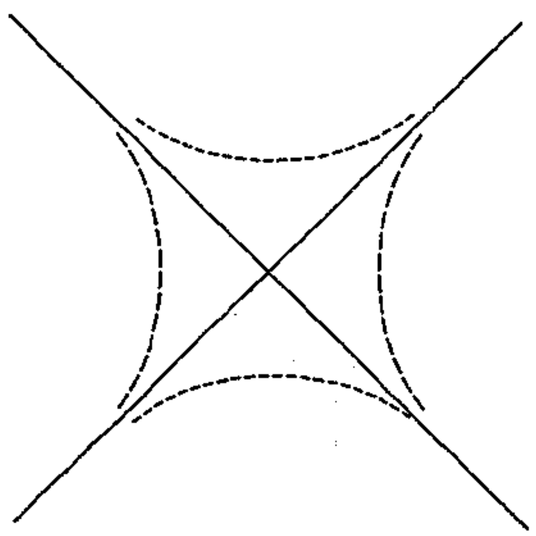

Figure 2. Two possible Brakke flows out of a cross.

In Brakke's approach, one considers the evolution of a surface by studying how an *n-varifold* evolves according to an equation describing a weak evolution by mean curvature, called the *Brakke flow*. The flow is designed to "jump" when a singularity develops. However, this jumping gives rise to non-uniqueness and suddenly vanishing. For example, the flow out of a cross could evolve in either of two directions (figure 2), and a homothetic spoon will evolve until it becomes a half line, at which point it vanishes instantaneously (figure 3).

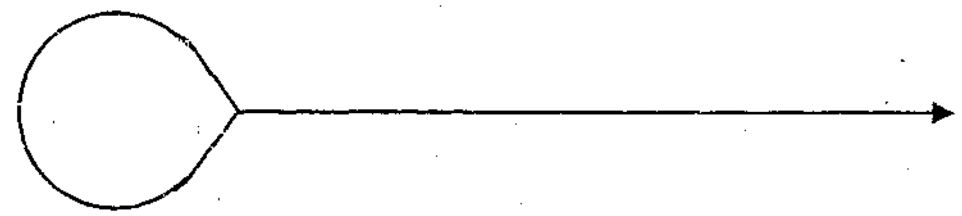

Figure 3. The homothetic spoon.

In 1994, Ilmanen showed existence of Brakke flows using an especially elegant method called *elliptic regularisation*. This was inspired by a similar method used by Evans & Spruck for the level set flow [ES]. The idea is to turn the parabolic problem into an elliptic problem and study the elliptic solutions. The elliptic solutions turn out to be *translative soliton solutions* and approximate the space time track of a weak evolution of a specified initial hypersurface by its mean curvature. Making the approximation more accurate, one obtains a solution to the weak mean curvature flow (level set flow or Brakke flow).

The first regularity result for the Brakke flow was obtained by Brakke [B]. Under an assumption of unit density, he was able to prove that at almost-every time during a weak motion a varifold by its mean curvature is smooth almost-everywhere. Using this powerful result, Ilmanen proved that, if a level set flow doesn't fatten, then the flow is smooth almost-everywhere at almost every time [I2]. Such fattening occurs in the example depicted in figure 2. Instantaneously, the level set flow becomes the region bounded by the dotted lines and evolves outwards.

An alternate approach to regularity has been used in [E], [I3], and [W1,2]. There one obtains a regularity theory for smooth curvature flows that develop a singularity at some time. The theory uses a quantity called the *gaussian density* which, for a smooth mean curvature flow $\mathcal{M} = \{\Gamma_t\}_{0\leq t<T}$, is given by

$$\Theta(\mathcal{M}, y, T) := \lim_{t\uparrow T} \int_{\Gamma_t} \rho_{y,T}(x,t)\, d\mathcal{H}^n(x)$$

where

$$\rho_{y,T}(x,t) := \frac{1}{(4\pi(T-t))^{n/2}} e^{-|x-y|^2/4(T-t)}.$$

If the gaussian density at a point is sufficiently close to unity, then the surface is smooth near the point. This is a parabolic analogue of Allard's regularity theorem [A].

The organisation of this thesis is as follows. In Part I, §1 introduces some notation and some basic geometric measure theory which will be used throughout the thesis. Notation and conventions can be found after the Appendix. In §2, we introduce the Brakke flow by motivating it for a smooth eternal flow $\mathcal{M} = \{\Gamma_t\}_{t\geq 0}$. We also list some interesting results relating to the Brakke flow, such as the compactness theorem for Brakke flows (Theorem

2.3) proved by Ilmanen in [I2]. Analogues will be proved for forced Brakke flows later in the thesis. An outline of the elliptic regularisation method is then given in §3. There we include some properties of the minimisers to F^{ε}.

Part II begins by defining *general Brakke functionals* and *general Brakke flows* in §4. These are defined with a minimal number of assumptions required to obtain a compactness theorem (Theorem 5.1) which we prove in §5. The compactness theorem allows for the possibility of the sequences being general Brakke flows not necessarily satisfying the same equation, but rather allowing for the possibility that the general Brakke flows are defined by a sequence of general Brakke functionals. This is a necessary aspect required later in the thesis, not merely a matter of art for art's sake (as we shall shortly expand upon). In §4, we also give some examples of general Brakke flows including White's *K-almost Brakke flows* [W1] and *forced Brakke flows*, a weak version of (fMCF). The forced Brakke flow is the main subject of this thesis.

Next, in Part III, we adapt the method of elliptic regularisation to the forced Brakke flow. This presents its own problems not found in the Brakke flow since the forcing term is only assumed to be L^p at best. In fact, the problems encountered are still relevant in even the simplest extension where the forcing term is C_c^{∞}. As with the Brakke flow, we begin in §6 by defining a related elliptic problem, namely some prescribed mean curvature problem related to the forcing term in (fMCF). This reflects an underlying principle: *when using elliptic regularisation, one should choose an elliptic problem that reflects the parabolic problem.*

Unlike the Brakke flow, the elliptic solutions turn out to satisfy a forced Brakke flow that only *approximate* the forced Brakke flow we would like to solve (see equation (8.1). Incidentally, if the forcing term were a constant as in [DER], existence would follow by an easy adaptation of the arguments in [I2]). The method developed here extends the standard elliptic regularisation argument by using the compactness theorem from §5. This illustrates the necessity for proving a compactness theorem that allows for the possibility that the general Brakke flows are defined by a sequence of general Brakke functionals. This method could be used to prove existence of a weak volume preserving mean curvature flow (see, for example, [H2] and [Ath]) if one could properly define the average mean curvature.

As a slight interlude, we prove a *cylindrical monotonicity formula* (Lemma 7.1) in §7. This is applicable to the solutions from §6 and those used for the Brakke flow in §3 [I2]. We use the cylindrical monotonicity formula to obtain a nice geometric property relating the height of the minimisers from §6 over an $(n + 1)$-dimensional plane $S \times \mathbf{R}$ in $\mathbf{R}^{n+2}$ (Lemma 7.5). This could point the way to a new method for obtaining Brakke's regularity theorem [B 6.12] by using the methods from [S §§20-22].

In order to apply the compactness theorem, in §8 we must obtain local mass estimates for the solutions to the elliptic problem defined in §6 (Lemma 8.6). Due to the forcing terms, the local mass estimates require further extensions to their counterparts used for the Brakke flow. The local mass bounds are then used in §9 together with an alternate version of the compactness theorem that links the approximate forced Brakke flow

(equation (8.1)) to the forced Brakke flow (Theorem 9.3) to obtain existence of forced Brakke flows (Theorem 9.6).

In §10, we prove some properties of forced Brakke flows obtained using elliptic regularisation. These include some lemmata that characterise the area-ratio and tilt-excess in terms of similar quantities for the minimisers from §6 for certain forced Brakke flows obtained using the elliptic regularisation methods in §§6-9.

Finally, Part IV is dedicated to a regularity theory for the forced Brakke flow. We begin in §11 by proving a monotonicity formula (Lemma 11.2) analogous to Huisken's famous monotonicity formula [H1] for smooth mean curvature flows $\mathcal{M} = \{\Gamma_t\}_{0 \le t < T}$:

$$\frac{d}{dt}\int_{\Gamma_t} \rho_{y,T}(x,t)\, d\mathcal{H}^n(x) = -\int_{\Gamma_t} \left| \vec{H}_{\Gamma_t}(x) + \frac{(x-y)^{\perp}}{2(T-t)} \right|^2 \rho_{y,T}(x,t)\, d\mathcal{H}^n(x)$$

This is used to show existence of a localised version of the gaussian density for a forced Brakke flow (Proposition 11.4). A version of Brakke's clearing out lemma [B] is also obtained (Proposition 11.6). §12 concludes with a proof of a local regularity theorem for the singular time of a smooth forced mean curvature flow (Theorem 12.1). In the subsequent Appendices, we consider a model of the aforementioned ferronemetics and a model of "biased" search patterns.

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Oh, and thanks to nature for providing inspiration and life...

Part I – Introduction

This part is devoted to the background material. We first introduce some notation and basic geometric measure theory for which the standard texts are [F] and [S]. See also [Mo] for a beautiful introduction. The remaining sections form an overview of the Brakke flow (see [B] and [I2]) and elliptic regularisation (see [I2]). These are used as motivation for Parts II and III.

1 Notation and Preliminaries

To study the evolution of a surface after the onset of singularities we use the language of geometric measure theory. We will always denote an evolution by $\mathcal{M}$ and smooth evolutions will be written $\{\Gamma_t\}_{t\in I}\,(=\mathcal{M})$ where I is an interval in $\mathbf{R}$.

Radon Measures: A *Borel regular measure* on $\mathbf{R}^{n+k}$ is an (outer) measure on $\mathbf{R}^{n+k}$ such that all Borel sets (all open sets and all closed sets on $\mathbf{R}^{n+k}$) are measurable and every set $A\subset\mathbf{R}^{n+k}$ is contained in some Borel set B of equal measure. A *Radon measure* on $\mathbf{R}^{n+k}$ is a Borel measure for which compact sets have finite measure. We denote the space of all Radon measures by $\mathcal{M}(\mathbf{R}^{n+k})$. Radon measures will be denoted by μ. Special examples are the *d-dimensional Hausdorff measure*, denoted by $\mathcal{H}^d$, and the *Lebesgue measure*, written $\mathcal{L}^{n+k}$.

Let $A\subset\mathbf{R}^{n+k}$. We define the *restriction* of μ to A by

$$(\mu\llcorner A)(B) := \mu(A\cap B),$$

whenever $B\subset\mathbf{R}^{n+k}$ and denote the *mass* of μ by

$$\mathbf{M}(\mu) := \mu(\mathbf{R}^{n+k}).$$

Let φ be locally μ-integrable. Then define $\mu\llcorner\varphi$ by

$$(\mu\llcorner\varphi)(B) := \int_B \varphi(x)\,d\mu(x),$$

where $B\subset\mathbf{R}^{n+k}$.

If the limit

$$\Theta^n(\mu,a) := \lim_{\rho\to 0}\omega_n^{-1}\rho^{-n}\mu(B_\rho(a))$$

exists, the we call it the *n-dimensional density* of μ at $a\in\mathbf{R}^{n+k}$.

Topology of Radon Measures: Let $\mu\in\mathcal{M}(\mathbf{R}^{n+1})$ and define

$$\mu(\varphi) := \int \varphi(x)\, d\mu(x),$$

where $\varphi \in C_c^0(\mathbf{R}^{n+k}, \mathbf{R})$. The topology on $\mathcal{M}(\mathbf{R}^{n+k})$ is given by the condition that $\mu_i \to \mu$ if and only if $\mu_i(\varphi) \to \mu(\varphi)$ for any $\varphi \in C_c^0(\mathbf{R}^{n+k}, \mathbf{R})$.

Approximate Tangent plane: Define the Radon measure $\mu_{a,\lambda}$ by

$$\mu_{a,\lambda}(A) := \lambda^{-n} \mu(\lambda A + a).$$

Let S be an n-dimensional plane in $\mathbf{R}^{n+k}$ containing 0 (i.e. an n-dimensional vector subspace of $\mathbf{R}^{n+k}$), and $\theta > 0$. If, for any $\varphi \in C_c^0(\mathbf{R}^{n+k}, \mathbf{R})$,

$$\lim_{\lambda \downarrow 0} \mu_{a,\lambda}(\varphi) = \theta(a) \int_S \varphi(x)\, d\mathcal{H}^n(x),$$

then we say that S is the *n-dimensional approximate tangent plane* of μ at a with multiplicity θ. We write this as $T_a\mu$. If $M \subset \mathbf{R}^{n+k}$, then we define the *n-dimensional approximate tangent plane* of M at a by

$$T_a M := T_a(\mathcal{H}^n \llcorner M),$$

if it exists.

Rectifiable sets: Let $X \subset \mathbf{R}^{n+k}$ and suppose

$$X \subset C_0 \cup (\bigcup_{i \geq 1} C_i),$$

where $\mathcal{H}^n(C_0) = 0$ and each C_i is an embedded C^1 n-dimensional submanifold. Then we call X *countably n-rectifiable*. If X has finite $\mathcal{H}^n$-measure on compact sets, then we say that X is *locally n-rectifiable*.

If X is locally n-rectifiable and $\mathcal{H}^n$-measurable, then $T_a X$ exists $\mathcal{H}^n \llcorner X$-*a.e.*

Rectifiable Radon Measures: Let $X \subset \mathbf{R}^{n+k}$ be $\mathcal{H}^n$-measurable and let $\theta : \mathbf{R}^{n+k} \to \mathbf{N}$ be locally $\mathcal{H}^n$-integrable such that $X = \{\theta > 0\}$ $\mathcal{H}^n$-*a.e.* Define the Radon measure $\mu(X,\theta)$ by

$$\mu(X, \theta) := \mathcal{H}^n \llcorner \theta.$$

We say that μ is an *n-rectifiable Radon measure* if either
(i) μ has an n-dimensional tangent plane μ-*a.e.*, or
(ii) $\mu = \mu(X,\theta)$ for some $\mathcal{H}^n$-measurable, countably n-rectifiable set X, and some locally $\mathcal{H}^n$-integrable function $\theta : \mathbf{R}^{n+k} \to [0, \infty)$.

We denote the space of n-rectifiable Radon measures on $\mathbf{R}^{n+k}$ by $\mathcal{M}_n(\mathbf{R}^{n+k})$.

We say that μ is an *integer n-rectifiable Radon measure* if either
(i) μ has an n-dimensional tangent plane μ-*a.e.* with positive integer multiplicity, or
(ii) $\mu = \mu(X,\theta)$ for some $\mathcal{H}^n$-measurable, countably n-rectifiable set X, and some locally $\mathcal{H}^n$-integrable function $\theta : \mathbf{R}^{n+k} \to \mathbf{N}$.

We denote the space of integer n-rectifiable Radon measures on $\mathbf{R}^{n+k}$ by $I\mathcal{M}_n(\mathbf{R}^{n+k})$.

General Varifolds: Define *grassmanian* by

$$G_n(\mathbf{R}^{n+k}) := \{(x, S) : x \in \mathbf{R}^{n+k}, S \text{ is an } n\text{-plane in } \mathbf{R}^{n+k} \text{ with } 0 \in S\}.$$

Let V be a Radon measure on $G_n(\mathbf{R}^{n+k})$. Then V is called a *general n-varifold* on $\mathbf{R}^{n+k}$. We denote the space of all general n-varifolds on $\mathbf{R}^{n+k}$ by $\mathbf{V}_n(\mathbf{R}^{n+k})$ and give it the topology of Radon measure convergence. Let $\varphi \in C_c^0(G_n(\mathbf{R}^{n+k}), \mathbf{R})$. Then write

$$V(\varphi) := \int \varphi(x, S)\, dV(x, S).$$

Let $V \in \mathbf{V}_n(\mathbf{R}^{n+k})$ and define the Radon measure μ_V by

$$\mu_V(\varphi) := \int (\varphi \circ \pi)(x, S)\, dV(x, S),$$

where $\pi : G_n(\mathbf{R}^{n+k}) \to \mathbf{R}^{n+k}$ and $\varphi \in C_c^0(\mathbf{R}^{n+k}, \mathbf{R})$. Likewise, if $\mu \in \mathcal{M}_n(\mathbf{R}^{n+k})$, then associated to μ is the n-varifold V_μ defined by

$$V_\mu(\varphi) := \int \varphi(x, T_x\mu)\, d\mu(x),$$

for any $\varphi \in C_c^0(G_n(\mathbf{R}^{n+k}), \mathbf{R})$. This makes sense since $T_x\mu$ exists μ-*a.e.* We call varifolds of this form *integer rectifiable n-varifolds* whenever $\mu \in I\mathcal{M}_n(\mathbf{R}^{n+k})$, and denote the space of integer rectifiable n-varifolds by $\mathbf{IV}_n(\mathbf{R}^{n+k})$.

The First Variation of a Varifold: Let S be an n-dimensional subspace of $\mathbf{R}^{n+k}$. We also use S to denote projection onto S.

Let $V \in \mathbf{V}_n(\mathbf{R}^{n+k})$. Let $\Phi : \mathbf{R}^{n+k} \to \mathbf{R}^{n+k}$ be a diffeomorphism and define the *push-forward* $\Phi_\#(V)$ by

$$\Phi_\#(V)(\varphi) := \int \varphi(x, S) J_S \Phi(x)\, dV(x, S),$$

for any $\varphi \in C_c^0(G_n(\mathbf{R}^{n+k}), \mathbf{R})$. Here $J_S \Phi(x)$ is the *Jacobian* and equals

$$J_S\Phi(x) = \sqrt{\det(d\Phi(x)|_S)^{\mathrm{T}}(d\Phi(x)|_S)}\,.$$

Let $\{\Phi_t\}_{-\delta<t<\delta}$ be a family of compactly supported diffeomorphisms on $\mathbf{R}^{n+k}$ satisfying

$$\Phi_0(x) = x \text{ and } \left.\frac{\partial}{\partial t}\right|_{t=0} \Phi_t(x) = X(x),$$

for some C_c^1 vectorfield X on $\mathbf{R}^{n+k}$. Then it is a straight forward exercise to show that

$$\left.\frac{\partial}{\partial t}\right|_{t=0} J_S\Phi_t(x) = \operatorname{div}_S X(x),$$

where $\operatorname{div}_S X(x) := \operatorname{trace}(\nabla^S X(x))$ and ∇^S is the covariant derivative on S. Consequently,

$$\delta V(X) := \left.\frac{d}{dt}\right|_{t=0} (\Phi_t)_\#(V)(\mathbf{R}^{n+k}) = \int \operatorname{div}_S X(x)\, dV(x,S)\,.$$

We call δV the *first variation* of V.

When $\mu \in \mathcal{M}_n(\mathbf{R}^{n+k})$ we will write

$$\delta V_\mu(X) = \int \operatorname{div}_\mu X(x)\, d\mu(x) = \int \operatorname{div}_{T_x\mu} X(x)\, d\mu(x),$$

and when $\mu = \mu(M,\theta)$ for some $\mathcal{H}^n$-measurable, countably n-rectifiable set M, and some locally $\mathcal{H}^n$-integrable function $\theta : \mathbf{R}^{n+k} \to [0,\infty)$, we will write

$$\delta V_\mu(X) = \int \operatorname{div}_M X(x)\, d\mu(x)\,.$$

Let $U \subset \mathbf{R}^{n+k}$ be open. Define the *total first variation* of V by

$$|\,\delta V\,|(U) := \sup\{\delta V(X) : X \in C_c^1(U,\mathbf{R}^{n+k}), |\,X\,| \le 1\}\,.$$

If $|\,\delta V\,| \in \mathcal{M}(\mathbf{R}^{n+k})$, the Riesz representation theorem implies that we may decompose δV as

$$\delta V(X) = -\int \vec{H}\cdot X d\mu_V + \delta V_{\mathrm{sing}}(X)$$

where $\vec{H}$ is a locally μ_V-integrable vectorfield and $\mu_V(\operatorname{spt}\delta V_{\mathrm{sing}}) = 0$.

Theorem 1.1 (Compactness theorem for $I\!M_n(\mathbf{R}^{n+k})$ [A]): *Let $\{\mu_i\}_{i\ge1}$ be a sequence in $I\!M_n(\mathbf{R}^{n+k})$ with*

$$\sup_i(\mu_i(K) + |\,\delta V_{\mu_i}\,|(K)) < \infty$$

for any $K \subset\subset \mathbf{R}^{n+k}$. Then there exists a $\mu \in I\!M_n(\mathbf{R}^{n+k})$ and a subsequence $\{\mu_{i'}\}_{i'\ge1}$ such that

(i) $\mu_{i'} \to \mu$,

(ii) $V_{\mu_{i'}} \to V_\mu$,

(iii) $\delta V_{\mu_{i'}} \to \delta V_\mu$,

(iv) $|\,\delta V_\mu\,|(K) \le \liminf_{i'\to\infty} |\,\delta V_{\mu_{i'}}\,|(K)$.

Currents: Denote the set of n-vectors on $\mathbf{R}^{n+k}$ by $\Lambda_n\mathbf{R}^{n+k}$, and the set of n-forms by $\Lambda^n\mathbf{R}^{n+k}$. We denote the space of all differential n-forms by $\mathcal{D}^n(\mathbf{R}^{n+k})$. If T is a continuous linear functional on $\mathcal{D}^n(\mathbf{R}^{n+k})$, then we say that T is an *n-current*. We write $\mathcal{D}_n(\mathbf{R}^{n+k})$ for the space of all n-currents on $\mathbf{R}^{n+k}$ and equip it with the topology given by

$$T_i(\alpha) \to T(\alpha),$$

for any $\alpha \in \mathcal{D}^n(\mathbf{R}^{n+k})$. Define the *mass measure* of T by

$$\mu_T(U) := \sup\{T(\alpha) : \alpha \in \mathcal{D}^n(U), |\,\alpha\,| \le 1\},$$

where $U \subset \mathbf{R}^{n+k}$ is open. The *mass* of T is given by

$$\mathbf{M}(T) := \mu_T(\mathbf{R}^{n+k})\,.$$

If μ_T is finite on compact sets, then we say that T has *locally finite mass*. Note that if $T_i \to T$ in $\mathcal{D}_n(\mathbf{R}^{n+k})$ then

$$\mu_T(K) \le \liminf_{i\to\infty} \mu_{T_i}(K),$$

for all $K \subset\subset \mathbf{R}^{n+1}$.

Suppose T has locally finite mass. Then μ_T is a Radon measure and, by the Riesz representation theorem we can find a locally μ_T-integrable $\xi \in \Lambda_n\mathbf{R}^{n+k}$

$$T(\alpha) = \int <\alpha(x), \xi(x)> d\mu_T(x),$$

where $<\cdot,\cdot>$ is the dual pairing.

Now, suppose $\mu = \mu(M,\theta)$ is an integer n-rectifiable Radon measure and assume $T_x M = \operatorname{span}\{\tau_1(x),\ldots,\tau_n(x)\}$ (whenever it exists). Let $\xi \in \Lambda_n \mathbf{R}^{n+k}$ be given so that, for μ-*a.e.* x,

$$\xi(x) = \tau_1(x) \wedge \cdots \wedge \tau_n(x),$$

then we call ξ an *orientation* for $T_x M$.

Define the current $\tau(M,\theta,\xi)$ by

$$\tau(M,\theta,\xi)(\alpha) = \int < \alpha(x), \xi(x) > d\mu(x),$$

where $\alpha \in \mathcal{D}^n(\mathbf{R}^{n+k})$. We call such a current a *locally integer rectifiable n-current* and denote the space of all such currents by $\mathcal{R}_n^{loc}(\mathbf{R}^{n+k})$.

If M is an oriented smooth submanifold, then we can associate to it a locally integral current, $[M]$, by

$$[M](\alpha) := \tau(M, \chi_M, *\nu_M)(\alpha)$$

where ν_M is the unit normal to M.

Boundary of a Current: By Stoke's theorem, we have that, for any $\alpha \in \mathcal{D}^n(\mathbf{R}^{n+k})$,

$$\int_D d\alpha = \int_{\partial D} \alpha,$$

for some domain D. We use this as an analogy to define the *boundary* of a current $T \in \mathcal{D}_n(\mathbf{R}^{n+k})$ as the current $\partial T \in \mathcal{D}_{n-1}(\mathbf{R}^{n+k})$ given by

$$\partial T(\alpha) = T(d\alpha)$$

for any $\alpha \in \mathcal{D}^n(\mathbf{R}^{n+k})$.

We call T a *cycle* if $\partial T = 0$.

If $T \in \mathcal{R}_n^{loc}(\mathbf{R}^{n+k})$ and $\partial T \in \mathcal{R}_{n-1}^{loc}(\mathbf{R}^{n+k})$, then we say that T is a *locally integral n-current* and denote the space of such currents by $\mathbf{I}_n^{loc}(\mathbf{R}^{n+k})$. Note that the boundary of a locally rectifiable n-current may wriggle so much that it doesn't even have locally finite mass (see figure 4). So $\mathbf{I}_n^{loc}(\mathbf{R}^{n+k})$ is a proper subspace of $\mathcal{R}_n^{loc}(\mathbf{R}^{n+k})$.

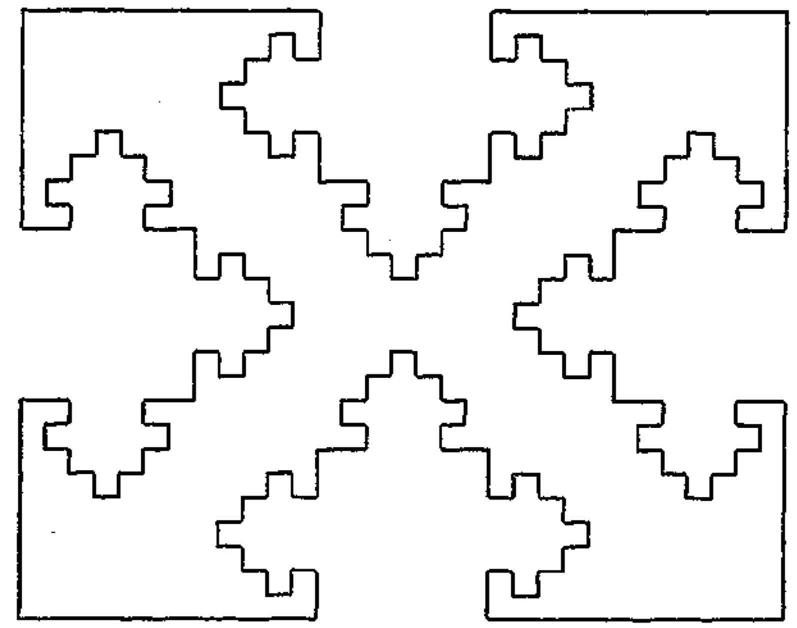

Figure 4. Cantor-like construction by removing squares.

Theorem 1.2 (Compactness theorem for $\mathbf{I}_n^{loc}(\mathbf{R}^{n+k})$ [F]): *Let $\{T_i\}_{i\geq 1}$ be a sequence in $\mathbf{I}_n^{loc}(\mathbf{R}^{n+k})$ satisfying*

$$\sup_{i\geq 1}(\mu_{T_i}(K) + \mu_{\partial T_i}(K)) < \infty$$

for any $K \subset\subset \mathbf{R}^{n+k}$. Then there is a $T \in \mathbf{I}_n^{loc}(\mathbf{R}^{n+k})$ and a subsequence $\{T_{i'}\}_{i'\geq 1}$ such that $T_{i'} \to T$.

Slicing a Current: Let $T \in \mathcal{D}_n(\mathbf{R}^{n+k})$. Define the *restriction* of T to $A \subset \mathbf{R}^{n+k}$ by

$$(T \llcorner A)(\alpha) = T(\alpha|_A)$$

for each $\alpha \in \mathcal{D}^n(\mathbf{R}^{n+k})$. We define the *slice* of T with respect to $f : \mathbf{R}^{n+k} \to \mathbf{R}$ at $r \in \mathbf{R}$ by

$$< T, f, r > := \partial(T \llcorner \{f > r\}) - \partial T \llcorner \{f > r\}.$$

We will often be slicing $T \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+1} \times \mathbf{R})$ with $\partial T \in \mathbf{I}_n^{loc}(\mathbf{R}^{n+1} \times \{0\})$ at height $z \in \mathbf{R}$. Define

$$T_z := \partial(T \llcorner (\mathbf{R}^{n+1} \times [z,\infty))).$$

We have the following useful result:

Lemma 1.3 (Co-Area formula [S 28.1]): *Let $T \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+1} \times \mathbf{R})$. Then, for any compactly supported, μ_T-integrable function g on $\mathbf{R}^{n+1} \times \mathbf{R}$, we have*

$$\int g(x,z)\,|\,\omega^{\mathrm{T}}(x,z)\,|\,d\mu_T(x,z)=\iint g(x,z)\,d\mu_{T_z}(x)\,dz$$

where $\omega^{\mathrm{T}}(x,z)$ *is the projection of* e_{n+2} *onto* $T_{(x,z)}\mu_T$. *Moreover, for a.e.* $z\in\mathbf{R}$, *we have that*

$$T_z=<T,\pi^{\perp},z>\in \mathbf{I}_n^{loc}(\mathbf{R}^{n+1}\times\{z\})$$

Parabolic norms: We will be using $C^{1,\alpha}$, $C^{2,\alpha}$ and $W^{2,p}$ norms of a smooth evolution $\mathcal{M}=\{\Gamma_t\}_{t<s}$ at a point $(x,t)\in\mathcal{M}$. These will be denoted $K_{1,\alpha}(\mathcal{M},x,t)$, $K_{2,\alpha}(\mathcal{M},x,t)$ and $J_{2,p}(\mathcal{M},x,t)$ (respectively).

Suppose that $(0,0)\in\mathcal{M}$ and that we can rotate $\mathcal{M}$ to get a new set $\overline{\mathcal{M}}$ for which the intersection

$$\overline{\mathcal{M}}\cap(B_1(0)\times(-1,1)\subset \operatorname{graph}u$$

for some function $u:B_1^n(0)\times(-1,1)\to\mathbf{R}$ whose parabolic $C^{2,\alpha}$ norm

$$\sup_{(x,t)\in B_1^n(0)\times(-1,1)}|\,u(x,t)\,|+\sum_{j+2k\le 2}\ \sup_{(x,t),(y,s)\in B_1^n(0)\times(-1,1),(x,t)\neq(y,s)}\frac{|\,D^j(\partial_t)^k u(x,t)-D^j(\partial_t)^k u(y,s)\,|}{(\max\{|\,x-y\,|,|\,t-s\,|^{1/2}\})^{\alpha}}$$

is no greater than 1. Then we will say that

$$K_{2,\alpha}(\mathcal{M},x,t)=K_{2,\alpha}(\mathcal{M},0,0)\le 1.$$

Otherwise $K_{2,\alpha}(\mathcal{M},0,0)>1$.

More generally, we let

$$K_{2,\alpha}(\mathcal{M},0,0)=\sup\{\lambda>0:K_{2,\alpha}(\mathcal{M}_{0,0}^{\lambda},0,0)\le 1\},$$

where $\mathcal{M}_{0,0}^{\lambda}=\{\Gamma_{\tau}^{\lambda,(0,0)}\}_{\tau<0}$ is given by $\Gamma_{\tau}^{\lambda,(0,0)}:=\dfrac{1}{\lambda}\Gamma_{\lambda^2\tau}$.

Finally, define

$$K_{2,\alpha}(\mathcal{M},x,t)=K_{2,\alpha}(\mathcal{M}-(x,t),0,0).$$

The norms $K_{1,\alpha}(\mathcal{M},x,t)$ and $J_{2,p}(\mathcal{M},x,t)$ are similarly defined.

2 The Brakke flow

The Brakke flow is an integral form of the mean curvature flow and is obtained by integration by parts over test-functions. We begin by considering a smooth eternal mean curvature flow $\mathcal{M}=\{\Gamma_t\}_{t\ge 0}$ as a model for our definition of the Brakke flow.

Recall that the area element $d\mu_t:=d(\mathcal{H}^n\llcorner\Gamma_t)$ satisfies the evolution equation (see [H1])

$$\frac{\partial}{\partial t}d\mu_t=-H_{\Gamma_t}^2\,d\mu_t.$$

This implies the well-known area decay formula:

$$\frac{d}{dt}\mathcal{H}^n(\Gamma_t)=-\int H_{\Gamma_t}^2(x)\,d\mu_t(x).$$

The Brakke flow will be motivated by the local version of this formula.

In order to study the local version of the area decay formula, we need to first understand how the first variation of the functional F_{φ} given by

$$F_{\varphi}(M)=\int_M\varphi(x)\,d\mathcal{H}^n(x),$$

where $\varphi\in C_c^2(\mathbf{R}^{n+1})$. Recall the formula for the first variation of an n-rectifiable Radon measure $\mu=\mu(M,\theta)$:

$$\delta V_{\mu}(X)=\int \operatorname{div}_M X(x)\,d\mu(x).$$

Fix a C_c^1 vectorfield X compactly supported in $\mathbf{R}^{n+1}-\partial M$ and let $\{\Phi_t\}_{-\delta<t<\delta}$ be a family of diffeomorphisms satisfying

$$\Phi_0(x)=x,\quad \left.\frac{\partial}{\partial t}\right|_{t=0}\Phi_t(x)=X(x).$$

Then we have

$$\begin{aligned}\delta F_{\varphi}(M)(X)&:=\left.\frac{d}{dt}\right|_{t=0}\int\varphi(\Phi_t(x))(J_{T_x\mu_M}\Phi_t)(x)\,d\mu(x)\\&=\int(D\varphi\cdot X+\varphi\operatorname{div}_M X)\,d\mu\\&=\int(\operatorname{div}_M(\varphi X)+\nabla^{\perp}\varphi\cdot X)\,d\mu\\&=\delta V_{\mu}(\varphi X)+\int\nabla^{\perp}\varphi\cdot X\,d\mu.\end{aligned}$$

If $|\,\delta V_{\mu_M}\,|<<\mu_M$ (which is true for a smooth hypersurface), then $(\delta V_{\mu_M})_{\text{sing}}=0$ and

$$\delta F_\varphi(M)(X) = \int (-\varphi \vec{H} + \nabla^\perp \varphi) \cdot X d\mu_M \tag{2.1}$$

for any C_c^1 vectorfield X compactly supported in $\mathbf{R}^{n+1} \sim \partial M$. If M is a smooth hypersurface, we may smoothly extend the mean curvature vector off M and choose X = smooth extension of $\vec{H}_M$ as our vectorfield giving

$$\delta F_\varphi(M)(\vec{H}_M) = \int (-\varphi H_M^2 + \nabla^\perp \varphi \cdot \vec{H}_M)\, d\mu_M .$$

For our smooth mean curvature flow $\mathcal{M}$ we have

$$\frac{d}{dt} F_\varphi(\Gamma_t) = \delta F_\varphi(\Gamma_t)(\vec{H}_{\Gamma_t}),$$

or

$$\frac{d}{dt} \int_{\Gamma_t} \varphi \, d\mathrm{H}^n = \int_{\Gamma_t} (-\varphi H^2 + \nabla^\perp \varphi \cdot \vec{H})\, d\mathrm{H}^n, \tag{2.2}$$

for any $\varphi \in C_c^2(\mathbf{R}^{n+1})$.

This will be our starting point for our definition of a weak version of the mean curvature flow. It will be useful to think of the (weak) evolution as a family of Radon measures rather than hypersurfaces. We begin by taking care to define the right hand side of (2.2) for general Radon measures:

Definition ($\mathcal{B}(\mu;\varphi)$): *Let $\mu \in \mathcal{M}(\mathbf{R}^{n+1})$ and assume $\varphi \in C_c^2(\mathbf{R}^{n+1},[0,\infty))$. If one of the following cases holds, set $\mathcal{B}(\mu;\varphi) = -\infty$:*

(i) $\mu \llcorner \{\varphi > 0\} \notin \mathcal{M}_n(\mathbf{R}^{n+1})$,

(ii) $|\delta V| \llcorner \{\varphi > 0\} \notin \mathcal{M}(\{\varphi > 0\})$ *where* $V := V_\mu \llcorner \{\varphi > 0\}$,

(iii) $\delta V_{\mathrm{sing}} \llcorner \{\varphi > 0\} \neq 0$,

(iv) $\int \varphi H^2 d\mu = \infty$.

Otherwise, we define

$$\mathcal{B}(\mu;\varphi) = \int (-\varphi H^2 + \nabla^\perp \varphi \cdot \vec{H})\, d\mu .$$

Remark: *We will call $C_c^2(\mathbf{R}^{n+1},[0,\infty))$ the space of admissible test-functions on $\mathbf{R}^{n+1}$.*

Here are some properties of $\mathcal{B}(\mu;\varphi)$:

Lemma 2.1 (properties of $\mathcal{B}(\mu;\varphi)$ [B 3.4] [I2 7.3]): *Let $\mu \in \mathcal{M}(\mathbf{R}^{n+1})$ and let φ be a test-function. Then*

(i) $\mathcal{B}(\mu;\varphi) \le c(\varphi)\mu\{\varphi > 0\} < \infty$

(ii) *if* $\mu(\varphi) \le c$ *and* $\mathcal{B}(\mu;\varphi) \ge -c$, *then* $\int \varphi H^2 d\mu \le \tilde{c}(c,\varphi) < \infty$,

(iii) *if $\{\mu_i\}_{i\ge1}$ is a sequence of Radon measures converging to μ, then*

$$\limsup_{i\to\infty} \mathcal{B}(\mu_i;\varphi) \le \mathcal{B}(\mu;\varphi).$$

(iv) *if $\{\mu_i\}_{i\ge1}$ is a sequence in $I\mathcal{M}_n(\mathbf{R}^{n+1})$ satisfying*

$$\inf_i \mathcal{B}(\mu_i;\varphi) \ge -C, \quad \sup_i \mu_i(\{\varphi > 0\}) \le C,$$

then there is a subsequence $\{i'\}_{i'\ge1}$ and a $\mu \in I\mathcal{M}_n(\mathbf{R}^{n+1})$, measure such that

$$\mu_{i'} \llcorner \{\varphi > 0\} \to \mu, \quad V_{\mu_{i'} \llcorner \{\varphi > 0\}} \to V_\mu .$$

Remark: *By applying the Cauchy-Schwarz inequality and using (ii), it is easy to see that if $\mu(\varphi) \le c$ and $\mathcal{B}(\mu;\varphi) \ge -c$, then we actually have $|\delta V_\mu|(U) \le C(c,\varphi,U)$.*

There may be times during the (weak) flow where the derivative isn't defined, for example when the flow must jump in the case $\mathcal{B}(\mu_t;\varphi) = -\infty$. To take care of such cases, we use the *upper derivative*, which is always well defined:

Definition (upper derivative): *Let $f : \mathbf{R} \to \mathbf{R}$. Then*

$$\overline{D}_t f(t_0) := \limsup_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0},$$

is called the upper derivative of f at t_0.

Finally, we define our *Brakke flow*. Taking into account the upper-semicontinuity of $\mathcal{B}(\mu;\varphi)$ (Lemma 2.1 (iii)), we make the following definition so that the flow will have nice compactness properties:

Definition (Brakke flow, integer Brakke flow): *Let $\mathcal{M} = \{\mu_t\}_{t\ge0}$ be a family of Radon measures on $\mathbf{R}^{n+1}$. If*

$$\overline{D}_t \mu_t(\varphi) \le \mathcal{B}(\mu_t;\varphi),$$

for any test-function φ on $\mathbf{R}^{n+1}$, then we call $\mathcal{M} = \{\mu_t\}_{t\ge0}$ a Brakke flow. If $\mu_t \in I\mathcal{M}_n(\mathbf{R}^{n+1})$ for a.e. $t \ge 0$, then we call $\mathcal{M} = \{\mu_t\}_{t\ge0}$ an integer Brakke flow.

Note that it is straightforward to generalise the definition to include the possibility that the ambient space is a general $(n + k)$-dimensional differentiable manifold [I1]. The subsequent analysis will also apply in that case.

Lemma 2.2 (some properties of Brakke flows [B 3.7, 3.10, 4.18], [I2 6.8]): *Let $\mathcal{M} = \{\mu_t\}_{t\ge0}$ be a Brakke flow. Let φ be a test-function on $\mathbf{R}^{n+1}$. Then,*

(i) *if* $\int_{t_0}^{t} \mathbf{M}(\mu_s)\, ds < \infty$, $t \ge t_0$ *then* $\mathbf{M}(\mu_t) \le \mathbf{M}(\mu_{t_0})$,

(ii) *if* $\mu_0(\mathbf{R}^{n+1} - B_R(0)) = 0$*, then* $\mu_t(\mathbf{R}^{n+1} - B_{\sqrt{R^2-2nt}}(0)) = 0$ *for all* $t \geq 0$*,*

Assume $\mu_t(K) < \infty$*,* $K \subset\subset \mathbf{R}^{n+1}$ *for all* $t \geq 0$*. Then,*

(iii) *the left and right limits always exist and satisfy*

$$\lim_{s \uparrow t} \mu_s(\varphi) \geq \mu_t(\varphi) \geq \lim_{s \downarrow t} \mu_s(\varphi),$$

(iv) $\overline{D}_t \mu_t(\varphi) > -\infty$ *for a.e.* $t \geq 0$*,*

(v) $\mu_t \in \mathcal{M}_n(\mathbf{R}^{n+1})$ *for a.e.* $t \geq 0$*,*

We also have a compactness theorem for integer Brakke flows which was proved using ideas from [B]:

Theorem 2.3 (compactness theorem [I2 7.1]): *Suppose* $\mathcal{M}^i = \{\mu_t^i\}_{t\geq 0}$*,* $i = 1,2,\ldots$ *is a sequence of integer Brakke flows satisfying*

$$\sup_{i,t} \mu_t^i(K) \leq c(K) < \infty,$$

for any $K \subset\subset \mathbf{R}^{n+1}$*. Then*

(i) *there is a subsequence* $\mathcal{M}^{i'} = \{\mu_t^{i'}\}_{t\geq 0}$*,* $i' = 1,2,\ldots$ *and an integer Brakke flow* $\mathcal{M} = \{\mu_t\}_{t\geq 0}$ *such that* $\mu_t^{i'} \to \mu_t$ *for every* $t \geq 0$*,*

(ii) *there is a further subsequence (also denoted* i'*)* $\mathcal{M}^{i'} = \{\mu_t^{i'}\}_{t\geq 0}$*,* $i' = 1,2,\ldots$ *(depending on t) such that* $V_{\mu_t^{i'}} \to V_{\mu_t}$ *for a.e.* $t \geq 0$*.*

This is the most important ingredient in the proof of existence by elliptic regularisation.

3 Elliptic Regularisation for the Brakke flow

Here we give an outline of the proof for existence of integer Brakke flows using the method of Elliptic Regularisation. This is a summary of the first chapter in [I2] for the special case where the ambient space is $\mathbf{R}^{n+1}$. We will be adapting the main ideas from this method to prove existence of weak solutions to the *forced mean curvature flow.*

Definition (Initial surface): *Assume* $M_0 \in \mathbf{I}_n^{loc}(\mathbf{R}^{n+1} \times \{0\})$ *is a cycle of finite mass. Then we say that* M_0 *is an initial surface and we call* μ_{M_0} *the initial data.*

We now aim to turn the parabolic problem of existence of an integer Brakke flow with initial data μ_0 into a fixed boundary elliptic problem in a higher dimensional ambient space. The idea is to try to approximate the (stretched out) space time track of the Brakke flow by a locally integral n-current M^ε minimising an appropriate functional. Since the minimiser is approximating the (stretched) space time track, we hope that μ_{M^ε} will be a downward translating soliton solution for the Brakke flow. Therefore, we would like M^ε to satisfy the Euler-Lagrange equation

$$\vec{H}_{M^\varepsilon} = -\frac{1}{\varepsilon}\omega^\perp,$$

where $\omega := e_{n+2}$.

Consider the functional given by

$$F^\varepsilon(M) := \frac{1}{\varepsilon}\int e^{-z/\varepsilon}\, d\mu_M(x,z), \quad (x,z) \in \mathbf{R}^{n+1} \times \mathbf{R},$$

for any $M \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+1} \times \mathbf{R})$ with boundary in $\mathbf{I}_n^{loc}(\mathbf{R}^{n+1} \times \{0\})$. Then, whenever $|\delta V_{\mu_M}| << \mu_M$ we have, by (2.1),

$$\delta F^\varepsilon(M)(X) = -\frac{1}{\varepsilon}\int (\vec{H}_M + \frac{1}{\varepsilon}\omega_M^\perp) \cdot X e^{-z/\varepsilon}\, d\mu_M(x,z),$$

for any C_c^1-vectorfield X compactly supported in $\mathbf{R}^{n+1} \times (\mathbf{R} - \{0\})$.

Lemma 3.1 (Euler-Lagrange equation [I2 2.6]): *Suppose* $M \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+1} \times \mathbf{R})$ *a stationary point for* F^ε*. Then, for each* C_c^1*-vectorfield* X *compactly supported in* $\mathbf{R}^{n+1} \times (\mathbf{R} - \{0\})$

$$0 = \int (\vec{H}_M + \frac{1}{\varepsilon}\omega_M^\perp) \cdot X\, d\mu_M,$$

and

(i) $\vec{H}_M^\varepsilon := \vec{H}_M + \frac{1}{\varepsilon}\omega_M^\perp = 0$*,* $\mu_M - a.e.$*,*

(ii) $|\vec{H}_M| \le \frac{1}{\varepsilon}$, $\mu_M - a.e.$,

(iii) $\vec{H}_M^{\perp} = \vec{H}_M$, $\mu_M - a.e.$,

(iv) $1 = \varepsilon^2 H_M^2 + |\omega_M^{\mathrm{T}}|^2$, $\mu_M - a.e.$.

The existence of F^{ε}-minimisers can be proved using a standard argument.

Lemma 3.2 (existence of F^{ε}-minimisers [I2 3.2]): *Let $M_0 \in \mathbf{I}_n^{loc}(\mathbf{R}^{n+1} \times \{0\})$ be an initial surface. There exists $M^{\varepsilon} \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+1} \times \mathbf{R})$ minimising F^{ε} with $\partial M^{\varepsilon} = M_0$ and so that,*

(i) $\mathrm{spt} M^{\varepsilon} \subset \mathbf{R}^{n+1} \times [0,\infty)$,

(ii) $F^{\varepsilon}(M^{\varepsilon}) \le \mathbf{M}(M_0)$,

Now, define the family of locally integral n-currents $\{M^{\varepsilon}(t)\}_{t\ge 0}$ by

$$M^{\varepsilon}(t) := (\sigma_{-t/\varepsilon})_{\#}(M^{\varepsilon}),$$

where $\sigma_{-t/\varepsilon}(x,z) = (x, z - t/\varepsilon)$. Note that each $M^{\varepsilon}(t)$ is F^{ε}-minimising in $\mathbf{R}^{n+1} \times (0,\infty)$ with $\partial(M^{\varepsilon}(t) \llcorner (\mathbf{R}^{n+1} \times [0,\infty))) = M^{\varepsilon}_{t/\varepsilon}$. Otherwise, we could deform M^{ε} to decrease F^{ε}.

Fix $t \ge 0$. Then, on the set

$$W^{\varepsilon} := \{(x,z,s) : z > -s/\varepsilon,\ s \ge 0\}$$

we have, for any test-function φ on $\mathbf{R}^{n+1} \times \mathbf{R}$ such that $\mathrm{spt}\varphi \times \{t\} \subset W^{\varepsilon}$,

$$\begin{aligned}\frac{d}{dt}\mu_{M^{\varepsilon}(t)}(\varphi) &= \int (-\varphi \vec{H} + \nabla^{\perp}\varphi) \cdot (-\frac{1}{\varepsilon}\omega)\, d\mu_{M^{\varepsilon}(t)} \\ &= \mathcal{B}(\mu_{M^{\varepsilon}(t)};\varphi),\end{aligned}$$

by Lemma 3.1 (i). Hence $\mathcal{M}^{\varepsilon} = \{\mu_{M^{\varepsilon}(t)}\}_{t\ge 0}$ is an integer Brakke flow on W^{ε}, that is $\mu_{M^{\varepsilon}}$ is a translative soliton solution to the Brakke flow.

Note that, since $e^{-z/\varepsilon}$ dies off so quickly for $\varepsilon \ll 1$, M^{ε} will be very tall (of order $1/\varepsilon$). In other words, M^{ε} will be close to a cylinder in any fixed finite region. So, for ε very small,

$$\vec{H}_{M^{\varepsilon}} \approx \vec{H}_{M_z^{\varepsilon}},\ \mu_{M^{\varepsilon}} - a.e.$$

Thus, when ε is small, the motion of M^{ε} by $-\frac{1}{\varepsilon}\omega$ nearly produces motion of M_z^{ε} by its mean curvature. This motivates us to send $\varepsilon \to 0$.

Recalling Theorem 2.3, we know that in order to send $\varepsilon \to 0$ we need area bounds of the form

$$\sup_{i,t} \mu_{M^{\varepsilon_i}(t)}(K) < \infty,$$

where $\varepsilon_i \downarrow 0$. For this we have:

Lemma 3.3 (Local mass bound [I2 5.1]): *Let $A \subset \mathbf{R}$ be measurable. Then*

$$\mathbf{M}(M^{\varepsilon} \llcorner (\mathbf{R}^{n+1} \times A)) \le (\mathcal{L}^1(A) + \varepsilon)\mathbf{M}(M_0).$$

To motivate this, assume for the moment that T is the space time track of a smooth evolution $\mathcal{M} = \{\Gamma_t\}_{t\ge 0}$. By the co-area formula (Lemma 1.3) we have,

$$\mathcal{H}^{n+1}(T \cap (\mathbf{R}^{n+1} \times (t,t+\delta))) = \int_t^{t+\delta}\int_{\Gamma_s} \frac{1}{|\nabla \tau(x,s)|}\, d\mathcal{H}^n(x) ds.$$

where $\tau(x,t) = t$. If dw is a line element pointing in the direction of fastest time increase, we have

$$\frac{1}{|\nabla\tau|} = \frac{dw}{dt} = \sqrt{(dx/dt)^2 + 1} = \sqrt{H^2 + 1},$$

by Pythagoras' theorem. Using the area decay formula

$$\mathcal{H}^n(\Gamma_t) + \int_0^t \int_{\Gamma_s} H^2\, d\mathcal{H}^n ds = \mathcal{H}^n(\Gamma_0), \tag{3.1}$$

we have

$$\begin{aligned}\mathcal{H}^{n+1}(T \cap (\mathbf{R}^{n+1} \times (t,t+\delta))) &= \int_t^{t+\delta}\int_{\Gamma_s} \sqrt{H^2+1}\, d\mathcal{H}^n ds \\ &\le \sqrt{\int_0^t\int_{\Gamma_s}(H^2+1) d\mathcal{H}^n ds}\sqrt{\int_t^{t+\delta}\mathcal{H}^n(\Gamma_s) ds} \\ &\le \sqrt{1+\delta}\sqrt{\delta}\mathcal{H}^n(\Gamma_0).\end{aligned}$$

It is possible to derive an estimate analogous to (3.1) so that one will obtain Lemma 3.3. The estimate is:

Lemma 3.4 [I2 4.5]: *For a.e. a,b with $0 \le a \le b$, we have*

$$\int |\omega^{\mathrm{T}}|\, d\mu_{M_b^{\varepsilon}} + \int_{\mathbf{R}^{n+1}\times(a,b)} \varepsilon H^2 d\mu_{M^{\varepsilon}} \le \int |\omega^{\mathrm{T}}|\, d\mu_{M_a^{\varepsilon}} \le F^{\varepsilon}(M^{\varepsilon}) \le \mathbf{M}(M_0).$$

Thus, for all $t \geq 0$,

$$\mu_{M^\varepsilon(t)}(\mathbf{R}^{n+1} \times (r,s)) \leq (s - r + \varepsilon)\mathbf{M}(M_0), \tag{3.2}$$

and hence, for any $K \subset\subset \mathbf{R}^{n+1} \times \mathbf{R}$ we can find a constant c such that

$$\sup_{t\geq 0, \varepsilon>0} \mu_{M^\varepsilon(t)}(K) \leq c(K) < \infty .$$

Therefore, we can find a sequence $\varepsilon_i \downarrow 0$ and an integer Brakke flow $\hat{\mathcal{M}} = \{\hat{\mu}_t\}_{t\geq 0}$ such that $\mu_{M^{\varepsilon}(t)} \to \hat{\mu}_t$ for each $t \geq 0$.

We now use $\hat{\mathcal{M}}$ to define a Brakke flow $\mathcal{M} = \{\mu_t\}_{t\geq 0}$ with initial data $\mu_0 = \mu_{M_0}$.

According to Lemma 2.2 (iv),(v), the $\hat{\mu}_t$ are (vertically) translationally invariant for all but countably many $t \geq 0$. Fix a $\theta \in C_c^2((0,\infty),[0,\infty))$ such that $\int \theta dz = 1$ and define $\{\mu_t\}_{t\geq 0}$ by

$$\mu_t(\varphi) = \hat{\mu}_t(\theta\varphi),$$

for any test-function φ on $\mathbf{R}^{n+1}$.

Defining $\mathcal{M}$ in this way, one may show that, whenever $\hat{\mu}_t$ is translationally invariant, $\hat{\mu}_t = \mu_t \times \mathcal{L}^1$ ($\hat{\mu}_t = \mu_t \times \mathcal{L}^1 \llcorner (0,\infty)$ if $t = 0$) and

$$\mathcal{B}(\mu_t;\varphi) = \mathcal{B}(\hat{\mu}_t;\theta\varphi).$$

Therefore $\mathcal{M}$ is an integer Brakke flow.

Finally one must show that $\mathcal{M}$ has initial data $\mu_0 = \mu_{M_0}$. On one hand one has $\mathbf{M}(\mu_t) \leq \mathbf{M}(M_0)$ by (3.2). While on the other hand, it can be shown that $\mu_t \geq \mu_{M_t}$. So we in fact have $\mu_0 = \mu_{M_0}$. We have outlined the proof of:

Theorem 3.5 (Existence of integer Brakke flows [I2 8.1]): *Let $M_0 \in \mathbf{I}_n^{loc}(\mathbf{R}^{n+1})$ be an initial surface. Then there exists an integer Brakke flow $\mathcal{M} = \{\mu_t\}_{t\geq 0}$ and a current $M \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+2})$ such that*

(i) $\partial M = M_0$,

(ii) $\mathbf{M}_{K\times B}(M) << \mathcal{L}^1(B)$ *for each* $K \subset \mathbf{R}^{n+1}, B \subset \mathbf{R}$,

(iii) $\mu_0 = \mu_{M_0}, \mu_t \geq \mu_{M_t}$,

(iv) $\mathbf{M}_K(\mu_t) \leq \mathbf{M}_K(\mu_0)$,

where $M_t := \partial(M \llcorner [t,\infty))$.

This motivates the following definition:

Definition (Enhanced motion, overflow, undercurrent): *Let $M_0 \in \mathbf{I}_n^{loc}(\mathbf{R}^{n+1})$ be an initial surface. Suppose $\mathcal{M} = \{\mu_t\}_{t\geq 0}$ is an integer Brakke flow and $M \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+2})$ so that the conclusions of Theorem 3.4 are satisfied. Then we call the pair $(M, \mathcal{M})$ an enhanced motion with initial condition M_0. We also say that $\mathcal{M}$ is the overflow and M is the undercurrent.*

Whenever the discrepancy between μ_t and μ_{M_t} is zero, we then say the enhanced motion is a *matching motion*:

Definition (Matching motion): *Let $M_0 \in \mathbf{I}_n^{loc}(\mathbf{R}^{n+1})$ be an initial surface. We call a current $M \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+2})$ a matching motion for M_0 if*

(i) $\partial M = M_0$, *and*

(ii) the family of measures given by $\mu_t = \mu_{M_t}$ *is a Brakke flow.*

Ilmanen has proved the following interesting fact about enhanced motions:

Lemma 3.5 (Existence of matching motions [I2 9.2]): *$M_0 \in \mathbf{I}_n^{loc}(\mathbf{R}^{n+1})$ be an initial surface. Suppose every enhanced motion with initial condition M_0 has the same undercurrent $M \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+2})$. Then M is a matching motion for M_0.*

Part II – General Compactness theorem

In this part we define a generalised version of the Brakke flow before proving a compactness theorem for the generalised flow. This will allow us to use elliptic regularisation for a wide range of evolution equations, in particular a Brakke flow with forcing term.

4 General Brakke flows

We now define a *general Brakke flow* $\mathcal{M} = \{\mu_t\}_{t\geq 0}$ of Radon measures on $\mathbf{R}^{n+1}$. This is done in such a way that the compactness theorem for the flow may be proved with a minimum number of assumptions on the flow. We make the ansatz that the flow takes the form

$$\overline{D}_t\mu_t(\varphi) \leq \mathcal{G}(\mu_t,\varphi),$$

for any test function φ.

Firstly, for any fixed test function φ we assume that $\mathcal{G}(\cdot,\varphi)$ is upper-semicontinuous, since continuity would be too strong an assumption. Furthermore, assuming continuity would exclude the Brakke flow.

Since the family $\{\mu_t\}_{t\geq 0}$ are Radon measures, the proof of the compactness theorem would rely on Theorem 1.1. Thus we require, in addition to upper-semicontinuity, that $|\mathcal{G}(\mu,\varphi)|<\infty$ implies $\mu\llcorner\{\varphi>0\}\in\mathcal{M}_n(\mathbf{R}^{n+1})$, $|\delta V|\llcorner\{\varphi>0\}\in\mathcal{M}_n(\mathbf{R}^{n+1})$, and $|\delta V|\llcorner\{\varphi>0\}<\infty$, where $V:=V_\mu$. These form the basis of the singular conditions.

Finally, we would like the general Brakke flow to jump down rather than up when $\mu(\{\varphi>0\})<\infty$. So we assume $\mathcal{G}(\mu,\varphi)\leq C(\varphi,\mu(\{\varphi>0\});\mathcal{G})$ where C satisfies

$$\mu(\{\varphi>0\})\leq c_1 \Rightarrow C(\varphi,\mu(\{\varphi>0\});\mathcal{G})\leq C(\varphi,c_1;\mathcal{G}) \tag{4.1}$$

and the third parameter (written as $\mathcal{G}$) indicates constants that depend on global properties of $\mathcal{G}$ such as forcing terms, etc. For example, if we were considering the Brakke flow, the third parameter would be zero as the Brakke flow is purely local. We will illustrate this with a few examples. As we shall see in Lemma 4.1, $\mathcal{G}(\mu,\varphi)\leq C(\varphi,\mu(\{\varphi>0\});\mathcal{G})$ will imply certain continuity properties of the flow other than no sudden increases in mass.

Thus we define:

Definition (general Brakke functional): *Let φ be a test function on $\mathbf{R}^{n+1}$. Suppose*
$\mathcal{G}:\mathcal{M}(\mathbf{R}^{n+1})\times C_c^2(\mathbf{R}^{n+1};[0,\infty))\to[-\infty,\infty]$
satisfies the following conditions:

(i) if $\mu\llcorner\{\varphi>0\}\notin\mathcal{M}_n(\mathbf{R}^{n+1})$, then $-\infty=\mathcal{G}(\mu,\varphi)$,

(ii) if $|\delta V|\llcorner\{\varphi>0\}\notin\mathcal{M}(\{\varphi>0\})$, then $-\infty=\mathcal{G}(\mu,\varphi)$, where $V:=V_\mu\llcorner\{\varphi>0\}$,

(iii) if $|\delta V|\llcorner\{\varphi>0\}$ is singular, then $-\infty=\mathcal{G}(\mu,\varphi)$, where $V:=V_\mu\llcorner\{\varphi>0\}$,

(iv) $\mathcal{G}(\cdot,\varphi)$ *is upper-semicontinuous,*
(v) $\mathcal{G}(\mu,\varphi) \le C(\varphi,\mu(\{\varphi>0\});\mathcal{G})$ *where the constant C satisfies (4.1).*
Then we call $\mathcal{G}$ *a general Brakke functional.*

Remark: *If we were considering, say, area maximising hypersurfaces, then the inequality would be reversed, the upper-semicontinuity condition would be replaced with a lower-semicontinuity condition, and the upper bound would be a lower bound. Such considerations might be made if one were looking at flow by mean curvature on a Lorentz manifold (see [Ba]) or flow by inverse mean curvature (see [HI]).*

Examples: *In the following examples,* φ *is a test-function on* $\mathbf{R}^{n+1}$, $U \subset \mathbf{R}^{n+1}$ *is bounded, and* $\mu\llcorner\{\varphi>0\} \in \mathcal{M}_n(\mathbf{R}^{n+1})$ *satisfies none of the singular conditions ((i)-(iii) in the definition) and the additional singular condition* $\int \varphi H^2 d\mu < \infty$.

(i) *The Brakke functional* $\mathcal{B}$ *of §2 is a general Brakke functional.*
(ii) *B. White's K-almost Brakke flows [W1, §11] have a functional with the same singular conditions as* $\mathcal{B}$, *and*

$$\mathcal{G}(\mu,\varphi) := \int(-\varphi H^2 + D^\perp\varphi\cdot\vec{H})d\mu + K\int|-\varphi\vec{H} + D^\perp\varphi|\,d\mu$$

is a general Brakke functional. In this case, the global parameter in the constant C from part (v) of the definition will depend on K, i.e. $C = C(\varphi,\mu(\{\varphi>0\});K)$.

(iii) *Fix* $p \ge 2$, *and suppose* $g \in L^p(\mu)$. *Then with the same singular conditions as* $\mathcal{B}$,

$$\mathcal{G}(\mu,\varphi) := \int(-\varphi H^2 + D^\perp\varphi\cdot\vec{H})d\mu + \int|g||-\varphi\vec{H} + D^\perp\varphi|\,d\mu$$

is a general Brakke functional. In this case, the global parameter in the constant from part (v) will depend on n, p, and g, i.e. $C = C(\varphi,\mu(\{\varphi>0\});n,p,g)$. *Later, in §9 we will call this the* forced Brakke functional.

The *K*-almost Brakke flow comes from some natural generalisations of the Brakke flow, such as flow on a sphere or volume preserving mean curvature flow (as in [H2] and [Ath]). However, the *K*-almost Brakke flow assumes the additional forcing terms remain *uniformly* bounded, which, unfortunately, may rule out the volume preserving mean curvature flow because the average mean curvature may blow up when singularities arise, as in the rotationally symmetric case [Ath]. It is an open question whether the average mean curvature does blow up at the onset of singularities.

The forced Brakke functional relaxes the assumption of uniformly bounded forcing terms. If *g* is bounded by *K*, then the forced Brakke functional becomes the general Brakke functional from example (ii). So, *K*-almost Brakke flows are *forced Brakke flows* with $p = \infty$. Although the *K*-almost Brakke flow arises in some very natural *geometric* situations, the forced Brakke flow arises in some very natural *physical* situations (as mentioned in the introduction to this thesis).

Definition (general Brakke flow, general integer Brakke flow): *Let* $\mathcal{M} = \{\mu_t\}_{t\ge0}$ *be a family of Radon measures on* $\mathbf{R}^{n+1}$ *and suppose* $\mathcal{G}$ *is a General Brakke functional. If*

$$\overline{D}_t\mu_t(\varphi) \le \mathcal{G}(\mu_t;\varphi),$$

for any test-function φ *on* $\mathbf{R}^{n+1}$, *then we call* $\mathcal{M}$ *a general Brakke flow. If* $\mu_t \in I\mathcal{M}_n(\mathbf{R}^{n+1})$ *for a.e.* $t \ge 0$, *then we call* $\mathcal{M}$ *a general integer Brakke flow.*

In the next section we will prove a compactness theorem for general integer Brakke flows.

Lemma 4.1 (Some continuity properties of general Brakke flows): *Let* φ *be a test-function on* $\mathbf{R}^{n+1}$. *Let* $\mathcal{G}$ *be a general Brakke functional and suppose* $\mathcal{M} = \{\mu_t\}_{t\ge0}$ *is a general Brakke flow satisfying*

$$\mu_t(K) \le c_1(K),$$

for all $t \ge 0$. *Then*

(i) $\mu_t(\varphi) - C(\varphi,c_1(\{\varphi>0\});\mathcal{G})t$ *is non-increasing at each* $t \ge 0$,
(ii) *the limits* $\lim_{s\uparrow t}\mu_s(\varphi)$ *and* $\lim_{s\downarrow t}\mu_s(\varphi)$ *exist and satisfy the inequality* $\lim_{s\uparrow t}\mu_s(\varphi) \ge \mu_t(\varphi) \ge \lim_{s\downarrow t}\mu_s(\varphi)$ *for each* $t \ge 0$,
(iii) *there is a co-countable set* $\mathcal{T}_1$ *of time where* μ_t *is continuous at each* $t \in \mathcal{T}_1$,
(iv) *there is a full measure set* $\mathcal{T}_2$ *of time (i.e. the complement of* $\mathcal{T}_2$ *has zero measure) where* $\overline{D}_t\mu_t(\varphi) > -\infty$ *at each* $t \in \mathcal{T}_2$,
(v) $\mu_t \in \mathcal{M}_n(\mathbf{R}^{n+1})$ *for a.e.* $t \ge 0$.

Remark: *Note that (ii) does not imply decrease of mass under a general Brakke flow. It does however say that the mass can never locally jump up (an artefact of the definition of the general Brakke flow). Whenever* μ_t *is continuous, it could be possible for the mass to increase, unlike the Brakke flow.*

Proof of Lemma 4.1: Since $\mathcal{G}$ is a General Brakke functional, we have

$$\overline{D}_t\mu_t(\varphi) \le \mathcal{G}(\mu_t,\varphi) \le C(\varphi,\mu_t(\{\varphi>0\});\mathcal{G}) \le C(\varphi,c_1(\{\varphi>0\});\mathcal{G}),$$

from which (i) follows.

Let $s > t$. By (i) we have

$$\mu_s(\varphi) \le \mu_t(\varphi) - C(\varphi,c_1(\{\varphi>0\});\mathcal{G})(t-s).$$

Likewise for $s < t$ we have

$$\mu_s(\varphi) \ge \mu_t(\varphi) - C(\varphi,c_1(\{\varphi>0\});\mathcal{G})(t-s)$$

and (ii) follows.

The arguments for (iii) and (iv) are similar. For (iii), we let Ψ be a countable dense subset of $C_c^2(\mathbf{R}^{n+1};[0,\infty))$. Since $\mu_t(\varphi)-C(\varphi,c_1(\{\varphi>0\});\mathcal{G})t$ is non-increasing for all $t\geq 0$ it follows that, for any $\psi\in\Psi$, we may find a co-countable set (that is, the complement of a countable set) of times $\mathcal{T}_\psi$ where $\mu_t(\psi)$ is continuous. Otherwise, we could find a compact set of times (wlog assumed to be the interval $[0,1]$) on which, for any $\varepsilon>0$,

$$|\lim_{s\downarrow t}\mu_s(\psi)-\mu_t(\psi)|>\varepsilon \text{ or } |\lim_{s\uparrow t}\mu_s(\psi)-\mu_t(\psi)|>\varepsilon$$

for infinitely many times (wlog we consider only the case $|\lim_{s\downarrow t}\mu_s(\psi)-\mu_t(\psi)|>\varepsilon$). Let $\{t_i\}_{i\geq 1}$ be a decreasing sequence of such times. Since this is a bounded sequence, there is a time $t_0\leq 1$ and a subsequence such that $t_{i'}\downarrow t_0$. By (ii), there is an ℓ_0 such that, for any $\varepsilon_0>0$ we can find an $N\geq 1$ such that

$$i'\geq N\Rightarrow|\mu_{t_{i'}}(\psi)-\ell_0|\leq\varepsilon_0.$$

Fix $\varepsilon_0>0$. Then, for any $\varepsilon>0$, we can find an $N\geq 1$ such that

$$i'\geq N\Rightarrow\varepsilon_0+|\ell_0-\lim_{s\downarrow t_{i'}}\mu_s(\psi)|\geq|\mu_{t_{i'}}(\psi)-\lim_{s\downarrow t_{i'}}\mu_s(\psi)|>\varepsilon$$

a contradiction. Thus our assertion that $\mathcal{T}_\psi$ is co-countable is valid.

Define $\mathcal{T}_1:=\bigcap_{\psi\in\Psi}\mathcal{T}_\psi$. Then $\mu_t(\psi)$ is continuous for any $\psi\in\Psi$ at all $t\in\mathcal{T}_1$. For general test-functions φ, the property follows by uniform approximation of φ by sequences in Ψ.

By (iii), for any $\psi\in\Psi$ we can find a full measure set of times for which (iv) is true. Denoting this set by $\mathcal{T}'_\psi$ we define $\mathcal{T}_2:=\bigcap_{\psi\in\Psi}\mathcal{T}'_\psi$. Let φ be a test-function on $\mathbf{R}^{n+1}$, and choose $\psi\in\Psi$ so that $\psi\geq\varphi$. Then, for all $t\in\mathcal{T}_2$,

$$-\infty<\overline{D}_t\mu_t(\psi)\leq\overline{D}_t\mu_t(\varphi)+\overline{D}_t\mu_t(\psi-\varphi)\leq\overline{D}_t\mu_t(\varphi)+C(\psi-\varphi,c_1(\{\psi>\varphi\});\mathcal{G}),$$

by (i). Hence (iv) is true for any test-function on $\mathbf{R}^{n+1}$ and (v) follows from (iv) and the definition of $\mathcal{G}$. ■

5 The compactness theorem for general integer Brakke flows

The advantage of proving compactness for the general class of flows defined in the previous section is that we need only prove upper-semicontinuity and boundedness for any Brakke-type flow. The other conditions on $\mathcal{G}$ are singular conditions.

Theorem 5.1 (compactness theorem): *Suppose $\{\mathcal{G}^i\}_{i\geq 1}$ is a sequence of general Brakke functionals and assume there is a general Brakke functional $\mathcal{G}$ such that, for any Radon measure μ and any test-function φ,*

$$\limsup_{i\to\infty}\mathcal{G}^i(\mu,\varphi)\leq\mathcal{G}(\mu,\varphi).$$

Suppose $\mathcal{M}^i=\{\mu_t^i\}_{t\geq 0}$, $i=1,2,\ldots$ is a sequence of general integer Brakke flows (with general Brakke functional $\mathcal{G}^i$) satisfying

$$\sup_{i,t}\mu_t^i(K)\leq c_1(K)<\infty,$$

for any $K\subset\subset\mathbf{R}^{n+1}$. Then

(i) there is a subsequence $\mathcal{M}^{i'}$, $i'=1,2,\ldots$ and a general integer Brakke flow $\mathcal{M}=\{\mu_t\}_{t\geq 0}$ (with general Brakke functional $\mathcal{G}$) such that $\mu_t^{i'}\to\mu_t$ for all $t\geq 0$,

(ii) there is a further subsequence $\mathcal{M}^{i'}$, $i'=1,2,\ldots$ (also written i' but depending on t) such that if $\mathcal{G}^i(\mu;\varphi)\geq -C\Rightarrow|\delta V_\mu|(K)\leq C(C,\varphi,K)$ for each $i\geq 1$, then $V_{\mu_t^{i'}}\to V_{\mu_t}$ for a.e. $t\geq 0$.

Proof: The proof is similar to the proof of Theorem 2.3 by Ilmanen [I2] and is in four parts.

Part 1: Let B be a countable dense set of times. For every $t\in B$, the bound

$$\sup_{i,t}\mu_t^i(K)\leq c_1(K)<\infty$$

together with the compactness theorem for Radon measures (Theorem 1.1 (i) without the assumption of bounded first variation) allow us to find a subsequence $\{i'\}$ of $\{i\}$ and $\mu_t\in I\mathcal{M}_n(\mathbf{R}^{n+1})$ so that

$$\mu_t^{i'}\to\mu_t$$

weakly in $\mathcal{M}(\mathbf{R}^{n+1})$. Doing this for all $t\in B$ and diagonalising, we can find a single subsequence (also written $\{i'\}$) such that

$$\mu_t^{i'}\to\mu_t,\quad\forall t\in B$$

weakly in $\mathcal{M}(\mathbf{R}^{n+1})$.

Note that the condition $\limsup_{i\to\infty} \mathcal{G}^i(\mu,\varphi) \le \mathcal{G}(\mu,\varphi)$ and Lemma 4.1(i) implies that, for all sufficiently large i' and any test-function φ,

$$\mu_t^{i'}(\varphi) - C(\varphi, c_1(\{\varphi>0\}); \mathcal{G})t$$

is non-increasing for all $t \in B$ independent of i'. Sending $i' \to \infty$ we find that, for all $t \in B$,

$$\mu_t(\varphi) - C(\varphi, c_1(\{\varphi>0\}); \mathcal{G})t$$

is non-increasing.

Part 2: Let $t \notin B$ be some fixed time. Let $\{i''\}$ be any convergent subsequence (depending on t) of $\{i'\}$. Define

$$\mu_t := \lim_{i''\to\infty} \mu_t^{i''}$$

at t. Then μ_t is defined for all time.

Now apply Part 1 to $B \cup \{t\}$ for each $t \notin B$. Then

$$\mu_t(\varphi) - C(\varphi, c_1(\{\varphi>0\}); \mathcal{G})t$$

is non-increasing for all $t \ge 0$.

By Lemma 4.1 (iii), we may find a co-countable set of times $\mathcal{T}_1$ on which μ_t is continuous. For such times, μ_t is uniquely determined regardless of the subsequence chosen for t. Hence the full sequence converges on $\mathcal{T}_1$. In the complement of $\mathcal{T}_1$ we define μ_t in a similar manner as before: we define $\mu_t := \lim_{i''\to\infty} \mu_t^{i''}$ on $[0,\infty) - \mathcal{T}_1$ with $\{i''\}$ depending on t. For such t, replace the previous values of μ_t with these new ones.

Thus we have constructed a family $\mathcal{M} = \{\mu_t\}_{t\ge 0}$ of integral Radon measures such that, for all $t \ge 0$,

$$\mu_t^{i'} \to \mu_t \text{ weakly in } \mathcal{M}(\mathbf{R}^{n+1}),$$

and

$$\mu_t(\varphi) - C(\varphi, c_1(\{\varphi>0\}); \mathcal{G})t \ge \mu_s(\varphi) - C(\varphi, c_1(\{\varphi>0\}); \mathcal{G})s.$$

Part 3: We now show that the family $\mathcal{M}$ constructed in Part 2 is a general integer Brakke flow.

Fix $t_0 \ge 0$ and let φ be a fixed test-function on $\mathbf{R}^{n+1}$. For any function $f: \mathbf{R} \to \mathbf{R}$ define

$$D_t^+ f(t_0) := \limsup_{t\downarrow t_0} \frac{f(t)-f(t_0)}{t-t_0}$$

and

$$D_t^- f(t_0) := \limsup_{t\uparrow t_0} \frac{f(t)-f(t_0)}{t-t_0}.$$

Since

$$\overline{D}_t f(t_0) = \max\{D_t^+ f(t_0), D_t^- f(t_0)\},$$

it suffices to show that $\mathcal{M}$ satisfies

$$D_t^+ \mu_{t_0}(\varphi) \le \mathcal{G}(\mu_{t_0}, \varphi). \tag{5.1}$$

The proof for $D_t^- \mu_{t_0}(\varphi)$ is similar.

Assume wlog $-\infty < D_t^+ \mu_{t_0}(\varphi)$. Then we may find a sequence $\{t_q\}_{q\ge1}$ descending to t_0, and a sequence $\{\beta_q\}_{q\ge1}$ descending to 0, such that

$$D_t^+ \mu_{t_0}(\varphi) - \beta_q \le \frac{\mu_{t_q}(\varphi) - \mu_{t_0}(\varphi)}{t_q - t_0}.$$

For any q, we may select an i_q from the sequence $\{i'\}$ (assume $i_q \to \infty$ with q) such that

$$D_t^+ \mu_{t_0}(\varphi) - 2\beta_q \le \frac{\mu_{t_q}^{i_q}(\varphi) - \mu_{t_0}^{i_q}(\varphi)}{t_q - t_0}.$$

By Lemma 4.1, $\overline{D}_t \mu_s^{i_q}(\varphi) \le C((\varphi, c_1(\{\varphi>0\}; \mathcal{G})$ is $\mathcal{L}^1$ - measurable. So,

$$D_t^+ \mu_{t_0}(\varphi) - 2\beta_q \le \frac{1}{t_q - t_0} \int_{t_0}^{t_q} \overline{D}_t \mu_s^{i_q}(\varphi) ds \le C((\varphi, c_1(\{\varphi>0\}; \mathcal{G}).$$

It follows that there is at least one point $t_0 \le s_q \le t_q$ such that

$$D_t^+\mu_{t_0}(\varphi)-2\beta_q \le \frac{1}{t_q-t_0}\int_{t_0}^{t_q}\overline{D}_t\mu_s^{i_q}(\varphi)ds \le \overline{D}_t\mu_{s_q}^{i_q}(\varphi) \le \mathcal{G}(\mu_{s_q}^{i_q},\varphi).$$

Again, by the compactness theorem for Radon measures (Theorem 1.1(i) without the assumption of bounded first variation), we may find a subsequence $\{q'\}$ and a $\mu \in I\mathcal{M}_n(\{\varphi>0\})$ such that $\mu_{t_{q'}}^{i_{q'}} \llcorner \{\varphi>0\} \to \mu$ weakly on $\{\varphi>0\}$.

For any fixed $q' \ge 1$, we have by hypothesis,

$$\limsup_{j\to\infty}\mathcal{G}^j(\mu_{t_{q'}}^{i_{q'}},\varphi) \le \mathcal{G}(\mu_{t_{q'}}^{i_{q'}},\varphi),$$

while on the other hand, for any $j \ge 1$,

$$\limsup_{q'\to\infty}\mathcal{G}^j(\mu_{t_{q'}}^{i_{q'}},\varphi) \le \mathcal{G}^j(\mu,\varphi).$$

Therefore, by diagonalising we have

$$\limsup_{q'\to\infty}\mathcal{G}^{i_{q'}}(\mu_{t_{q'}}^{i_{q'}},\varphi) \le \mathcal{G}(\mu,\varphi).$$

Consequently we have

$$D_t^+\mu_{t_0}(\varphi) \le \limsup_{q'\to\infty}\mathcal{G}^{i_{q'}}(\mu_{t_{q'}}^{i_{q'}},\varphi) \le \mathcal{G}(\mu,\varphi). \tag{5.2}$$

Thus we need to show $\mu=\mu_{t_0}\llcorner\{\varphi>0\}$.

Let ψ be a test-function on $\{\varphi>0\}$ with $\psi \le \varphi$. Using the same trick as in the proof of Lemma 4.1 (iv),

$$-\infty < D_t^+\mu_{t_0}(\varphi) \le D_t^+\mu_{t_0}(\psi)+C(\psi,c_1(\{\varphi>\psi\});\mathcal{G}).$$

So $-\infty < D_t^+\mu_{t_0}(\psi)$ and therefore

$$\mu_{t_0}(\psi) \le \lim_{t\downarrow t_0}\mu_t(\psi).$$

Fix $s>t_0$. For sufficiently large q', $t_0<s_{q'}<s$. Since $\mu_t^{i'}(\varphi)-C(\varphi,c_1(\{\varphi>0\});\mathcal{G})t$ is non-increasing for all $t \ge 0$, we have, increasing q' if necessary,

$$\begin{aligned}\mu_{t_0}^{i_{q'}}(\psi)+C(\psi,c_1(\{\psi>0\});\mathcal{G})(s_{q'}-t_0) &\ge \mu_{s_{q'}}^{i_{q'}}(\psi)\\ &\ge \mu_s^{i_{q'}}(\psi)+C(\psi,c_1(\{\psi>0\});\mathcal{G})(s_{q'}-s).\end{aligned}$$

Therefore, sending $q'\to\infty$,

$$\mu_{t_0}(\psi) \ge \mu(\psi) \ge \mu_s(\psi)+C(\psi,c_1(\{\psi>0\});\mathcal{G})(t_0-s),$$

for any $s>t_0$. Thus, sending $s\downarrow t_0$

$$\mu_{t_0}(\psi) \ge \mu(\psi) \ge \lim_{s\downarrow t_0}\mu_s(\psi) \ge \mu_{t_0}(\psi).$$

Therefore $\mu=\mu_{t_0}\llcorner\{\varphi>0\}$. Hence μ_{t_0} satisfies (5.1).

Part 4: We now show that we have varifold convergence under the hypothesis that

$$\mathcal{G}^i(\mu;\varphi) \ge -C \Rightarrow |\delta V_\mu|(K) \le C(C,\varphi,K) \quad \forall i \ge 1.$$

Assume t is a time when $\overline{D}_t\mu_t(\varphi)>-\infty$ for a test-function φ. Then, by (5.2) we can find a further subsequence $\{i''\}_{i''\ge1}$ (depending on t) such that

$$\mathcal{G}^{i''}(\mu_t^{i''};\varphi) \ge -C$$

where $C=C(t,\varphi)$. Therefore, by Allard's compactness theorem 1.1 and the hypothesis, we can find a further subsequence (labelled $\{i'\}_{i'\ge1}$) such that (ii) follows. ■

Corollary 5.2: *Let $\mathcal{G}$ be a general Brakke functional. Suppose $\mathcal{M}^i=\{\mu_t^i\}_{t\ge0}$, $i=1,2,\ldots$ is a sequence of general integer Brakke flows satisfying*

$$\sup_{i,t}\mu_t^i(K) \le c_1(K) < \infty,$$

for any $K \subset\subset \mathbf{R}^{n+1}$. Then

(i) there is a subsequence $\mathcal{M}^{i'}$, $i'=1,2,\ldots$ and a general integer Brakke flow $\mathcal{M}=\{\mu_t\}_{t\ge0}$ such that $\mu_t^{i'}\to\mu_t$ for all $t\ge0$,

(ii) there is a further subsequence $\mathcal{M}^{i''}$, $i'=1,2,\ldots$ (also written i' but depending on t) such that if $\mathcal{G}(\mu;\varphi)\ge -C \Rightarrow |\delta V_\mu|(K) \le C(C,\varphi,K)$ for each $i \ge 1$, then $V_{\mu_t^{i''}} \to V_{\mu_t}$ for a.e. $t\ge0$.

Proof: Since any fixed general Brakke functional satisfies the hypothesis

$$\limsup_{i\to\infty}\mathcal{G}(\mu,\varphi) \le \mathcal{G}(\mu,\varphi)$$

this is a trivial consequence of Theorem 5.1. ■

Part III – Forced Brakke flows

We will first consider an appropriate elliptic problem to apply elliptic regularisation to the examples given in §4. In particular, we would like to study the equation

$$\overline{D}_t\mu_t(\varphi) \leq \int(-\varphi H^2 + \nabla^\perp\varphi\cdot\vec{H})d\mu_t + \int g\,|-\varphi\vec{H} + \nabla^\perp\varphi\,|\,d\mu_t \qquad (*)$$

with appropriate singular conditions. In §9 we call this a *forced Brakke flow*. The solutions to the related elliptic problem turn out to be translating soliton solutions to a forced Brakke flow that only *approximates* (*). To prove existence of solutions, we require local mass bounds for the solutions to the elliptic problem so that we may apply the compactness theorem 5.1. Then, for given initial data, we will construct a solution to (*) using the solutions obtained for the related elliptic problem.

Existence for Brakke flows with a forcing term has never been proved before. In [W1 §11], White indicates that the existence of K-almost Brakke flows (see example (ii) of §4) should follow from minor modifications to Ilmanen's existence proof for Brakke flows. This claim is furhter substantiated in this part (in particular the remark on page 30 preceeding Lemma 6.1 and the remark on page 46 preceeding Lemma 8.1). However, the existence proof for forced Brakke flows doesn't follow as easily. Although the methods here are based on elliptic regularisation, the application of elliptic regularisation gives rise to the need for forced Brakke flows approximating (*). In the case of the Brakke flow and K-almost Brakke flow, the elliptic solutions translate vertically according to the Brakke flow or K-almost Brakke flow (respectively).

6 The elliptic problem

The elliptic problem related to (*) is a prescribed mean curvature problem for the metric giving the functional used in §3: namely $F^\varepsilon(M) := \frac{1}{\varepsilon}\int e^{-z/\varepsilon}d\mu_M(x,z)$. This problem has been extensively studied for the standard metric on $\mathbf{R}^{n+1}$. The following is by no means an exhaustive list of contributions: [DF], [DS1], [DS2], [Fu], [Gi], [GMT], and [Ma].

The functional will be defined on currents $M \in \mathbf{I}^{loc}_{n+1}(\mathbf{R}^{n+1}\times\mathbf{R})$ with $\partial M \in \mathbf{I}^{loc}_n(\mathbf{R}^{n+1}\times\{0\})$. Fix $\Omega_0 \in \mathbf{I}^{loc}_{n+1}(\mathbf{R}^{n+1}\times\{0\})$ so that $\partial\Omega_0 = \partial M$. We let $\Omega \in \mathbf{I}^{loc}_{n+2}(\mathbf{R}^{n+1}\times\mathbf{R})$ be the (unique) $(n+2)$ – current satisfying $\partial\Omega = M + \Omega_0$.

The "forcing" term in (*) will define the prescribed mean-curvature for our elliptic problem. Let $U \subset \mathbf{R}^{n+1}$ have finite measure and suppose $g : \mathbf{R}^{n+1}\times[0,\infty)\to[0,\infty)$ satisfies $g(x,t) \equiv 0 \ \ \forall x \notin U$,

$$g(\cdot,t) \in L^1(U)$$

for all $t \geq 0$, and

$$g(x,t) \equiv 0 \quad \forall t \geq \tau\,,$$

for some $\tau \in (0,\infty)$.

From the next section onwards we will also use the following "time-like" continuity assumption: for any $t \geq 0$,

$$\lim_{s \to t} g(x,s) = g(x,t) \ a.e.\ x \in U.$$

This assumption is not required for existence.

Define $g^\varepsilon : \mathbf{R}^{n+1} \times [0,\infty) \to [0,\infty)$ by

$$g^\varepsilon(x,z) := \begin{cases} g(x,\varepsilon z), & z \geq 0 \\ 0, & z < 0 \end{cases}.$$

Remark: *In the case of K-almost Brakke flows, we would have $g \leq K$ and we would choose g^ε to be given by $g^\varepsilon(x,z) = K$. As we shall point out later, this implies that the solutions obtained here will translate vertically according to the K-almost Brakke flow.*

Let $K \subset\subset \mathbf{R}^{n+1}$ (not to be confused with the K from the definition of K-almost Brakke flows). We consider the functional given by

$$G_K^\varepsilon(M) := \frac{1}{\varepsilon} \int_{K \times \mathbf{R}} e^{-z/\varepsilon} d\mu_M(x,z) - \frac{1}{\varepsilon} \int_{K \times \mathbf{R}} g^\varepsilon(x,z) e^{-z/\varepsilon} d\mu_\Omega(x,z).$$

Stationary points for this functional will have prescribed mean curvature given by $-(g^\varepsilon + \omega \cdot \nu / \varepsilon)$ (Lemma 6.1).

We will now consider the first variation of G_K^ε. Let X be a C_c^1-vectorfield compactly supported in $K \times (\mathbf{R} - \{0\})$ and let $\{\Phi_s\}_{-\delta<s<\delta}$ be a family of compactly supported diffeomorphisms mapping $\mathbf{R}^{n+2} \to \mathbf{R}^{n+2}$ and satisfying

$$\Phi_0(x,z) = (x,z), \left.\frac{\partial}{\partial s}\right|_{s=0} \Phi_s = X.$$

Let $f \in C_c^\infty(U \times [0,\infty),[0,\infty))$. Then

$$\begin{aligned} \left.\frac{d}{ds}\right|_{s=0} \int_{K \times \mathbf{R}} J\Phi_s (f \circ \Phi_s) e^{-\Phi_s \cdot \omega/\varepsilon} d\mu_\Omega &= \int_{K \times \mathbf{R}} (\operatorname{div} X f + \overline{D} f \cdot X - \frac{1}{\varepsilon} \omega \cdot X f) e^{-z/\varepsilon} d\mu_\Omega \\ &\quad + \int_{K \times \mathbf{R}} f \nu_M \cdot X e^{-z/\varepsilon} d\mu_M \\ &= \int_{K \times \mathbf{R}} (\operatorname{div}(f e^{-z/\varepsilon} X) d\mu_\Omega + \int_{K \times \mathbf{R}} f \nu_M \cdot X e^{-z/\varepsilon} d\mu_M \\ &= \int_{K \times \mathbf{R}} f \nu_M \cdot X e^{-z/\varepsilon} d\mu_M \end{aligned}$$

Therefore, by approximating g^ε by a sequence of $C_c^\infty(U \times [0,\infty),[0,\infty))$ functions $f_i \leq g^\varepsilon$ we have

$$\left.\frac{d}{ds}\right|_{s=0} \int_{K \times \mathbf{R}} J\Phi_s (g^\varepsilon \circ \Phi_s) e^{-\Phi_s \cdot \omega/\varepsilon} d\mu_\Omega = \int_{K \times \mathbf{R}} g^\varepsilon \nu_M \cdot X e^{-z/\varepsilon} d\mu_M$$

since $\int_{K \times \mathbf{R}} g^\varepsilon \nu_M \cdot X e^{-z/\varepsilon} d\mu_M < \infty$. Hence, whenever $|\delta V_{\mu_M}| << \mu_M$, we have

$$\delta G_K^\varepsilon(M)(X) = -\frac{1}{\varepsilon} \int_{K \times \mathbf{R}} (\vec{H}_M + \frac{1}{\varepsilon} \omega^\perp + g^\varepsilon \nu_M) \cdot X e^{-z/\varepsilon} d\mu_M,$$

for any C_c^1 - vectorfield X compactly supported in $K \times (\mathbf{R} - \{0\})$. Therefore we have:

Lemma 6.1 (Euler-Lagrange equation): *Let $K \subset\subset \mathbf{R}^{n+1}$ and suppose $M \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+1} \times \mathbf{R})$ is a stationary point for G_K^ε with fixed boundary in K. Then, for any C_c^1 - vectorfield X compactly supported in $K \times (\mathbf{R} - \{0\})$,*

$$0 = \int_{K \times \mathbf{R}} (\vec{H}_M + \frac{1}{\varepsilon} \omega^\perp + g^\varepsilon \nu_M) \cdot X e^{-z/\varepsilon} d\mu_M$$

and

(i) $\vec{H}_M^\varepsilon := \vec{H}_M + \frac{1}{\varepsilon} \omega^\perp = -g^\varepsilon \nu_M$, μ_M *- a.e. in* $(K - \partial K) \times \mathbf{R}$,

(ii) $\vec{H}_M^\perp = \vec{H}_M$, μ_M *- a.e. in* $(K - \partial K) \times \mathbf{R}$, *and*

(iii) $$\begin{aligned} 1 &= \varepsilon^2 |\vec{H}_M + g^\varepsilon \nu_M|^2 + |\omega^\top|^2 \\ &\leq \varepsilon^2 (2\vec{H}_M \cdot (\vec{H}_M + g^\varepsilon \nu_M) + (g^\varepsilon)^2) + |\omega^\top|^2, \ \mu_M - a.e.\ in\ (K - \partial K) \times \mathbf{R} \end{aligned}$$

Recall the definition of an initial surface from §3:

Definition (Initial surface): *Assume $M_0 \in \mathbf{I}_n^{loc}(\mathbf{R}^{n+1} \times \{0\})$ is a cycle of finite mass. Then we say that M_0 is an initial surface and we call μ_{M_0} the initial data.*

The following is the analogue to Lemma 3.2 for cycles of locally finite mass:

Lemma 6.2 (Existence of minimisers): *Let $K \subset\subset \mathbf{R}^{n+1}$, let $M_0 \in \mathbf{I}_n^{loc}(\mathbf{R}^{n+1}\times\{0\})$ be a cycle of locally finite mass and let $\Omega_0 \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+1}\times\{0\})$ denote the (unique) current with $\partial\Omega_0 \llcorner K = M_0 \llcorner K$. Then there exists an $M^\varepsilon \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+1}\times\mathbf{R})$ minimising G_K^ε with $\partial M^\varepsilon \llcorner K = M_0 \llcorner K$ and so that,*

(i) $\operatorname{spt} M^\varepsilon \cap (K\times\mathbf{R}) \subset K\times[0,\infty)$,

(ii) $G_K^\varepsilon(M^\varepsilon) \le \mathbf{M}(M_0\llcorner K) - \frac{1}{\varepsilon}\int_0^\infty\int_K g(x,\varepsilon z)e^{-z/\varepsilon}d\mu_{\Omega_0}dz$, *and*

(iii) $F^\varepsilon(M^\varepsilon \llcorner (K\times\mathbf{R})) \le \mathbf{M}(M_0\llcorner K) + \sup_{t\ge0}\| g(\cdot,t)\|_{L^1(U)}$.

Remarks: *As in [Ma], existence only requires $g(\cdot,\cdot)\in L^1_{loc}(\mathbf{R}^{n+2})$. If we let $g(\cdot,t)\in L^p(U)$ for some $p\ge1$, then it is easy to see that $g(\cdot,t)\in L^1(U)$. Later we will be looking at regularity when $p>n$. If $g(\cdot,\cdot)\in L^\infty(\mathbf{R}^{n+2})$ then the resulting flow will be a version of the K-Brakke flow (see [W1,2]).*

One could consider geometric assumptions such as those considered in [Fu] and [DF]. In [Fu] the prescribed mean curvature satisfies $\| g\|_{L^{n+1}(U)} < (n+1)\omega_{n+1}^{1/(n+1)}$ and one minimises $\mathbf{M}(\partial\Omega) - \int g\, d\mu_\Omega$ in the class of Caccioppoli sets. The upper bound on $\| g\|_{L^{n+1}(U)}$ comes from the application of the isoperimetric inequality. The related version for our circumstances will be mentioned in the following proof. The problem explored in [DF] is the higher co-dimensional case.

Note that, since

$$\int g^\varepsilon(x,z)e^{-z/\varepsilon}\, d\mu_\Omega(x,z) \le \int_U g^\varepsilon(x,z)e^{-z/\varepsilon}\, d\mathcal{L}^{n+1}(x,z) < \infty$$

it follows that G_K^ε is bounded from below. Contrast this with capillary surfaces where there is no such lower bound.

Proof of Lemma 6.2: Define

$$\mathcal{S} := \{S \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+2}) : \partial S\llcorner(K\times\mathbf{R}) = M_0\llcorner K\}.$$

Then $M_0\times[(0,\infty)]\in\mathcal{S}$ and

$$\begin{aligned}\inf_{S\in\mathcal{S}} G_K^\varepsilon(S) &\le G_K^\varepsilon(M_0\times[(0,\infty)])\\ &= \mathbf{M}(M_0\llcorner K) - \frac{1}{\varepsilon}\int_0^\infty\int_K g(x,\varepsilon z)e^{-z/\varepsilon}d\mu_{\Omega_0}(x)dz\\ &\le \mathbf{M}(M_0\llcorner K),\end{aligned}$$

since $g\ge0$. Therefore, if M^ε exists,

$$\begin{aligned}\mathbf{M}(M_0\llcorner K) &\ge F^\varepsilon(M^\varepsilon\llcorner K) - \frac{1}{\varepsilon}\int_0^\infty\int_K g(x,\varepsilon z)e^{-z/\varepsilon}d\mu_{\Omega_t^\varepsilon}(x)dz\\ &\ge F^\varepsilon(M^\varepsilon\llcorner K) - \frac{1}{\varepsilon}\int_0^\infty\int_{K\cap U} g(x,\varepsilon z)e^{-z/\varepsilon}d\mathcal{L}^n(x)dz\\ &\ge F^\varepsilon(M^\varepsilon\llcorner K) - \sup_{t\ge0}\| g(\cdot,t)\|_{L^1(U)},\end{aligned}$$

where Ω^ε is the (unique) $(n+2)$ – current satisfying

$$\partial\Omega^\varepsilon\llcorner(K\times\mathbf{R}) = (M^\varepsilon + \Omega_0)\llcorner(K\times\mathbf{R}).$$

We will briefly digress to explore an alternate, more geometric approach to obtaining an upper bound on $F^\varepsilon(M^\varepsilon\llcorner K)$:
At this point one could have used the isoperimetric inequality to obtain

$$\begin{aligned}&\mathbf{M}(M_0\llcorner K)\\ &\quad\ge F^\varepsilon(M^\varepsilon\llcorner K) - ((n+1)^{-1-1/n}\omega_{n+1}^{-1/n})^{1-1/p}\sup_{t\ge0}\| g(\cdot,t)\|_{L^p(U)} F^\varepsilon(M^\varepsilon\llcorner K)^{(1-1/p)(1+1/n)},\end{aligned}$$

where $p\ge1$. If $p<n+1$, we could therefore find a constant such that

$$F^\varepsilon(M^\varepsilon\llcorner(K\times\mathbf{R})) \le c(\mathbf{M}(M_0\llcorner K),n,p,g) < \infty$$

because $0\le(1+1/n)(1-1/p)<1$ and $x-ax^\delta\le c$ implies $0<x\le c(c,a,\delta)$ whenever $\delta<1$ (if $\delta>1$ then there is no upper bound on x).

Likewise, if we were to assume $\| g(\cdot,t)\|_{L^{n+1}(U)} < (n+1)\omega_{n+1}^{1/(n+1)}$, the isoperimetric inequality would imply

$$\mathbf{M}(M_0\llcorner K) \ge F^\varepsilon(M^\varepsilon\llcorner K) - (n+1)^{-1}\omega_{n+1}^{-1/(n+1)}\sup_{t\ge0}\| g(\cdot,t)\|_{L^{n+1}} F^\varepsilon(M^\varepsilon\llcorner K) > 0$$

and again we could find a constant such that

$$F^\varepsilon(M^\varepsilon\llcorner(K\times\mathbf{R})) \le c(\mathbf{M}(M_0\llcorner K),n,g) < \infty.$$

This case gives a similar result to [Fu].

Returning to the proof of the lemma, let $\{S_i\}_{i\ge1}$ be a sequence in $\mathcal{S}$ satisfying

$$G_K^\varepsilon(S_i)\downarrow\inf_{S\in\mathcal{S}} G_K^\varepsilon(S).$$

This implies that we can obtain local mass estimates for the S_i and therefore, by applying the compactness theorem, we can find a subsequence $\{i'\}$ and an $M \in \mathcal{S}$ such that

$$S_{i'} \to M$$

and, by approximating g^ε by C_c^0 functions,

$$G_K^\varepsilon(M) \le \liminf_{i' \to \infty} G_K^\varepsilon(S_i) = \inf_{S \in \mathcal{S}} G_K^\varepsilon(S).$$

Hence M minimises G_K^ε in $\mathcal{S}$ and satisfies (ii) and (iii).

Suppose $\mu_M(\mathbf{R}^{n+1} \times (-\infty, 0)) > 0$. Then

$$\begin{aligned} G_K^\varepsilon(M) &= G_K^\varepsilon(M \llcorner (\mathbf{R}^{n+1} \times (-\infty, 0])) + G_K^\varepsilon(M \llcorner (\mathbf{R}^{n+1} \times (0, \infty))) \\ &= F^\varepsilon(M \llcorner (K \times (-\infty, 0])) + G_K^\varepsilon(M \llcorner (\mathbf{R}^{n+1} \times (0, \infty))) \\ &> G_K^\varepsilon(\pi_\#^+(M)), \end{aligned}$$

where $\pi^+(x, z) = (x, \max\{z, 0\})$. Hence M satisfies (i). ∎

For the remainder of this thesis we will concentrate on minimisers M^ε of the functional

$$G^\varepsilon(M) := \frac{1}{\varepsilon} \int e^{-z/\varepsilon} d\mu_M(x, z) - \frac{1}{\varepsilon} \int g^\varepsilon(x, z) e^{-z/\varepsilon} d\mu_\Omega(x, z)$$

satisfying $\partial M^\varepsilon = M_0$, where M_0 is a given initial surface. For a given initial surface the existence of a solution, M^ε, is guaranteed by Lemma 6.2.

The case where the initial surface has *locally* finite mass is an easy extension of the ensuing analysis making use of Lemma 6.2 where necessary. The more general assumption of local finiteness clutters the main ideas and doesn't provide any additional insight.

7 Cylindrical Monotonicity

In this section we derive a *cylindrical monotonicity formula* and some consequences for $(n+1)$-dimensional currents $M \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+1} \times \mathbf{R})$ satisfying

$$\Big(\frac{1}{\varepsilon} \int_{C_R(0)} |\vec{H}|^p \, e^{-z/\varepsilon} d\mu_M\Big)^{1/p} \le \Lambda$$

where $C_R(0) = B_R^{n+1} \times \mathbf{R}$. By Lemma 6.1, minimisers of G^ε satisfy this condition.

In a private conversation, Ilmanen has told me that DeGiorgi has suggested that one could use elliptic regularisation to prove a version of Brakke's regularity theorem. It may be possible to apply the formula obtained here to the methods in §§20-22 of [S] to obtain a Brakke regularity theorem for flows arising from elliptic regularisation.

We finish by proving a "height" lemma for the minimisers of G^ε.

Lemma 7.1 (cylindrical monotonicity formula): *Suppose $M \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+1} \times \mathbf{R})$ satisfies $|\delta V_{\mu_M}| << \mu_M$. Then in the sense of distributions we have, for any $a \in \mathbf{R}^{n+1}$ and all $\rho > 0$,*

$$\begin{aligned} \frac{d}{d\rho}\Big[\rho^{-n} \frac{1}{\varepsilon} \int_{C_\rho(a)} e^{-z/\varepsilon} d\mu_M\Big] &= \rho^{-n} \frac{d}{d\rho} \frac{1}{\varepsilon} \int_{C_\rho(a)} |\nabla^\perp r|^2 \, e^{-z/\varepsilon} d\mu_M \\ &+ \rho^{-(n+1)} \frac{1}{\varepsilon} \int_{C_\rho(a)} (\operatorname{div} x - n) e^{-z/\varepsilon} d\mu_M \ , \\ &+ \rho^{-(n+1)} \frac{1}{\varepsilon} \int_{C_\rho(a)} \vec{H} \cdot (x - a) e^{-z/\varepsilon} d\mu_M \end{aligned}$$

where $C_\rho(a) := B_\rho^{n+1}(a) \times \mathbf{R}$.

Proof: Let $\zeta : \mathbf{R} \to [0, \infty)$ be C_c^1 and set $r = |x|$, $X = \zeta(r)x$. Then, for any $(n+1)$-dimensional plane $S = \operatorname{span}\{\tau_1, \cdots, \tau_{n+1}\}$,

$$\operatorname{div}_S X = \zeta(r) \operatorname{div}_S x + \zeta'(r) r |\nabla^S r|^2.$$

Writing $S^\perp$ for the line orthogonal to S, we find

$$r\zeta'(r) + n\zeta(r) = r\zeta'(r) |\nabla^{S^\perp} r|^2 - \zeta(r)(\operatorname{div}_S x - n + \vec{H} \cdot x)$$

a.e. in $\mathbf{R}^{n+1} \times \mathbf{R}$ since $|\delta V_{\mu_M}| << \mu_M$.

Fix $\rho > 0$ and consider a cut-off function $\varphi : \mathbf{R} \to [0,1]$ satisfying

$$\varphi'(t) \le 0, \varphi(t) = 1 \text{ on } (-\infty, \tfrac{1}{2}], \varphi(t) = 0 \text{ on } (1,\infty).$$

Assuming $\zeta(r) = \varphi(r/\rho)$, we have

$$r\zeta'(r) = \frac{r}{\rho}\varphi'(r/\rho) = -\rho\frac{\partial}{\partial\rho}\varphi(r/\rho),$$

and

$$\int (\rho\frac{\partial\varphi}{\partial\rho} - n\varphi)e^{-z/\varepsilon}d\mu_M = \int (\rho\frac{\partial\varphi}{\partial\rho}|\nabla^\perp r|^2 + \varphi(\operatorname{div}x - n + \vec{H}\cdot x))e^{-z/\varepsilon}d\mu_M,$$

or, multiplying through by $\rho^{-(n+1)}$,

$$\begin{aligned}\frac{d}{d\rho}[\rho^{-n}\int \varphi e^{-z/\varepsilon}d\mu_{M^\varepsilon}] &= \rho^{-n}\frac{d}{d\rho}\int \varphi|\nabla^\perp r|^2 e^{-z/\varepsilon}d\mu_{M^\varepsilon} \\ &+ \rho^{-(n+1)}\int \varphi(\operatorname{div}x - n + \vec{H}\cdot x)e^{-z/\varepsilon}d\mu_{M^\varepsilon}.\end{aligned}$$

Sending φ to $\chi_{(-\infty,1]}$, we obtain the lemma. ■

Lemma 7.2: *Let $R, \Lambda \in (0,\infty)$ and $p > n$. Suppose $M \in \mathbf{I}^{loc}_{n+1}(\mathbf{R}^{n+1}\times\mathbf{R})$ satisfies*

$$(\frac{1}{\varepsilon}\int_{C_R(0)} |\vec{H}|^p e^{-z/\varepsilon}d\mu_M)^{1/p} \le \Lambda.$$

Then, in the sense of distributions we have that, for any $a \in B^{n+1}_R(0)$ and all $\rho \in (0, R-|a|)$

(i) $\quad 0 \le \frac{d}{d\rho}[e^{p\Lambda\rho^{1-n/p}/(p-n)}\rho^{-n}\frac{1}{\varepsilon}\int_{C_\rho(a)} e^{-z/\varepsilon}d\mu_{M^\varepsilon}]^{1/p}.$

In particular, for all $0 < \tau \le \rho$,

(ii) $\quad (\tau^{-n}\frac{1}{\varepsilon}\int_{C_\tau(a)} e^{-z/\varepsilon}d\mu_M)^{1/p} + \frac{\Lambda}{p-n}\tau^{1-n/p} \le (\rho^{-n}\frac{1}{\varepsilon}\int_{C_\rho(a)} e^{-z/\varepsilon}d\mu_M)^{1/p} + \frac{\Lambda}{p-n}\rho^{1-n/p}.$

Remark: *Note that, whenever M^ε is smooth, $\lim_{\rho\to 0}\rho^{-n}\frac{1}{\varepsilon}\int_{C_\rho(a)} e^{-z/\varepsilon}d\mu_{M^\varepsilon} = 0$. This presents significant problems if one were to apply the arguments from §§20-22 in [S] because*

$\lim_{\rho\to 0}\rho^{-n}\frac{1}{\varepsilon}\int_{C_\rho(a)} e^{-z/\varepsilon}d\mu_{M^\varepsilon} = 0$ indicates a upper limit on the amount of "resolution" available for the Lipschitz approximation argument.

Proof: We assume wlog $a = 0$. By Lemma 7.1 we have

$$\begin{aligned}\frac{d}{d\rho}[\rho^{-n}\int_{C_\rho(0)} e^{-z/\varepsilon}d\mu_M] &\ge -\rho^{-n}\int_{C_\rho(0)} |\vec{H}|e^{-z/\varepsilon}d\mu_M \\ &\ge -\rho^{-n}\Lambda(\int_{C_\rho(0)} e^{-z/\varepsilon}d\mu_M)^{1-1/p}.\end{aligned} \tag{7.1}$$

Therefore

$$\begin{aligned}\frac{d}{d\rho}[\rho^{-n}\int_{C_\rho(0)} e^{-z/\varepsilon}d\mu_M]^{1/p} &= \frac{1}{p}(\rho^{-n}\int_{C_\rho(0)} e^{-z/\varepsilon}d\mu_M)^{1/p-1}\frac{d}{d\rho}[\rho^{-n}\int_{C_\rho(0)} e^{-z/\varepsilon}d\mu_M] \\ &\ge -\frac{\Lambda}{p}\rho^{-n/p}.\end{aligned} \tag{7.2}$$

Multiplying by the integrating factor $e^{\Lambda\rho^{1-n/p}/(p-n)}$ gives (i).

Finally, integrating (7.2) completes the proof. ■

The following version of the monotonicity formula is of some interest, especially if one were to use covering arguments for the G^ε-minimisers.

Lemma 7.3 (truncated cylindrical monotonicity formulae): *Let $M \in \mathbf{I}^{loc}_{n+1}(\mathbf{R}^{n+1}\times\mathbf{R})$ and suppose $|\delta V_{\mu_M}| << \mu_M$. Then, in the sense of distributions we have that, for any $a \in \mathbf{R}^{n+2}$, and all ρ and $\sigma > 0$,*

$$\begin{aligned}\frac{\partial}{\partial\rho}[\rho^{-n}\frac{1}{\varepsilon}\int_{C^\sigma_\rho(a)} e^{-z/\varepsilon}d\mu_M] &= \rho^{-n}\frac{\partial}{\partial\rho}\frac{1}{\varepsilon}\int_{C^\sigma_\rho(a)} |\nabla^\perp r|^2 e^{-z/\varepsilon}d\mu_M \\ &+ \rho^{-(n+1)}\frac{1}{\varepsilon}\int_{C^\sigma_\rho(a)} (\operatorname{div}x - n)e^{-z/\varepsilon}d\mu_M \\ &- \rho^{-(n+1)}\frac{\partial}{\partial\sigma}\frac{1}{\varepsilon}\int_{C^\sigma_\rho(a)} r\nabla r\cdot\nabla s e^{-z/\varepsilon}d\mu_M \\ &+ \rho^{-(n+1)}\frac{1}{\varepsilon}\int_{C^\sigma_\rho(a)} \vec{H}\cdot(x-a)e^{-z/\varepsilon}d\mu_M\end{aligned}$$

where $C^\sigma_\rho(a) := B^{n+1}_\rho(\pi(a))\times(\omega\cdot a - \sigma, \omega\cdot a + \sigma)$.

Furthermore, let $R, \Lambda \in (0,\infty)$ and $p > n$. Suppose $M \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+1} \times \mathbf{R})$ satisfies

$$\Big(\frac{1}{\varepsilon}\int_{C_R(0)} |\vec{H}|^p\, e^{-z/\varepsilon} d\mu_M\Big)^{1/p} \le \Lambda .$$

Then, in the sense of distributions we have, for any $a \in C_R(0)$ and all $\rho \in (0, R-|a|)$,

(i) $$0 \le \frac{d}{d\rho}\Big[e^{p\Lambda\rho^{1-n/p}/(p-n)}\rho^{-n}\frac{1}{\varepsilon}\int_{C_\rho^\rho(a)} e^{-z/\varepsilon} d\mu_{M^\varepsilon}\Big]^{1/p}.$$

In particular, for all $0 < \tau \le \rho$,

(ii) $$\Big(\tau^{-n}\frac{1}{\varepsilon}\int_{C_\tau^\tau(a)} e^{-z/\varepsilon} d\mu_M\Big)^{1/p} + \frac{\Lambda}{p-n}\tau^{1-n/p} \le \Big(\rho^{-n}\frac{1}{\varepsilon}\int_{C_\rho^\rho(a)} e^{-z/\varepsilon} d\mu_M\Big)^{1/p} + \frac{\Lambda}{p-n}\rho^{1-n/p}.$$

Proof: Consider variations by the vectorfield given by $X := \zeta(r) f(s) x$ where

$$r = |x|,\ s = |z| \text{ and } \zeta, f \in C_c^1(\mathbf{R};[0,\infty)).$$

Then, for any $(n+1)$ – dimensional plane S,

$$\operatorname{div}_S X = (r\zeta'(r)|\nabla^S r|^2 + \zeta(r)\operatorname{div}_S x) f(s) + \zeta(r) f'(s) \nabla^S s \cdot x$$

Fix $\sigma > 0$ and consider a cut-off function $h : \mathbf{R} \to [0,1]$ satisfying

$$h'(t) \le 0,\ h(t) = 1 \text{ on } (-\infty, \tfrac{1}{2}],\ h(t) = 0 \text{ on } (1,\infty).$$

Assuming $f(s) = h(s/\sigma)$, we have

$$f'(s) = \frac{1}{\sigma}h'(s/\sigma) = -\frac{\sigma}{s}\frac{\partial}{\partial\sigma}h(s/\sigma)$$

and consequently

$$\operatorname{div}_S X = (r\zeta'(r)|\nabla^S r|^2 + \zeta(r)\operatorname{div}_S x) h(s/\sigma) - \sigma\zeta(r)\frac{\partial}{\partial\sigma}h(s/\sigma)\frac{\nabla^S s}{s}\cdot x \qquad (7.3)$$

Replacing $r\zeta'(r)|\nabla^S r|^2 + \zeta(r)\operatorname{div}_S x$ with (7.3) and then using the same methods as those used for Lemma 7.1 and letting h become $\chi_{(-\infty,\sigma)}$, the truncated Monotonicity formula follows.

Note that,

$$\Big|\int_{C_\rho^\sigma(a)} r\nabla r \cdot \nabla s\, e^{-z/\varepsilon} d\mu_M\Big| \le \rho \int_{C_\rho^\sigma(a)} |\omega^{\mathrm{T}}|\, e^{-z/\varepsilon} d\mu_M .$$

Therefore.

$$-\rho\frac{\partial}{\partial\sigma}\int_{C_\rho^\sigma(a)} e^{-z/\varepsilon} d\mu_M \le \frac{\partial}{\partial\sigma}\int_{C_\rho^\sigma(a)} r\nabla r\cdot\nabla s\, e^{-z/\varepsilon} d\mu_M \le \rho\frac{\partial}{\partial\sigma}\int_{C_\rho^\sigma(a)} |\omega^{\mathrm{T}}|\, e^{-z/\varepsilon} d\mu_M .$$

Finally, if we use

$$\frac{d}{d\rho}\Big[\rho^{-n}\frac{1}{\varepsilon}\int_{C_\rho^\rho(a)} e^{-z/\varepsilon} d\mu_{M^\varepsilon}\Big] = \frac{\partial}{\partial\sigma}\Big|_{\sigma=\rho}\Big[\sigma^{-n}\frac{1}{\varepsilon}\int_{C_\sigma^\rho(a)} e^{-z/\varepsilon} d\mu_{M^\varepsilon} + \rho^{-n}\frac{1}{\varepsilon}\int_{C_\rho^\sigma(a)} e^{-z/\varepsilon} d\mu_{M^\varepsilon}\Big]$$

we have

$$\begin{aligned}\frac{d}{d\rho}\Big[\rho^{-n}\int_{C_\rho^\rho(0)} e^{-z/\varepsilon} d\mu_M\Big] &\ge -\rho^{-n}\int_{C_\rho^\rho(0)} |\vec{H}|\, e^{-z/\varepsilon} d\mu_M \\ &\ge -\rho^{-n}\Lambda\Big(\int_{C_\rho^\rho(0)} e^{-z/\varepsilon} d\mu_M\Big)^{1-1/p}.\end{aligned}$$

The proof is completed by using this in place of (7.1) and following the proof of Lemma 7.2. ∎

We present the following technical lemma analogous to [S 19.5]. This will be used to obtain a height lemma for the minimisers of G^ε.

Lemma 7.4: *Let $R, \Lambda \in (0,\infty)$, $\ell, \beta \in (0,1)$ and $p > n$. Suppose $M \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+1} \times \mathbf{R})$ satisfies*

$$\Big(\frac{1}{\varepsilon}\int_{C_R(0)} |\vec{H}|^p\, e^{-z/\varepsilon} d\mu_M\Big)^{1/p} \le \Lambda$$

and assume $\Lambda R^{1-n/p} \le 1 - n/p$. Let $a, b \in C_{\beta R}(0) \cap \operatorname{spt} M$ satisfy

$$|\pi(a-b)| \ge \beta R/2, \text{ and } |e_{n+1}\cdot(a-b)| > \ell\,|\pi(a-b)|.$$

If $\sigma \in (0, \ell\beta R/4]$ is chosen so that

$$\eta \le \omega_n^{-1}\sigma^{-n}\frac{1}{\varepsilon}\int_{C_\sigma^\sigma(\xi)} e^{-z/\varepsilon} d\mu_M,\quad \xi = a \text{ or } b,$$

for some $\eta > 0$, then

$$2\eta \le (1 + c\Lambda R^{1-n/p})(1-\beta)^{-n} R^{-n} \frac{1}{\varepsilon} \int_{C_R(0)} e^{-z/\varepsilon} d\mu_M$$

$$+ c(\ell\beta)^{-(n+1)} R^{-n} \frac{1}{\varepsilon} \int_{C_R(0)} | e_{n+1}^{\mathrm{T}} | e^{-z/\varepsilon} d\mu_M$$

where $c = c(n, p)$.

Proof: We consider the vectorfield $X := h(x,z)\zeta(r)x$ where h is a non-negative, C_c^1 function. Then

$$\mathrm{div}_S X = h(x,z)[r\zeta'(r) | \nabla^S r |^2 + \zeta(r)\mathrm{div}_S x] + \zeta(r)\nabla^S h(x,z) \cdot x$$

for any $(n + 1)$ – dimensional plane S. So, keeping h fixed and applying the proof of Lemma 7.1, we obtain (with $r := | x - \pi(\xi) |$,)

$$\frac{d}{d\rho}[\rho^{-n} \frac{1}{\varepsilon} \int_{C_\rho(\xi)} he^{-z/\varepsilon} d\mu_M] = \rho^{-n} \frac{d}{d\rho} \frac{1}{\varepsilon} \int_{C_\rho(\xi)} h | \nabla^\perp r |^2 e^{-z/\varepsilon} d\mu_M$$

$$+ \rho^{-(n+1)} \frac{1}{\varepsilon} \int_{C_\rho(\xi)} (\mathrm{div}x - n)he^{-z/\varepsilon} d\mu_M$$

$$+ \rho^{-(n+1)} \frac{1}{\varepsilon} \int_{C_\rho(\xi)} (x - \pi(\xi)) \cdot (h\vec{H} + \nabla h)e^{-z/\varepsilon} d\mu_M$$

$$\ge -\rho^{-n} \frac{1}{\varepsilon} \int_{C_\rho(\xi)} (h | \vec{H} | + | \pi(\nabla h) |)e^{-z/\varepsilon} d\mu_M$$

the inequality following from the fact that

$$| \int_{C_\rho(\xi)} (x - \pi(\xi)) \cdot \nabla he^{-z/\varepsilon} d\mu_M | \le \rho \int_{C_\rho(\xi)} | \pi(\nabla h) | e^{-z/\varepsilon} d\mu_M .$$

Now since

$$\frac{1}{\varepsilon} \int_{C_\rho(\xi)} h | \vec{H} | e^{-z/\varepsilon} d\mu_M \le \Lambda(\int_{C_\rho(\xi)} he^{-z/\varepsilon} d\mu_M)^{1-1/p}$$

we have, again as in the proof of Lemma 7.2,

$$\frac{d}{d\rho}[e^{p\Lambda\rho^{1-n/p}/(p-n)} \rho^{-n} \frac{1}{\varepsilon} \int_{C_\rho(\xi)} he^{-z/\varepsilon} d\mu_M]^{1/p} = \ge -e^{p\Lambda\rho^{1-n/p}/(p-n)} \rho^{-n} \frac{1}{\varepsilon} \int_{C_\rho(\xi)} | \pi(\nabla h) | e^{-z/\varepsilon} d\mu_M .$$

Thus, whenever $0 < \tau \le \rho$,

$$\tau^{-n} \frac{1}{\varepsilon} \int_{C_\tau(\xi)} he^{-z/\varepsilon} d\mu_M \le e^{p\Lambda(\rho^{1-n/p} - \tau^{1-n/p})/(p-n)} \rho^{-n} \frac{1}{\varepsilon} \int_{C_\rho(\xi)} he^{-z/\varepsilon} d\mu_M$$

$$+ e^{-p\Lambda\tau^{1-n/p}/(p-n)} \frac{1}{\varepsilon} \int_\tau^\rho \sigma^{-n} e^{p\Lambda\sigma^{1-n/p}/(p-n)} \int_{C_\sigma(\xi)} | \pi(\nabla h) | e^{-z/\varepsilon} d\mu_M d\sigma$$

$$\le e^{p\Lambda(\rho^{1-n/p} - \tau^{1-n/p})/(p-n)} \rho^{-n} \frac{1}{\varepsilon} \int_{C_\rho(\xi)} he^{-z/\varepsilon} d\mu_M$$

$$+ \frac{1}{\varepsilon} \tau^{-n} e^{p\Lambda(\rho^{1-n/p} - \tau^{1-n/p})/(p-n)} \rho \int_{C_\rho(\xi)} | \pi(\nabla h) | e^{-z/\varepsilon} d\mu_M .$$

Suppose $h(x,z) = f(e_{n+1} \cdot (x - \xi))$ where

$$f(t) = 1 \text{ if } | t | \le \frac{\ell\beta R}{8}, f(t) = 0 \text{ if } | t | > \frac{\ell\beta R}{4}, | f'(t) | \le \frac{10}{\ell\beta R} \ \forall t \ge 0$$

where $0 < \ell, \beta < 1, R > 0$ are fixed for the moment. For any $(n + 1)$-dimensional plane S, we have,

$$| \pi(\nabla^S (e_{n+1} \cdot (x - \xi))) | \le | e_{n+1}^S |$$

Define the set $P_\xi = \{(x,z) \in \mathbf{R}^{n+1} \times \mathbf{R} : | e_{n+1} \cdot (x - \xi) | \le \ell\beta R / 4\}$. Then,

$$\tau^{-n} \frac{1}{\varepsilon} \int_{C_\tau(\xi)} e^{-z/\varepsilon} d\mu_M \le e^{p\Lambda(\rho^{1-n/p} - \tau^{1-n/p})/(p-n)} \rho^{-n} \frac{1}{\varepsilon} \int_{C_\rho(\xi) \cap P_\xi} e^{-z/\varepsilon} d\mu_M$$

$$+ \frac{c}{\ell\beta R} e^{p\Lambda(\rho^{1-n/p} - \tau^{1-n/p})/(p-n)} \tau^{-n} \rho \frac{1}{\varepsilon} \int_{C_\rho(\xi)} | e_{n+1}^{\mathrm{T}} | e^{-z/\varepsilon} d\mu_M$$

By Lemma 7.2 (ii) we have, for any $0 < \sigma \le \tau$,

$$\sigma^{-n}\frac{1}{\varepsilon}\int_{C_\sigma(\xi)} e^{-z/\varepsilon}d\mu_M \le e^{p\Lambda\tau^{1-n/p}/(p-n)}[e^{p\Lambda(\rho^{1-n/p}-\tau^{1-n/p})/(p-n)}\rho^{-n}\frac{1}{\varepsilon}\int_{C_\rho(\xi)\cap P_\xi} e^{-z/\varepsilon}d\mu_M$$
$$+\frac{c}{\ell\beta R}e^{p\Lambda(\rho^{1-n/p}-\tau^{1-n/p})/(p-n)}\tau^{-n}\rho\frac{1}{\varepsilon}\int_{C_\rho(\xi)}|e_{n+1}^{\mathrm{T}}|e^{-z/\varepsilon}d\mu_M$$
$$= e^{p\Lambda\rho^{1-n/p}/(p-n)}\rho^{-n}\frac{1}{\varepsilon}\int_{C_\rho(\xi)\cap P_\xi} e^{-z/\varepsilon}d\mu_M$$
$$+\frac{c}{\ell\beta R}e^{p\Lambda\rho^{1-n/p}/(p-n)}\tau^{-n}\rho\frac{1}{\varepsilon}\int_{C_\rho(\xi)}|e_{n+1}^{\mathrm{T}}|e^{-z/\varepsilon}d\mu_M$$
$$\le (1+\frac{2p\Lambda}{p-n}R^{1-n/p})\rho^{-n}\frac{1}{\varepsilon}\int_{C_\rho(\xi)\cap P_\xi} e^{-z/\varepsilon}d\mu_M$$
$$+\frac{c}{\ell\beta R}\tau^{-n}\rho\frac{1}{\varepsilon}\int_{C_\rho(\xi)}|e_{n+1}^{\mathrm{T}}|e^{-z/\varepsilon}d\mu_M$$

if $\Lambda R^{1-n/p} \le 1-n/p$.

Let $\tau = \ell\beta R/4$ and $\rho = (1-\beta)R$. Then we obtain

$$\sigma^{-n}\frac{1}{\varepsilon}\int_{C_\sigma(\xi)} e^{-z/\varepsilon}d\mu_M \le (1+\frac{2p\Lambda}{p-n}R^{1-n/p})(1-\beta)^{-n}R^{-n}\frac{1}{\varepsilon}\int_{C_{(1-\beta)R}(\xi)\cap P_\xi} e^{-z/\varepsilon}d\mu_M$$
$$+c(\ell\beta)^{-(n+1)}R^{-n}\frac{1}{\varepsilon}\int_{C_{(1-\beta)R}(\xi)}|e_{n+1}^{\mathrm{T}}|e^{-z/\varepsilon}d\mu_M \tag{7.4}$$

Let $a,b\in C_{\beta R}(0)$ satisfy $|\pi(a-b)|\ge \beta R/2$ and $|e_{n+1}\cdot(a-b)|>\ell|\pi(a-b)|$. Then

$$C_{\ell\beta R/2}(a)\cap C_{\ell\beta R/2}(b)=\varnothing$$

and

$$C_{(1-\beta)R}(a)\cup C_{(1-\beta)R}(b)\subset C_R(0).$$

Note that $|e_{n+1}\cdot(a-b)|-\ell\beta R/2>0$ and $\mathrm{dist}(P_a,P_b)\ge|e_{n+1}\cdot(a-b)|-\ell\beta R/2>0$. Hence

$$(C_{(1-\beta)R}(a)\cap P_a)\cap(C_{(1-\beta)R}(b)\cap P_b)=\varnothing.$$

Therefore, combining the expressions for (7.4) when ξ is a and b, we find

$$\sigma^{-n}\frac{1}{\varepsilon}\int_{C_\sigma(a)} e^{-z/\varepsilon}d\mu_M + \sigma^{-n}\frac{1}{\varepsilon}\int_{C_\sigma(b)} e^{-z/\varepsilon}d\mu_M$$
$$\le (1+\frac{2p\Lambda}{p-n}R^{1-n/p})(1-\beta)^{-n}R^{-n}\frac{1}{\varepsilon}\int_{C_R(0)} e^{-z/\varepsilon}d\mu_M + c(\ell\beta)^{-(n+1)}R^{-n}\frac{1}{\varepsilon}\int_{C_R(0)}|e_{n+1}^{\mathrm{T}}|e^{-z/\varepsilon}d\mu_M$$

whenever $a,b\in C_{\beta R}(0), |\pi(a-b)|\ge \beta R/2, |e_{n+1}\cdot(a-b)|>\ell|\pi(a-b)|$ and $\sigma\le\ell\beta R/4$, as required. ■

Before we present the height lemma, we need a few definitions first.

Definition ((η,α) - unstacked about a): *Fix $R>0$ and $\ell\in(0,1)$. Let $M\in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+1}\times\mathbf{R})$ and let $a\in\mathbf{R}^{n+1}$. Suppose that we can find an $\eta>0$ and $\alpha\in(0,1)$ that*

$$\omega_n^{-1}R^{-n}\frac{1}{\varepsilon}\int_{C_R(a)} e^{-z/\varepsilon}d\mu_M \le 2\eta(1-\alpha)$$

Then we say that M is (η,α) - unstacked about a.

This definition just says that, for sufficiently small $\eta>0$, M doesn't have more than one layer over a vertical plane anywhere inside $C_R(a)$ (see figure 5). On the other hand, M can stack up in multiple layers over $\mathbf{R}^{n+1}\times\{0\}$. This corresponds to the resulting Brakke flow (after elliptic regularisation) moving through a given region several times.

Bearing in mind the fact that our cylindrical monotonicity formula is expressed inside cylinders, we define the F^ε analogue of the tilt excess (see, for example, [S 22.1] and [HS 1.4]). In this case, because the F^ε-minimisers will become very tall, we need to take special care of the vertical direction. So, we consider $(n+1)$-dimensional subspaces containing ω.

Definition (ε-Tilt-Excess over $\mathbf{C}_\rho(a)$ relative to S): *Let $M\in\mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+1}\times\mathbf{R})$, $\rho>0$, and let S be an $(n+1)$-dimensional plane containing ω. Define the ε-Tilt-Excess over $C_\rho(a)$ relative to S by*

$$E^\varepsilon(a,\rho,S):=\rho^{-n}\frac{1}{\varepsilon}\int_{C_\rho(a)}|\nu_S^{\mathrm{T}}|^2 e^{-z/\varepsilon}d\mu_M$$

where ν_S is a unit normal to S. We will also write $E=E^\varepsilon(0,R,\{x\cdot e_{n+1}=0\})$ for simplicity.

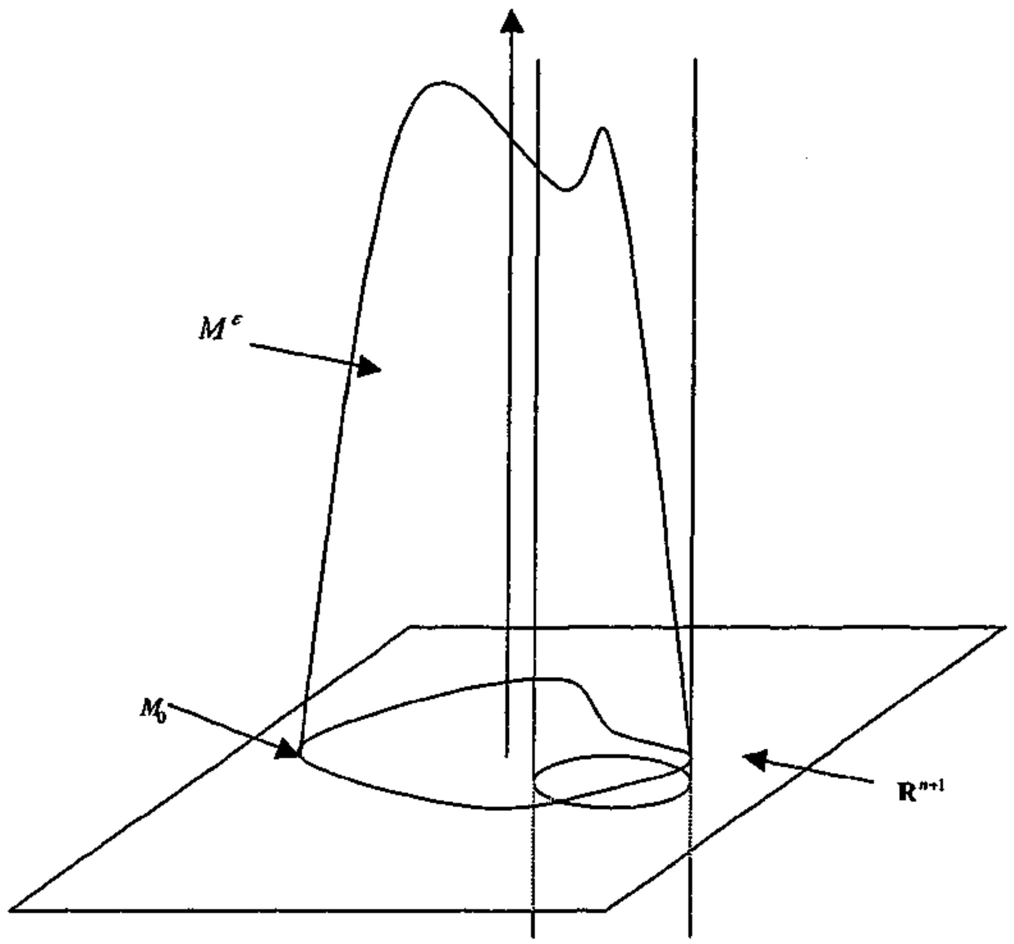

Figure 5.

Hardt and Simon have used a cylidrical tilt-excess to study boundary regularity for the oriented Plateau problem. There the boundary of an absolutely area-minimising locally rectifiable current is assumed to be $C^{1,\alpha}$. The cylindrical tilt-excess is used to control the excess of the surface in a neighborhood of the boundary. One should expect a similar result to hold for the minimisers of F^{ε}.

We now prove an interesting height estimate for (η,α) - unstacked G^{ε}-minimisers M^{ε}.

Lemma 7.5: *Let $R > 0$, $\ell \in (0,1)$, $a \in \mathbf{R}^{n+1}$ and let S be an n-dimensional plane containing ω. Suppose M^{ε} is (η,α)- unstacked about a for some $\eta > 0$ and $\alpha \in (0,1)$. There exists a $\beta \in (0,1)$ such that if we can find a $\sigma \in (0, \ell\beta R/4]$ such that*

$$\omega_n^{-1}\sigma^{-n}\frac{1}{\varepsilon}\int_{C_\sigma^\sigma(\xi)} e^{-z/\varepsilon}\,d\mu_{M^\varepsilon} \geq \eta$$

for all $\xi \in C_{\beta R}^R(a) \cap \operatorname{spt} M^{\varepsilon}$ then

$$|\nu_S \cdot (a-\xi)| \leq \tilde{c}E(a,R,S)^{1/(2n+2)}R,$$

for all $\xi \in C_{\beta R}^R(a) \cap \operatorname{spt} M^{\varepsilon}$, where $\tilde{c} = \tilde{c}(n,\alpha)$.

Proof: We assume wlog that $a = 0$ and $S = \{x \cdot e_{n+1} = 0\}$. Let $\beta \in (0,1)$. To begin with we assume that, for some $\delta > 0$, $E < \delta^2$ and write $\ell = (\delta^{-2}E)^{1/(2n+2)} < 1$. Take $a,b \in \operatorname{spt} M^{\varepsilon} \cap C_{\beta R}^R(0)$. Then, by Lemma 4.4,

$$\begin{aligned}
2\eta &\leq (1+c\Lambda R^{1-n/p})(1-\beta)^{-n}R^{-n}\frac{1}{\varepsilon}\int_{C_R(0)} e^{-z/\varepsilon}\,d\mu_{M^\varepsilon} + c(\ell\beta)^{-(n+1)}R^{-n}\frac{1}{\varepsilon}\int_{C_R(0)} |e_{n+1}^{\mathsf{T}}|\,e^{-z/\varepsilon}\,d\mu_{M^\varepsilon} \\
&\leq 2(1+c\Lambda R^{1-n/p})\eta(1-\alpha)(1-\beta)^{-n} + c\eta^{1/2}(1-\alpha)^{1/2}(\ell\beta)^{-(n+1)}E^{1/2} \\
&= 2(1+c\Lambda R^{1-n/p})\eta(1-\alpha)(1-\beta)^{-n} + c\eta^{1/2}(1-\alpha)^{1/2}\delta^{1/2}\beta^{-(n+1)}
\end{aligned}$$

Choosing δ, β appropriately, we obtain a contradiction. Thus,

$$|e_{n+1} \cdot (a-b)| \leq \ell\beta R/2 = \tilde{c}E^{1/(2n+2)}R \tag{7.5}$$

$\forall a,b \in C_{\beta R}^R(0) \cap \operatorname{spt} M^{\varepsilon}$ satisfying $|\pi(a-b)| \geq \beta R/2$.

In particular, choosing $b = 0$, we have,

$$|e_{n+1} \cdot a| \leq \tilde{c}E^{1/(2n+2)}R$$

$\forall a \in (C_{\beta R}^R(0) - C_{\beta R/2}(0)) \cap \operatorname{spt} M^{\varepsilon}$. Bootstrapping this gives

$$|e_{n+1} \cdot a| \leq \tilde{c}E^{1/(2n+2)}R$$

$\forall a \in (C_{\beta R}^R(0) - C_{\sigma}(0)) \cap \operatorname{spt} M^{\varepsilon}$.

Suppose $a \in C_{\sigma}^R(0) \cap \operatorname{spt} M^{\varepsilon}$. Then we trivially have

$$|e_{n+1} \cdot a| \leq \ell\beta R/4 \leq \tilde{c}E^{1/(2n+2)}R$$

where $\tilde{c}$ is the constant from (7.5). Hence the lemma is true whenever $E < \delta^2$.

On the other hand, if $\delta^2 \leq E$, then the inequality is trivial with $\tilde{c}$ chosen sufficiently large. ■

8 Local mass bounds

Here we derive local mass bounds for the minimisers of G^ε. We begin by motivating the need for such bounds by showing that the minimisers are a translating soliton solution to a forced Brakke flow approximating the flow given by

$$\overline{D}_t\overline{\mu}_t(\varphi) \le \int(-\varphi H^2 + \overline{\nabla}^\perp\varphi\cdot\vec{H})\, d\overline{\mu}_t + \int g(x,t)\,|-\varphi\vec{H} + \overline{\nabla}^\perp\varphi\,|\, d\overline{\mu}_t$$

with appropriate singular conditions. We then use Lemma 6.1 to obtain the local mass bounds.

Note that the minimisers M^ε of G^ε satisfy an approximate forced Brakke flow. Indeed, define the mapping $\sigma_{-t/\varepsilon}(x,z) := (x, z - t/\varepsilon)$ and consider the currents given by

$$P^\varepsilon(t) := (\sigma_{-t/\varepsilon})_\#(M^\varepsilon).$$

For any $t \ge 0$, $P^\varepsilon(t)$ is just the current M^ε translated vertically down by t/ε. Fix $t \ge 0$ and let $\overline{\varphi}$ be a test-function on $\mathbf{R}^{n+2}$ with

$$\mathrm{spt}\overline{\varphi}\times\{t\} \subset \{(x,z,s) : z > -\frac{s}{\varepsilon},\ s \ge t\}.$$

Then, since $(\mathrm{spt}(\partial P^\varepsilon(t))\times\{t\}) \cap \{(x,z,s) : z > -s/\varepsilon, s \ge t\} = \varnothing$, we have

$$\begin{aligned}
\frac{d}{dt}\mu_{P^\varepsilon(t)}(\overline{\varphi}) &= \frac{d}{dt}\int \overline{\varphi}\, d(\sigma_{-t/\varepsilon})_\#(\mu_{M^\varepsilon}) \\
&= \int(-\overline{\varphi}\vec{H} + \overline{\nabla}^\perp\overline{\varphi})\cdot(-\frac{1}{\varepsilon}\omega)\, d\mu_{P^\varepsilon(t)} \\
&= \int(-\overline{\varphi}\vec{H} + \overline{\nabla}^\perp\overline{\varphi})\cdot(\vec{H} + g(x, \varepsilon z + t)\nu)\, d\mu_{P^\varepsilon(t)} \\
&\le \int(-\overline{\varphi}H^2 + \overline{\nabla}^\perp\overline{\varphi}\cdot\vec{H} + g(x,\varepsilon z + t)\,|-\overline{\varphi}\vec{H} + \overline{\nabla}^\perp\overline{\varphi}\,|)\, d\mu_{P^\varepsilon(t)}.
\end{aligned} \tag{8.1}$$

This and the condition that $\mathrm{spt}\overline{\varphi}\times\{t\} \subset \{(x,z,s) : z > -s/\varepsilon,\ s \ge t\}$ says that $\{\mu_{P^\varepsilon(t)}\}_{t\ge0}$ is a forced Brakke flow on the set $W^\varepsilon := \{(x,z,t) : z > -t/\varepsilon,\ t \ge 0\}$.

Remark: *If we were considering K-almost Brakke flows (see example (ii) in §4 and, in particular, the remark preceeding Lemma 6.1), then instead of (8.1) we would have*

$$\frac{d}{dt}\mu_{P^\varepsilon(t)}(\overline{\varphi}) \le \int(-\overline{\varphi}H^2 + \overline{\nabla}^\perp\overline{\varphi}\cdot\vec{H} + K\,|-\overline{\varphi}\vec{H} + \overline{\nabla}^\perp\overline{\varphi}\,|)\, d\mu_{P^\varepsilon(t)}.$$

Therefore, the family $\{\mu_{P^\varepsilon(t)}\}_{t\ge0}$ *would be a K-almost Brakke flow, simplifying the proof considerably.*

As in §3, we aim to pass to the limit $\varepsilon \downarrow 0$. To do this, we must show that the $P^\varepsilon(t)$ have locally bounded mass. Then the compactness theorem and the "time-like" continuity of g will show that the limit $\mu_{P^\varepsilon(t)} \to \overline{\mu}_t$ exists for each $t \ge 0$ and satisfies

$$\overline{D}_t\overline{\mu}_t(\varphi) \le \int(-\overline{\varphi}H^2 + \overline{\nabla}^\perp\overline{\varphi}\cdot\vec{H})\, d\overline{\mu}_t + \int g(x,t)\,|-\overline{\varphi}\vec{H} + \overline{\nabla}^\perp\overline{\varphi}\,|\, d\overline{\mu}_t$$

(with the appropriate singular conditions given in §4). The family $\{\overline{\mu}_t\}_{t\ge0}$ will then be used to define a forced integer Brakke flow with initial data $\mu_0 = \mu_{M_0}$.

Lemma 8.1: *Let ξ be Lipschitz with* $\mathrm{spt}\xi \subset\subset [0,\infty)$. *Then*

$$\int(\frac{d\xi}{dz}|\omega^\mathrm{T}|^2 - \frac{1}{\varepsilon}\xi)e^{-z/\varepsilon}d\mu_{M^\varepsilon} = \int g^\varepsilon\xi\nu\cdot\omega e^{-z/\varepsilon}d\mu_{M^\varepsilon}.$$

Proof: Suppose the lemma is true for any $f \in C_c^1([0,\infty))$. Approximate ξ by $C_c^1([0,\infty))$ functions f_i satisfying

$$f_i \le \xi,\ f_i \to \xi \text{ uniformly}, \frac{df_i}{dz} \to \frac{d\xi}{dz} \text{ weakly} - * \text{in } L^\infty.$$

Then terms linear in f_i converge and, by Lemma 1.3,

$$\int\frac{d\xi}{dz}|\omega^\mathrm{T}|^2 e^{-z/\varepsilon}d\mu_{M^\varepsilon} = \int_0^\infty\frac{d\xi}{dz}\int|\omega^\mathrm{T}|e^{-z/\varepsilon}d\mu_{M_z^\varepsilon}dz,$$

so terms linear in df_i/dz converge since $\int|\omega^\mathrm{T}|e^{-z/\varepsilon}d\mu_{M_z^\varepsilon}$ is an L^1 function of z. Therefore, the lemma holds for Lipschitz ξ if it is true for $f \in C_c^1([0,\infty))$.

We assume $f \in C_c^1([0,\infty))$ and wlog $0 \in \mathrm{spt}M^\varepsilon$. Choose $\sigma_R \in C_c^1(\mathbf{R}^{n+1},[0,1])$ satisfying

$$\sigma_R = 1 \text{ in } B_R(0),\ \sigma_R = 0 \text{ off } B_{2R}(0),\ |D\sigma_R| \le \frac{2}{R}.$$

Let $X(x,z) = \sigma_R(x)f(z)\omega$. Then, by Lemma 6.1, we have

$$\begin{aligned}
0 &= \int(\mathrm{div}X - \frac{1}{\varepsilon}\omega\cdot X - \tilde{g}\nu\cdot X)e^{-z/\varepsilon}d\mu_{M^\varepsilon} \\
&= \int(f\nabla\sigma_R\cdot\omega + \sigma_R(\frac{df}{dz}|\omega^\mathrm{T}|^2 - \frac{1}{\varepsilon}f - \tilde{g}f\nu\cdot\omega)e^{-z/\varepsilon}d\mu_{M^\varepsilon}.
\end{aligned}$$

Note that

$$|\int f\nabla\sigma_R\cdot\omega e^{-z/\varepsilon}d\mu_{M^\varepsilon}|\leq\frac{2\varepsilon}{R}\sup|f|F^\varepsilon(M^\varepsilon)<\infty,$$

and

$$\sigma_R|\frac{df}{dz}|\omega^\top|^2-\frac{1}{\varepsilon}f-g^\varepsilon f\nu\cdot\omega|e^{-z/\varepsilon}\leq\frac{1}{\varepsilon}(\sup|\frac{df}{dz}|+\sup|f|(1+\varepsilon|g^\varepsilon|))e^{-z/\varepsilon}$$

which is a fixed L^1 function (by assumption on g^ε). Thus we may send $R\to\infty$ to obtain,

$$0=\int(\frac{df}{dz}|\omega^\top|^2-\frac{1}{\varepsilon}f-g^\varepsilon f\nu\cdot\omega)e^{-z/\varepsilon}d\mu_{M^\varepsilon},$$

by dominated convergence. Hence the lemma is true whenever $f\in C_c^1([0,\infty))$, as required. ■

Corollary 8.2: *Let ξ be Lipschitz with $\operatorname{spt}\xi\subset\subset[0,\infty)$. Then*

$$\begin{aligned}\int g^\varepsilon\xi\nu\cdot\omega d\mu_{M^\varepsilon}&=\int(\frac{d\xi}{dz}|\omega^\top|^2-\frac{1}{\varepsilon}\xi|\omega^\perp|^2)d\mu_{M^\varepsilon}\\&=\int(\frac{d\xi}{dz}|\omega^\top|^2-\varepsilon\xi|\vec{H}+g^\varepsilon\nu|^2)d\mu_{M^\varepsilon}.\end{aligned}$$

Proof: The first equality follows by replacing ξ by $\xi e^{-z/\varepsilon}$ in Lemma 8.1. The second is an application of Lemma 6.1 (i). ■

We now apply Lemma 8.1 and Corollary 8.2 to two choices of ξ. These applications will give us an estimate analogous to the formula

$$\mathcal{H}^n(\Gamma_t)+\int_0^t\int_{\Gamma_s}H_{\Gamma_s}(H_{\Gamma_s}+g)\,d\mathcal{H}^n ds=\mathcal{H}^n(\Gamma_0)$$

which is valid for a smooth forced mean curvature flow $\mathcal{M}=\{\Gamma_t\}_{t\geq0}$.

Lemma 8.3: *For any $\delta>0$, we have*

$$\frac{1}{\delta}\int_a^{a+\delta}e^{-z/\varepsilon}\int|\omega^\top|d\mu_{M_s^\varepsilon}ds\leq F^\varepsilon(M^\varepsilon\llcorner(\mathbf{R}^{n+1}\times(a,\infty))+\int g^\varepsilon|\omega^\perp|e^{-z/\varepsilon}d\mu_{M^\varepsilon\llcorner(\mathbf{R}^{n+1}\times(a,\infty))}$$

and in particular

$$\lim_{\delta\to0}\frac{1}{\delta}\int_a^{a+\delta}e^{-z/\varepsilon}\int|\omega^\top|d\mu_{M_s^\varepsilon}ds\leq F^\varepsilon(M^\varepsilon\llcorner(\mathbf{R}^{n+1}\times(a,\infty))+\int g^\varepsilon|\omega^\perp|e^{-z/\varepsilon}d\mu_{M^\varepsilon\llcorner(\mathbf{R}^{n+1}\times(a,\infty))}$$

Proof: Let $L>0$ and define ξ by

$$\xi(z)=\begin{cases}0 & 0\leq z<a\\1 & z=a+\delta\\0 & a+\delta+L<z<\infty\end{cases}$$

and linearly interpolated between. By Lemma 8.1, we compute

$$\begin{aligned}\frac{1}{\delta}\int_a^{a+\delta}e^{-z/\varepsilon}\int|\omega^\top|d\mu_{M_s^\varepsilon}ds&=\frac{1}{L}\int_{a+\delta}^{a+\delta+L}e^{-z/\varepsilon}\int|\omega^\top|d\mu_{M_s^\varepsilon}ds+\int\xi(\frac{1}{\varepsilon}+g^\varepsilon\nu\cdot\omega)e^{-z/\varepsilon}d\mu_{M^\varepsilon}\\&\leq(1+\frac{\varepsilon}{L})F^\varepsilon(M^\varepsilon\llcorner(\mathbf{R}^{n+1}\times(a,\infty))+\int g^\varepsilon|\omega^\perp|e^{-z/\varepsilon}d\mu_{M^\varepsilon\llcorner(\mathbf{R}^{n+1}\times(a,\infty))}.\end{aligned}$$

Since, by Lemma 6.2, $F^\varepsilon(M^\varepsilon)<\infty$, it follows that $(1+\varepsilon/L)F^\varepsilon(M^\varepsilon)\to F^\varepsilon(M^\varepsilon)$ when $L\to\infty$. Therefore

$$\frac{1}{\delta}\int_a^{a+\delta}e^{-z/\varepsilon}\int|\omega^\top|d\mu_{M_s^\varepsilon}ds\leq F^\varepsilon(M^\varepsilon\llcorner(\mathbf{R}^{n+1}\times(a,\infty))+\int g^\varepsilon|\omega^\perp|e^{-z/\varepsilon}d\mu_{M^\varepsilon\llcorner(\mathbf{R}^{n+1}\times(a,\infty))}$$

and, since the right hand side is finite, we have in particular

$$\lim_{\delta\to0}\frac{1}{\delta}\int_a^{a+\delta}e^{-z/\varepsilon}\int|\omega^\top|d\mu_{M_s^\varepsilon}ds\leq F^\varepsilon(M^\varepsilon\llcorner(\mathbf{R}^{n+1}\times(a,\infty))+\int g^\varepsilon|\omega^\perp|e^{-z/\varepsilon}d\mu_{M^\varepsilon\llcorner(\mathbf{R}^{n+1}\times(a,\infty))}$$

completing the proof. ■

Now consider the Lipschitz function ξ given by

$$\xi(z)=\begin{cases}0 & 0\leq z<a\\1 & a+\delta<z<b\\0 & b+\delta<z<\infty\end{cases}$$

and linearly interpolated between. Applying Lemma 1.3 and Corollary 8.2 we find

$$\begin{aligned}&\varepsilon\int\xi|\vec{H}+g^\varepsilon\nu|^2\,d\mu_{M^\varepsilon}+\int\xi g^\varepsilon\omega\cdot\nu\,d\mu_{M^\varepsilon}\\&\quad=\frac{1}{\delta}\int|\omega^\top|^2\,d\mu_{M^\varepsilon\llcorner(\mathbf{R}^{n+1}\times(a,a+\delta))}-\frac{1}{\delta}\int|\omega^\top|^2\,d\mu_{M^\varepsilon\llcorner(\mathbf{R}^{n+1}\times(b,b+\delta))}\\&\quad=\frac{1}{\delta}\int_a^{a+\delta}\int|\omega^\top|d\mu_{M_s^\varepsilon}ds-\frac{1}{\delta}\int_b^{b+\delta}\int|\omega^\top|d\mu_{M_s^\varepsilon}ds.\end{aligned}\tag{8.2}$$

We will write

$$f_\delta(z):=\frac{1}{\delta}\int_z^{z+\delta}\int|\omega^\top|d\mu_{M_s^\varepsilon}ds.$$

Lemma 8.4: *For any measurable set A*

$$\int |\omega^{\mathrm{T}}|^2 \, d\mu_{M^\varepsilon \llcorner (\mathbf{R}^{n+1}\times A)} \le \Big(F^\varepsilon(M^\varepsilon) + \int g^\varepsilon |\omega^\perp| e^{-z/\varepsilon} d\mu_{M^\varepsilon} + \int g^\varepsilon \omega\cdot \nu d\mu_{M^\varepsilon \llcorner (\mathbf{R}^{n+1}\times(0,\sup A))}\Big)\mathcal{L}^1(A)$$

in particular $f_\delta(0) \le F^\varepsilon(M^\varepsilon) + \int g^\varepsilon |\omega^\perp| e^{-z/\varepsilon} d\mu_{M^\varepsilon} + \int g^\varepsilon \omega\cdot\nu d\mu_{M^\varepsilon \llcorner (\mathbf{R}^{n+1}\times(0,\delta))}$.

Proof: By Lemma 1.3 and (8.2) we have that, for any sufficiently small open interval $(a, a+\delta)$,

$$\int |\omega^{\mathrm{T}}|^2 \, d\mu_{M^\varepsilon \llcorner (\mathbf{R}^{n+1}\times(a,a+\delta))} - \delta \int \xi g^\varepsilon \omega\cdot\nu d\mu_{M^\varepsilon} \le \int |\omega^{\mathrm{T}}|^2 \, d\mu_{M^\varepsilon \llcorner (\mathbf{R}^{n+1}\times(0,\delta))}$$

where ξ is the function from (8.2) with (a,b) replaced by $(0,a)$. Therefore

$$\int |\omega^{\mathrm{T}}|^2 \, d\mu_{M^\varepsilon \llcorner (\mathbf{R}^{n+1}\times(a,a+\delta))} \le \delta\Big[F^\varepsilon(M^\varepsilon) + \int g^\varepsilon |\omega^\perp| e^{-z/\varepsilon} d\mu_{M^\varepsilon} + \int g^\varepsilon \omega\cdot\nu d\mu_{M^\varepsilon \llcorner (\mathbf{R}^{n+1}\times(0,\sup A))}\Big] \tag{8.3}$$

by Lemma 8.3.

If *A* were open, then we could decompose it into many small intervals and use (8.3) to obtain

$$\int |\omega^{\mathrm{T}}|^2 \, d\mu_{M^\varepsilon \llcorner (\mathbf{R}^{n+1}\times A)} \le \Big(F^\varepsilon(M^\varepsilon) + \int g^\varepsilon |\omega^\perp| e^{-z/\varepsilon} d\mu_{M^\varepsilon} + \int g^\varepsilon \omega\cdot\nu d\mu_{M^\varepsilon \llcorner (\mathbf{R}^{n+1}\times(0,\sup A))}\Big)\mathcal{L}^1(A)$$

Therefore it is true for any measurable set *A*. The estimate for $f_\delta(0)$ is obtained by setting $A = (0,\delta)$. ■

Therefore, by combining Lemmas 8.3 and 8.4 with (8.2) we find:

Lemma 8.5 (area estimate): *There is a full measure set Z such that, for any* $a, b \in Z$, $a \le b$,

$$\int |\omega^{\mathrm{T}}| d\mu_{M_b^\varepsilon} + \varepsilon \int |\vec{H} + g^\varepsilon \nu|^2 \, d\mu_{M^\varepsilon \llcorner (\mathbf{R}^{n+1}\times(a,b))} = \int |\omega^{\mathrm{T}}| d\mu_{M_a^\varepsilon} - \int g^\varepsilon \omega\cdot\nu d\mu_{M^\varepsilon \llcorner (\mathbf{R}^{n+1}\times(a,b))}$$
$$\le F^\varepsilon(M^\varepsilon) + \int g^\varepsilon |\omega^\perp| (1 + e^{-z/\varepsilon}) d\mu_{M^\varepsilon}$$

Proof: By (8.2) we have

$$f_\delta(z) \le f_\delta(0) + \int g^\varepsilon \omega\cdot\nu d\mu_{M^\varepsilon \llcorner (\mathbf{R}^{n+1}\times(0,z+\delta))}$$
$$= \frac{1}{\delta}\int_0^\delta \int |\omega^{\mathrm{T}}| d\mu_{M_s^\varepsilon} ds + \int g^\varepsilon \omega\cdot\nu d\mu_{M^\varepsilon \llcorner (\mathbf{R}^{n+1}\times(0,z+\delta))}$$
$$\le c < \infty$$

where *c* is independent of δ.

By the Lebesgue differentiation theorem we can find a full measure set *Z* such that

$$\lim_{\delta\to 0} f_\delta(z) = \int |\omega^{\mathrm{T}}| d\mu_{M_z^\varepsilon},$$

for any $z \in Z$. Therefore, by Lemmas 8.3 and 8.4 we have

$$\int |\omega^{\mathrm{T}}| d\mu_{M_b^\varepsilon} + \varepsilon \int |\vec{H} + g^\varepsilon \nu|^2 \, d\mu_{M^\varepsilon \llcorner (\mathbf{R}^{n+1}\times(a,b))} = \int |\omega^{\mathrm{T}}| d\mu_{M_a^\varepsilon} - \int g^\varepsilon \omega\cdot\nu d\mu_{M^\varepsilon \llcorner (\mathbf{R}^{n+1}\times(a,b))}$$
$$\le F^\varepsilon(M^\varepsilon) + \int g^\varepsilon |\omega^\perp| e^{-z/\varepsilon} d\mu_{M^\varepsilon} + \int g^\varepsilon \omega\cdot\nu d\mu_{M^\varepsilon}$$
$$\le F^\varepsilon(M^\varepsilon) + \int g^\varepsilon |\omega^\perp| (1 + e^{-z/\varepsilon}) d\mu_{M^\varepsilon}$$

as desired. ■

Before proceeding we present a technical lemma:

Lemma 8.6: *For any* $p \ge 1$, *if* $\sup_{t\ge 0} \| g(\cdot,t)\|_{L^p(U)} < \infty$ *then there is a constant* $C = C(p,g)$ *such that*

$$J_p^\varepsilon(g) := \int (g^\varepsilon)^p d\mu_{M^\varepsilon} \le C < \infty.$$

Remark: *If the functional* G^ε *is solved in the setting of Cacciopoli sets, then the constant C is* $\sup_{t\ge 0} \| g(\cdot,t)\|_{L^p(U)}$.

Proof: We may assume wlog that $g \not\equiv 0$ *a.e.* Suppose the lemma is false. Then, we may find a sequence $\varepsilon_i \downarrow 0$ such that

$$J_p^{\varepsilon_i}(g) \ge i.$$

Let $N^\varepsilon := \pi_\#(M^\varepsilon \llcorner (\mathbf{R}^{n+1} \times (0, \tau/\varepsilon)))$ where $\pi(x,z) = x$. By the "time-like" continuity assumption we have

$$\lim_{i\to\infty} J_p^{\varepsilon_i}(g) = \lim_{i\to\infty} \int (g(x,0))^p d\mu_{N^{\varepsilon_i}}(x).$$

By passing to a subsequence we may consider the following two cases:

Case 1: Suppose $N^{\varepsilon_i} \in \mathbf{I}^{n+1}_{loc}(\mathbf{R}^{n+1}\times\{0\})$ for all $i \geq 1$. Then

$$\| g(\cdot,t)\|_{L^p(\mu_{N^{\varepsilon_i}})} \leq \text{ess sup}\,\theta_{M^{\varepsilon_i}\llcorner(\mathbf{R}^{n+1}\times(0,\tau/\varepsilon_i))} \| g(\cdot,t)\|_{L^p(U)} \leq c < \infty .$$

This is independent of i otherwise $F^{\varepsilon_i}(M^{\varepsilon_i}) > \mathbf{M}(M_0) + \sup_{t\geq 0} \| g(\cdot,t)\|_{L^1(U)}$ for some i. Therefore, for sufficiently large $i \geq 1$,

$$i \leq J^{\varepsilon_i}_p(g) \leq c$$

a contradiction.

Case 2: Suppose $N^{\varepsilon_i} \in \mathbf{I}^{n}_{loc}(\mathbf{R}^{n+1}\times\{0\})$ for all $i \geq 1$. Then $\omega^{\perp} = 0$ *a.e.* in $\text{spt}M^{\varepsilon_i}$. Therefore, by Lemma 6.1 it follows that g^{ε_i} is a constant *a.e.*, violating the "time-like" continuity assumption and $g \not\equiv 0$. ∎

By combining Lemmas 6.1, 8.4, and 8.5 we have

Lemma 8.7: *For any measurable set $A \subset \mathbf{R}$, there is a constant $c_g = c_g(g)$ such that*

$$\mathbf{M}(M^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times A)) \leq (\mathcal{L}^1(A)+\varepsilon)(\mathbf{M}(M_0)+c_g),$$

Proof: By Lemma 6.1 (iii) we have

$$\mathbf{M}(M^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times A)) \leq \varepsilon^2 \int |\vec{H} + g^{\varepsilon}\nu|^2 d\mu_{M^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times A)} + \int |\omega^{\mathrm{T}}|^2 \, d\mu_{M^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times A)} .$$

By Lemma 8.5 we have

$$\varepsilon^2 \int |\vec{H} + g^{\varepsilon}\nu|^2 \, d\mu_{M^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times A)} \leq \varepsilon (F^{\varepsilon}(M^{\varepsilon}) + \int g^{\varepsilon}(e^{-z/\varepsilon}+1)|\omega^{\perp}|\, d\mu_{M^{\varepsilon}})$$

and by Lemma 8.4 we have

$$\int |\omega^{\mathrm{T}}|^2 \, d\mu_{M^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times A)} \leq (F^{\varepsilon}(M^{\varepsilon}) + \int g^{\varepsilon}(e^{-z/\varepsilon}+1)|\omega^{\perp}| d\mu_{M^{\varepsilon}})\mathcal{L}^1(A).$$

Combining these we have

$$\begin{aligned}\mathbf{M}(M^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times A)) &\leq (\mathcal{L}^1(A)+\varepsilon)(F^{\varepsilon}(M^{\varepsilon}) + \int g^{\varepsilon}|\omega^{\perp}|(1+e^{-z/\varepsilon})d\mu_{M^{\varepsilon}})\\ &\leq (\mathcal{L}^1(A)+\varepsilon)(F^{\varepsilon}(M^{\varepsilon})+c(g))) \qquad (8.4)\\ &\leq (\mathcal{L}^1(A)+\varepsilon)(\mathbf{M}(M_0)+\sup_{t\geq 0}\| g(\cdot,t\|_{L^1(U)} + c(g))\end{aligned}$$

by Lemmas 6.2 and 8.6. ∎

Remark: *As noted in the proof of Lemma 6.2, one could use the Hölder inequality to obtain*

$$\begin{aligned}&\mathbf{M}(M^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times A))\\ &\quad\leq (\mathcal{L}^1(A)+\varepsilon)(F^{\varepsilon}(M^{\varepsilon}) + (J^{\varepsilon}_p(g))^{1/p}(F^{\varepsilon}(M^{\varepsilon})^{1-1/p} + \mathbf{M}(M^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times A))^{1-1/p})).\end{aligned}$$

If $p \geq 1$, this leads to an equation of the form $x - rbx^{\delta} \leq rc$ where $\delta < 1$ and $r = \mathcal{L}^1(A)+\varepsilon$. Hence we can find a constant $C = C(\mathcal{L}^1(A), n, p, g, \mathbf{M}(M_0))$ such that

$$\mathbf{M}(M^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times A)) \leq C.$$

Now, consider the equation $x - rbx^{\delta} \leq rc$ where $\delta \leq 1$. Dividing through by $\sigma \leq 1$, gives

$$\begin{aligned}(r/\sigma)c &\geq (x/\sigma) - (r/\sigma)bx^{\delta}\\ &\geq (x/\sigma) - (r/\sigma)b(x/\sigma)^{\delta}.\end{aligned}$$

Therefore, the constant C satisfies

$$\sigma C(\mathcal{L}^1(A)/\sigma, n, p, g, \mathbf{M}(M_0)) \leq C(\mathcal{L}^1(A), n, p, g, \mathbf{M}(M_0))$$

for any $\sigma \leq 1$.

The following lemma will be used to obtain an undercurrent (cf. §3):

Lemma 8.8: *Let $\kappa_{\varepsilon}(x,z) = (x, \varepsilon z)$. Define $T^{\varepsilon} := (\kappa_{\varepsilon})_{\#}(M^{\varepsilon})$. Then, for any measurable set $B \subset \mathbf{R}$,*

$$\mathbf{M}(T^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times B)) \leq ((\mathcal{L}^1(B)+\varepsilon^2) + (\mathcal{L}^1(B)+\varepsilon^2)^{1/2})(\mathbf{M}(M_0)+c_g).$$

Remark: *If one were to use the same method outlined in the remark after the proof of Lemma 8.7, then one would obtain an estimate of the form*

$$\mathbf{M}(T^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times B)) \leq C$$

where $C = C(\mathcal{L}^1(B), n, p, g, \mathbf{M}(M_0))$. Furthermore, the constant C will vanish with $\mathcal{L}^1(B)$.

Incidentally, prior to a very productive discussion with Maria Athanassenas, these were the original estimates I had for Lemmas 8.7 and 8.8.

Proof of Lemma 8.8: Firstly,

$$\begin{aligned}\mathbf{M}(T^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times B)) &= \int J\kappa_{\varepsilon}d\mu_{M^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times B/\varepsilon)} \\ &= \int\sqrt{|J\pi|^2+\varepsilon^2|\omega^{\mathrm{T}}|^2}\,d\mu_{M^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times B/\varepsilon)} \\ &\le \int(J\pi+\varepsilon)d\mu_{M^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times B/\varepsilon)}\end{aligned}$$

where $\pi(x,z)=x$. Note that

$$1=|J(\mathrm{id})|^2=|J\pi|^2+|\omega^{\mathrm{T}}|^2$$

so, by Lemma 6.1 (iii),

$$|J\pi|^2=\varepsilon^2\,|\vec{H}_M+g^{\varepsilon}\nu_M|^2,\ \mu_M-a.e.$$

Therefore, by Lemma 8.7 we can find a constant c depending only on g such that

$$\begin{aligned}\int J\pi d\mu_{M^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times B/\varepsilon)} &\le (\varepsilon\mathbf{M}(M^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times B/\varepsilon)))^{1/2}\cdot \\ &\quad\cdot(\varepsilon\int|\vec{H}_M+g^{\varepsilon}\nu_M|^2d\mu_{M^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times B/\varepsilon)})^{1/2} \\ &\le((\mathcal{L}^1(B)+\varepsilon^2)(\mathbf{M}(M_0)+c_g))^{1/2}\cdot \\ &\quad\cdot(\varepsilon\int|\vec{H}_M+g^{\varepsilon}\nu_M|^2d\mu_{M^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times B/\varepsilon)})^{1/2}.\end{aligned}\tag{8.5}$$

By Lemma 8.6 we have

$$\varepsilon\int|\vec{H}+g^{\varepsilon}\nu|^2\,d\mu_{M^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times B/\varepsilon)}\le F^{\varepsilon}(M^{\varepsilon})+\int g^{\varepsilon}(1+e^{-z/\varepsilon})d\mu_{M^{\varepsilon}}.$$

Combining this with (8.5) and using (8.4) we have

$$\begin{aligned}\int J\pi d\mu_{M^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times B/\varepsilon)} &\le((\mathcal{L}^1(B)+\varepsilon^2)(\mathbf{M}(M_0)+c_g))^{1/2}(F^{\varepsilon}(M^{\varepsilon})+\int g^{\varepsilon}(1+e^{-z/\varepsilon})d\mu_{M^{\varepsilon}})^{1/2} \\ &\le(\mathcal{L}^1(B)+\varepsilon^2)^{1/2}(\mathbf{M}(M_0)+c_g).\end{aligned}$$

Therefore we obtain

$$\mathbf{M}(T^{\varepsilon}\llcorner(\mathbf{R}^{n+1}\times B))\le((\mathcal{L}^1(B)+\varepsilon^2)+(\mathcal{L}^1(B)+\varepsilon^2)^{1/2})(\mathbf{M}(M_0)+c_g).$$

from which the lemma follows. ■

9 Existence of forced Brakke flows

We now construct a solution to the most general example of a forced Brakke flow given in §4. We begin by taking care to define the flow. Since we will be using Cauchy's inequality to obtain certain bounds, we will require $g\in L^2(U)$. As noted in the remark after Lemma 6.2, we easily have $\|g\|_{L^1(U)}\le\|g\|_{L^2(U)}(\mathcal{L}^1(U))^{1/2}<\infty$, so the results from §§6-8 still hold.

Definition: *Let $\mu\in\mathcal{M}(\mathbf{R}^{n+1})$ and assume φ is a test function. If one of the following cases holds, set $\mathcal{G}(\mu;\varphi)=-\infty$:*

(i) $\mu\llcorner\{\varphi>0\}\notin\mathcal{M}_n(\mathbf{R}^{n+1})$,

(ii) $|\delta V|\llcorner\{\varphi>0\}\notin\mathcal{M}(\{\varphi>0\})$ *where* $V:=V_{\mu}\llcorner\{\varphi>0\}$,

(iii) $\delta V_{\mathrm{sing}}\llcorner\{\varphi>0\}\neq0$,

(iv) $\int\varphi H^2d\mu=\infty$.

Otherwise, we define

$$\mathcal{G}(\mu,\varphi)=\int(-\varphi H^2+\nabla^{\perp}\varphi\cdot\vec{H})\,d\mu+\int g\,|-\varphi\vec{H}+\nabla^{\perp}\varphi|\,d\mu,$$

where $g:\mathbf{R}^{n+1}\to[0,\infty)$ satisfies $g(x)\equiv0$ for any $x\in U\subset\mathbf{R}^{n+1}$ (U with finite measure), and $g\in L^2(\mu)$ for all $\mu\in\mathcal{M}_n(\mathbf{R}^{n+1})$.

Remark: *The assumption $g\in L^2(\mu)$ for all $\mu\in\mathcal{M}_n(\mathbf{R}^{n+1})$ implies that $g\in L^2(U)$ and $\dim_{\mathcal{H}}\{x\in U:|g|=\infty\}<n$. For example,*

$$g(x)=\begin{cases}|x|^{-1/4}, & x\in B_R(0)\\ 0, & \text{otherwise}\end{cases}$$

satisfies $g\in L^2(\mu)$ for all $\mu\in\mathcal{M}_n(\mathbf{R}^{n+1})$.

Lemma 9.1: *$\mathcal{G}$ is a general Brakke functional.*

Proof: We need only show that conditions (iv) and (v) in the definition of a general Brakke functional (§4) are satisfied.

To show that $\mathcal{G}(\cdot,\varphi)$ is upper-semicontinuous, we let $\{\mu^i\}_{i\ge1}$ be a sequence of Radon measures on $\{\varphi>0\}$ converging to a Radon measure μ and that $\sup_{i\ge1}\mu^i\{\varphi>0\}\le C_0<\infty$. If $\limsup_{i\to\infty}\mathcal{G}(\mu^i;\varphi)=-\infty$ then we are done. Otherwise we may assume, by taking a subsequence if necessary, $\lim_{i\to\infty}\mathcal{G}(\mu^i;\varphi)$ exists and is finite. We begin by showing that this implies that we may assume $V_{\mu^i\llcorner\{\varphi>0\}}\to V_{\mu}$ (after relabelling).

By Cauchy's inequality we have

$$\mathcal{G}(\mu^i,\varphi) \le -\frac{1}{4}\int \varphi H^2 \, d\mu^i + \frac{1}{2}\int g^2 \varphi \, d\mu^i + \frac{3}{4}\int \frac{|D\varphi|^2}{\varphi} \, d\mu^i$$

and hence

$$\int \varphi H^2 d\mu^i \le C(C_0, \varphi; n, g). \tag{9.1}$$

Thus, by Cauchy-Schwarz we have, for any $K \subset\subset \{\varphi > 0\}$,

$$|\delta V_{\mu^i}|(K) \le C(C_0, \varphi, K; n, g). \tag{9.2}$$

Hence the compactness theorem 1.1 implies $V_{\mu^i \llcorner \{\varphi>0\}} \to V_\mu$ after relabelling.

The proof of upper-semicontinuity for the first two parts of $\mathcal{G}$ is exactly as it is presented in [I2] since it is the Brakke functional. We include it here for completeness.

Let $\psi \in C_c^2(\{\varphi > 0\}, [0,\infty))$ Then, by (9.1) and (9.2) both $\int \psi H^2 d\mu^i$ and $|\int \nabla^\perp \psi \cdot \vec{H} \, d\mu^i|$ are bounded independent of i. We next show that $\int \psi H^2 d\mu^i$ is lower-semicontinuous.

By approximation by C_c^1 vectors we have

$$\left(\int \psi H^2 d\mu\right)^{1/2} = \sup\left\{\int \psi^{1/2} \vec{H} \cdot X \, d\mu : \ \|X\|_{L^2} = 1, X \in C_c^1(\{\varphi > 0\})\right\}.$$

Since $\psi \in C_c^2$ we have

$$\begin{aligned}
\int \psi^{1/2} \vec{H} \cdot X \, d\mu &= -\delta V_\mu(\psi^{1/2} X) \\
&= -\lim_{i\to\infty} \delta V_{\mu^i}(\psi^{1/2} X) \\
&= \lim_{i\to\infty} \int \psi^{1/2} \vec{H} \cdot X \, d\mu^i \\
&\le \liminf_{i\to\infty} \left(\int \psi H^2 d\mu^i\right)^{1/2} \left(\int |X|^2 \, d\mu^i\right)^{1/2} \\
&\le \liminf_{i\to\infty} \left(\int \psi H^2 d\mu^i\right)^{1/2}
\end{aligned}$$

from which it follows that

$$\int \psi H^2 d\mu \le \liminf_{i\to\infty} \int \psi H^2 d\mu^i .$$

Now we concentrate on the second term of the Brakke functional. We may find an $X \in C_c^1(\{\varphi > 0\})$ such that

$$\|\nabla^\perp \psi - X\|_{L^2} \le \delta .$$

Therefore, by the bound on $\int \psi H^2 d\mu^i$ and lower-semicontinuity, we have

$$\begin{aligned}
\frac{1}{2}\left|\int \nabla^\perp \psi \cdot \vec{H} \, d\mu - \lim_{i\to\infty}\int \nabla^\perp \psi \cdot \vec{H} \, d\mu^i\right|^2 &\le \left|\int (\nabla^\perp \psi - X) \cdot \vec{H} \, d\mu\right|^2 \\
&\quad + |\delta V_\mu(X) - \lim_{i\to\infty} \delta V_{\mu^i}(X)|^2 + \lim_{i\to\infty}\left|\int (\nabla^\perp \psi - X) \cdot \vec{H} \, d\mu^i\right|^2 \\
&\le \delta^2 \sup_{\{\psi>0\}} \frac{1}{\varphi} \int \varphi H^2 d\mu \\
&\quad + \sup_{\{\psi>0\}} \frac{1}{\varphi} \sup_{i\ge 1} \int \varphi H^2 d\mu^i \lim_{i\to\infty} \int |\nabla^\perp \psi - X|^2 \, d\mu^i |^2 \\
&\le \delta^2 C(n, p, g, \varphi, \psi) < \infty .
\end{aligned}$$

Sending $\delta \to 0$ gives

$$\int \nabla^\perp \psi \cdot \vec{H} \, d\mu = \lim_{i\to\infty} \int \nabla^\perp \psi \cdot \vec{H} \, d\mu^i$$

and consequently, using the lower-semicontinuity proved above, we have

$$\int -\psi H^2 + \nabla^\perp \psi \cdot \vec{H} d\mu \le \limsup_{i\to\infty} \int -\psi H^2 + \nabla^\perp \psi \cdot \vec{H} d\mu^i ,$$

for any $\psi \in C_c^2(\{\varphi > 0\}, [0,\infty))$.

Let $\{\psi_j\}_{j\ge 1} \subset C_c^2(\{\varphi > 0\}, [0,\infty))$ be chosen so that $\psi_j \le \varphi$ and $\psi_j \to \varphi$ in C^2. By the dominated convergence theorem we have that,

$$\lim_{j\to\infty} \int \psi_j H^2 d\mu = \int \varphi H^2 d\mu .$$

Furthermore, since $\int \varphi H^2 d\mu < \infty$ and $\psi_j \to \varphi$ in C^2

$$\left|\lim_{j\to\infty} \int \nabla^\perp \psi_j \cdot \vec{H} \, d\mu - \int \nabla^\perp \varphi \cdot \vec{H} \, d\mu\right| \le \lim_{j\to\infty} \int \frac{|D\psi_j - D\varphi|^2}{|\psi_j - \varphi|} d\mu \int |\psi_j - \varphi| H^2 d\mu = 0 .$$

Consequently we have

$$\int -\varphi H^2 + \nabla^\perp \varphi \cdot \vec{H} \, d\mu = \lim_{j\to\infty} \int -\psi_j H^2 + \nabla^\perp \psi_j \cdot \vec{H} \, d\mu .$$

Since the same argument applies with μ replaced by μ^i we have

$$\begin{aligned}\limsup_{i\to\infty}\mathcal{B}(\mu^i,\varphi)&=\limsup_{i\to\infty}\lim_{j\to\infty}\mathcal{B}(\mu^i,\psi_j)\\&\le\lim_{j\to\infty}\limsup_{i\to\infty}\mathcal{B}(\mu^i,\psi_j)\\&\le\lim_{j\to\infty}\mathcal{B}(\mu,\psi_j)=\mathcal{B}(\mu,\varphi)\end{aligned}$$

Now we turn to the remainder of $\mathcal{G}$. Since we are assuming $\lim_{i\to\infty}\mathcal{G}(\mu^i;\varphi)$ exists and is finite, by (9.1) we have that $\int\varphi H^2 d\mu^i$ is bounded independent of i. Then, by lower-semicontinuity of $\int\varphi H^2 d\mu$ we have $\int\varphi^2 H^2 d\mu<\infty$ and

$$\begin{aligned}\int g\,|-\varphi\vec{H}+\nabla^\perp\varphi|\,d\mu&\le\frac{1}{2}\int_{\{\varphi>0\}}g^2 d\mu+\frac{1}{4}\int\varphi^2H^2d\mu+\frac{1}{4}\int|D\varphi|^2\,d\mu\\&\le\frac{1}{2}\|g\|^2_{L^2(\mu)}+\frac{1}{4}\int\varphi^2H^2d\mu+\frac{1}{4}\int|D\varphi|^2\,d\mu\end{aligned}$$

which is finite. Hence $g\,|-\varphi\vec{H}+\nabla^\perp\varphi|\in L^1_{loc}$ and, for any $\delta>0$, we can find an $f\in C^0_c$ such that

$$\|\,g\,|-\varphi\vec{H}+\nabla^\perp\varphi|-f\,\|_{L^1}\le\delta$$

Therefore

$$\begin{aligned}&\left|\int g\,|-\varphi\vec{H}+\nabla^\perp\varphi|\,d\mu-\lim_{i\to\infty}\int g\,|-\varphi\vec{H}_i+\nabla^\perp\varphi|\,d\mu^i\right|\\&\le\left|\int(g\,|-\varphi\vec{H}+\nabla^\perp\varphi|-f)d\mu\right|+\left|\int f d\mu-\lim_{i\to\infty}\int f d\mu^i\right|\\&+\left|\lim_{i\to\infty}\int(g\,|-\varphi\vec{H}_i+\nabla^\perp\varphi|-f)d\mu^i\right|\\&\le 2\delta\end{aligned}$$

since $\mu^i\to\mu$. Finally, send $\delta\to 0$ to obtain

$$\int g\,|-\varphi\vec{H}+\nabla^\perp\varphi|\,d\mu=\lim_{i\to\infty}\int g\,|-\varphi\vec{H}_i+\nabla^\perp\varphi|\,d\mu^i \tag{9.3}$$

whenever $\sup_{i\ge1}\mu^i\{\varphi>0\}<\infty$ and none of the singular conditions on $\mathcal{G}$ are satisfied. Hence $\mathcal{G}$ is upper-semicontinuous.

Finally, assume $\mathcal{G}(\mu;\varphi)>-\infty$. Then

$$\begin{aligned}&\int(-\varphi H^2+\nabla^\perp\varphi\cdot\vec{H})d\mu+\int g(\cdot,t)\,|-\varphi\vec{H}+\nabla^\perp\varphi|\,d\mu\\&\le\int\Big(-\varphi\Big|\vec{H}-\frac{1}{2}\frac{D\varphi}{\varphi}\Big|^2+\frac{1}{4}\frac{|D\varphi|^2}{\varphi}\Big)d\mu+\frac{1}{2}\int\Big(\varphi g^2+\varphi\Big|\vec{H}-\frac{D\varphi}{\varphi}\Big|^2\Big)d\mu\\&\le\int\Big(-\varphi\Big|\vec{H}-\frac{1}{2}\frac{D\varphi}{\varphi}\Big|^2+\frac{1}{4}\frac{|D\varphi|^2}{\varphi}\Big)d\mu+\frac{1}{2}\int\Big(\varphi g^2+\frac{1}{2}\varphi\Big|\vec{H}-\frac{1}{2}\frac{D\varphi}{\varphi}\Big|^2+\frac{1}{2}\frac{|D\varphi|^2}{\varphi}\Big)d\mu\\&\le\sup|D^2\varphi|\,\mu\{\varphi>0\}+\frac{1}{2}\sup|\varphi|\,\|g\|^2_{L^2(\mu)}\end{aligned}$$

which is non-decreasing in $\mu\{\varphi>0\}$. Hence $\mathcal{G}$ is a general Brakke functional. ∎

In order to define our forced Brakke flows, we will need the following version of $\mathcal{G}$, which is a general Brakke functional by Lemma 9.1:

Definition: *Let $t\ge0$, $\mu\in\mathcal{M}(\mathbf{R}^{n+1})$ and assume φ is a test function. If one of the following cases holds, set $\mathcal{G}_t(\mu;\varphi)=-\infty$:*

(i) $\mu\llcorner\{\varphi>0\}\notin\mathcal{M}_n(\mathbf{R}^{n+1})$,
(ii) $|\delta V|\llcorner\{\varphi>0\}\notin\mathcal{M}(\{\varphi>0\})$ *where* $V:=V_\mu\llcorner\{\varphi>0\}$,
(iii) $\delta V_{sing}\llcorner\{\varphi>0\}\neq0$,
(iv) $\int\varphi H^2 d\mu=\infty$.

Otherwise, we define

$$\mathcal{G}_t(\mu,\varphi)=\int(-\varphi H^2+\nabla^\perp\varphi\cdot\vec{H})\,d\mu+\int g(\cdot,t)\,|-\varphi\vec{H}+\nabla^\perp\varphi|\,d\mu,$$

where $g:\mathbf{R}^{n+1}\times[0,\infty)\to[0,\infty)$ *satisfies*

(v) $g(x,t)\equiv0$ *for any* $x\in U\subset\mathbf{R}^{n+1}$ *and all* $t\ge0$,
(vi) *for any* $t\ge0$, $\lim_{s\to t}g(x,s)=g(x,t)$ *for a.e.* $x\in U$,
(vii) $g(\cdot,t)\in L^2(\mu)$ *for all* $\mu\in\mathcal{M}_n(\mathbf{R}^{n+1})$,
(viii) $\sup_{t\ge0}\|g(\cdot,t)\|_{L^2(U)}<\infty$, *and*
(ix) *for some* $\tau\in(0,\infty)$, $g(\cdot,t)\equiv0\quad\forall t\ge\tau$,

where U has finite measure.

Definition (forced Brakke flow, forced integer Brakke flow): *Let $\{\mu_t\}_{t\ge0}$ be a family of Radon measures on $\mathbf{R}^{n+1}$ and suppose $\mathcal{G}_t$ is as defined above. If*

$$\overline{D}_t\mu_t(\varphi)\le\mathcal{G}_t(\mu_t;\varphi)$$

for any test-function φ on $\mathbf{R}^{n+1}$, then we call $\{\mu_t\}_{t\ge0}$ a forced Brakke flow. If $\mu_t\in I\mathcal{M}_n(\mathbf{R}^{n+1})$ for a.e. $t\ge0$, then we call $\{\mu_t\}_{t\ge0}$ a forced integer Brakke flow.

Before proceeding with the existence proof, we need to prove a consequence of the Compactness Theorem 5.1 that will be used in the proof. This will require a simple lemma concerning the relationship between $\mathcal{G}_t$ and the following functional:

Definition: *Let $t \ge 0$, $\mu \in \mathcal{M}(\mathbf{R}^{n+2})$ and assume $\bar{\varphi} \in C_c^2(\mathbf{R}^{n+2},[0,\infty))$. If one of the following cases holds, set $\mathcal{G}_t^{\varepsilon}(\mu;\varphi) = -\infty$:*

(i) $\mu\lfloor\{\bar{\varphi} > 0\} \notin \mathcal{M}_{n+1}(\mathbf{R}^{n+2})$,

(ii) $|\delta V|\lfloor\{\bar{\varphi} > 0\} \notin \mathcal{M}(\{\bar{\varphi} > 0\})$ where $V := V_\mu\lfloor\{\bar{\varphi} > 0\}$,

(iii) $\delta V_{\text{sing}}\lfloor\{\bar{\varphi} > 0\} \neq 0$,

(iv) $\int \bar{\varphi} H^2 d\mu = \infty$.

Otherwise, we define

$$\mathcal{G}_t^{\varepsilon}(\mu,\bar{\varphi}) = \int (-\bar{\varphi} H^2 + \bar{\nabla}^{\perp}\bar{\varphi}\cdot\vec{H})\, d\mu + \int g^{\varepsilon}(\cdot\,, z + t/\varepsilon)\,|-\bar{\varphi}\vec{H} + \bar{\nabla}^{\perp}\bar{\varphi}|\, d\mu,$$

where g^{ε} is defined by

$$g^{\varepsilon}(x,z) := \begin{cases} g(x,\varepsilon z), & z \ge 0 \\ 0, & z < 0 \end{cases}$$

where g satisfies conditions (v)-(ix) in the definition of $\mathcal{G}_t$.

Lemma 9.2: *For any $\varepsilon > 0$ and any $t \ge 0$, $\mathcal{G}_t^{\varepsilon}$ is a general Brakke functional. Furthermore, if $\varepsilon_i \downarrow 0$, then $\lim_{i\to\infty} \mathcal{G}_t^{\varepsilon_i}(\mu,\bar{\varphi}) = \mathcal{G}_t(\mu,\bar{\varphi})$ for any $t \ge 0$ and any Radon measure μ.*

Proof: By Lemma 9.1 we see that $\mathcal{G}_t^{\varepsilon}$ is a general Brakke functional and the continuity $\lim_{i\to\infty} \mathcal{G}_t^{\varepsilon_i}(\mu,\bar{\varphi}) = \mathcal{G}_t(\mu,\bar{\varphi})$ follows from the continuity of g. ■

By using Lemmas 8.7 and 9.2 and (8.1) we have:

Theorem 9.3: *There exists a sequence $\{\varepsilon_i\}_{i\ge1}$ be a descending to zero and an integer forced Brakke flow $\bar{\mathcal{M}} = \{\bar{\mu}_t\}_{t\ge0}$ such that*

$$\mu_{P^{\varepsilon_i}(t)} \to \bar{\mu}_t \text{ and } V(\mu_{P^{\varepsilon_i}(t)}) \to V(\bar{\mu}_t)$$

and $\bar{\mu}_t(\mathbf{R}^{n+1}\times(a,b)) \le (b-a)(\mathbf{M}(M_0) + c_g)$.

Proof: By (8.1) we know that, for any $i \ge 1$, $\mathcal{M}^{\varepsilon_i} = \{\mu_{P^{\varepsilon_i}(t)}\}_{t\ge0}$ is a general Brakke flow with $\mathcal{G}_t^{\varepsilon_i}$ as the general Brakke functional. Therefore, we may use Lemmas 8.6 and 9.2

together with Theorem 5.1(i) to find an integer forced Brakke flow $\bar{\mathcal{M}} = \{\bar{\mu}_t\}_{t\ge0}$ such that $\mu_{P^{\varepsilon_i}(t)} \to \bar{\mu}_t$.

By using the same argument leading to (9.2) we have

$$\mathcal{G}_t^{\varepsilon_i}(\mu;\bar{\varphi}) \ge -C_1 \Rightarrow |\delta V_\mu|(K) \le C(C_1,\bar{\varphi},K) \quad \forall i \ge 1$$

for all $t \ge 0$. Combining this with Theorem 5.1 (ii) completes the proof. ■

The next step in showing existence is to show that the forced Brakke flow obtained in Theorem 9.3 is translationaly invariant for *a.e.* $t \ge 0$. Then we will show that, for any $\theta \in C_c^2(\mathbf{R})$ satisfying $\int \theta(z)dz = 1$, the family $\{\mu_t\}_{t\ge0}$ given by

$$\mu_t(\varphi) := \bar{\mu}_t(\theta\varphi) \quad \forall \varphi \in C_c^2(\mathbf{R}^{n+1},[0,\infty))$$

is an integer forced Brakke flow with initial data given by $\mu_0 = \mu_{M_0}$.

Lemma 9.4: *Let $t \ge 0$. Then*

$$(\sigma_\tau)_{\#}(\bar{\mu}_t) = \bar{\mu}_t \quad \forall \tau \in \mathbf{R}$$

for all $t \in \mathcal{T}_1$, where $\sigma_\tau(x,z) = (x, z+\tau)$ and $\mathcal{T}_1$ is the set from Lemma 4.1.

Proof: Let $\bar{\varphi} \in C_c^2(\mathbf{R}^{n+1}\times(-t/\varepsilon,\infty),[0,\infty))$ and let $\tau \ge 0$. Define

$$\bar{\varphi}^{\tau}(x,z) := \begin{cases} \bar{\varphi}(x, z-\tau) & \text{if } z - \tau > t/\varepsilon \\ 0 & \text{otherwise} \end{cases}.$$

Then, for any $\varepsilon > 0$, $\mu_{P^{\varepsilon}(t)}(\bar{\varphi}^{\tau}) = \mu_{P^{\varepsilon}(t+\varepsilon\tau)}(\bar{\varphi})$. By Lemmas 4.1 and 8.7, there is a constant such that

$$\mu_{P^{\varepsilon}(t)}(\bar{\varphi}) - C(\bar{\varphi},n,g)t$$

is non-increasing. Fix $s > t$ and pass to the limit as $\varepsilon_i \downarrow 0$ to find that, for any $\bar{\varphi} \in C_c^2(\mathbf{R}^{n+1}\times\mathbf{R},[0,\infty))$,

$$\bar{\mu}_t(\bar{\varphi}) - C(\bar{\varphi},n,g)t \ge \bar{\mu}_t(\bar{\varphi}^{\tau}) - C(\bar{\varphi},n,g)t \ge \bar{\mu}_s(\bar{\varphi}) - C(\bar{\varphi},n,g)s,$$

by Theorem 9.3. Therefore, sending $s \downarrow t$, we have

$$\bar{\mu}_t(\bar{\varphi}) \ge \bar{\mu}_t(\bar{\varphi}^{\tau}) \ge \lim_{s\downarrow t} \bar{\mu}_s(\bar{\varphi}). \tag{9.4}$$

Similarly, whenever $\tau \leq 0$ and $t > 0$,

$$\overline{\mu}_t(\overline{\varphi}) \leq \overline{\mu}_t(\overline{\varphi}^\tau) \leq \lim_{s \uparrow t} \overline{\mu}_s(\overline{\varphi}). \tag{9.5}$$

Finally, by Lemma 4.1 (iii) we can find a co-countable set $\mathcal{T}_1$ of times such that equality holds in both (9.4) and (9.5), that is $(\sigma_\tau)_\#(\overline{\mu}_t) = \overline{\mu}_t$ for all $\tau \in \mathbf{R}$, as desired. ∎

Lemma 9.5: *Let $\overline{\mu} \in \mathcal{M}(\mathbf{R}^{n+1} \times (z_0, \infty))$ and suppose that, for each $\overline{\varphi} \in C_c^0(\mathbf{R}^{n+1} \times (z_0, \infty), [0, \infty))$ and each $\tau \geq 0$, $\overline{\mu}$ satisfies*

$$\overline{\mu}(\overline{\varphi}^\tau) = \overline{\mu}(\overline{\varphi}),$$

where

$$\overline{\varphi}^\tau(x, z) := \begin{cases} \overline{\varphi}(x, z - \tau) & \text{if } z - \tau > t/\varepsilon \\ 0 & \text{otherwise} \end{cases}.$$

Let $\theta \in C_c^0((z_0, \infty))$ be chosen so that $\int \theta(z)dz = 1$. Then

(i) the Radon measure given by $\mu(\varphi) := \overline{\mu}(\theta\varphi)$ for any $\varphi \in C_c^0(\mathbf{R}^{n+1}, [0, \infty))$ is independent of the choice of θ and $\overline{\mu} = \mu \times \mathcal{L}^1 \llcorner (z_0, \infty)$,

(ii) $\overline{\mu} \in \mathcal{M}_{n+1}(\mathbf{R}^{n+1} \times (z_0, \infty))$ implies $\mu \in \mathcal{M}_n(\mathbf{R}^{n+1})$, and $\overline{\mu} \in I\mathcal{M}_{n+1}(\mathbf{R}^{n+1} \times (z_0, \infty))$ implies $\mu \in I\mathcal{M}_n(\mathbf{R}^{n+1})$, and

(iii) if $\theta \in C_c^2((z_0, \infty))$ then

$$\mathcal{G}_t(\overline{\mu}, \theta\varphi) = \mathcal{G}_t(\mu, \varphi),$$

for any $\varphi \in C_c^2(\mathbf{R}^{n+1}, [0, \infty))$ and any $t \geq 0$.

Proof: By extending $\overline{\mu}$ to a translation-invariant measure on all of $\mathbf{R}^{n+1} \times \mathbf{R}$, we assume $z_0 = -\infty$. The proofs of (i) and (ii) are in [I2]. We include them here for completeness. The proof of (iii) is similar to the proof of [I2 8.5 (iii)].

Let $K := \{\theta \in C_c^0(\mathbf{R}) : \int \theta dz = 1\}$. For each $\theta \in K$, let $K(\theta)$ be the set of all finite convex combinations

$$\theta' = \sum a_\tau \theta^\tau, \quad \sum a_\tau = 1, \quad a_\tau \geq 0.$$

Since $\overline{\mu}$ is translationally invariant, $\overline{\mu}(\theta'\varphi) = \overline{\mu}(\theta''\varphi)$ for any $\theta', \theta'' \in K(\theta)$. Any two fixed functions $\theta_1, \theta_2 \in K$ can be approximated uniformly and arbitrarily closely by $\theta_1', \theta_2' \in K(\hat{\theta})$ for some sufficiently tall, narrow $\hat{\theta} \in K$. Therefore

$$\overline{\mu}(\theta_1\varphi) = \overline{\mu}(\theta_2\varphi),$$

for any $\varphi \in C_c^0(\mathbf{R}^{n+1}, [0, \infty))$. Therefore, for any $\vartheta \in C_c^0(\mathbf{R}, [0, \infty))$,

$$\overline{\mu}(\vartheta\varphi) = \mu(\varphi)\mathcal{L}^1(\vartheta)$$

i.e. $\overline{\mu} = \mu \times \mathcal{L}^1$.

Assume $\overline{\mu} \in \mathcal{M}_{n+1}(\mathbf{R}^{n+2})$. Note that $\omega \in T_{(x,z)}\overline{\mu}$ $\overline{\mu} - a.e.$. Otherwise, by the rectifiability of $\overline{\mu}$ we could find (x, z) and a small C^1 $(n + 1)$-submanifold C transverse to ω containing (x, z) but so that $\overline{\mu}(C) > 0$. However, by the translational invariance of $\overline{\mu}$, we could translate C along the z-axis to contradict the local finiteness of $\overline{\mu}$.

Since $\overline{\mu}$ is rectifiable, we have, for any $\varphi \in C_c^0(\mathbf{R}^{n+1}, [0, \infty))$ and $\overline{\mu}$-*a.e.* $(x, z) \in U \times \mathbf{R}$

$$\begin{aligned}
\lim_{\lambda \downarrow 0} \mu_{x,\lambda}(\varphi) &= \lim_{\lambda \downarrow 0} \overline{\mu}_{(x,z),\lambda}(\theta\varphi) \\
&= \theta_{\overline{\mu}}(x, z) \int_{T_{(x,z)}\overline{\mu}} \theta(z)\varphi(x) d\mathcal{H}^{n+1}(x, z) \\
&= \theta_{\overline{\mu}}(x, z) \int_{T_{(x,0)}\overline{\mu}} \varphi(x) d\mathcal{H}^{n+1}(x)
\end{aligned}$$

since $\omega \in T_{(x,z)}\overline{\mu}$. Therefore $\mu \in \mathcal{M}_n(\mathbf{R}^{n+1})$ and $T_{(x,z)}\overline{\mu} = T_x\mu \oplus \operatorname{span}\omega$ for μ-*a.e.* x.

For (iii), note that, by (ii), $\mu \llcorner \{\varphi > 0\} \in \mathcal{M}_n(\mathbf{R}^{n+1})$ iff $\overline{\mu} \llcorner (\{\varphi > 0\} \times \mathbf{R}) \in \mathcal{M}_{n+1}(\mathbf{R}^{n+2})$. Suppose this is the case. Let $X \in C_c^1(\{\varphi > 0\} \times \mathbf{R})$ and define $\tilde{X}(x) := \int T_{(x,z)}\mu(X(x, z))dz$. Then we have

$$\begin{aligned}
\delta V_\mu(\tilde{X}) &= \int \operatorname{div}_\mu \left(\int T_{(x,z)}\mu(X(x, z))dz\right) d\mu \\
&= \int \operatorname{div}_\mu T_{(x,z)}\mu(X(x, z)) d\mu dz \\
&= \int \operatorname{div}_{\overline{\mu}} X(x, z) d\overline{\mu} \\
&= \delta V_{\overline{\mu}}(X)
\end{aligned}$$

since $\overline{\mu} = \mu \times \mathcal{L}^1$ and $T_{(x,z)}\overline{\mu} = T_x\mu \oplus \operatorname{span}\omega$ μ-*a.e.* Hence

$$\delta V_{\overline{\mu}} \llcorner (\{\varphi > 0\} \times \mathbf{R}) = \delta V_\mu \llcorner \{\varphi > 0\} \times \mathcal{L}^1$$

and consequently the singular conditions on $\mathcal{G}_t$ hold for $\mu \llcorner \{\varphi > 0\}$ iff they hold for $\overline{\mu} \llcorner (\{\varphi > 0\} \times \mathbf{R})$. Furthermore, for μ-*a.e.* $x \in \{\varphi > 0\}$ and all z, $\vec{H}_\mu(x) = \vec{H}_{\overline{\mu}}(x, z)$ and $\omega_{\overline{\mu}}^{\perp} = 0$ (since $\omega \in T_{(x,z)}\overline{\mu}$). Hence, if $\mathcal{G}_t(\overline{\mu}, \theta\varphi) > -\infty$, we have

$$\begin{aligned}\mathcal{G}_t(\overline{\mu},\theta\varphi) &= \int(-\theta\varphi H_{\overline{\mu}}^2 + \overline{\nabla}^{\perp}(\theta\varphi)\cdot\vec{H}_{\overline{\mu}})d\overline{\mu} + \int g(x,t)\,|-\theta\varphi\vec{H}_{\overline{\mu}} + \overline{\nabla}^{\perp}(\theta\varphi)|\,d\overline{\mu} \\ &= \int\theta\mathcal{G}_t(\mu,\varphi)dz + \int\varphi\theta'\omega^{\perp}\cdot\vec{H}_{\overline{\mu}}d\overline{\mu} \\ &= \mathcal{G}_t(\mu,\varphi)\end{aligned}$$

as desired. ■

Theorem 9.6 (Existence of forced integer Brakke flows): *Let $M_0 \in \mathbf{I}_n^{loc}(\mathbf{R}^{n+1})$ be an initial surface. Then there exists a $M \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+1}\times[0,\infty))$ and a forced integer Brakke flow $\mathcal{M} = \{\mu_t\}_{t\geq 0}$ such that*

(i) $\partial M = M_0$,

(ii) $\mathbf{M}(M\llcorner(\mathbf{R}^{n+1}\times A)) \leq (\mathcal{L}^1(A) + \mathcal{L}^1(A)^{1/2})(\mathbf{M}(M_0) + c_g)$, *and*

(iii) $\mu_{M_t} \leq \mu_t$, $\mu_0 = \mu_{M_0}$, $\mathbf{M}(\mu_t) \leq \mathbf{M}(M_0) + c_g$

where $M_t := \partial(M\llcorner(\mathbf{R}^{n+1}\times[t,\infty)))$.

Proof: The existence of $\{\mu_t\}_{t\geq 0}$ follows from Lemmas 9.3, 9.4, and 9.5 by defining

$$\mu_t(\varphi) = \overline{\mu}_t(\theta\varphi),$$

for all $t \geq 0$.

Define $T^{\varepsilon} := (\kappa_{\varepsilon})_{\#}(M^{\varepsilon})$ where $\kappa_{\varepsilon}(x,z) := (x,\varepsilon z)$. By Lemmas 6.2 and 8.8 we have, by the compactness theorem 1.2, that there exists a sequence $\{\varepsilon_i\}_{i\geq 1}$ and a current $M \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+1}\times[0,\infty))$ such that

$$\partial M = M_0 \text{ and } T^{\varepsilon_i} \to M.$$

Furthermore, Lemma 8.8 implies (ii).

Now, fix $t \geq 0$ and let $\varphi \in C_c^0(\mathbf{R}^{n+1},[0,\infty))$. Let $\delta > 0$ and choose a cut-off function $\eta \in C^{\infty}(\mathbf{R},[0,1])$ such that

$$\eta = 0 \text{ on } (-\infty, t-\delta],\ \eta = 1 \text{ on } [t,\infty).$$

For all τ and any $\alpha \in \mathcal{D}_{n+1}(\mathbf{R}^{n+2})$ we have, by Lemma 8.8,

$$\begin{aligned}&|M(\eta\alpha) - \lim_{i\to\infty} T^{\varepsilon_i}\llcorner(\mathbf{R}^{n+1}\times[t+\varepsilon_i\tau,\infty))(\alpha)| \\ &= |M(\eta\alpha) - \lim_{i\to\infty} T^{\varepsilon_i}\llcorner(\mathbf{R}^{n+1}\times[t+\varepsilon_i\tau,\infty))(\eta\alpha)| \\ &\leq |M(\eta\alpha) - \lim_{i\to\infty} T^{\varepsilon_i}(\eta\alpha)| + |\lim_{i\to\infty} T^{\varepsilon_i}\llcorner(\mathbf{R}^{n+1}\times[0,t+\varepsilon_i\tau))(\eta\alpha)| \\ &\leq \max|\alpha|(\delta+\varepsilon_i^2 + (\delta+\varepsilon_i^2)^{1/2})(\mathbf{M}(M_0)+c_g) < \infty.\end{aligned}$$

Sending $\delta \to 0$ we obtain

$$T^{\varepsilon_i}\llcorner(\mathbf{R}^{n+1}\times[t+\varepsilon_i\tau,\infty)) \to M\llcorner(\mathbf{R}^{n+1}\times[t,\infty)),$$

in particular,

$$T_{t+\varepsilon_i\tau}^{\varepsilon_i} \to M_t \text{ and } \liminf_{i\to\infty} \mu_{T_{t+\varepsilon_i\tau}^{\varepsilon_i}} \geq \mu_{M_t}.$$

Therefore

$$\begin{aligned}\mu_t(\varphi) &= \overline{\mu}_t(\theta\varphi) \\ &= \lim_{i\to\infty}\mu_{P^{\varepsilon_i}(t)}(\theta\varphi) \\ &\geq \liminf_{i\to\infty}\int\theta\varphi\,|\omega^{\mathrm{T}}|\,d\mu_{P^{\varepsilon_i}(t)} \\ &= \liminf_{i\to\infty}\int_0^{\infty}\theta\int\varphi\,d\mu_{\partial(P^{\varepsilon_i}(t)\llcorner[z,\infty))}dz \\ &= \liminf_{i\to\infty}\int_0^{\infty}\theta\int\varphi\,d\mu_{M_{t/\varepsilon_i+z}^{\varepsilon_i}}dz \\ &= \liminf_{i\to\infty}\int_0^{\infty}\theta\int\varphi\,d\mu_{T_{t+z\varepsilon_i}^{\varepsilon_i}}dz \\ &\geq \int_0^{\infty}\theta\liminf_{i\to\infty}\mu_{T_{t+z\varepsilon_i}^{\varepsilon_i}}(\varphi)dz \\ &\geq \mu_{M_t}(\varphi).\end{aligned}$$

Also, approximating θ by step functions and using Lemma 9.3, we have

$$\mathbf{M}(\mu_t) \leq \mathbf{M}(M_0) + c_g$$

where c_g is the constant from Lemma 8.7. Finally we show that $\lim_{i\to\infty}\mu_{M^{\varepsilon_i}}(\theta\varphi) = \mu_{M_0}(\varphi)$. By Lemma 6.2 we have

$$F^{\varepsilon}(M^{\varepsilon}) \leq \mathbf{M}(M_0) - \frac{1}{\varepsilon}\int_0^{\infty}\int g(x,\varepsilon z)e^{-z/\varepsilon}d\mu_{\Omega_0}dz + \frac{1}{\varepsilon}\int_0^{\infty}\int g(x,\varepsilon z)e^{-z/\varepsilon}d\mu_{\Omega_z^{\varepsilon}}dz. \quad (9.6)$$

Furthermore, Lemma 8.7 and the isoperimetric inequality provide local mass bounds to ensure (passing to a subsequence if necessary) the existence of some $\Omega \in \mathbf{I}_{n+2}^{loc}(\mathbf{R}^{n+1}\times[0,\infty))$ such that

$$\Omega^{\varepsilon_i} \to \Omega \text{ and } \partial\Omega = \Omega_0 + \overline{M},$$

where $\overline{M} = \lim_{i\to\infty} M^{\varepsilon_i}$. Now, by lower-semicontinuity of mass we have, for any fixed $j \geq 1$,

$$\liminf_{i\to\infty}\frac{1}{\varepsilon_j}\int_0^\infty\int g(x,\varepsilon_j z)e^{-z/\varepsilon_j}d\mu_{\Omega_z^{\varepsilon_i}}dz\le\frac{1}{\varepsilon_j}\int_0^\infty\int g(x,\varepsilon_j z)e^{-z/\varepsilon_j}d\mu_{\Omega_z}dz.$$

Let $f\in C_c^1([0,\infty))$. Then

$$|f(0)-\frac{1}{\varepsilon}\int_0^\infty e^{-z/\varepsilon}f(z)dz|=|\int_0^\infty e^{-z/\varepsilon}f'(z)dz|\le\varepsilon\sup|f'|\to 0,$$

so

$$\lim_{j\to\infty}\frac{1}{\varepsilon_j}\int_0^\infty e^{-z/\varepsilon_j}f(z)dz=f(0)\quad\forall f\in C_c^1([0,\infty)).$$

Therefore, approximating $u\in L_{loc}^1([0,\infty))$ by C_c^1 functions, we have

$$\lim_{j\to\infty}\frac{1}{\varepsilon_j}\int_0^\infty e^{-z/\varepsilon_j}u(z)dz=u(0). \tag{9.7}$$

Hence, for any fixed $i\ge 1$,

$$\lim_{j\to\infty}\frac{1}{\varepsilon_j}\int_0^\infty\int g(x,\varepsilon_j z)e^{-z/\varepsilon_j}d\mu_{\Omega_z^{\varepsilon_i}}=\int g(x,0)d\mu_{\Omega_0}.$$

By diagonalising we obtain

$$\liminf_{i\to\infty}\frac{1}{\varepsilon_i}\int_0^\infty\int g(x,\varepsilon_j z)e^{-z/\varepsilon_i}d\mu_{\Omega_z^{\varepsilon_i}}dz\le\int g(x,,0)d\mu_{\Omega_0}.$$

Since $\mu_{M^{\varepsilon_i}}\llcorner\theta\to\mu_0$ for any $\theta\in C_c^2((0,\infty))$ satisfying $\int\theta dz=1$, we have

$$\begin{aligned}\mathbf{M}(\mu_0)&\le\mathbf{M}(M_0)-\lim_{i\to\infty}\frac{1}{\varepsilon_i}\int_0^\infty\int g(x,\varepsilon_i z)e^{-z/\varepsilon_i}d\mu_{\Omega_0}dz\\&+\liminf_{i\to\infty}\frac{1}{\varepsilon_i}\int_0^\infty\int g(x,\varepsilon_i z)e^{-z/\varepsilon_i}d\mu_{\Omega_0}dz\\&\le\mathbf{M}(M_0).\end{aligned}$$

However, since $\mu_0\ge\mu_{M_0}$ we actually have equality. ■

Definition (enhanced forced motion): *If the pair $(M,\mathcal{M})$ satisfies the conclusions of Theorem 8.6, then we call them an enhanced forced motion with initial condition M_0. We also say that M is the undercurrent and $\mathcal{M}$ is the forced overflow.*

As with the Brakke flow, we may restart enhanced forced motions [I2, 8.4]:

Lemma 9.7 (restartability): *Let $M_0\in\mathbf{I}_n^{loc}(\mathbf{R}^{n+1})$ be an initial surface and suppose $(M,\mathcal{M})$ is an enhanced forced motion with initial condition M_0. Then, for all $t\ge 0$, M_t is an initial surface and $\{M_t\}_{t\ge 0}$ is weakly continuous.*

Proof: By Theorem 9.6 (iii) we have

$$\mathbf{M}(M_t)\le\mathbf{M}(\mu_t)\le\mathbf{M}(M_0)+c_g<\infty \tag{9.8}$$

Furthermore, by Theorem 9.8 (ii), $\{M_t\}_{t\ge 0}$ is continuous in the weak topology. Therefore, by the compactness theorem 1.2 and (9.8), $M_t\in\mathbf{I}_n^{loc}(\mathbf{R}^{n+1}\times\{t\})$. Hence, for all $t\ge 0$, M_t is an initial surface. ■

10 Some properties of enhanced forced motions

Recall that in an enhanced forced motion we have the discrepency between μ_t and μ_{M_t}. Suppose we have an enhanced forced motion $(M,\mathcal{M})$ with initial condition M_0 satisfying $\mu_t > \mu_{M_t}$ at some time $t > 0$. Define $M(t) := (\sigma_{-t})_\# M\llcorner(\mathbf{R}^{n+1}\times(t,\infty))$ and $\mathcal{M}(t) := \{\mu_s\}_{s\geq t}$, where $\mu_s \in \mathcal{M}$ for all $s \geq t$. Because of the equality $\mu_0 = \mu_{M_0}$ required in the definition for enhanced forced motions, the pair $(M(t),\mathcal{M}(t))$ will not be an enhanced forced motion. In the special case when we have equality $\mu_t = \mu_{M_t}$ for all time, then we will say that the enhanced forced motion is a *forced matching motion*. We begin by proving existence of forced matching motions under the hypothesis of uniqueness of the undercurrent. We will conclude with some lemmata characterising the area ratio and tilt-excess for forced matching motions obtained using elliptic regularisation.

As in [I2 9.1] we formally define a forced matching motion as follows:

Definition (forced matching motion): *Let $M_0 \in \mathbf{I}_n^{loc}(\mathbf{R}^{n+1})$ be an initial surface. We call a current $M \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+1}\times[0,\infty))$ a forced matching motion for M_0 if*

(i) $\partial M = M_0$,

(ii) $\mathbf{M}(M\llcorner(\mathbf{R}^{n+1}\times A)) \leq (\mathcal{L}^1(A)+\mathcal{L}^1(A)^{1/2})(\mathbf{M}(M_0)+c_g)$, *and*

(iii) $\mathcal{M} = \{\mu_{M_t}\}_{t\geq 0}$ *is a forced Brakke flow.*

Suppose every enhanced forced motion $(M,\mathcal{M})$ with initial surface M_0 has the same undercurrent. Then we will say that M is the *unique undercurrent for* M_0. If we assume such uniqueness, we have:

Lemma 10.1 (existence of forced matching motions): *Let $M_0 \in \mathbf{I}_n^{loc}(\mathbf{R}^{n+1})$ be an initial surface and suppose that $M \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+1}\times[0,\infty))$ is the unique undercurrent for M_0. Then M is a forced matching motion for M_0.*

Proof: The most vital ingredient for the proof is Theorem 8.6 (iii). With that, the proof runs almost identically to the proof of Lemma 3.5 [I2, 9.2].

We must show that $\{\mu_{M_t}\}_{t\geq 0}$ is a forced Brakke flow. Let $\delta > 0$. We first construct an enhanced forced motion $(M,\mathcal{M}^\delta)$ satisfying

$$\mathbf{M}(\mu_t^\delta) < \mathbf{M}(M_t)+\delta,$$

for all $t \geq 0$. Then we will use the compactness theorem for general Brakke flows to obtain a sequence $\delta_i \downarrow 0$ and a forced Brakke flow $\mathcal{M}' = \{\mu_t'\}_{t\geq 0}$ such that

$$\mu_t^{\delta_i} \to \mu_t' \text{ and } \mu_t' \geq \mu_{M_t}.$$

However, since $\mathbf{M}(\mu_t^\delta) < \mathbf{M}(M_t)+\delta$, we will then have $\mathbf{M}(\mu_t') \leq \mathbf{M}(M_t)$ for all $t \geq 0$, i.e. $\mu_t' = \mu_{M_t}$ for all $t \geq 0$.

Define the set $T_\delta := \{t \geq 0 : \mathbf{M}(\mu_t) \geq \mathbf{M}(M_t)+\delta\}$. If $T_\delta = \emptyset$, then set $\mu_t^\delta = \mu_t$. Otherwise, let $t_0 := \inf T_\delta$. By Lemma 8.7, $\{M_t\}_{t\geq 0}$ is weakly continuous. By Lemma 4.1 (iii), we have $\lim_{t\downarrow t_0}\mathbf{M}(\mu_t) \leq \mathbf{M}(\mu_{t_0})$. Therefore

$$\mathbf{M}(M_{t_0})+\delta \leq \liminf_{t\downarrow t_0}\mathbf{M}(M_t)+\delta \leq \lim_{t\downarrow t_0}\mathbf{M}(\mu_t) \leq \mathbf{M}(\mu_{t_0}),$$

i.e. $t_0 \in T_\delta$.

Now, by Lemma 8.7, we may restart the flow at t_0 with M_{t_0} as the initial surface. Then Theorem 8.6 ensures the existence of an enhanced forced motion $(\tilde{M},\tilde{\mathcal{M}})$ satisfying

$$\tilde{M} \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+1}\times[t_0,\infty)),\quad \partial\tilde{M} = M_{t_0},\quad \tilde{\mu}_{t_0} = \mu_{M_{t_0}}.$$

We define the pair $(\hat{M},\hat{\mathcal{M}})$ by

$$\hat{M} = M - M\llcorner(\mathbf{R}^{n+1}\times[t_0,\infty)) + \tilde{M},\quad \hat{\mu}_t = \begin{cases}\mu_t & 0 \leq t < t_0\\ \tilde{\mu}_t & t_0 \leq t\end{cases},$$

and show that $(\hat{M},\hat{\mathcal{M}})$ is an enhanced motion for M_0.

Firstly, note that

$$\begin{aligned}\partial\hat{M} &= \partial M - \partial(M\llcorner(\mathbf{R}^{n+1}\times[t_0,\infty))) + \partial\tilde{M}\\ &= M_0 - M_{t_0} + M_{t_0} = M_0\end{aligned}.$$

Since

$$\mathbf{M}(M\llcorner(\mathbf{R}^{n+1}\times A)) \leq (\mathcal{L}^1(A)+\mathcal{L}^1(A)^{1/2})(\mathbf{M}(M_0)+c_g)$$

and

$$\mathbf{M}(\tilde{M}\llcorner(\mathbf{R}^{n+1}\times A)) \leq (\mathcal{L}^1(A)+\mathcal{L}^1(A)^{1/2})(\mathbf{M}(M_0)+c_g)$$

it follows that $\mathbf{M}(\hat{M}\llcorner(\mathbf{R}^{n+1}\times A)) \leq (\mathcal{L}^1(A)+\mathcal{L}^1(A)^{1/2})(\mathbf{M}(M_0)+c_g)$. Likewise we have

$$\mathbf{M}(\hat{\mu}_t) \le \mathbf{M}(M_0) + c_g$$

by (8.8). We also trivially have $\mu_{M_t} \le \hat{\mu}_t$, $\hat{\mu}_0 = \mu_{M_0}$. So all that remains to be checked is that $\hat{\mathcal{M}} = \{\hat{\mu}_t\}_{t\ge 0}$ is a forced Brakke flow, which is clear whenever $t \ne t_0$ and whenever $t \downarrow t_0$. Therefore we show that, for any $\varphi \in C_c^2(\mathbf{R}^{n+1},[0,\infty))$

$$D_t^- \hat{\mu}_{t_0}(\varphi) \le \mathcal{G}_{t_0}(\hat{\mu}_{t_0},\varphi),$$

where

$$D_t^- f(t_0) := \limsup_{t\uparrow t_0} \frac{f(t_0)-f(t)}{t_0 - t}.$$

Since we have

$$\hat{\mu}_{t_0} = \tilde{\mu}_{t_0} = \mu_{M_{t_0}} \le \mu_{t_0},$$

we may assume wlog that $\hat{\mu}_{t_0}(\varphi) = \mu_{t_0}(\varphi)$ (otherwise $D_t^-\hat{\mu}_{t_0}(\varphi) = -\infty$). Then

$$D_t^- \hat{\mu}_{t_0}(\varphi) = D_t^- \mu_{t_0}(\varphi) \le \mathcal{G}_{t_0}(\mu_{t_0},\varphi) = \mathcal{G}_{t_0}(\hat{\mu}_{t_0},\varphi).$$

Hence $\hat{\mathcal{M}} = \{\hat{\mu}_t\}_{t\ge 0}$ is a forced Brakke flow.

By assumption we have that M is the unique undercurrent, so $\hat{M} = M$. As before, we have

$$\lim_{t\downarrow t_0} \mathbf{M}(\hat{\mu}_t) \le \mathbf{M}(\hat{\mu}_{t_0}) = \mathbf{M}(M_{t_0}) \le \liminf_{t\downarrow t_0} \mathbf{M}(M_t),$$

that is, we have increased $\inf T_\delta$. Moreover

$$\mathbf{M}(\hat{\mu}_{t_0}) = \mathbf{M}(M_{t_0}) \le \mathbf{M}(\mu_{t_0}) - \delta \le \mathbf{M}(M_0) + c_g - \delta.$$

Therefore, if we repeat the above process, it will terminate after no more than $[(\mathbf{M}(M_0)+c_g)/\delta]$ iterations. Thus, after iterating the above process we will be left with an enhanced forced motion $(M, \mathcal{M}^\delta)$ satisfying

$$\mathbf{M}(\mu_t^\delta) < \mathbf{M}(M_t) + \delta,$$

for all $t \ge 0$.

Finally, as outlined above we may apply the compactness theorem for general Brakke flows to find a sequence $\delta_i \downarrow 0$ and a forced Brakke flow $\mathcal{M}' = \{\mu_t'\}_{t\ge 0}$ such that

$$\mu_t^{\delta_i} \to \mu_t' \text{ and } \mu_t' \ge \mu_{M_t}.$$

However, since $\mathbf{M}(\mu_t^\delta) < \mathbf{M}(M_t) + \delta$ for each $\delta > 0$, we in fact have

$$\mathbf{M}(\mu_t') \le \mathbf{M}(M_t),$$

for all $t \ge 0$, i.e. $\mu_t' = \mu_{M_t}$ for all $t \ge 0$. Hence $\mathcal{M} = \{\mu_{M_t}\}_{t\ge 0}$ is a forced Brakke flow, that is, M is a forced matching motion. ■

We now turn our attention to some geometric properties of forced matching motions. To do this, we will need to construct a new family of M^ε's.

Suppose $M \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+2})$ is a forced matching motion for the initial surface M_0. Let $t \ge 0$ and define the functional

$$G_t^\varepsilon(M) := \frac{1}{\varepsilon}\int e^{-z/\varepsilon} d\mu_M(x,z) - \frac{1}{\varepsilon}\int g^\varepsilon(x, z - \varepsilon t) e^{-z/\varepsilon} d\mu_\Omega(x,z).$$

By Lemma 6.2 there exists a G_t^ε-minimiser $M^\varepsilon(t)$ with initial surface M_t. So we definine the family of G_t^ε-minimisers $\{M^\varepsilon(t)\}_{t\ge 0}$:

Definition: *Let $M_0 \in \mathbf{I}_n^{loc}(\mathbf{R}^{n+1})$ be an initial surface and suppose $M \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+2})$ is a matching motion for M_0. For each $t \ge 0$, we define $M^\varepsilon(t)$ to be a G_t^ε- minimiser with initial surface M_t.*

Now for some observations.

Lemma 10.2: *Suppose $\mu_{M^{\varepsilon_i}(t)} \to \mu_t \times \mathcal{L}^1 \llcorner (0,\infty)$. Then*

$$\lim_{i\to\infty} \mu_{M^{\varepsilon_i}(t)} \llcorner \left(\frac{1}{\varepsilon_i} e^{-z/\varepsilon_i}\right) = \mu_t.$$

Remark: *If we have the convergence $\lim_{i\to\infty} \mu_{M^{\varepsilon_i}(t)} \llcorner (\frac{1}{\varepsilon_i} e^{-z/\varepsilon_i}) = \mu_t$, it follows that*

$$\mu_t(K) \le \liminf_{i\to\infty} \frac{1}{\varepsilon_i} \int_{K\times(0,\infty)} e^{-z/\varepsilon_i} d\mu_{M^{\varepsilon_i}(t)}$$

by lower-semicontinuity of mass. On the other hand, we have

$$\limsup_{i\to\infty}\frac{1}{\varepsilon_i}\int_{K\times(0,\infty)} e^{-z/\varepsilon_i}d\mu_{M^{\varepsilon_i}(t)} \le \mu_t(K)$$

by Lemma 6.2(ii) and (9.7) as computed in the proof of Theorem 9.6. Therefore, whenever $\mu_{M^{\varepsilon_i}(t)} \to \mu_t \times \mathcal{L}^1 \llcorner (0,\infty)$ *it follows that*

$$\lim_{i\to\infty}\frac{1}{\varepsilon_i}\int_{K\times(0,\infty)} e^{-z/\varepsilon_i}d\mu_{M^{\varepsilon_i}(t)} = \mu_t(K).$$

Proof of Lemma 10.2: We assume wlog $t = 0$. As indicated in the remark, the mass bound $G^\varepsilon(M^\varepsilon) \le G^\varepsilon(M_0 \times [[0,\infty)])$ and (9.7) implies

$$\limsup_{i\to\infty}\frac{1}{\varepsilon_i}\int_{K\times(0,\infty)} e^{-z/\varepsilon_i}d\mu_{M^{\varepsilon_i}} \le \mu_{M_0}(K) \tag{10.1}$$

On the other hand,

$$\liminf_{i\to\infty} d\mu_{M^{\varepsilon_i}} \ge d\mu_{M_0} d(\mathcal{L}^1\llcorner(0,\infty))$$

by the assumption $\mu_{M^{\varepsilon_i}(t)} \to \mu_t \times \mathcal{L}^1\llcorner(0,\infty)$ and lower-semicontinuity of mass. So, for any $\varphi \in C_c^0(\mathbf{R}^{n+1}\times(0,\infty))$ and for any $j \ge 0$,

$$\int \frac{1}{\varepsilon_j} e^{-z/\varepsilon_j}\varphi d\mu_{M_0} d\mathcal{L}^1 \le \liminf_{i\to\infty}\frac{1}{\varepsilon_j}\int \varphi e^{-z/\varepsilon_j} d\mu_{M^{\varepsilon_i}}.$$

Thus, by diagonalising,

$$\begin{aligned}\mu_{M_0}(\varphi) &= \lim_{j\to\infty}\int\frac{1}{\varepsilon_j}e^{-z/\varepsilon_j}\varphi d\mu_{M_0} d\mathcal{L}^1\\ &\le \lim_{j\to\infty}\liminf_{i\to\infty}\frac{1}{\varepsilon_j}\int\varphi e^{-z/\varepsilon_j}d\mu_{M^{\varepsilon_i}}\\ &= \liminf_{j\to\infty}\inf_{i\ge j}\frac{1}{\varepsilon_j}\int\varphi e^{-z/\varepsilon_j}d\mu_{M^{\varepsilon_i}}\\ &= \lim_{j\to\infty}\frac{1}{\varepsilon_j}\int\varphi e^{-z/\varepsilon_j}d\mu_{M^{\varepsilon_j}}\end{aligned}$$

since, for each j,

$$\inf_{i\ge j}\frac{1}{\varepsilon_j}\int\varphi e^{-z/\varepsilon_j}d\mu_{M^{\varepsilon_i}} = \frac{1}{\varepsilon_j}\int\varphi e^{-z/\varepsilon_j}d\mu_{M^{\varepsilon_j}}$$

by definition. Hence

$$\mu_{M_0} \le \lim_{i\to\infty}\mu_{M^{\varepsilon_i}}\llcorner\left(\frac{1}{\varepsilon_i}e^{-z/\varepsilon_i}\right)$$

and equality holds by (10.1). ■

We now introduce some new terminology to indicate when we are looking at a flow constructed using elliptic regularisation.

Definition (Regularised enhanced motion): *If an enhanced forced motion has been obtained using the methods of §§6-9, then we say that it is a regularised enhanced forced motion.*

Lemma 10.3: *Suppose* $M \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+2})$ *is a regularised forced matching motion. Then, for a.e.* $t \ge 0$,

(i) $\mu_{M^{\varepsilon_i}(t)} \to \mu_t \times \mathcal{L}^1\llcorner(0,\infty)$, *and*

(ii) there is a subsequence $\{i'\}$ *such that* $V_{\mu_{M^{\varepsilon_{i'}}(t)}} \to V_{\mu_t\times\mathcal{L}^1\llcorner(0,\infty)}$.

Remark: *This is really a version of the area continuity hypothesis* $\lim_{s\uparrow t}\mu_s = \mu_t$ *restated for a regularised forced matching motion.*

Proof of Lemma 10.3: By Lemma 4.1, μ_t is continuous at t and $\overline{D}_t\mu_t(\varphi) > -\infty$, for all $t \in \mathcal{T}_2$ where $\mathcal{T}_2$ is the set from Lemma 4.1. We assume t is such a time. Let $s > 0$ and define

$$P^\varepsilon(s,t) := (\sigma_{-s/\varepsilon})_\#(M^\varepsilon(t))\llcorner(\mathbf{R}^{n+1}\times(s,\infty))$$

where $\sigma_{-s/\varepsilon}(x,z) = (x, z - s/\varepsilon)$. Then, by Lemma 9.4,

$$\mu_{P^{\varepsilon_i}(s,t)} \to \nu_s \times \mathcal{L}^1\llcorner(0,\infty)$$

whenever ν_s is continuous at s. However $\nu_0 = \mu_t$ and $P^\varepsilon(0,t) = M^\varepsilon(t)$ for each $\varepsilon > 0$. Thus, by the continuity of μ_t at t, we have $\mu_{M^{\varepsilon_i}(t)} \to \mu_t \times \mathcal{L}^1\llcorner(0,\infty)$.

Furthermore, since $\overline{D}_t\mu_t(\varphi) > -\infty$, the definition of a general Brakke flow and (i) imply the existence of a subsequence such that $V_{\mu_{M^{\varepsilon_{i'}}(t)}} \to V_{\mu_t\times\mathcal{L}^1\llcorner(0,\infty)}$ as desired. ■

Combining Lemmas 10.2 and 10.3 we find:

Corollary 10.4: *Suppose $M \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^{n+2})$ is a regularised forced matching motion with initial surface M_0. Then, for a.e. $t \geq 0$,*

(i) $\lim_{i\to\infty} \mu_{M^{\varepsilon_i}(t)} \llcorner (\frac{1}{\varepsilon_i} e^{-z/\varepsilon_i}) = \mu_t$,

(ii) there is a subsequence $\{i'\}$ such that $\lim_{i'\to\infty} V_{\mu_{M^{\varepsilon_{i'}}(t)}} \llcorner (\frac{1}{\varepsilon_{i'}} e^{-z/\varepsilon_{i'}}) = V_{\mu_t}$, *and*

(iii) in particular $\lim_{i'\to\infty} \frac{1}{\varepsilon_{i'}} \int_{C_\rho(a_{i'})} |\nu_S{}^{\mathrm{T}}|^2 e^{-z/\varepsilon_{i'}} d\mu_{M^{\varepsilon_{i'}}(t)} = \int_{B_\rho^{n+1}(a)} |\nu_S{}^{\mathrm{T}}|^2 d\mu_t$,

whenever S is an n-dimensional subspace in $\mathbf{R}^{n+1} \times \{0\}$ and $a_i \to a$.

Remark: *Note that (iii) says that the tilt-excess of μ_t over $B_\rho^{n+1}(a)$ is the limit of the ε-tilt excess from §7:*

$$E^\varepsilon(a,\rho,S) = \rho^{-n} \frac{1}{\varepsilon} \int_{C_\rho(a)} |\nu_S^{\mathrm{T}}|^2 e^{-z/\varepsilon} d\mu_{M^\varepsilon(t)}.$$

Proof of Corollary 10.4: The first and second consequence follow by applying Lemmas 10.2 and 10.3.

Let $\psi \in C_c^0(G^{n+1}(\mathbf{R}^{n+2}))$. Then, by Lemma 7.2 we can find a subsequence $\{i'\}$ such that, at *a.e.* $t \geq 0$,

$$\mu_{M^{\varepsilon_{i'}}(t)} \llcorner (\frac{1}{\varepsilon_{i'}} e^{-z/\varepsilon_{i'}}) \to \mu_t \text{ and } V_{\mu_{M^{\varepsilon_{i'}}(t)}} \to V_{\mu_t \times \mathcal{L}^1 \llcorner (0,\infty)}$$

Therefore, by diagonalising,

$$\begin{aligned}\lim_{i'\to\infty} \frac{1}{\varepsilon_{i'}} \int \psi((x,z), T_{(x,z)}\mu_{M^{\varepsilon_{i'}}(t)}) e^{-z/\varepsilon_{i'}} d\mu_{M^{\varepsilon_{i'}}(t)} &= \int \psi((x,z), T_x\mu_t \times \mathcal{L}^1) d\mu_t \\ &= V_{\mu_t}(\psi)\end{aligned}$$

Let $a_i \in B_\delta^{n+1}(a)$. Then

$$\int_{B_\rho^{n+1}(a_i)} |\nu_S^{\mathrm{T}}|^2 d\mu_t \leq \int_{B_{\rho+\delta}^{n+1}(a)} |\nu_S^{\mathrm{T}}|^2 d\mu_t$$

and

$$\int_{B_\rho^{n+1}(a)} |\nu_S^{\mathrm{T}}|^2 d\mu_t \leq \int_{B_{\rho+\delta}^{n+1}(a_i)} |\nu_S^{\mathrm{T}}|^2 d\mu_t$$

Therefore, sending $\delta \to 0$, we obtain

$$\int_{B_\rho^{n+1}(a)} |\nu_S^{\mathrm{T}}|^2 d\mu_t = \lim_{i\to\infty} \int_{B_\rho^{n+1}(a_i)} |\nu_S^{\mathrm{T}}|^2 d\mu_t$$

Likewise, for each $k \geq 1$,

$$\frac{1}{\varepsilon_k} \int_{B_\rho^{n+1}(a)} |\nu_S^{\mathrm{T}}|^2 e^{-z/\varepsilon_k} d\mu_{M^{\varepsilon_k}(t)} = \lim_{i\to\infty} \frac{1}{\varepsilon_k} \int_{B_\rho^{n+1}(a_i)} |\nu_S^{\mathrm{T}}|^2 e^{-z/\varepsilon_k} d\mu_{M^{\varepsilon_k}(t)}$$

Hence using (iii) and diagonalising we obtain (iv). ■

Part IV – Regularity

In this part we prove a monotonicity formula related to the flow. This is the forced Brakke flow analogue to Huisken's monotonicity formula [H1]. The aim is to obtain some results concerning the Caussian density of the forced Brakke flow. We then use methods from [E], [I3], and [W2] to obtain a local regularity result in §12.

11 Another monotonicity formula

In [H1] Huisken proved the well known monotonicity formula for the mean curvature flow:

$$\frac{d}{dt}\int \rho_{y,s}\, d\mu_t = -\int \left| \vec{H} + \frac{(x-y)^{\perp}}{2(s-t)} \right|^2 \rho_{y,s}\, d\mu_t\,,$$

for all $t < s$ and where

$$\rho_{y,s}(x,t) := \frac{1}{(4\pi(s-t))^{n/2}} e^{-|x-y|^2/4(s-t)}\,.$$

In [E], [I3], and [W2] this has been used to obtain a local regularity theorem similar to Allard's regularity theorem [A], [S]. Together with Brakke's clearing out lemma [B 6.3], the local regularity theorem has been used in [E] and [I3] to prove a version of Brakke's famous regularity theorem [B 6.12].

Recall from the introduction that we call $\mathcal{M} = \{\Gamma_t\}_{0\le t<s}$ a smooth forced mean curvature flow if

$$\frac{\partial x}{\partial t} = \vec{H}_{\Gamma_t}(x) + g(x,t)\nu_{\Gamma_t}(x), \quad x \in \Gamma_t\,.$$

We have the following lemma for smooth forced mean curvature flows:

Lemma 11.1: *Let $s < \infty$. Suppose $\mathcal{M} = \{\Gamma_t\}_{0\le t<s}$ is a smooth forced mean curvature flow. Then, for any $R > 0$, there are constants $c_1 = c_1(p,g,s,R)$ and $c_2 > 0$ such that*

$$\mathcal{H}^n(\Gamma_t \cap B_{R/2}(y))^{2/p} \le 8(\mathcal{H}^n(\Gamma_0 \cap B_R(y))^{2/p} + c_1 R^2((1-e^{-2c_2 t/pR^2}))e^{2c_2 t/pR^2}\,,$$

for all $t \in [0, \min\{s, R^2/8n\}]$.

Remark: *Suppose we have the uniform bound*

$$\sup_{x\in\mathbb{R}^{n+1}} \sup_{R\ge 1} \mathcal{H}^n(\Gamma_0 \cap B_R(x))R^{-n} \le A\,.$$

Then Lemma 11.1 implies

$$\sup_{x\in\mathbf{R}^{n+1}}\sup_{R\ge 1}(\mathcal{H}^n(\Gamma_t\cap B_{R/2}(x))^{2/p}R^{-2n/p})\le c(A,n,p)(1+c_1\sup_{R\ge 1}(R^{2-2n/p}(e^{2c_2 s/pR^2}-1)))$$
$$=c(A,n,p,g,s),$$

for all $t\in[0,\min\{s,R^2/8n\}]$.

Hence Lemma 11.1 allows us to make the uniform assumption

$$\sup_{t<T}\sup_{x\in\mathbf{R}^{n+1}}\sup_{R\ge\max\{1,\sqrt{2ns}\}}\mathcal{H}^n(\Gamma_t\cap B_R(x))R^{-n}\le A_0.$$

We will follow Ecker's lead [E] and call this the area condition. *More generally, if we have* $\mathcal{M}=\{\Gamma_t\}_{t\in I}$ *for some bounded interval* $I\subset\mathbf{R}$, *the area condition is*

$$\sup_{t\in I}\sup_{x\in\mathbf{R}^{n+1}}\sup_{R\ge R_I}\mathcal{H}^n(\Gamma_t\cap B_R(x))R^{-n}\le A_0$$

for some $R_I<\infty$.

Proof: Define
$$\varphi_{y,s,R}(x,t):=(1-(|x-y|^2+2n(s-t))R^{-2})_+^3,$$

where $(a)_+=\max\{0,a\}$. Note that, for any n-dimensional subspace S and any time s

$$(\frac{\partial}{\partial t}-\operatorname{div}_S D)\varphi_{y,s,R}\le 0.$$

Then we have

$$\begin{aligned}\frac{d}{dt}\int_{\Gamma_t}\varphi_{y,s,R}\,d\mathcal{H}^n&=\int_{\Gamma_t}(H+g)(-H+D\varphi_{y,s,R}\cdot\nu/\varphi_{y,s,R})\varphi_{y,s,R}\,d\mathcal{H}^n+\int_{\Gamma_t}\frac{\partial\varphi_{y,s,R}}{\partial t}d\mathcal{H}^n\\&\le\int_{\Gamma_t}(-H^2+g(-H+D\varphi_{y,s,R}\cdot\nu/\varphi_{y,s,R}))\varphi_{y,s,R}\,d\mathcal{H}^n\\&\le\int_{\Gamma_t}(-\frac{1}{2}H^2+\frac{1}{2}g^2+g\,|D\varphi_{y,s,R}|/\varphi_{y,s,R})\varphi_{y,s,R}\,d\mathcal{H}^n\\&\le\int_{\Gamma_t}(-\frac{1}{2}H^2+g^2+|D\varphi_{y,s,R}|^2/\varphi_{y,s,R}^2)\varphi_{y,s,R}d\mathcal{H}^n\\&\le-\frac{1}{2}\int_{\Gamma_t}H^2d\mathcal{H}^n+\frac{1}{2}\max|D^2\varphi_{y,s,R}|\int_{\Gamma_t}\varphi_{y,s,R}\,d\mathcal{H}^n\\&\quad+\sup_{t<T}\|g(\cdot,t)\|^2_{L^p(\Gamma_t\cap B_R(y))}(\int_{\Gamma_t}\varphi_{y,s,R}\,d\mathcal{H}^n)^{1-2/p}.\end{aligned}$$

Let $u(t):=\int_{\Gamma_t}\varphi_{y,s,R}d\mathcal{H}^n$. Then

$$u'-c_1u^{1-2/p}-c_2u\le 0,$$

where $c_1=\sup_{t<s}\|g(\cdot,t)\|^2_{L^p(\Gamma_t\cap B_R(y))}$ and $c_2=R^2\max|D^2\varphi_{y,s,R}|/2\le 15$. Using the integrating factor e^{-c_2t} we have

$$\frac{d}{dt}(e^{-c_2t}u)^{2/p}\le\frac{2}{p}c_1e^{-2c_2t/pR^2},$$

or

$$e^{-2c_2t/p}u(t)^{2/p}\le u(0)^{2/p}+R^2\frac{c_1}{c_2}(1-e^{-2c_2t/pR^2}).$$

Therefore, setting $c=c_1/c_2$ we have

$$\mathcal{H}^n(\Gamma_t\cap B_{R/2}(y))^{2/p}\le 8(\mathcal{H}^n(\Gamma_0\cap B_R(y))^{2/p}+cR^2((1-e^{-2c_2t/pR^2}))e^{2c_2t/pR^2},$$

for all $t\in[0,\min\{s,R^2/8n\}]$ as desired. ∎

In this Part, we will be considering forced Brakke flows given by $\mathcal{M}=\{\mu_t\}_{t<s}$. Existence is guaranteed by Theorem 9.6.

Lemma 11.2 (monotonicity formula): *Let* $y\in\mathbf{R}^{n+1}$ *and* $s\in\mathbf{R}$. *Suppose* $\mathcal{M}=\{\mu_t\}_{t<s}$ *is a forced integer Brakke flow satisfying the area condition. Then, whenever* $|s-t|\le R$ *and* φ *is a test-function satisfying*

$$\varphi=1\text{ in }B_R(y),\ \varphi=0\text{ off }B_{2R}(y),\text{ and }R|D\varphi|+R^2|D^2\varphi|\le c_0,$$

we have (in the distributional sense)

$$\begin{aligned}\frac{d}{dt}\int\varphi\rho_{y,s}\,d\mu_t&\le-\frac{1}{2}\int|\bar{H}+\frac{(x-y)^\perp}{2(s-t)}|^2\,\varphi\rho_{y,s}\,d\mu_t\\&\quad+\Lambda(\int\varphi\rho_{y,s}\,d\mu_t)^{1-2/p}+\frac{c_3}{R^2},\end{aligned}$$

where $c_3=c_1(n,c_0,A)$ *and* $\Lambda=\sup_{t<s}\|g(\cdot,t)\|^2_{L^p(\mu_t\llcorner B_{2R}(y))}e^{-n/p}(8\pi R^2/n)^{-n/p}/2$.

Remark: *Other similar monotonicity formulas for weak curvature flows include Ilmanen's local version of Huisken's monotonicity formula (with* $g\equiv 0$, *see [I3 Lemma 7]), and White's monotonicity formula for K-almost Brakke flows (where* $g=K<\infty$, *see [W1 §11]). This version is the parabolic counterpart of the classic monotonicity formula for varifolds [S §17].*

Proof: Wlog we consider only $(y,s)=(0,0)$ and write $\rho=\rho_{0,0}$. For the proof we will require the identity

$$\frac{\partial \rho}{\partial t}+\operatorname{div}_S D\rho+\frac{|\nabla^{S^\perp}\rho|^2}{\rho}=0 \tag{11.1}$$

which is valid for any n - dimensional hyperplane S. This can be computed as follows:

$$\begin{aligned}\frac{\partial \rho}{\partial t}+\operatorname{div}_S D\rho+\frac{|\nabla^{S^\perp}\rho|^2}{\rho}&=\frac{1}{(4\pi)^{n/2}}e^{|x|^2/4t}[-\frac{n}{2(-t)^{n/2+1}}-\frac{|x|^2}{4(-t)^{n/2}t^2}+\frac{2n}{(-t)^{n/2}4t}\\&\quad+\frac{4|x^T|^2}{(-t)^{n/2}(4t)^2}+\frac{4|x^\perp|^2}{(-t)^{n/2}(4t)^2}]\\&=0.\end{aligned}$$

Now, suppose φ is a time-varying test-function and assume wlog $\overline{D}_t\mu_t(\varphi)>-\infty$. Then it is easy to see that

$$\overline{D}_t\mu_t(\varphi)\le \mathcal{G}_t(\mu_t,\varphi)+\mu_t(\varphi_{,t}),$$

where $\varphi_{,t}=\dfrac{\partial\varphi}{\partial t}$.

Let $\varphi=1$ in $B_R(y)$, $\varphi=0$ off $B_{2R}(y)$, and $R|D\varphi|+R^2|D^2\varphi|\le c_0$. Then, in the sense of distributions, we have

$$\begin{aligned}\frac{d}{dt}\int\varphi\rho_{y,s}\,d\mu_t&\le\int(-\varphi\rho_{y,s}H^2+\vec{H}\cdot\nabla^\perp(\varphi\rho_{y,s}))\,d\mu_t\\&+\int g\,|-\varphi\rho_{y,s}\vec{H}+\nabla^\perp(\varphi\rho_{y,s})|\,d\mu_t\\&+\int\frac{\partial}{\partial t}(\varphi\rho_{y,s})\,d\mu_t\\&\le-\int|\vec{H}-\frac{\nabla^\perp\rho_{y,s}}{\rho_{y,s}}|^2\varphi\rho_{y,s}\,d\mu_t\\&+\int(\varphi\frac{|\nabla^\perp\rho_{y,s}|^2}{\rho_{y,s}}-\varphi\vec{H}\cdot\nabla^\perp\rho_{y,s}+\rho_{y,s}\vec{H}\cdot\nabla^\perp\varphi)\,d\mu_t\\&+\int g^2\varphi\rho_{y,s}\,d\mu_t+\frac{1}{2}\int(|\vec{H}-\frac{\nabla^\perp\rho_{y,s}}{\rho_{y,s}}|^2\varphi\rho_{y,s}+\rho_{y,s}\frac{|\nabla^\perp\varphi|^2}{\varphi})\,d\mu_t\\&+\int(\frac{\partial\varphi}{\partial t}\rho_{y,s}+\varphi\frac{\partial\rho_{y,s}}{\partial t})\,d\mu_t\end{aligned}$$

$$\begin{aligned}&\le-\frac{1}{2}\int|\vec{H}+\frac{(x-y)^\perp}{2(s-t)}|^2\varphi\rho_{y,s}\,d\mu_t+\frac{1}{2}\int g^2\varphi\rho_{y,s}\,d\mu_t\\&+\int\varphi(\frac{\partial\rho_{y,s}}{\partial t}-\vec{H}\cdot\nabla^\perp\rho_{y,s}+\frac{|\nabla^\perp\rho_{y,s}|^2}{\rho_{y,s}})\,d\mu_t\\&+\int(\rho_{y,s}\frac{\partial\varphi}{\partial t}-\operatorname{div}_{\mu_t}(\rho_{y,s}\nabla^\perp\varphi)+\rho_{y,s}|D^2\varphi|)\,d\mu_t\\&\le-\frac{1}{2}\int|\vec{H}+\frac{(x-y)^\perp}{2(s-t)}|^2\varphi\rho\,d\mu_t+\frac{1}{2}\int g^2\varphi\rho_{y,s}\,d\mu_t\\&+\int\rho_{y,s}(\frac{\partial\varphi}{\partial t}-\operatorname{div}_{\mu_t}D\varphi+2|D^2\varphi|)\,d\mu_t\\&+\int\varphi|D^2\rho_{y,s}|\,d\mu_t\end{aligned}$$

where we have used the Cauchy-Schwarz inequality.

Now, for any n – dimensional hyperplane S we have

$$\frac{\partial\varphi}{\partial t}-\operatorname{div}_S D\varphi+2|D^2\varphi|\le\frac{4c_0}{R^2}.$$

Furthermore, we have

$$\sup_{x\in B_{2R}(y)}\rho_{y,s}\le\frac{1}{(4\pi(s-t))^{n/2}}e^{-R^2/(s-t)}.$$

It is easy to compute that

$$\frac{1}{(4\pi(s-t))^{n/2}}e^{-R^2/(s-t)}\le\frac{1}{(8\pi R^2/n)^{n/2}}e^{-n/2}. \tag{11.2}$$

Therefore, by the area condition we have

$$\int\rho_{y,s}(\frac{\partial\varphi}{\partial t}-\operatorname{div}_{\mu_t}D\varphi+2|D^2\varphi|)\,d\mu_t\le\frac{4Ac_0}{(8\pi/n)^{n/2}R^2}. \tag{11.3}$$

Similarly, we have that

$$\sup_{x\in B_{2R}(y)}|D^2\rho_{y,s}|\le\frac{1}{(4\pi(s-t))^{n/2}(s-t)}e^{-R^2/(s-t)}(1+\frac{R^2}{(s-t)})$$

which can easily shown to be bounded by $(4\pi)^{-n/2}(R^2/c_n)^{-n/2-1}e^{-c_n}$ where $c_n=\frac{1}{4}(n+2+\sqrt{(n+6)(n+2)})$. Therefore, again by the area condition, we have

$$\int \varphi |D^2\rho_{y,s}|\,d\mu_t \le \frac{Ac_n^{n/2+1}e^{-c_n}}{(4\pi)^{n/2}R^2}. \tag{11.4}$$

Finally, using (11.2) and the Hölder inequality, we have

$$\begin{aligned}\int g^2\varphi\rho_{y,s} &\le \left(\int \varphi\rho_{y,s}g^p\right)^{2/p}\left(\int \varphi\rho_{y,s}\right)^{1-2/p}\\ &\le \left(\int \varphi\rho_{y,s}\right)^{1-2/p}\sup_{B_{2R}(y)}\left[(\rho_{y,s})^{2/p}\left(\int_{B_{2R}(y)} g^p\right)^{2/p}\right]\\ &\le 2\Lambda\left(\int \varphi\rho_{y,s}\right)^{1-2/p}.\end{aligned}$$

Combining this with (11.3) and (11.4) completes the proof. ■

Corollary 11.3: *Let $y\in\mathbf{R}^{n+1}$ and $s\in\mathbf{R}$. Suppose $\mathcal{M}=\{\mu_t\}_{t<s}$ is a forced integer Brakke flow satisfying the area condition. Let φ be a test-function satisfying*

$$\varphi = 1 \text{ in } B_R(y),\ \varphi=0 \text{ off } B_{2R}(y), \text{ and } R|D\varphi|+R^2|D^2\varphi|\le c_0.$$

Then $[\int \varphi\rho_{y,s}d\mu_t + \frac{c_3}{R^2}(s-t)]^{2/p} + \frac{2\Lambda}{p}(s-t)$ is a non-increasing function of t on the interval $[s-R^2,s]$.

Proof: By Lemma 11.2 we have, in the sense of distributions,

$$\frac{d}{dt}\int \varphi\rho_{y,s}d\mu_t - \Lambda\left(\int\varphi\rho_{y,s}d\mu_t\right)^{1-2/p} - \frac{c_3}{R^2}\le 0$$

whenever $t\in[s-R^2,s]$.

Therefore

$$\frac{d}{dt}\left[\int\varphi\rho_{y,s}d\mu_t + \frac{c_3}{R^2}(s-t)\right]^{2/p}\le\frac{2\Lambda}{p}$$

since $t<s$. ■

Recall the function from the proof of Lemma 11.1:

$$\varphi_{y,s,R}(x,t) := (1-(|x-y|^2+2n(s-t))R^{-2})_+^3$$

where $(a)_+ = \max\{0,a\}$. We will write

$$\Phi_{y,s,R}(x,t) := \varphi_{y,s,R}(x,t)\rho_{y,s}(x,t)$$

and

$$\Phi_R(x,t) := \varphi_{0,0,R}(x,t)\rho_{0,0}(x,t).$$

The following definition essentially says that the point $(y,s)\in\mathbf{R}^{n+1}\times\mathbf{R}$ is in the support of $\mathcal{M}$.

Definition: *Let $s\in\mathbf{R}$, $y\in\mathbf{R}^{n+1}$, and suppose $\mathcal{M}=\{\Gamma_t\}_{t<s}$ is a smooth forced mean curvature flow. Suppose there exist sequences $t_i\uparrow s$ and $\{x_i\}_{i\ge1}$ with $x_i\in\Gamma_{t_i}$ and $x_i\to y$. Then we say that $\mathcal{M}$ reaches y at time s.*

Proposition 11.4: *Let $y\in\mathbf{R}^{n+1}$ and $s\in\mathbf{R}$. Suppose $\mathcal{M}=\{\mu_t\}_{t<s}$ is a forced integer Brakke flow satisfying the area condition. Then the gaussian density*

$$\Theta(\mathcal{M},y,s) := \lim_{t\uparrow s}\int\Phi_{y,s,R}(x,t)d\mu_t(x)$$

exists and is upper-semicontinuous. Moreover, if $\mathcal{M}=\{\Gamma_t\}_{t<s}$ is a smooth forced Brakke flow that reaches y at time s, then $\Theta(\mathcal{M},y,s)\ge1$.

Proof: Existence follows from Corollary 11.3.

For upper-semicontinuity, we wish to show that if $(x_i,t_i)\to(y,s)$ then

$$\Theta(\mathcal{M},y,s)\ge\limsup_{i\to\infty}\Theta(\mathcal{M},x_i,t_i).$$

Let $\tau<s$ be fixed. Let N be chosen so that $\tau<t_i<t$ for $i\ge N$. Then, Corollary 11.3 implies

$$\Theta(\mathcal{M},x_i,t_i)^{2/p}\le\left[\int\Phi_{x_i,t_i,R}(x,t)d\mu_t(x)+c_3R^{-2}(t_i-t)\right]^{2/p}+\frac{2\Lambda}{p}(t_i-t),$$

for any $t<t_i$. By the continuity of $\varphi\rho_{y,s}$ we have $\varphi\rho_{x_i,t_i}\to\varphi\rho_{y,s}$. Hence, taking limsup on both sides we have

$$\limsup_{i\to\infty}\Theta(\mathcal{M},x_i,t_i)^{2/p}\le\left[\int\Phi_{y,s,R}(x,t)d\mu_t(x)+c_3R^{-2}(s-t)\right]^{2/p}+\frac{2\Lambda}{p}(s-t).$$

Sending $t\uparrow s$ on the right hand side we obtain the upper-semicontinuity.

Suppose $\mathcal{M}=\{\Gamma_t\}_{t<s}$ is a smooth forced mean curvature flow that reaches y at time s. Assume that $\mathcal{M}$ is smooth near y. Consider the rescaling $x=\lambda\bar{x}+y$ and $t=\lambda\tau+s$ where $s>0$ and $\lambda>0$. Then the family $\mathcal{M}^\lambda_{y,s}=\{\Gamma_\tau^{\lambda,(y,s)}\}_{\tau<0}$ given by

$$\Gamma_\tau^{\lambda,(y,s)} := \frac{1}{\lambda}(\Gamma_{\lambda^2\tau+s}-y)$$

is a smooth forced mean curvature flow.

Since $\mathcal{M}$ is smooth near y we have

$$\lim_{\lambda \downarrow 0} \Gamma_\tau^{\lambda,(y,s)} = T_y \Gamma_s .$$

In general, the limit defines the *tangent flow to $\mathcal{M}$ at* (y, s). More on this in §12.

Note that

$$\begin{aligned}\Phi_{\lambda^{-1}R}(\bar{x},\tau) &= \Phi_{\lambda^{-1}R}(\lambda^{-1}(x-y), \lambda^{-2}(t-s)) \\ &= \Phi_{y,s,R}(x,t)\end{aligned}$$

from which it follows

$$\int \Phi_{y,s,R}(x,t) d\mu_t(x) = \int \Phi_{\lambda^{-1}R}(x,\tau) d\mu_\tau^{\lambda,(y,s)}(x) .$$

Therefore, for every $\tau < 0$,

$$\Theta(\mathcal{M}, y, s) = \lim_{\lambda \downarrow 0} \int \Phi_{\lambda^{-1}R}(x,\tau) d\mu_\tau^{\lambda,(y,s)}(x) = \int_{T_y\Gamma_s} \rho_{0,0}(x) d\mathcal{H}^n(x) = 1 .$$

Now, let

$$A := \{ y \in \mathbf{R}^{n+1} : \mathcal{M} \text{ reaches } y \text{ at } s, \Theta(\mathcal{M}, y, s) < 1 \} .$$

By the upper-semicontinuity of the gaussian density, it follows that A is relatively open in the set $\{\mathcal{M} \text{ reaches } y \text{ at } s\}$. So if $A \neq \varnothing$ it follows that $\mu_s(A) > 0$. But, since $\mathcal{M}$ is smooth, $\Theta(\mathcal{M}, y, s) = 1$ for *a.e.* y that $\mathcal{M}$ reaches at s. Hence $A = \varnothing$. ■

Corollary 11.5: *Let $y \in \mathbf{R}^{n+1}$ and $s \in \mathbf{R}$. Suppose $\mathcal{M} = \{\Gamma_t\}_{t<s}$ is a smooth forced mean curvature flow that reaches y at time s and satisfies the area condition. If $\Lambda(s-t) \le p/4$ then*

$$e^{2^{1+2/p}\Lambda(s-t)} \left(\int \Phi_{y,s,R}(x,t) d\mu_t(x) + \frac{c_3}{2^{1+2/p}\Lambda} R^{-2} \right)$$

is a non-increasing function of t on the interval $[s - mR^2, s]$ where $m := \min\{1, c_3^{-1} 2^{-1-p/2}\}$.

Proof: By Corollary 11.3 and Lemma 11.4 we have

$$\int \Phi_{y,s,R}(x,t)\, d\mu_t(x) \ge (1 - 2\Lambda(s-t)/p)^{p/2} - c_3 R^{-2}(s-t) .$$

Therefore, if $\Gamma(s-t) \le p/4$ and if $c_3(s-t)R^{-2} \le 2^{-1-p/2}$,

$$\begin{aligned}\int \Phi_{y,s,R}(x,t)\, d\mu_t(x) &\ge 2^{-p/2} - c_3 R^{-2}(s-t) \\ &\ge 2^{-1-p/2} .\end{aligned}$$

Applying this to Lemma 11.2 we have

$$\frac{d}{dt} \int \Phi_{y,s,R}(x,t)\, d\mu_t(x) - c_3 R^{-2} - 2^{1+2/p} \Lambda \int \Phi_{y,s,R}(x,t)\, d\mu_t(x) \le 0 .$$

Multiplying through by the integrating factor $e^{2^{1+2/p}\Lambda(s-t)}$ completes the proof. ■

Proposition 11.6 (clearing out): *Let $s \in \mathbf{R}$ and suppose $\mathcal{M} = \{\Gamma_t\}_{t<s}$ is a smooth forced mean curvature flow satisfying the area condition. Suppose $\mathcal{M}$ reaches y at time s. Then, for any $0 < \beta < 1/2n$ there exists a $0 < \theta = \theta(n,\beta) < 1$ such that for all $(1-2n\beta)^{-1/2} < \rho \le (p/(4\Lambda\beta))^{1/2}$,*

$$\mu_{s-\beta\rho^2}(B_\rho(y)) \ge \begin{cases} e^{-2^{1+2/p}\Lambda\beta\rho^2} \theta\rho^n - \dfrac{c_3}{2^{1+2/p}\Lambda}, & \Lambda > 0 \\ \theta\rho^n, & \Lambda = 0 \end{cases} .$$

Equivalently, if for some $0 < \beta < 1/2n$ and some $\rho \in ((1-2n\beta)^{-1/2}, (p/(4\Lambda\beta))^{1/2}]$

$$\mu_{s-\beta\rho^2}(B_\rho(y)) < \begin{cases} e^{-2^{1+2/p}\Lambda\beta\rho^2} \theta\rho^n - \dfrac{c_3}{2^{1+2/p}\Lambda}, & \Lambda > 0 \\ \theta\rho^n, & \Lambda = 0 \end{cases} ,$$

then there exists a $\delta > 0$ such that $\mu_t(B_\delta(y)) = 0$ for all $t \in (s - \delta^2, s)$.

Remarks: *Note that the interval $((1-2n\beta)^{-1/2}, (p/(4\Lambda\beta))^{1/2}]$ is non-empty iff $\beta < p(4\Lambda + 2np)^{-1}$. In the case when $\Lambda = 0$ this just says $\beta < 1/2n$.*

Suppose $\mathcal{M}$ is the volume preserving mean curvature flow of a sphere. Then we trivially have

$$\mu_{s-\beta\rho^2}(B_\rho(y)) > \omega_n \rho^n > e^{-2^{1+2/p}\Lambda\beta\rho^2} \theta\rho^n - \frac{c_1}{2^{1+2/p}\Lambda} ,$$

for any $\rho > 0$ and all $y \in S^n$.

Proof of Lemma 11.6: We assume $\Lambda \neq 0$ since that is the case for the mean curvature flow (see [E] for details). By Corollary 11.4 we have that

$$1 \le e^{2^{1+2/p}\Lambda(s-t)}\left(\int \Phi_{y,s,R}(x,t)\,d\mu_t(x) + \frac{c_3}{2^{1+2/p}\Lambda}R^{-2}\right)$$
$$\le e^{2^{1+2/p}\Lambda(s-t)}\left(\frac{(1+(s-t)R^{-2})^3}{(4\pi(s-t))^{n/2}}\mu_t(B_{\sqrt{R^2+2n(s-t)}}(y)) + \frac{c_3}{2^{1+2/p}\Lambda}R^{-2}\right),$$

whenever $\Lambda(s-t) \le p/4$. Let $\alpha > 0$ and set $t = s - \alpha R^2$. Then

$$\mu_t(B_{\sqrt{1+2n\alpha}R}(y)) \ge e^{-2^{1+2/p}\Lambda\alpha R^2}\frac{(4\pi\alpha)}{(1+2n\alpha)^3}R^n - \frac{c_3}{2^{1+2/p}\Lambda},$$

since $R \ge 1$. Set $\rho = \sqrt{1+2n\alpha}R$ and $\alpha R^2 = \beta\rho^2$ to obtain

$$\mu_{s-\beta\rho^2}(B_\rho(y)) \ge e^{-2^{1+2/p}\Lambda\beta\rho^2}\theta\rho^n - \frac{c_3}{2^{1+2/p}\Lambda}$$

for some $0 < \theta = \theta(n,\beta) < 1$. The fact that ρ must be in the interval $((1-2n\beta)^{-1/2}, (p/(4\Lambda\beta))^{1/2}]$ can be seen as follows:
Firstly $1+2n\alpha = (1-2n\beta)^{-1}$, so the condition $\Lambda(s-t) \le p/4$ is satisfied iff

$$\frac{p}{4} \ge \Lambda\beta\rho^2,$$

or

$$\rho \le (p/(4\Lambda\beta))^{1/2}.$$

The lower bound follows from $\rho = \sqrt{1+2n\alpha}R$ and $1+2n\alpha = (1-2n\beta)^{-1}$. ■

12 Local regularity

Recall the $C^{1,\alpha}$, $C^{2,\alpha}$ and $W^{2,p}$ norms of a smooth mean curvature flow $\mathcal{M} = \{\Gamma_t\}_{t<s}$ at a point $(x,t) \in \mathcal{M}$: $K_{1,\alpha}(\mathcal{M},x,t)$, $K_{2,\alpha}(\mathcal{M},x,t)$ and $J_{2,p}(\mathcal{M},x,t)$ (respectively) as defined at the end of §1.

In this section we prove the following local regularity theorem:

Theorem 12.1: *Let $p \in [2,\infty) \cap (n,\infty)$. There exists a $\delta_0 > 0$ and a $C_0 > 0$ such that for any $(y,s) \in \mathbf{R}^{n+1} \times \mathbf{R}$ and any smooth forced mean curvature flow $\mathcal{M} = \{\Gamma_t\}_{t<s}$ satisfying, for some $\rho > 0$*

$$\int \Phi_{x,t,R}\,d\mu_\tau \le 1 + \delta_0$$

for all $(x,t) \in B_\rho(y) \times (s-\rho^2, s)$ and all $\tau \in (t-\rho^2, t)$, then, for any $0 < \sigma < \rho$,

$$\sigma^2 \sup_{t\in[s-(\rho-\sigma)^2,s)}\ \sup_{x\in B_{\rho-\sigma}(y)\cap\Gamma_t}[J_{2,p}(\mathcal{M},x,t) + K_{1,\alpha}(\mathcal{M},x,t)] \le C_0(1+\rho^2\Lambda)$$

where $\Lambda = \sup_{t<s}\| g(\cdot,t)\|^2_{L^p(\mu_t \llcorner B_{2R}(y))} e^{-n/p}(8\pi R^2/n)^{-n/p}/2 < \infty$ and $\alpha = 1 - n/p$.

This is similar to [W2 6.1]. There, White defines what he calls a *Brakke operator* to act as a forcing term. For a given flow $\mathcal{M}$, if the mean curvature vector exists at x at time t, the Brakke operator $\beta(\mathcal{M}): U \times (-\infty,s) \times G_n(U) \to \mathbf{R}^{n+1}$ is defined by

$$\beta(\mathcal{M})(x,t,S) = \nu(x,t) - \vec{H}(x,t),$$

where $G_n(U)$ is the Grassmanian on the set U (see §1) and ν is the normal velocity of the flow at x. Examples (from [W2 §4]) include:

(i) The Brakke operator for a compact embedded hypersurface M moving by the gradient flow for the functional (area – volume) is $\beta(\mathcal{M})(x,t,T_xM_t) = \nu_{M_t}(x)$, where $\nu_{M_t}(x)$ is the outward pointing unit normal to the surface at time t.

(ii) The mean curvature flow on the unit sphere has $\beta(\mathcal{M})(x,t,S) = nx$ for its Brakke operator, where S is an n-dimensional subspace.

Concentrating on regularity, White doesn't prove existence for flows with non-zero Brakke operator.

Since the smooth forced mean curvature flow is the gradient flow for the functional (area – $\int_\Omega g\,d\mathcal{L}^{n+1}$), where Ω denotes the region enclosed by the evolving hypersurface, the Brakke operator for the smooth forced mean curvature flow is

$$\beta(x,t,S) := g(x,t)\nu_S(x)$$

for any n-dimensional subspace S. This requires an orientation on the flow and therefore isn't applicable to varifolds, unlike the more general approach we've taken so far. Furthermore, only the case $g \in L^\infty$ is treated in [W2 6.1], corresponding to a K-almost Brakke flow [W1 §11] with

$$K = \sup_t \sup_x |g(x,t)|.$$

As a slight interlude, we will first prove an interesting property of the *tangent flows* used in the proof of Proposition 11.4. To do this, we formally define the tangent flow as follows:

Let $\mathcal{M} = \{\mu_t\}_{t<s}$ be a forced Brakke flow. Let $(x_0,t_0) \in \mathbf{R}^{n+1} \times (-\infty,s)$ and let $\lambda > 0$. Set $t = \lambda^2\tau + t_0$ and $y = \lambda x + x_0$. Define $\mathcal{M}^\lambda_{x_0,t_0} = \{\mu_\tau^{\lambda,(x_0,t_0)}\}_{\tau<0}$ where

$$\int f(y,\tau)\, d\mu_\tau^{\lambda,(x_0,t_0)}(y) = \frac{1}{\lambda^n}\int f(\lambda^{-1}(x-x_0), \lambda^{-2}(t-t_0))\, d\mu_{\lambda^2\tau+t_0}(x).$$

Note that

$$\begin{aligned}\rho(y,\tau) &:= \rho_{0,0}(\lambda^{-1}(x-x_0), \lambda^{-2}(t-t_0)) \\ &= \rho_{x_0,t_0}(x,t)\end{aligned}$$

from which it follows

$$\int \rho_{x_0,t_0}\, d\mu_t = \int \rho\, d\mu_\tau^{\lambda,(x_0,t_0)}. \tag{12.1}$$

Therefore, for every $\tau < 0$,

$$\lim_{\lambda\downarrow 0}\int \rho\, d\mu_\tau^{\lambda,(x_0,t_0)} = \Theta(\mathcal{M}, x_0, t_0).$$

Now, by Theorem 5.1, we can find a sequence $\lambda_i \downarrow 0$ and a forced Brakke flow $\mathcal{M}' = \{\mu'_\tau\}_{\tau<0}$ such that

$$\mu_\tau^{\lambda_i,(x_0,t_0)} \to \mu'_\tau,$$

for any $\tau < 0$. We call $\mathcal{M}'$ a *tangent flow to $\mathcal{M}$ at (x_0,t_0)*.

Lemma 12.2: *Let $p > n$ and let $\mathcal{M} = \{\mu_t\}_{t<s}$ be a forced Brakke flow. Let $(x_0,t_0) \in \mathbf{R}^{n+1} \times (-\infty,s)$. Then any tangent flow to $\mathcal{M}$ at (x_0,t_0) is a Brakke flow.*

Proof: For any $\lambda > 0$, $\mathcal{M}^\lambda_{x_0,t_0} = \{\mu_\tau^{\lambda,(x_0,t_0)}\}_{\tau<0}$ is a forced Brakke flow with forcing term given by

$$g^\lambda(y,\tau) = \lambda g(\lambda y + x_0, \lambda^2\tau + t_0).$$

Now, suppose wlog $(x_0,t_0) = (0,0)$. We have

$$\begin{aligned}\| g^\lambda(\cdot,\tau)\|_{L^p(\mu_\tau^{\lambda,(0,0)})} &= \lambda(\lambda^{-n}\int g(x,\lambda^2\tau)^p\, d\mu_{\lambda^2\tau}(x))^{1/p} \\ &= \lambda^{1-n/p}\| g(\cdot,\lambda^2\tau)\|_{L^p(\mu_{\lambda^2\tau})} \\ &\le \lambda^{1-n/p}\sup_{t<0}\| g(\cdot,t)\|_{L^p(\mu_t)}.\end{aligned}$$

Since $p > n$, we therefore have $g^\lambda \to 0$ in L^p. Consequently, any tangent flow to $\mathcal{M}$ at (x_0,t_0) is a Brakke flow. ■

We recall the following local regularity result for smooth mean curvature flow which is, of course, valid for tangent flows to forced Brakke flows:

Theorem 12.3 [W2 3.1]: *For $0 < \alpha < 1$, there exists a $\delta_0 > 0$ and a $C_0 > 0$ such that for any $(y,s) \in \mathbf{R}^{n+1} \times \mathbf{R}$ and any smooth mean curvature flow $\mathcal{M} = \{\Gamma_t\}_{t<s}$ satisfying*

$$\Theta(\mathcal{M}, y, s) \le 1 + \delta_0.$$

Then, there exists a $\rho > 0$ such that, for any $0 < \sigma < \rho$,

$$\sigma^2 \sup_{t\in[s-(\rho-\sigma)^2,s)}\ \sup_{x\in B_{\rho-\sigma}(y)\cap\Gamma_t} K_{2,\alpha}(\mathcal{M},x,t) \le C_0.$$

Now to the proof of Theorem 12.1:

Proof of Theorem 12.1: We may assume the flow is smooth up to time 0, since we could first prove the theorem for s replaced by $s - \delta^2$ and then let $\delta \to 0$. We assume wlog that $(y,s) = (0,0)$ and that $\mathcal{M}$ reaches 0 at time 0. By Morrey's lemma ([G] or [GT]), we need only show the estimate

$$\sigma^2 \sup_{t\in[s-(\rho-\sigma)^2,s)}\ \sup_{x\in B_{\rho-\sigma}(y)\cap\Gamma_t} J_{2,p}(\mathcal{M},x,t) \le C_0(1+\rho^2\Lambda)$$

is true.

Now, suppose the theorem is false. Then, for any $i \ge 1$, we can find a smooth forced Brakke flow $\mathcal{M}^i = \{\Gamma^i_t\}_{t\le 0}$ and $\rho_i > 0$ such that

$$1 \le \int \Phi_{x,t,R}\, d\mu_t \le 1 + \frac{1}{i},$$

for all $(x,t) \in B_{\rho_i}(0) \times (-\rho_i^2, 0)$ and all $\tau \in (t-\rho_i^2, t)$ but so that, for some $0 < \sigma_i < \rho_i$

$$\eta_i^2 := \sigma_i^2 \sup_{t\in[-(\rho_i-\sigma_i)^2,0)}\ \sup_{x\in B_{\rho_i-\sigma_i}(0)\cap\Gamma^i_t} \frac{J_{2,p}(\mathcal{M}^i,x,t)}{1+\rho_i^2\Lambda_i} \ge i. \tag{12.2}$$

We can find times $-(\rho_i - \sigma_i)^2 \le t_i \le 0$ and points $x_i \in \overline{B}_{\rho_i-\sigma_i}(0) \cap \Gamma^i_{t_i}$ such that

$$\sigma_i^2 \frac{J_{2,p}(\mathcal{M}^i, x_i, t_i)}{1+\rho_i^2 \Lambda_i} = \eta_i^2 .$$

Note that,

$$\frac{\sigma_i^2}{4} \sup_{t\in[-(\rho_i-\sigma_i/2)^2,0)} \sup_{x\in B_{\rho_i-\sigma_i/2}(0)\cap\Gamma^i_t} \frac{J_{2,p}(\mathcal{M}^i,x,t)}{1+\rho_i^2\Lambda_i} \le \eta_i^2$$

which gives

$$\sup_{t\in[-(\rho_i-\sigma_i/2)^2,0)} \sup_{x\in B_{\rho_i-\sigma_i/2}(0)\cap\Gamma^i_t} J_{2,p}(\mathcal{M}^i,x,t) \le 4J_{2,p}(\mathcal{M}^i,x_i,t_i) .$$

Now, since $t_i \ge -(\rho_i - \sigma_i)^2 \ge \sigma_i^2/4 - (\rho_i - \sigma_i/2)^2$ and since $x_i \in \overline{B}_{\rho_i-\sigma_i}(0)$ we have

$$t_i - \frac{\sigma_i^2}{4} \ge -(\rho_i - \frac{\sigma_i}{2})^2 \text{ and } B_{\sigma_i/2}(x_i) \subset \overline{B}_{\rho_i- \;\; i/2}(0)$$

so that

$$\sup_{t\in[t_i-(\sigma_i/2)^2,t_i)} \sup_{x\in B_{\sigma_i/2}(x_i)\cap\Gamma^i_t} J_{2,p}(\mathcal{M}^i,x,t) \le 4J_{2,p}(\mathcal{M}^i,x_i,t_i) .$$

Let $\lambda_i = 1/\sqrt{J_{2,p}(\mathcal{M},x_i,t_i)}$ and define

$$\widetilde{\Gamma}^i_s := \frac{1}{\lambda_i}(\Gamma^i_{\lambda_i^2 s+t_i} - x_i)$$

for $s \in [-\sigma_i^2/(4\lambda_i^2), 0]$. Note that it is possible for λ_i to remain finite when we send $i \to \infty$ so what we obtain after sending $i \to \infty$ won't necessarily be a tangent flow. With this definition we have that, for all $i \ge 1$, $\widetilde{\mathcal{M}}^i = \{\widetilde{\Gamma}^i_s\}$ is a smooth forced mean curvature flow satisfying

$$0 \in \widetilde{\Gamma}^i_0,\ J_{2,p}(\widetilde{\mathcal{M}}^i,0,0) = 1$$

and

$$\sup_{s\in[-(\sigma_i/2\lambda_i)^2,0)} \sup_{x\in B_{\sigma_i/2\lambda_i}(x_i)\cap\widetilde{\Gamma}^i_s} J_{2,p}(\widetilde{\mathcal{M}}^i,x,s) \le 4,$$

for all $i \ge 1$.

Since $\sigma_i^2\lambda_i^{-2} \ge \eta_i^2 \ge i$ it follows that, for every $R > 0$, we can find an $N_R \ge 1$ such that

$$\sup_{s\in[-R^2,0)} \sup_{x\in B_R(0)\cap\widetilde{\Gamma}^i_s} J_{2,p}(\widetilde{\mathcal{M}}^i,x,s) \le 4, \tag{12.3}$$

for all $i \ge N_R$. Furthermore, by (12.1) we have

$$1 \le \int \Phi_R d\widetilde{\mu}^i_\tau \le 1 + \frac{1}{i}, \tag{12.4}$$

for all $s \in (-\rho_i^2\lambda_i^{-2}, 0)$.

By the compactness theorem we can find a subsequence and a integer forced Brakke flow $\widetilde{\mathcal{M}} = \{\widetilde{\Gamma}_t\}_{t<0}$ such that

$$\widetilde{\mathcal{M}}^i \to \widetilde{\mathcal{M}}$$

and, by the Arzela-Ascoli theorem [W2 2.6],

$$J_{2,p}(\widetilde{\mathcal{M}},x,s) \le 4 \text{ and } J_{2,p}(\widetilde{\mathcal{M}},0,0) = 1 . \tag{12.5}$$

We now show that $\lim_{i\to\infty}\Lambda_i = 0$. If this is the case, then Lemma 9.1 and (12.4) implies that $\widetilde{\mathcal{M}}$ is a homothetically shrinking mean curvature flow. The convergence is easily shown as follows:

$$\Lambda_i \le \frac{1+\Lambda_i\rho_i^2}{\sigma_i^2} \le \frac{4}{\mu_i^2} \le \frac{4}{i} \to 0 \tag{12.6}$$

by (12.2) and (12.3). This calculation wouldn't be possible if the right hand side of the estimate in the conclusion of the theorem was $C_0(1+\Lambda)$.

By Morrey's Lemma and (12.5), it follows that $K_{1,\alpha}(\widetilde{\mathcal{M}},x,t) < \infty$ where $\alpha = 1 - n/p \in (0,1)$. However, since $\widetilde{\mathcal{M}}$ is a homothetically shrinking mean curvature flow smooth up to $s = 0$, it follows that $\widetilde{\Gamma}_{-1}$ is a hyperplane. Therefore $K_{1,\alpha}(\widetilde{\mathcal{M}},x,t) = 0$ for all x and all t.

Using the definition of the parabolic $C^{1,\alpha}$ norm and the $C^{1,\alpha}$ convergence, we can find a sequence $r_i \to \infty$ and functions $u_i : B^n_{r_i}(0) \times (-r_i^2, r_i^2) \to \mathbf{R}$ such that

$$\operatorname{graph} u_i = \widetilde{\mathcal{M}}^i \cap (B_{r_i}(0) \times (-r_i^2, r_i^2))$$

and so that $\| u_i \|_{1,\alpha} \to 0$. The u_i are weak solutions to

$$\frac{\partial u_i}{\partial t} - \Delta u_i = g_i + f_i$$

where

$$f_i := \sum_{j,k=1}^{n} \frac{D_j u_i D_k u_i}{1+|Du_i|^2} D_j D_k u_i .$$

Since $\| g_i \|_p \to 0$ and $\| f_i \|_p \to 0$ by (12.6) and the $C^{1,\alpha}$ convergence (respectively), we have, by [GT 9.12], that $\| u_i \|_{2,p} \to 0$. By the definition of $J_{2,p}(\widetilde{\mathcal{M}},0,0)$, this contradicts (12.5). ∎

Corollary 12.4: *Let* $p \in [2,\infty) \cap (n,\infty)$. *There exist constants* $\delta_0 > 0$ *and a* $C_0 > 0$ *such that for any* $(y,s) \in \mathbf{R}^{n+1} \times \mathbf{R}$ *and any smooth mean curvature flow* $\mathcal{M} = \{\Gamma_t\}_{t<s}$ *there exists a* $\rho > 0$ *such that if*

$$\Theta(\mathcal{M}, y, s) \leq 1 + \delta_0$$

then, for any $0 < \sigma < \rho$,

$$\sigma^2 \sup_{t \in [s-(\rho-\sigma)^2, s)} \sup_{x \in B_{\rho-\sigma}(y) \cap \Gamma_t} [J_{2,p}(\mathcal{M},x,t) + K_{1,\alpha}(\mathcal{M},x,t)] \leq C_0(1 + \rho^2 \Lambda),$$

where $\alpha = 1 - n/p$.

Proof: We begin by showing that $\Theta(\mathcal{M}, y, s) \leq 1 + \delta_0$ implies the existence of a $\rho > 0$ such that the hypotheses of Theorem 12.1 are satisfied. We assume y and s are 0.

Suppose $\mathcal{M}$ satisfies $\Theta(\mathcal{M}, 0, 0) \leq 1 + \delta_0$. Then, by the existence of the limit $\Theta(\mathcal{M},0,0) := \lim_{t \uparrow 0} e^{-2^{1+2/p}\Lambda t} \int \Phi_R d\mu_t$, we can find a $\rho_0 > 0$ such that

$$e^{2^{1+2/p}\Lambda \rho_0^2} \int \Phi_R \, d\mu_{-\rho_0^2} \leq 1 + \frac{4}{3}\delta_0.$$

The continuity of the map $(y,s) \mapsto \int \Phi_{y,s,R} d\mu_{-\rho_0^2}$ implies that there exists a $0 < \sigma_0 < \rho_0$ such that, for all $(y,s) \in B_{\sigma_0}(0) \times (-\sigma_0^2, 0)$

$$e^{2^{1+2/p}\Lambda \rho_0^2} \int \varphi_{(y,s),R} \rho_{y,s} \, d\mu_{-\rho_0^2} \leq 1 + \frac{5}{3}\delta_0.$$

By reducing σ_0 if necessary we may assume $(s - \sigma_0^2, s) \subset (-\rho_0^2, 0)$ and, reducing ρ_0 if necessary, we may also assume $\Lambda \rho_0^2 \leq p/4$. This, together with the assumption $(s - \sigma_0^2, s) \subset (-\rho_0^2, 0)$, implies $\Lambda(s - t) \leq p/4$ for all $t \in (s - \sigma_0^2, s)$.

By Corollary 11.5 we have

$$\begin{aligned} e^{2^{1+2/p}\Lambda(s-\tau)} &\Big(\int \varphi_{(y,s),R} \rho_{y,s} \, d\mu_\tau + \frac{c_3}{2^{1+2/p}\Lambda} R^{-2}\Big) \\ &\leq e^{2^{1+2/p}\Lambda(s+\rho_0^2)} \Big(\int \varphi_{(y,s),R} \rho_{y,s} \, d\mu_{-\rho_0^2} + \frac{c_3}{2^{1+2/p}\Lambda} R^{-2}\Big) \\ &\leq e^{2^{1+2/p}\Lambda s} \Big(1 + \frac{5}{3}\delta_0 + e^{2^{1+2/p}\Lambda \rho_0^2} \frac{c_3}{2^{1+2/p}\Lambda} R^{-2}\Big) \end{aligned}$$

for all $\tau \in [s - \sigma_0^2, s]$. Rewriting this we obtain

$$\int \varphi_{(y,s),R} \rho_{y,s} \, d\mu_\tau \leq e^{2^{1+2/p}\Lambda \tau} \Big(1 + \frac{5}{3}\delta_0 + \frac{c_3}{2^{1+2/p}\Lambda} R^{-2} (e^{2^{1+2/p}\Lambda \rho_0^2} - 1)\Big)$$

for all $(y,s) \in B_{\sigma_0}(0) \times (-\sigma_0^2, 0)$ and all $\tau \in [s - \sigma_0^2, s]$. Further reduction of ρ_0 (if necessary) gives

$$\int \varphi_{(y,s),R} \rho_{y,s} \, d\mu_\tau \leq 1 + 2\delta_0$$

for all $(y,s) \in B_{\sigma_0}(0) \times (-\sigma_0^2, 0)$ and all $\tau \in [s - \sigma_0^2, s]$. Such a ρ_0 exists since

$$\frac{c_3}{2^{1+2/p}\Lambda} R^{-2} (e^{2^{1+2/p}\Lambda \rho_0^2} - 1) \leq \frac{1}{3}\delta_0$$

has a positive solution for any $\delta_0 > 0$.

We complete the proof by applying Theorem 12.1. ∎

Appendix A: Ferronematics

In [ZI], Zubarev & Iskakova consider an initially flat layer (say of depth d) of a liquid crystal containing ferromagnetic particles, called a *ferronematic* (see figure A). The surface is $M_0 = [\mathbf{R}^2 \times \{0\}]$ which is locally finite. The applied magnetic field is assumed to be in the vertical direction. The ferromagnetic particles in the crystal produce a magnetic field $\bar{h} = h_i e_i$ that adds vectorially with the external magnetic field. This is called the *demagnetisation* of the ferronematic.

To model this, we will construct an evolution of M_0. We make the simplifying assumption that the layer is infinitely deep. Previously, the demagnetisation vector has been assumed to be smooth [ZI, p187].

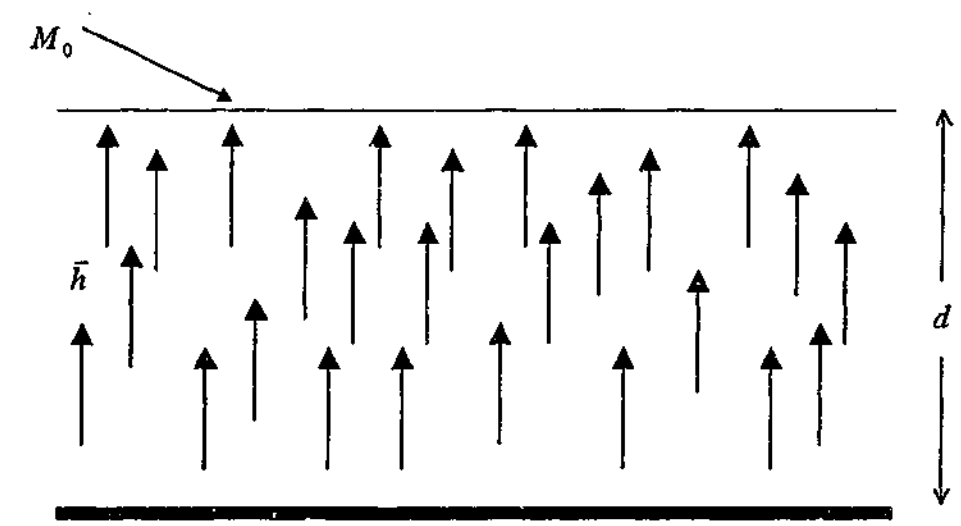

Figure A

Let $\alpha : U \to \mathbf{R}$, $U \subset \mathbf{R}^3$, be the magnetic potential of $\bar{h}$. By Maxwell's equations we have

$$\Delta\alpha + \lambda^2 \frac{\partial^2 \alpha}{\partial z^2} = 0,$$

where $\lambda^2 \geq 1$. By [GT 9.13] we have that if $\alpha \in L^p(U)$ then $\alpha \in W^{2,p}(K')$, for any $K' \subset\subset U$. Therefore, for any $K' \subset\subset U$, $h_i \in W^{1,p}(K')$ if $\alpha \in L^p(U)$. If we make the reasonable physical assumption $\alpha \in L^p(U)$ then our demagnetisation vector is $W^{1,p}$.

If the external magnetic field is sufficiently strong, the demagnetisation alligns itself with the external magnetic field. Thus, as time progresses, a family of demagnetisation vectors $\bar{h}_t(x, y, z) = h(x, y, z, t)$ is produced. There exists a time when the demagnetising vector will be equal to the external magnetic field $m_0 e_3$.

We consider the forcing term given by

$$g(x,y,z,t)=\frac{m_0}{2\sigma}h_3(x,y,z,t),$$

where σ is the co-efficient of the surface tension, and m_0 is the strength of the external magnetic field.

Let $T=\inf\{t>0:h_3=m_0e_3\}$. We will call this the *drying time.*

Let $\delta>0$. We redefine g as

$$g(x,y,z,t)=\begin{cases}\frac{m_0}{2\sigma}h_3(x,y,z,t), & t\le T\\ 0, & t\ge T+\delta\end{cases}$$

and linearly interpolated (*a.e.*) between.

Lemma A1: *There exists an enhanced forced motion* $(M,\{\mu_t\}_{0\le t<T})$ *(a ferronematic flow, if you will) with initial surface* M_0.

Proof: As outlined above, we know that $g(\cdot,t)\in L^p(U)$ and that $\sup_{t\ge0}\|g(\cdot,t)\|_{L^p(U)}<\infty$. Also, since the demagnetising vector is vertical, we also have $g(\cdot,t)\in L^p(\mu)$ for all $\mu\in\mathcal{M}_n(\mathbf{R}^3)$. Since h_3 changes (in time) at the constant rate m_0, the forcing term is continuous in time $a.e.$ Therefore we can find an enhanced forced motion $(M,\{\mu_t\}_{t\ge0})$, in particular $(M,\{\mu_t\}_{0\le t<T})$ is an enhanced forced motion. ■

Remark: *The theory in [ZI] is closely related to the zebra rocks (figure 1) [M+]. So, it may be possible to construct a model for the zebra rocks using a forced Brakke flow. This would probably be best achieved by including a volume preserving term in the equations and evolving the interface between magnetic and neutral regions. However, it isn't clear* a priori *if such a term is in bounded up until the drying time (see the discussion after the examples in §4).*

Appendix B: Biased search patterns

This application was inspired by a conference seminar given by Justin Beck on 2nd May, 2002 at the Defence Operations Analysis Symposium in Adelaide. Hellman [He] has considered the problem of the optimal search for a target moving randomly in a region $U\subset\mathbf{R}^n$. This problem is relevant in a wide number of cases ranging from anti-submarine warfare and airborne early warning, to, say, pest control and swatting a fly.

Let $u_0:\mathbf{R}^n\to\mathbf{R}$ be the initial probability density function vanishing outside U. The search pattern is given by a PDE, for example the diffusion equation, though the equation need not include any spacial derivatives. One might use the diffusion equation if the target were moving randomly. In general, the probability density of a moving target satisfies

$$\frac{\partial u}{\partial t}(x,t)=L(x,t,u),$$

where L is a linear operator independent of time [He].

If at a later time t a search is made in some region by an operator and the target is not found, then the probability density at time t would be altered to include this knowledge. That is, in order to include active searching, one would introduce a forcing term to the equations:

$$\frac{\partial u}{\partial t}(x,t)=L(x,t,u)+g(x,t).$$

The solution u allows one to calculate the effectiveness of the search.

One drawback to using the diffusion equation is that it tends to "smear" any information on the whereabouts of the target, tantamount to assuming the target is constantly on the move. That is, the method assumes the target's motion is limited in complexity. Furthermore, suppose the operator were biased in their thinking that an initial guess to the location of the target was "pretty close" (as we are loath to accept total control by a computer). Then, in the mind of the operator, the shape of the probability density should be preserved for as long as possible.

Consider the case where

$$L=\Delta u-\frac{D_iuD_ju}{1+|Du|^2}D_iD_ju.$$

This gives rise to a forced mean curvature flow of the graph of u. This is chosen because the mean curvature flow preserves the overall shape of the probability density better than a Laplacian. This shape preservation could be interpreted as a representation of the aforementioned bias in an operator that the initial probability density was a good approximation to the location of the target.

Let g be the active search pattern as controlled by an operator. Our assumptions in §8 on the forcing term can be interpreted as follows:

(i) $\lim_{s \to t} g(\cdot, s) = g(\cdot, t)$ *a.e.* says that the search cannot be instantly changed everywhere,

(ii) the condition that g vanishes after some time is a restriction on the amount of time that can be spent actively searching for the target, and

(iii) the L^p conditions are restrictions on how much effort can be spent on an active search: a violation is equivalent to an intense search that would be likely to use more energy than is available to the operator.

If an active search pattern satisfies the assumptions of §8 we say it is a *pragmatic search.*

Lemma B1: *Let g be a pragmatic search and assume $u_0 : \mathbf{R}^n \to \mathbf{R}$ is an initial probability density function vanishing outside $U \subset \mathbf{R}^n$. Then there exists an enhanced forced motion $(M, \{\mu_t\}_{0 \le t < T})$ (a biased search pattern) with initial surface $M_0 = [\text{graph} u_0]$.*

...people who can change and change again are so much more reliable and happier than those who can't.

Stephen Fry

Notation and Convention

Notation/ Convention		Page
$\mathcal{M}(\mathbf{R}^{n+k})$	Space of Radon measures on $\mathbf{R}^{n+k}$	1
$\mathcal{M}_n(\mathbf{R}^{n+k})$	Space of n-rectifiable Radon measures on $\mathbf{R}^{n+k}$	3
$I\mathcal{M}_n(\mathbf{R}^{n+k})$	Space of integer n-rectifiable Radon measures on $\mathbf{R}^{n+k}$	3
$\mathbf{V}_n(\mathbf{R}^{n+k})$	Space of n-Varifolds on $\mathbf{R}^{n+k}$	3
$\mathbf{IV}_n(\mathbf{R}^{n+k})$	Space of integer rectifiable n-Varifolds on $\mathbf{R}^{n+k}$	3
$\mathcal{D}_n(\mathbf{R}^{n+k})$	Space of n-Currents on $\mathbf{R}^{n+k}$	5
$\mathbf{I}_n^{loc}(\mathbf{R}^{n+k})$	Space of locally integral n-Currents on $\mathbf{R}^{n+k}$	7
$\Lambda_n\mathbf{R}^{n+k}$	Space of n-vectors on $\mathbf{R}^{n+k}$	5
$\Lambda^n\mathbf{R}^{n+k}$	Space of n-forms on $\mathbf{R}^{n+k}$	5
$\mathcal{D}^n(\mathbf{R}^{n+k})$	Space of differential n-forms on $\mathbf{R}^{n+k}$	5
μ	Radon measure	1
$\mu\llcorner A$	Restriction of a Radon measure to A	1
$\mathbf{M}(\mu)$	Mass of μ	1
$\Theta^n(\mu,a)$	n-dimensional density of a Radon measure	1
$T_a\mu$	Approximate tangent plane of a Radon measure	2
T_aM	Approximate tangent plane of $M\subset\mathbf{R}^{n+k}$	2
$V,\ V_\mu$	Varifold, varifold associated with a Radon measure	3
$\Phi_\#(V)$	Push forward of a varifold by a diffeomorphism	3
δV	First variation of a varifold	4
$\lvert\delta V\rvert$	First variation measure of a varifold	4
μ_T	Mass measure of a current	5
$\mathbf{M}(T)$	Mass of a current T	5
∂T	Boundary of a current	6
$T\llcorner A$	Restriction of a current to A	7
$<T,f,r>$	Slice of a current by the function f	7
T_z	Slice of a current at height z	7
π	Projection onto $\mathbf{R}^{n+1}\times\{0\}$	
ω	e_{n+2}	
Γ	Smooth hypersurface	
S	n-dimensional subspace of $\mathbf{R}^{n+k}$ (also represent projection onto the subspace)	
$S(v),\ v^{\mathrm{T}},v^{\perp}$	Projection of the vector v onto S, $T_a\mu$, $(T_a\mu)^{\perp}$ (resp.)	
$J_S\Phi(x)$	Jacobian of a diffeomorphism relative to the plane S at x	3

$D, \widetilde{D}$	Covariant derivative on $\mathbf{R}^{n+1}$, $\mathbf{R}^{n+1} \times \mathbf{R}$ (resp.)	
$\nabla^S, \nabla, \nabla^\perp$	Covariant derivative on S, $T_a\mu$, $(T_a\mu)^\perp$	
$\mathrm{div}_S, \mathrm{div}_\mu$	Divergence on S, μ (resp.)	4
$K_{1,\alpha}(\mathcal{M},x,t)$	Parabolic $C^{1,\alpha}$ norm of a curvature flow	8
$K_{2,\alpha}(\mathcal{M},x,t)$	Parabolic $C^{2,\alpha}$ norm of a curvature flow	8
$J_{2,p}(\mathcal{M},x,t)$	Parabolic $W^{2,p}$ norm of a curvature flow	8

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