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Static Spherically Symmetric Electrovac Brans-Dicke Spacetimes

by

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This thesis includes 1 original paper published in a peer reviewed journal and 2 unpublished papers. The core theme of the thesis is the Stability of Static, Spherically Symmetric, Brans-Dicke, Electrovac Spacetimes. The ideas, development and writing up of all the papers in the thesis were the principal responsibility of myself, the candidate, working within the School of Mathematical Sciences under the supervision of Anthony Lun.

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"In the creation of the heavens and of the earth, and in the alternation of night and day, are signs for people who know."

Ayat 90, Surah Ali-Imraan, Holy Quran

STATIC SPHERICALLY SYMMETRIC ELECTROVAC BRANS-DICKE SPACETIMES

by Maya Watanabe

Abstract

We investigate the stability of static spherically symmetric electrovac BD spacetimes under an electrostatic perturbation by a point charge. The field equations are integrated directly, and we are able to give, for the first time, a solution describing a general static spherically symmetric charged Brans-Dicke (CBD) spacetimes that is reducible to all known BD spacetimes. We find there are nine classes of independent solutions. We are able to give the physical interpretation of the parameters contained within the metric and shed light on not only the CBD spacetimes but the BD spacetimes as well.

We investigate the stability of the CBD spacetimes by electrostatically perturbing it with a point charge. By extending a method first introduced by [1] we are able to convert the partial differential equation on the electrostatic potential generated by the point charge into a solvable ordinary differential equation. In this way we are able to give an exact, closed-form solution for the electrostatic potential generated by a point charge in a CBD spacetime. We introduce a boundary condition, based on Gauss' divergence theorem, that enables us to determine the constants of integration such that the solution is representative of a single charge.

Furthermore, we introduce a method by which the CBD metric can be converted from isotropic to Schwarzschild-type coordinates. In Schwarzschild-type coordinates we find that the CBD Class I solution (also referred to as the Brans-Dicke Reissner-Nordström or BDRN solution) exhibits an extra S^2 singularity in addition to the generalized inner and outer "horizons". This additional singularity and the behaviour of the electrostatic potential is investigated by graphically representing in isotropic and Schwarzschild-type coordinates the equipotential surfaces generated by the perturbing charge in the four backgrounds (BDRN, Brans-Dicke, Reissner-Nordström and Schwarzschild), alongside Copson's perturbed Schwarzschild solution of [1].

We find that all nine classes of the formal, generalized CBD solution are stable under electrostatic perturbations.

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I thank my parents, for fostering my curiosity and encouraging me to strive for excellence. The little seed that you planted and so diligently nurtured, cared for and loved has all grown up and has spread her branches to fruition, but none of this would have been possible without all that you did. To my brothers and sisters, my best friends, my role models. I may be the last one of us six to get a higher degree, but I did it!

To my little family. To the ones that give my life meaning; my four beautiful, amazing, incredible children. The four most wonderful distractions; Khalid (Ludi), Salma, Zaharah and Ayla. One day you will all be old enough to understand what your Mama has accomplished here, but know always that you four will always be my greatest accomplishment. I love you all to the end of the universe and back!

Finally, to the only person who's name should be on this thesis with mine, my dearest Saleh. This thesis is equally yours as it is mine. Without your sacrifices, your support and your belief in me, I could not have achieved any of this. It has been a long haul but you were there through all of it, encouraging me all the way. Thank you for so selflessly giving up your time, money and energy so that I could pursue my passion. Thank you for looking after our rugrats so that Mama could study. Words will never suffice and I look forward to returning the favor when you continue your Doctorate next year. Until then, here it is. Your thesis, jan.

Contents

To my little family:

Saleh

Ludi

Salma

Zaharah

and
Ayla

Chapter 1

Introduction

This chapter introduces the key topics and concepts that are required to gain an appreciation for the basis of this research. We begin with an introduction to scalar-tensor theory as an alternate theory to general relativity in Section 1.1. We give a brief history of scalar-tensor theory in regards to its synthesis and the motivation that led to the Brans-Dicke theory. The scope and applicability of scalar-tensor theories, ranging from explaining cosmological phenomena to creating unification theories such as string theory, are discussed with particular attention paid to recent developments in Section 1.2. In Section 1.3, we discuss our choice for investigating the stability of static spherically symmetric electrovac Brans-Dicke spacetimes and give a brief history of the development of charged Brans-Dicke spacetimes up to the present. In Section 1.4, perturbation theory and its history is discussed in the context of a "test" perturbing charge with some references made to Regge-Wheeler type perturbations. In Section 1.5, we give a brief overview of the astrophysical investigations that are of interest, such as the possibility of charged Brans-Dicke black holes and wormholes. Lastly, in Section 1.6 we conclude by briefly summarizing the key features of the research that makes up this thesis.

1.1 Brans-Dicke Theory

General relativity (GR) is one of the most well known and elegant theories to appear in the scientific world. Its elegance lies not only in its simplicity but also in its ability to withstand obvservational tests including, but not limited to, the precession of Mercury, gravitational lensing and gravitational redshift. However, in recent years it is becoming ever more evident that GR is not without its limitations. Its inability to reconcile with quantum mechanics has been a long standing problem and with the current acceleration of the universe being all but confirmed [2], it seems necessary to modify GR somewhat to accomodate these phenomena. One such alternate theory to GR is the scalar tensor theory (STT).

Scalar tensor theory can be traced back to Jordan [3] who coupled a scalar field to the Ricci scalar of the general Lagrangian as follows

$$\mathcal{L}_J = \sqrt{-g}\left[\varphi_J^{\gamma}\left(R - \omega_J \frac{1}{\varphi_J^2} g^{\mu\nu}\partial_\mu\varphi_J\partial_\nu\varphi_J\right) + L_{matter}(\varphi_J, \Psi)\right], \tag{1.1}$$

where φ_J is Jordan's scalar field, γ and ω are constants (the meanings of which will be made apparent later) and Ψ represents the matter fields (see [4]). Scalar tensor theories are classified by the presence of the nonminimal coupling term (the first term inside the square brackets of the RHS of Eq.(1.1)). The four dimensional scalar field, φ_J, enabled Jordan to describe the gravitational "constant" as being dependent on spacetime, a consequence that was in agreement with an earlier conjecture by Dirac [5]. However a consequence of coupling the scalar field to the matter term (the third term on the RHS of Eq.(1.1) is that the weak equivalence principle (WEP) is violated.

The WEP states that a gravitational acceleration at a given point is independent of mass [6]. A consequence of WEP violation is that gravitational acceleration would be dependent on the velocity of the body under consideration and thus, for example, a spinning body would have a smaller acceleration in a gravitational field than its identical non-spinning counterpart. Like Einstein before them, Brans and Dicke found WEP violation unappealing. To reconcile the issue they decoupled the scalar field from the matter part [7], [8] to produce the following Lagrangian which did not violate WEP

$$\mathcal{L}_B D = \sqrt{-g}\left[\varphi R - \omega\frac{1}{\varphi}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + L_{matter}(\Psi)\right]. \tag{1.2}$$

(for a more thorough history on the motivation behind their modification see [6]). We point out that the scalar field has $\gamma = 1$ dependency as a direct consequence of decoupling the matter part of the Lagrangian from the scalar field and ω is the only parameter in

Brans-Dicke (BD) theory (we omit a factor of 16π that appeared in the original BD theory [7] for aesthetic purposes).

A quick comparison with the Einstein-Hilbert term of GR reveals that, like Jordan's general STT, Brans-Dicke theory also possesses a spacetime dependent gravitational "constant" which is characterized by the scalar field and is called the effective gravitational constant, G_{eff}.

Interestingly it was not the nonminimal coupling term that was, and still is, the centre of discussion regarding the the BD theory but the dimensionless coupling parameter ω. The sign of ω determines the type of energy that a normal field will possess: that is a positive ω corresponds to a normal field possessing positive energy and a negative ω corresponds to a "ghost" field possessing negative energy. At the time of the formulation of the theory, negative energy was deemed physically unacceptable as it appeared to violate the weak energy condition. However, we know now that there are instances where a negative ω need not violate the weak energy condition. Due to the presence of the nonminimal coupling term in the Lagrangian the dominating energy density may still be positive. This is the case where the scalar field arises from the size of compactified internal space in Kaluza-Klein (KK) theory. The Kaluza-Klein theory, a predecessor to STT and string theory, describes a five-dimensional spacetime to which GR is applied. Four of the spatial dimensions extend to infinity and make up the world as we see it, however one dimension is assumed to be "compactified" to a circle so small that no phenomena can detect it. The scalar field that arises in this compactified region is a ghost field. However, the overall system remains positive, due to the presence of the nonminimal coupling term, see [4].

Another example where ghost fields do not imply physical inconsistencies is in the case of dilatons in string theory. String theory, which has gained particular momentum shows that a kind of second-rank tensor or "closed string" behaves like the spacetime metric at low-energies. Additionally, interaction among strings occurs in a fashion similar to the way gravitons are proposed to interact with each other in GR. A companion to the graviton in STT is a scalar field called the "dilaton", see [4]. Although the dilaton possesses a negative value for ω, the overall positivity of the physical mode is assured as it is easy to restrict other parameters in order for the "diagnolized" scalar field to be a nonghost field. One last example is in brane theory where in order to exist in a negative-tension brane (which is required to solve the hierarchy problem), ω must be negative but larger than $-\frac{3}{2}$ which again does not violate the positivity condition as the "diagnolized" scalar field can still be expressed as a nonghost field. Even the extreme circumstance where $\omega = 0$ need not be eliminated from consideration [4].

Moreover, even if the overall energy were to be negative we now know that this is not the impasse it once appeared to be. There are several instances where negative energy has been theorized to exist such as squeezed light states [9], the Casimir effect [10] and moving boundaries [11]. While for the above theories researchers have only been able to measure the indirect effects of negative energy, it may be possible to detect them directly using atomic spins [12]. Another famous example of negative energy is the Hawking prediction that black holes evaporate by emitting radiation [13] which would manifest as an absorption of negative energy. Lastly, in [14], the authors found that in order for a wormhole to be traversable, it is in fact *necessary* for the wormhole throat to possess negative energy.

What about observtional constraints on ω? From the relativistic perihelion rotation rate of Mercury, Brans and Dicke were able to place the acceptable value of the coupling parameter at $\omega \simeq 6$. This was in stark contrast to obvervational data that placed the value at $\omega \geq 1000$ from the Shapiro time-delay measurement of the Viking Project of the 1970s. Even when the possible oblateness of the sun was taken into consideration, the observational constraints did not place ω anywhere in the vicinity of the value proposed by Brans and Dicke. The Cassini-Huygen's data of 2003 places the current accepted value of the coupling parameter at $\omega \geq 40000$. However even this problem can also be resolved if one recognizes that if the force range of the scalar force is smaller than the size of the solar system, and consequently no longer affects the perihelion advance of Mercury, then the observational constraints are irrelevant [4]. Thus there is no reason why any range on ω should be dismissed as unphysical without considering the physics of every proposed model of the STT.

1.2 Scalar Tensor Theories of Gravitation

There are several applications of STT that are outside of the scope the research included herein. In this section we have included only those investigations that are pertinent to the theme of this thesis.

There is no doubt that one of the reasons scalar tensor theory in general has enjoyed a renewed interest recently is due its broad range of applicability. Scalar tensor theories, and its most simple prototype, the BD theory, has the ability to model quantum phenomena such as Higgs bosons and dilatons, and cosmological phenomena such as the cosmological constant and the continued acceleration of the present universe, using for example inflatons. A consequence of this applicability hints at the possibility of a unified field theory.

Among these unification attempts has been string theory. In this theory, the graviton which has played a major role in unifying gravitation with quantum physics has a scalar companion called the dilaton. String theory predicts that the coupling parameter will be $\omega = -1$ which, as we saw from our discussion in the previous Section, is not unacceptable. We point out that like most other unification theories, string theory allows the scalar field to couple directly to matter, in contrast to the BD theory, and as such, the WEP is inevitably violated. There are physical instances however that suggest that the Equivalence Principle (EP) (one of the central pillars of GR) may still be observed even when the WEP is not. Of course the implications upon GR if the WEP is found to be unnecessary is still huge [15] but out of the scope of this thesis as we focus our attention on the WEP preserving BD theory.

Another prediction of string theory is the appearance of a nonlinear structure called the "brane" (which is a nomenclature of "membrane") [16], [17], [18]. The brane is essentially a boundary layer upon which the ends of open strings can "stand". Although at first glance, branes appear to only affect physics at a quantum level, it actually has implications even on a cosmological level, namely that we are living in a 3 dimensional hypersurface in a higher dimensional spacetime called a brane world. One type of scalar field in a brane world appears to be related to the "distance" between two branes and is thus referred to as a "radion". This massless scalar field would also appear in our four-dimensional world. It can be shown that the BD model can also accomodate branes when restrictions are placed upon the coupling parameter [19]. It follows from brane theory that we must live in a negative-tension brane and thus the constraint on the coupling parameter is $-\frac{3}{2} < \omega < 0$.

Another strength of STT over GR is the ability to accomodate the acceleration of the universe and the emergence of a nonzero cosmological constant [2], [20], [21], [22]. The expansion of the universe can be addressed by STT in several different ways; including through the existence of dark energy and quintessence.

We first look at the cosmological constant as a scalar field. Discussion regarding the cosmological constant, Λ, has been reignited since Einstein's famous "blunder" and since Hubble's discovery that the universe is expanding. Observational data from type Ia supernovae [2], data from high-redshift supernovae [23], the large-scale structure of galaxies [24], the number of galaxies [25] and gravitational lensing of distant objects [26] all suggest a nonzero cosmological constant. Most recently, data from the Planck mission of 2015 suggests strongly that the cosmological constant is indeed nonzero [27].

Shortly after the big bang, it is widely accepted that the universe experienced a very rapid expansion called an "inflation period". This exponential inflation can be attributed to Λ. This inflation period was orginally attributed to vacuum energy [28], [29], [30].

However "new inflation" models attribute the inflation to classical scalar fields such as those seen in STT and BD thoery. One of the greatest drawbacks of the original "old inflation" models with a true constant is that the inflation never ends, a result which is completely unrealistic. After the inflationary phase, the expansion that we see today must set in. Therefore a smooth transition must occur between these two phases. This is referred to as the "graceful exit". "New inflation" has a way of avoiding this graceful exit problem by adjusting the parameters of the effective potential. La and Steinhardt [31] introduced a new method by which the graceful exit problem could be avoided as an extension of the old inflation model that did away with the need to "fine tune" parameters.

One benefit of STT is the ability to either include or disclude Λ as required. Berman and Som [32] were able to show that the exponential inflation phase is possible in BD cosmology with both positive pressure and density, but for which the violated energy condition ($p = -\rho$) did not hold in the absence of Λ. They then included the cosmological constant into their theory and arrived at the same conclusion [33]. More recently they found a solution for exponential inflation in a generalized BD model with a varying coupling parameter [34]. They were able to show that the negativity of cosmic pressure implied an accelerated expansion of the universe even with a negative cosmological constant.

We now turn our attention to different candidates for the scalar field Λ. It is widely accepted that most of the matter density of the universe can be attributed to dark matter. However to reconcile with the accelerated expansion of the universe and from observational data [35], we know that another as-of-yet unknown and unclumpy component of energy must exist as well, such as the cosmological constant or a spatially uniform scalar field. This is called "dark energy". One of the requirements on dark energy is that it must possess negative pressure. As such it could replace the cosmological constant in discussions regarding the rate of expansion.

Another candidate for the cosmological constant is quintessence. Quintessence is often referred to as a decaying cosmological constant but is also used to imply a fifth element of the universe, representing an unclumpy distribution with negative pressure [36], [37]. The drawback of quintessence is the need for some fine tuning of the parameters contained within. Also, the universe as we know it today would not have evolved naturally from the quintessence model. If we instead consider our universe to be a brane world, we can do away with the need for fine tuning and perhaps arrive at a more compatible version of quintessence. This is indeed the case where quintessence in a brane world can meet crucial criteria for a physically realistic cosmological model such as nucleosynthesis and matter dominance.

If one is to assume a zero cosmological constant then some other mechanism must be introduced to account for the accelerated expansion of the universe. One such mechanism is the inclusion of a new massless scalar field called the “inflaton”. In some theories the inflaton behaves similarly to a dilaton whilst in others it behaves like the scalar field of older STT theories (see [6] and the references given within).

Lastly, we look at how STT reconciles with quantum theories. The investigations into quantum scalar fields are very different to those that drive research in scalar fields of macroscopic theories however there are certain forms of the quantum formalism that are similar to classical scalar fields. The most obvious example of this is the aforementioned scalar field called the “dilaton” that arises naturally from discussions regarding string theory. Another example, is the famous Higgs boson or “God particle” from which subatomic particles are endowed with mass. With the recent discovery of the Higgs boson [38] it is not surprising that STT has received yet another boost. There are some key aspects of the Higgs mechanism that we must point out: namely that the gravitational scalar field ϕ couples to the matter directly at the level of the Lagrangian in violation of the WEP (and in contrast to the BD model). Also, the scalar field representative of matter, Φ, is massless and no kinetic term is produced, thus the second term of the Lagrangian also appears differently to that of the BD model Eq.1.2.

We point out that in 2005, Brans posed the question as to why no fundamental scalar field has been observed as of yet [6] in spite of the fact that they have been speculated so widely. We feel he will be happy to know that a scalar field has finally been found!

1.3 Static Spherically Symmetric Electrovac Brans-Dicke Spacetimes

The strength of STT, and BD theory in particular, has been illustrated in the preceeding section and it is therefore self explainatory why further investigations are warranted. Here we give our motivations for studying in great detail, a specific class of BD spacetimes viz. the static spherically symmetric electrovac Brans-Dicke spacetimes.

There is strong reason to believe that a static black hole resulting from the gravitational collapse of a massive object, can be described by its mass and charge alone as the other characteristics are “radiated away” during the collapsing process [39]. It seems physically relevant therefore to study static electrovac BD spacetimes as opposed to vacuum BD spacetimes.

The first attempt to add charge to the standard vacuum BD model can be attributed to Mahanta [40] who found an approximate solution describing a static spherically symmetric metric due to a point charged mass in BD theory in 1972. We point out that the constants of integration contained therein were determined using the weak field approximation like that of the original BD theory [7]. This was quickly followed up by an exact solution by Buchdahl [41] who found ten classes of charged BD (CBD) solutions of which "at least five" were physically acceptable. One of these five classes of physically acceptable solutions reduced to the BD Branch I solution, while his Class IIIa solution "resembled" the BD Branch II solution. He was not concerned that his solution was unable to recover the other BD classes as he claimed that the other BD Classes could be excluded on the basis that the parameters (Λ and C) in those spacetimes were not real.

Also in 1972, Luke and Szamozi [42] found an exact solution by integrating the field equations directly, that described a charged BD spacetime. The solution could be classified into four branches depending on the sign of the parameter B $(> 0, < 0, = 0, = \infty)$. They were however, unable to give an expression for the constant of integration B as they assumed the only way to do so was by implementing the weak field approximation, a procedure for which they could find no justification. As a result they were only able to give the limit for B when the charge was allowed to vanish, and found that it was a function of the mass of the particle $B = \frac{G_0 M}{2c^2}$. The BD solutions were not recoverable even with $B = \frac{G_0 M}{2c^2}$. The only degeneracy they were able to achieve was to the Reissner-Nordström solution when the parameter $g_0 = 0$, which corresponds to a constant scalar field.

A short while later, Raychaudri et al [43], independently of work done previously by Buchdahl, also found a solution for a static, spherically symmetric electrovac BD spacetime. Their solution coincided with that of Buchdahl except for the fact that they were able to find that there were only five independent classes in lieu of Buchdahl's ten.

Singh and Rai [44] found a solution for a static axially symmetric electrovac BD spacetime for three types of electromagnetic fields (azimuthal, radial and longitudinal). Meanwhile, Reddy et al [45], [46] were able to find a solution for a static, spherically symmetric electrovac BD spacetime that was conformally flat. This was, by their own admission, astrophysically interesting though unrealistic.

In an extension of the work done in [46] and [45], Van den Bergh looked for the same type of solution for a static spherically symmetric electrovac BD spacetime which was not conformally flat. He limited his investigations to those where he thought the coupling parameter was indicative of "positive energy density", that is $\omega > -\frac{3}{2}$. He found three classes of solutions depending on the sign of the integration constant λ $(= 0, < 0, > 0)$, although like Luke and Szamosi before him, he was unable to explain the physical meaning of the integration constant. Unfortunately, the final form of the solutions were tediously complicated. He was however able to point out that in order for the effective gravitational "constant" to be non negative at infinity, the conformally flat solution of [46] is only valid for $\omega < -\frac{3}{2}$.

Around the same time that static, spherically symmetric electrovac BD spacetimes were being conceived by the likes of [40], [41], and [42], study into a new class of scalar-tensor theories was underway. Based on previous work by Bergmann [47] and Wagoner [48], Nordtvedt [49] proposed a class of STT similar to the BD theory where the coupling parameter ω was allowed to be an arbitrary function of the scalar field,ϕ, such that $\omega(\phi)$. Of course, the BD theory was a subclass of this more general theory. These types of scalar tensor theories are known as the Bergmann-Wagoner-Nordtvedt (BWN) theory. A special class of the BWN theory was proposed by Barker where the gravitational "constant" G was allowed to be independent of time [50].

Van den Bergh, using the method by [44] was able to find a solution to the BWN field equations for a stationary, axially symmetric electrovac spacetime. As an extension of this, Singh and Singh [51], were able to integrate the BWN field equations directly using the method introduced in [42] to give closed form solutions describing static, spherically symmetric, electrovac BWN spacetimes. More specifically their solutions were representative of the Barker and Schwinger theories.

Bronnikov [52] studied the stability of a class of BWN spacetimes in the context of non-trivial black holes with particular attention paid to BD spacetimes. He was able to find a class of electrovac BD spacetimes where the coupling parameter was constrained to $\omega < -\frac{3}{2}$. He did not give the degeneracy of his solution but we find it to degenerate to a limiting case of the BD Class III solution.

To the best of our knowledge, to date there has been no success in finding a solution for the BD field equations that describe a static, spherically symmetric, electrovac BD spacetime that degenerates to the four known BD spacetimes of [8] in the absence of an electromagnetic potential. All the previous investigations referred to above, have covered only a subset of the detailed solution presented in this thesis.

1.4 Perturbation Theory

If an electromagnetic test charge is dropped into a static BD black hole there are two possible outcomes: 1. the electromagnetic field of the perturbing charge creates a sufficiently large stress energy that destroys the horizon or makes it singular; 2. the horizon is not destroyed [53], [54], [55], [56]. If the former outcome is true then there can exist no BD black holes in nature since, even if one were formed, a charged particle at some point in time would enter it and destroy it. Thus a very basic test for the stability of a gravitating theory is by an electromagnetic perturbation (note that the same principle applies to scalar field perturbations, see for example [57], [39], [58], [59], [60]).

The electromagnetic perturbation in the context of this thesis refers to a "test" charge, where the effects of the charge are sufficiently small such that the background metric is unaffected by the presence of the perturbing charge. This is in contrast to gravitational Regge-Wheeler type perturbations where the electromagnetic potential alters the geometry of the background space as it couples directly to the metric [61]. Unless otherwise stated, in this thesis, a perturbing charge shall refer to a "test" charge. There are some instances where reference will be made to relevant Regge-Wheeler type perturbing massive electric charges and these shall be made explicit.

The study of electromagnetic perturbations date as far back as 1927 when Whittaker [62] found a multipole expansion solution describing the electric potential generated by a static perturbing charge in quasi-uniform and Schwarzschild spacetimes. For the former he was able to find an exact, closed-form solution (also known as an "algebraic" or "analytic" solution), while for the latter he was only able to find a multipole expansion solution (also referred to as "series expansion" solutions).

Shortly after, using a different method altogether, Copson [1] was able to find closed-form solutions for the electric potential generated by a static perturbing charge in both the quasi-uniform and Schwarzschild spacetimes, the former agreeing with the results of Whittaker. Copson used Hadamard's theory of "elementary" solutions (referred to in subsequent literature as "fundamental solutions") to find the first few terms of the fundamental solution. From the form of the first few terms, Copson found that he could use a substitution by which to convert the partial differential equation on the electrostatic potential into a solvable ordinary differential equation (the details of this method are given in the Introduction of Chapter 2). He was able to solve this ordinary differential equation to give a closed-form solution for the electrostatic potential. He plotted the equipotential surfaces of his solution in isotropic coordinates and found no anomalies. The constant of integration was chosen such that the resulting potential would be symmetrical in the radial coordinate r and the location of the perturbing charge b. This choice resulted in his solution being different to that of Whittaker when the exact solution was expanded in series form and a term by term comparison was made. Copson found that his solution contained an extra leading term while the remaining infinitely many terms were exactly the same! The physical implication of this extra term would not be made clear until nearly half a century later.

In the meantime, independently of the work done by [62], the electric potential of a static perturbing charge in a Schwarzschild spacetime was rederived by Cohen and Wald [53] and Hanni and Ruffini [63] who also used a multipole expansion method. Cohen and Wald and Hanni and Ruffini, found that when the charge was lowered into the Schwarzschild black hole, the electrostatic field remained well behaved and the multipole moments, except for the monopole, faded away. Both studies concluded that a Reissner-Nordström black hole was produced although it later came to light that this was not strictly true (see Introduction of Chapter 3). Hanni and Ruffini [63] plotted the electric field lines of the perturbing charge for decreasing values of the radial coordinate to gain an appreciation for the evolution of the charge as it is "lowered" into the Schwarzschild black hole. Both Hanni and Ruffini and Cohen and Wald implemented the boundary conditions that the electrostatic potential of the perturbing charge must be well behaved even as the charge approaches the horizon and that the flux through a surface enclosing the hole, but not the point charge, must be zero.

In a series of papers, Bicak et al [64] perturbed a Schwarzschild spacetime by electromagnetic charges dynamically also by using multipole expansions. Their solution at the static limit coincided with those found by [53] and [63] in that all multipole moments faded away leaving only a monopole which could be interpreted as representative of a Reissner-Nordström black hole.

Not long after, Linet [65] revisited Copson's earlier work [1] and using Gauss' theorem at infinity, found that the electrostatic potential of Copson's solution contained more than one source. This second source was attributed to a second charge located in the region not covered by the isotropic coordinates. We point out briefly that in isotropic coordinates the spherical surface at $r = B$ is not a horizon but rather a surface of inversion where the interior region is identical to the exterior under an inverse transformation. Thus the isotropic coordinate system contains two copies of the region exterior to the horizon in Schwarzschild-type coordinates while the region inside the horizon of the Scharzschild coordinates is completely excised from the isotropic coordinates (the details of this are given in Chapter 2). As Copson placed only one perturbing charge outside the surface of inversion (and indeed only one charge appears in his oringal plot), Linet found that there must exist a second charge lying within the surface of $r = B$, the region not covered by the isotropic coordinates. He went on to correct Copson's solution such that it would be representative of a single charge perturbation and to convert it into the usual Schwarzschild coordinates. The ammended solution was now in agreement with multipole expansions of [53], [63], and [62]. Harpaz [66] was able to graphically represent the solutions and showed that the electric field lines were comparable with those plotted in [63].

The extra charge that Copson inadvertantly introduced into his solution was a result of his choice of boundary condition. By requiring only that the solution be symmetrical in terms of the radial coordinate and the location of the charge, he was unable to ensure that the resulting solution was representative of a single perturbing charge. The Copson-Hadamard method in finding a closed-form solution itself was still valid and Linet et al. used it to find closed-form solutions describing the electric potential of a point charge in a Reissner-Nordström spacetime and in a BD spacetime. It turned out that the form of the partial differential equation on the electrostatic potential of the point charge in the Reissner-Nordström spacetime was identical to that of the Schwarzschild spacetime and hence the solution could be derived trivially from the Schwarzschild solution.

The partial differential equation on the electrostatic potential of a point charge in a BD spacetime of course was different, but Linet was able to find a substitution that allowed him to convert it into a solvable form. He was therefore able to find a closed-form solution for the electrostatic potential of a point charge in a Class I BD spacetime. The only boundary condition he placed was the boundary condition at infinity, which was insufficient in determening the constants of integration, as will be made clear in Chapter 2. Linet determined the constants of integration by looking at the first few terms of a multipole expansion. There is thus an obvious need to address this boundary condition problem and to introduce a method by which the constants of integration can be found directly and without looking at multipole expansion solutions, so that the resulting solution is truly representative of a single perturbing charge.

Adding to the growing body of literature, Molnar [67] found a Green's function and multipole expansion to describe the electrostatic potential generated by a point charge in a Schwarzschild spacetime and found that his solution was in agreement with that found earlier in [65]. In later research [68] he was further able to prove that the solution of [67] (and [65]) was indeed unique.

In [69], Linet attempted to extend the Copson-Hadamard method to Schwarzschild black holes in higher dimensions with little success. He used several different forms of the substitution in an attempt to convert the governing partial differential equation on the electrostatic potential and scalar field into solvable form but was able to find a closed-form solution only for the electrostatic potential and scalar field generated by a point source in spacetimes with non-degenerate horizons.

We now turn our attention to a few pertinent gravitational Regge-Wheeler type perturbations that have been made in the same field of research. In the following studies the charge is linearly perturbed directly to the background metric in the fashion of [61].

In [64], Bicak electromagnetically perturbed an extreme Reissner-Nordström spacetime using the Regge-Wheeler type perturbation and multipole expansion method. In a following paper they was able to find a multipole expansion solution for the magnetic field of a perturbing current loop of a Reissner-Nordström black hole [70]. They went on to numerically calculate the magentic lines force and plot them graphically.

In a similiar manner Bronnikov [52], perturbed the electrovac BD spacetime he derived under the restriction $\omega < -\frac{3}{2}$, with a time varying electric charge but focused on the infalling times and finiteness of the horizons in an attempt to ascertain the possibility of BWN and BD black holes. He found that his class of electrovac BD spacetime was stable under electrodynamic perturbations. The results of his investigations into the possibility of BD black holes are discussed in the next section.

Another question that is posed when discussing electrostatic perturbations is how the perturbing charge is able to be held in place without succumbing to the force acting upon it by the gravitating field of the spacetime. This so-called "equilibrium" problem has been a long standing one with several attempts at finding a model where equilibrium is achieved between two charged and/or gravitating objects [71], [72]. Essentially, the issue has revolved around finding a scenario where the electromagnetic field generated by either one, or both objects balances out the attractive force of the gravitational field produced by either one, or both, of them.

Alekseev and Belinski [73] were able to find an exact closed-form solution representing a Reissner-Nordström black hole in equilibrium with a Reissner-Nordström naked singularity. This was plotted graphically by Pizzi [74] and Paolino et al [75] for three different scenarios: where the two bodies had the same charges, different charges and when one was neutral.

In a series of papers, Bini et al [76], [77], using gravitational Regge-Wheeler type perturbations, were able to represent in closed-form a solution for a charged massive particle at rest in a Reissner-Nordström spacetime. They extended this study in [78] by representing the results of [76] and [77] graphically and found that their solution coincided with the Alekseev and Belinski in the Regge-Wheeler gauge. An attempt to find a corresponding equilibrium solution for the Schwarzschild spacetime resulted in the need to introduce a strut [76] to prevent the massive charge from "falling" into the Schwarzschild black hole due to gravitational attraction.

In the context of investigating the stability of a background spacetime, it has been argued that electrostatic perturbations need not necessarily be physical, that is, equilibrium need not necessarily be reached between the perturbing charge and the gravitating singularity. Perhaps this is why there have been, to date, only a few attempts to explain how the electric perturbing charge can remain static in the presence of an attractive gravitational field. Certainly no attempts have been made with respect to static spherically symmetric electrovac BD background spacetimes. There is no doubt that finding a method by which the perturbing charge can be held static would be very interesting.

In Chapter 2 and 3, we provide the mechanism whereby the perturbing electric charge is held static by the charge distribution inside the outer "horizon".

1.5 Astrophysical Interests

The search for BD black holes was initiated by Campanelli and Lousto [79] who studied whether black holes would arise in the BD theory of gravity as they would in GR. The authors first converted the BD metric from isotropic to Schwarzschild-type coordinates to give the following

$$ds^2 = -c^2 A(r_s)^{m_0+1} dt^2 + A(r_s)^{n_0-1} dr_s^2 + A(r_s)^{n_0} r^2 d\Omega^2,$$

where

$$A(r_s) = 1 - 2\frac{r_0}{r}, \tag{1.3}$$

and $\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

They found that the leading terms of the Kretschmann scalar invariant is regular everywhere if $n_0 \leq -1$ and also when $m_0 = n_0 = 0$; the former corresponds to the coupling constant being $\omega < 0$ while the latter being simply the conditions underwhich the Schwarzschild black hole solution is recovered. By studying the Killing vectors and the outgoing radial null geodesics when $n_0 \leq -1$ he found that the outer "horizon" $r = 2r_0$ would represent an event horizon whenever $m_0 - n_0 + 1 > 0$, which corresponds to $\omega < -3/2$.

Interestingly, it was not until two decades later that a closer look at the Campanelli-Lousto findings by Vanzo et al [80] revealed that the BD spacetime was representative not of a black hole but, according to the values of a parameter, of a wormhole or naked singularity. Taking a closer look at the horizon and in particular the areal radius revealed that when $n_0 \geq 0$ a bundle of radial outgoing null rays would experience an "hourglass" effect, where expansion of the cross-sectional area of the bundle would be positive up to r_{min}, where it vanishes, before becoming positive again. This effect is illustracted graphically in their paper. From this it can be inferred that the horizon $r = r_{min}$ is an apparent horizon of a wormhole. For all $n_0 < 0$ they found that the areal radius is always increasing, indicative of a spacetime with a naked singularity.

Their findings were in agreement with those found earlier by Agnese and LaCamera [81] in 1995 who found that the BD spacetimes gave rise to either a naked singularity or a wormhole depending on a parameter. Additionally they found that two-way traversable wormholes were possible when $\omega > -\frac{3}{2}$ and two-way traversable wormholes were possible for $\omega < -2$. After the Agnese-LaCamera paper, investigations into BD black holes expanded exponentially. We give here only an outline of the relevant papers.

In 1997, Nandi et al [82] found that of the four BD classes of solutions, three gave rise to wormholes. They were able to show that two-way traversable wormholes were permitted not just for negative values of the coupling parameter as stated by Agnese and LaCamera but were possible for any arbitrary positive value for ω. The authors quickly ammended their work in 1998, in [83] where they clarified that in fact, in the Jordan frame wormholes can exist in the very narrow interval of the coupling parameter that is $-\frac{3}{2} < \omega < -\frac{4}{3}$. Additionally they showed that due to the radial tidal acceleration being infinite at the throat of the wormhole, the wormhole was not traversable. In the Einstein frame, they concluded that wormholes do not exist at all unless energy conditions are violated.

However this would lead to some controversy. Bloomfield [84] did not agree that the weak field approximation would permit the range on the coupling parameter as given in [83] for which they claimed wormholes would be permissible in the Jordan frame. Besides having an issue with the weak field approximation being applied to the strong field wormhole phenomena under question at all, he found that the weak field approximation would allow wormholes only when $\omega < -2$ as originally stated in [81]. Moreover, he found that the radial tidal acceleration was not infinite at the throat either. He agreed that wormholes were not permissible in the Einstein frame, but by a different reasoning, that the energy condition requires $\omega > -\frac{3}{2}$ which is excluded from the weak field approximation.

In a response to Bloomfields critiques, Nandi [85] clarified that the form of the weak field approximation implemented in [85] was a misprint, missing an exponent of two. Thus the range $-\frac{3}{2} < \omega < -\frac{4}{3}$ was still applicable and representative of wormholes. He conceded however that the radial tidal acceleration was indeed finite and this implied that the wormholes were now traversable.

In a final reply, Bloomfield [84] was not convinced that the weak field approximation could permit the range $-\frac{3}{2} < \omega < -\frac{4}{3}$ and showed that the behaviour of the redshift function Φ (related to the metric coefficient g_{00}) of [83] was in contradiction to the redshift of the BD Class I which it supposedly represented.

There was no conclusion to this controversy except that Nandi would return to discuss wormholes in general STT later [86].

Research into BD black holes and wormholes continued and in 2002 He and Liu [87] claimed to have found two new traversable wormhole solutions in vacuum BD theory. Like Agnese and LaCamera before them one of these was permitted for the range $\omega < -2$ and the other when $-2 < \omega \leq 0$. Unfortunately, it was only a short while later that Bhadra et al [88] found that these solutions were not new but in fact just the BD Class I and II solutions in a different conformal gauge. Bhadra et al extended their study into BD wormholes in [89] and found that of the different classes of BD solutions, BD Class I solution allowed wormholes when $\omega < -\frac{3}{2}$ and excluding $\omega = 2$ and BD Class II solutions allowed wormholes when $-2 < \omega < -\frac{3}{2}$. For the radius of the wormhole throat to be large enough for travel, they found that ω had to be very large and of the order consistent with obversational findings $\simeq 40,000$. Additionally for the acceleration experienced by the traveller at the throat of the hole to be of a physically acceptable level, the mass of the wormhole would have to be $M > 4 \times 10^{12} M_{\odot}$. The journey time to cross a wormhole of that size would be extremely long and thus the authors found that though the wormhole may be traversable, they were not suitable for interstellar travel.

Bhadra et al then went on to study the possibility of BD black holes in [90]. They first concluded that of the four branches of BD solutions only two were independent (we discuss in Chapter 4 that this is not, in fact, the case). By matching interior to exterior scalar fields they found that Brans Class I solutions could represent an exterior metric for a nonsingular massive object when $-2 < \omega < -(2 + \frac{1}{\sqrt{3}})$. To understand the BD Class I spacetime they looked at the curvature invariants in the same manner as Campanelli and Lousto [79] before them. They found that when ω was negative then an outgoing null surface representative of an event horizon is possible. This range included that of their interior and exterior matching condition of $-2 < \omega < -(2 + \frac{1}{\sqrt{3}})$. It is thus somewhat surprising that they concluded that BD black holes were not possible as the small negative range for ω was in contradiction with observational data. They also concluded that the Class IV solution does not act as an exterior metric for any reasonable gravitating object (the reason for this is made clear in Chapter 4).

Study into a nonsingular BD black hole was conducted by Ismailov et al [91] and Bhattacharya et al [92] who both showed that the range on the coupling parameter that gives rise to a wormhole in the BD Class I spacetime given in [83] could be revised to represent a singularity free wormhole in the BD Class II spacetime when $-2 < \omega < -\frac{3}{2}$. Thus the problem of having a singularity within the wormhole could be avoided.

Interestingly, and as a concluding remark, we turn our attention to a study by Boisseau and Linet [93], where a wormhole spacetime (as given by Morris and Thorne in [14]) in n-dimensions was electrostatically perturbed. The closed-form solution describing the electrostatic potential generated by the perturbing charge was found using the Copson-Hadamard method.

1.6 Summary

In the preceeding sections we have made it explicitly clear why scalar tensor theories and BD theory in particular, are worthy of investigation. There is an obvious paucity within the litarature regarding solutions describing a static spherically symmetric electrovac BD spacetime which is reducible to the four known BD Classes of solutions.

The aim of this thesis is to investigate the stability of static spherically symmetric electrovac BD spacetimes by subjecting them to an electrostatic perturbing point charge and to address the inconsistencies and incompleteness of the literature. We outline the structure of our research as it appears in this thesis below.

In Chapter 2, we integrate the BD field equations directly, to find a class of static spherically symmetric electrovac BD spacetimes that is reducible to the BD Class I and Reissner-Nordström spacetimes which we term the BD Reissner-Nordström (BDRN) spacetime. As we have integrated the field equations directly we are able to give an interpretation on the physical meaning behind the constants of integration for the first time. As such we are able to shed light, not only on the BDRN spacetime but the BD spacetime as well. From the discussion in Section 1.5 there are some instances where the implementation of the weak field approximation is warranted however there is no a priori reason why it *must* be implemented. Thus we give the restrictions on the parameters if the weak field approximation is implemented, but emphasize that we have not implemented it in the derivation of the BDRN solution.

Closed-form solutions have clear advantages over multipole expansion solutions in that they are exact solutions, therefore in Chapter 2, we implement the Copson-Hadamard method to find a closed-form solution describing the electrostatic potential generated by the perturbing charge. As discussed in Section 1.4, the Copson-Hadamard substitution term which converts the partial differential equation on the electrostatic potential into a solvable ordinary differential equation, was found case-by-case and by looking to the first few terms of the series solution for inspiration. In this Chapter, we introduce a method by which one can decipher the exact form of the requisite substitution from the background metric alone. This does away with the need to “guess” at the form of the substitution term.

Additionally, in Chapter 2 we address the issue of a suitable boundary condition. As outlined in Section 1.4, the lack of an appropriate boundary condition led Copson to erroneously add an extra charge into his solution. Further study into exact, closed-form solutions were unable to produce a boundary condition that would resolve the issue once and for all. We introduce in this Chapter a boundary condition that is necessary and sufficient and ensures that the resulting solution is representative of a single perturbing charge. The boundary condition is based on Gauss' divergence theorem and states that an integration over any simply connected annular domain not containing the perturbing charge must be exactly equal to zero even if that region contains a horizon or surface of inversion. Correspondingly, an integration over a region containing the perturbing charge must be exactly $4\pi\epsilon_0$ where ϵ_0 is the magnitude of the charge at the singular point $r = b$. Another way of interpreting this boundary condition is that the net electric flux over any bounded region not containing the perturbing charge must be exactly zero.

With the ansatz on the Copson-Hadamard method and this sufficient boundary condition we are able to find an exact, closed-form solution describing the electrostatic potential generated by a perturbing charge in a BDRN spacetime. We find that the BDRN spacetime is stable under such electromagnetic perturbations.

While the isotropic coordinates are sufficient in most investigations regarding the exterior of the surface of inversion B, to understand a spacetime in its entirety, it is necessary to extend the region inside the surface of $r = B$. This can only be done by recasting a metric into the usual Schwarzschild-type coordinates.

In Chapter 3, we investigate the BDRN spacetime further by including the region not covered by the isotropic coordinates. This is done by introducing a transformation equation by which the metric can be converted into Schwarzschild-type coordinates. We convert the BDRN metric into Schwarzschild-type coordinates and astonishingly we find that the singularity one would expect to see at the origin as a point singularity, in the BDRN spacetime appears as an S^2 surface at r_0. This S^2 surface is in addition to the two known surfaces; the generalization of the inner and outer "horizons". The behavior of this additional singularity and the conditions underwhich it manifests are investigated and detailed.

Using the transformation equation we are also able to convert the electrostatic potential of a point charge derived in Chapter 1 into Schwarzschild-type coordinates. We find that in both coordinate systems our solution on the electrostatic potential generated by the perturbing charge is well behaved even when the charge is placed at the surface of inversion or outer "horizon". In this way we are able to prove that the boundary condition outlined in [53] is satisfied. An interesting discovery is that the electrostatic potential generated by an electrostatic perturbing charge placed at the horizon in a

BDRN spacetime is independent of the angle θ and is only a function of the radial coordinate r_s. This can be interpreted using the Schwarzchild-type coordinates and is as follows: the surface of $r_s = const$ inside the horizon of the uncharged spacetimes (BD and Schwarzschild) and the region in between the two "horizons" of the charged spacetimes $r_- < r_s < r_+$ (where r_- refers to the inner "horizon" and r_+ refers to the outer "horizon") are spacelike surfaces. In these regions the radial coordinate and time coordinate become timelike and spacelike respectively. Thus when the perturbing charge is placed on this spacelike surface of $r_s = r_+$ the charge is immediately distributed evenly across the entire surface of $r_s = r_+$ and is no longer dependent on the angle θ. This is in agreement with results found earlier when the Schwarzschild spacetime was perturbed electrostatically in [63].

To illustrate the differences between the isotropic and Schwarzschild-type coordinates, and to gain a greater understanding of the behaviour of the electrostatic potential in the BDRN, BD, Reissner-Nordström and Schwarzschild spacetimes, we graphically plot the equipotential surfaces in isotropic and Schwarzschild-type coordinates. Additionally we plot the equipotential surfaces generated by the perturbing charges of Copson's solution of [1] to illustrate the delicacy of choosing the right boundary conditions and prove that our boundary condition is indeed sufficient. We apply the boundary condition of [53] and find that the electrostatic potential of Copson's solution is also well behaved when the perturbing charge is allowed to approach the horizon and we thus show that the boundary condition though necessary, is insufficient in ensuring that the solution is representative of a single charge.

Furthermore, we find that in Schwarzschild-type coordinates from the region interior to the outer "horizon" we are able to prove that the static perturbing charge scenario is physical, in that equilibrium is achieved. We show that the charge configuration within the horizons of the BDRN, BD, Reissner-Nordström and Schwarzschild spacetimes allows the charge to remain static outside the horizon without the need for struts or strings. This is true even for the uncharged BD and Schwarzschild spacetimes. The region inside the horizon in these spacetimes takes on a dipole like configuration. The effect of this dipole is that a negative charge is induced at the horizon near the perturbing (negative) charge. A positive charge is induced at the other end of the horizon and the net charge over the entire surface of the horizon is maintained at zero, ie. the ingoing electric flux is equivalent to the outgoing electric flux. In the BDRN and Reissner-Nordström spacetimes, as the background is electromagntically charged itself, in order for the perturbing charge to be held in equilibrium, in addition to this dipole configuration, an image charge to the perturbing charge appears within the inner "horizon". As the perturbing charge is brought closer to the outer "horizon", the image charge approaches the inner "horizon". The effect is the same; the perturbing charge is able to be held static outisde the "horizon".

From the investigations it is evident that the BDRN spacetime proves to be very interesting as it is stable under electromagnetic perturbations and is able to hold a perturbing charge in equilibrium, we find it pertinent to study such static spherically symmetric electrovac BD spacetimes further.

In Chapter 4, we extend our study of static spherically symmetric electrovac BD spacetimes to include the entire class of solutions. We integrate the BD field equations for an electrovac spacetime and apply only the boundary condition at infinity. By applying no further restrictions or assumptions on the parameters we are able to find a general class of charged Brans-Dicke (CBD) spacetimes.

The spacetimes are classified according to three conditions and are thus divided into nine independent classes. The nine classes are classified as follows: the roman numeral (I, II and III/IV) represent respectively when the parameter ab is $> 0, < 0$ or $= 0$. The superscript $+, -$ and 0 represent respectively when the parameter κ is $> 0, < 0$ or $= 0$. This gives a total of nine classes viz. CBDI$^+$, CBDI$^-$, CBDI^0, CBDII$^+$, CBDII$^-$, CBDII^0, CBDIII/IV$^+$, CBDIII/IV$^-$, and CBDIII/IV^0.

We are able to prove that the BD Class III and IV are the inverse tranform of one another which we refer to as the BD Class III/IV solutions, and in contradiction to [90], we find that of the four BD classes of solutions, three are independent. We find that of the nine classes of CBD solutions, three reduce down to the three independent BD solutions, that is the CBDI$^+$ reduces to the BD Class I, CBDII$^+$ reduces to the BD Class II, and the CBDIII/IV$^+$ reduces to the CBD Class III/IV solutions.

In this way, we are able to, for the first time give a general solution representative of a charged BD spacetime that reduces to the known BD solutions. As mentioned previously, there is no a priori reason why the weak field approximation must be upheld however we find that the weak field approximation allows one to understand the physical interpretation of the spacetime parameters. We give the restrictions on the parameters should one wish to implement the weak field approximation.

The investigation into these nine classes of solutions would be incomplete if we did not study them in the Schwarzschild-type coordinates and include the region not covered by the isotropic coordinates. We therefore introduce a generalized form of the transformation equation of Chapter 2 that enables us to do so. We are able to express the CBD metric in Schwarzschild-type coordinates and show the degeneracy of the solutions involved. The issue of the parameter range on ω for the three Classes of the BD solution is put to rest with final and definitive ranges on ω allowed in each Class. Additionally we give the degeneracy of the CBD solution to the BD and Reissner-Nordström and Schwarzschild spacetimes but also to more exotic spacetimes such as the extreme Reissner-Nordström spacetime where $q = m$ and Reissner-Nordström-type spacetimes where $q > m$.

We are able to show that the physical implication of the parameter ab being $> 0, < 0$ or $= 0$ is the following: when $ab > 0$, representing the Class I spacetimes, $m > q$ where m and q are non-negative real constants and are identified, respectively, as mass measured in conventional units (kg) and charge measured in electrostatic units (e.s.u). When $ab < 0$, representing the Class II spacetimes, $m < q$ and is of interest in the particle physics gauge. Lastly, when $ab = 0$, representing the Class III/IV spacetimes, $m = q$ and represents the extreme case.

In order to understand the astrophysical implications of the CBD solution we look to the curvature invariants in the same method as Campanelli and Lousto [79] and Bhadra and Sarkar [90]. We find that only the CBDII$^-$ spacetime possesses a nonsingular horizon when $\omega < -2$ and thus may allow black holes or wormholes when the weak field approximation is not upheld. An investigation into the behaviour of the horizon in this nonsingular spacetime is warranted to determine the nature of the spacetime but is left for following paper. Moreover, we are able to show that the CBDIII/IV$^+$ and CBDIII/IV$^-$ spacetimes possess naked singularities.

Finally we electrostatically perturb the generalized CBD spacetime using the Copson-Hadamard ansatz introduced in Chapter 2. We find that this method is successful in converting the partial differential equation on the electrostatic potential generated by a point charge in a CBD spacetime into a solvable ordinary differential equation. The boundary condition introduced in Chapter 2 is applied to determine the constants of integration and we find that the boundary condition is robust and proves to be necessary and sufficient. In this way, we are able to solve this differential equation to find an exact closed-form solution on the electrostatic potential. We find the solution can be succinctly expressed using a single function $\Pi(\sigma)$ (where σ is a function of the coordinates r and θ). We are able to show that the general CBD spacetime is stable under electrostatic perturbations.

Declaration of Authorship

In the case of Chapter 2, the nature and extent of my contribution to the work was the following:

Nature of Contribution	Extent of Contribution
Initiation, key ideas, devised theories and methodology, contributed to discussion and analyses of results, prepared paper	67%

The following co-authors contributed to the work.

Name	Nature of contribution
Anthony W.C Lun	Key ideas, contributed to discussion and analyses of results

The undersigned hereby certify that the above declaration correctly reflects the nature and extent of the candidate's and co-authors' contributions to this work.

Candidate's
Signature: Date:

Main Supervisor's
Signature: Date:

Chapter 2

Electrostatic Potential of a Point Charge in a Brans-Dicke Reissner-Nordström Field

The historical context and literature regarding electrovac BD spacetimes has been discussed in Chapter 1, thus we will not reiterate it here. Instead we will go straight into the details of the most pertinent studies on perturbation theory.

In 1927, Whittaker [62] studied the effect of gravitation on electromagnetic phenomena according to general relativity. He discussed two kinds of gravitational fields: the Schwarzschild spacetime and a quasi-uniform field, where the gravitational force is uniform in the vicinity of the origin.

Using a series expansion method, he was able to find an exact solution to the partial differential equation on the electrostatic potential generated by a single electron in a quasi-uniform field. He was was unable to find an exact solution for an electron in a Schwarzschild spacetime and instead gave an infinite series solution to the phenomenon.

In 1928, Copson [1] approached the topic from a completely different angle. Using Hadamard's theory of "elementary solutions" [94] (referred to in subsequent literature as "fundamental solutions") he was able to construct an exact solution to a charge placed in both a quasi-uniform field and a Schwarzschild spacetime, the former of which agreed with the solution given earlier by [62] in isotropic coordinates.

Copson used Hadamard's formula to find the first order term of the elementary solution from which recurrent terms of higher order can be found using the recurrence relations of Hadamard's original theory. Copson's great contribution however was doing away with the need for recurrence relations, and applying the process by which they were obtained instead. By looking at the first three terms of the elementary solution, Copson was able to postulate the form of a substitution that would allow him to rewrite the partial differential equation on the electrostatic potential $u(r)$ in terms of an entirely new parameter $F(\gamma)$ where

$$u(r) = \frac{r}{(r+1)^2} F(\gamma) \tag{2.1}$$

and

$$\gamma(r, \theta) = \frac{\Gamma(r, \theta)}{r^2 - 1} \tag{2.2}$$

and $\Gamma(r, \theta)$ is the square of the distance of the charge to the origin. The result of the substitution was a solvable ordinary differential equation on $F(\gamma)$.

By solving for $F(\gamma)$ and substituting the solution back into Eq.(2.1) he was able to obtain a closed-form solution for the electrostatic potential generated by a charge placed in a Schwarzschild spacetime. The last step was to choose the integration constants of $u(r)$. Copson chose the integration constants such that the solution would be symmetrical in terms of r and the location of the perturbing charge $r = b, \theta = 0$. He plotted the equipotential surfaces generated by the point charge in isotropic coordinates but plotted only the region exterior to the surface of inversion. As a result there appeared only a single charge at $r = b$ as expected. He noted however, that there exists an image to the charge at b inside the surface of inversion as a consequence of the coordinate system. Most interestingly, a term by term comparison with Whittaker's series expansion solution for a charge placed in a Schwarzschild spacetime resulted in the appearance of an additional leading order term, while all the following terms were in agreement to infinity.

It was not until some fifty years later that the meaning of this additional term was understood. In 1976, Linet [65] revisited Copson's solution for a single charge residing in a Schwarzschild spacetime. By applying Gauss' law at infinity, Linet was able to prove that the spacetime contained more than one charge and that the additional term that arose in Copson's original paper was precisely this second charge. As Copson had placed only one charge outside the horizon at b (and it was clear from his equipotential surface plots that this was indeed the case) it was postulated that this second charge lay within the surface of inversion (and outside of the region covered by the isotropic coordinates).

In the meantime, Cohen and Wald [53] and Hanni and Ruffini [63] had, independently of the work done previously by Whittaker, also derived a multipole expansion solution for a charge placed in a Schwarzschild field and it is to their work that Linet compared his solution. When this additional charge was eliminated, Linet found that Copson's solution agreed with the series expansion solutions of [53] and [63] and naturally also [62].

Linet also managed to devise a transformation formula by which he was able to convert Copson's solution from isotropic coordinates into Schwarzschild coordinates.

In [95], Linet and co-authors found that a simple redefinition of the radial coordinate could recast the partial differential equation on the electrostatic potential generated by a charge placed in a Reissner-Nordström spacetime into a form identical to that of a charge placed in a Schwarzschild spacetime. As this partial differential equation was already solved in [1] and ammended by [65], the solution was immediately recoverable.

In [96], Linet and co-authors studied the electrostatic potential generated by a point charge in a Brans-Dicke spacetime. As Copson had before them, they postulated the form of a substitution that would convert the partial differential equation on the electrostatic potential into a solvable ordinary differential equation on a new variable. They were successful in converting the partial differential equation into a solvable form in terms of a new variable $F(\gamma)$ and were able to solve for $F(\gamma)$. Substituting this back into the original substitution equation the authors were able to find a closed-form solution for the electrostatic potential in terms of integration constants. The integration constants were chosen by conducting multipole expansions.

In [69], Linet used the Copson-Hadamard method to try to find a substitution that would convert the electrostatic potential generated by a point charge at rest in a Schwarzschild black hole in higher dimensions. He tried three different forms of the substitution but was unable to find a closed-form solution which would be the analogue of the ones already known in the Schwarzschild, Reissner-Nordström and Brans-Dicke spacetimes. He was however successful in finding a closed-form expresion for the electrostatic potential and static scalar field for a point source in a Reissner-Nordström black hole in four dimensions.

The choice of boundary conditions and consequently integration constants, proves to be crucial in deriving an accurate solution. It was evident that Copson's original choice of integration constants, chosen such that the solution would be symmetrical in terms of the radial coordinate r and the location of the charge b, resulted in the appearance of an additional charge inside the surface of inversion (the area not covered by the isotropic coordinates). Even a plot of the spacetime was not enough to identify the appearance of this additional charge. Although a term by term comparison with known multipole expansion solutions correctly exposed a discrepancy of the closed-form solution to multipole expansion solutions, the meaning behind this discrepancy was only made clear when Gauss' law at infinity was applied.

The accuracy and physical meaning of the final solution hinges on ones choice of boundary condition and perhaps it was for this reason that the authors of [96] used a complex but thorough multipole expansion method to determine the choice of integration constants.

In this paper we address several key areas of research that are lacking when it comes to closed-form solutions to electrostatic perturbations before extending our work to more general spacetimes.

Firstly, we find that although the Copson-Hadamard method is robust, it has the drawback that one must postulate the form of the substitution used which converts the partial differential equation on the electrostatic potential $V(r, \theta)$ into a solvable form in terms of a new variable $F(\gamma)$, in a case-by-case manner by looking at series solutions. We introduce an ansatz that allows one to find the requisite substitution from the background metric itself without having to look to the infinite series solution for inspiration. Using our method one can find the form of $V(r, \theta)$ in terms of $F(\gamma)$ and γ itself from the coefficients of the background metric.

Secondly, we find that after the ordinary differential equation on $F(\gamma)$ has been solved to give us a closed-form solution for $V(r, \theta)$ there still lies the problem of choosing appropriate boundary conditions and integration constants such that the resulting solution is actually representative of a single charge. We introduce a method based on Gauss' law which is both necessary and sufficient in ensuring that the final solution represents the electrostatic potential generated by a single charge placed in the background of one's choice and that does away with the need to look at multipole expansions.

In the first part of this paper, we construct the Brans-Dicke Reissner-Nordström background metric by integrating the electrovac BD field equations directly. We choose the integration constants such that the final solution reduces to the Brans-Dicke Class I and Reissner-Nordström spacetimes. It is for this reason that we label this spacetime the Brans-Dicke Reissner-Nordström (BDRN) spacetime. We give the spacetime metric in its most general form before applying the weak field approximation to gain insight into the integration constants and to give them a more concrete physical interpretation. In the second part of the paper we electrostatically perturb the BDRN spacetime and develop a method based on the Copson-Hadamard method by which we can find a closed-form solution for the electrostatic potential generated by a perturbing charge. We find that the BDRN spacetime is stable under electrostatic perturbations.

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Electrostatic potential of a point charge in a Brans-Dicke Reissner-Nordström field

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We consider the Brans-Dicke Reissner-Nordström spacetime in isotropic coordinates and the electrostatic field of an electric point charge placed outside its surface of inversion. We treat the static electric point charge as a linear perturbation on the Brans-Dicke Reissner-Nordström background. We develop a method based upon the Copson method to convert the governing Maxwell equation on the electrostatic potential generated by the static electric point charge into a solvable linear second-order ordinary differential equation. We obtain a closed-form fundamental solution of the curved-space Laplace equation arising from the background metric, which is shown to be regular everywhere except at the point charge and its image point inside the surface of inversion. We also develop a method that demonstrates that the solution does not contain any other charge that may creep into the region that lies beyond the surface of inversion and which is not covered by the isotropic coordinates. The Brans-Dicke Reissner-Nordström spacetime therefore is linearly stable under electrostatic perturbations. This stability result includes the three degenerate cases of the fundamental solution that correspond to the Brans Type 1, the Reissner-Nordström and the Schwarzschild background spacetimes.

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I. INTRODUCTION

The effect of gravitation on electromagnetic phenomena is of great interest due to its applications in both particle and astrophysics. To the astrophysicist, such study sheds light on phenomena occurring around black holes and other strong gravitational sources. Of particular interest to the authors of this paper are the closed-form fundamental solutions for electric potential that can be found for these situations. Such solutions, when they exist, are particularly interesting as they provide the basis for research on, amongst others, particle self-interaction (e.g. Refs. [1–4]) and electromagnetic phenomena around wormholes (e.g. Refs. [5–8]). See also Refs. [9–11]. This paper investigates the stability of a class of electrovac Brans-Dicke spacetimes linearly perturbed by a static electric point charge.

In 1927, Whittaker [12] investigated electric phenomena in gravitational fields including the study of the electrostatic potential generated by a static electric point charge in a Schwarzschild and a quasiuniform gravitational background. Using the method of separation of variables, he was able to find an infinite-series solution describing the former and a closed-form solution for the latter.

Shortly thereafter, in 1928, Copson [13] used Hadamard's [14] theory of "elementary solutions" (referred to in recent literature as fundamental solutions) of partial differential equations to not only reproduce Whittaker's original expression for the quasiuniform field but to go on and derive an exact closed-form expression for the potential generated by a static electric point charge in a Schwarzschild spacetime written in isotropic coordinates. Copson noted that his result differed from Whittaker's infinite-series solution by a zeroth-order term (see Sec. VI for further discussion on Whittaker's and Copson's solutions).

Independently, in the 1970s Cohen and Wald [15] and Hanni and Ruffini [16] used the method of separation of variables to express the electrostatic potential generated by a static electric point charge in a Schwarzschild background as an infinite series which concurred with Whittaker's earlier result.

Copson's result was ammended by Linet [17] who applied the boundary condition at infinity and an asymptotic expansion of Copson's solution to prove that it was for not one but two charges, the second residing within the horizon. Linet resolved this issue by excising the second charge, his result coinciding with those found using multipole expansions by Cohen and Wald [15], Hanni and Ruffini [16] and Whittaker [18]. Linet was also able to transform Copson's fundamental solution from isotropic coordinates into the usual Schwarzschild coordinates.

Using the Copson-Hadamard method Linet went on to derive expressions for the potential of a static electric point charge in the Reissner-Nordström field with Leaute [19] and in a Brans-Dicke-Schwarzschild field with Teyssandier [20]. Copson revisited his method of solution in 1978 [21], developing a closed-form solution for the potential generated by a static electric point charge (what he terms an "electron") in a Reissner-Nordström background field. His result again differed to that obtained by Leaute and Linet [19] due to a different choice of boundary conditions, which will be discussed in Sec. VI.

The Copson method for solving for the electric potential involves identifying a new independent variable that converts the governing Maxwell equation into a linear second-order ordinary differential equation. Linet and co-authors

[redacted]

adopted the form of Copson's independent variable to consider the Reissner-Nordström and Brans-Dicke-Schwarzschild cases and were able to find the corresponding governing equations for the above-mentioned cases as second-order ordinary differential equations (see also Ref. [22]). The results obtained by these authors were on a case-by-case basis. Here we introduce a method by which one is able to extract the form of the new independent variable and obtain the general second-order ordinary differential equation for all the electrovac spherically symmetric Brans-type solutions which are reducible to the Schwarzschild and Reissner-Nordström black hole solutions in the Einstein theory.

As Copson's method involves solving a linear second-order ordinary differential equation which would naturally produce two linearly independent solutions, it is necessary to impose appropriate boundary conditions that would allow one to determine the relationship between the two constant coefficients of the general solution. As mentioned earlier, it was due to a different choice of boundary condition that Copson's result differed from that found by Linet and coauthors, and thus it is clear to see how the interpretation of the solution hinges upon the choice of boundary condition. As a result of his choice of boundary conditions Copson's solution exhibited two charges, which was in contradiction to Hadamard's theory of "elementary" solutions that stipulates that there must exist only one singular point. Here we impose a boundary condition such that a Gauss' law-type integral over any closed surface in space not enclosing the perturbing charge must vanish even if that region contains a surface of inversion, which exists in all Brans Type I solutions which are reducible to the Schwarzschild and Reissner-Nordström black hole cases (see Theorem 1 in Sec. II). This boundary condition proves to be sufficient in determining the relationship between the constant coefficients such that they are in agreement with known multipole solutions found using the method of separation of variables [15,16,18] and those found using Hadamard's definition of "elementary" solutions [17,19,20].

In Sec. II we give a detailed overview of the Brans-Dicke Reissner-Nordström background which is the exact solution for the gravitational field generated by a point charge in a scalar-tensor field. The general Brans-Dicke electrovac solution has six constants of integration, two of which can be determined by scaling the coordinates r and t. Luke and Szamosi [23] showed that the remaining four constants of integration can be constrained such that the solution reduces to the Reissner-Nordström solution in Einstein's theory. The salient feature of these Brans-Dicke metrics is that in isotropic coordinates a surface of inversion separates the solution into two regions. This results in a double covering of the spacetime region corresponding to the exterior of a nonrotating black hole in general relativity. It is also important to detail the choice of constants and their subsequent physical interpretations as this will influence our choice of boundary condition (see Appendix A for more details). As demonstrated by Arnowitt, Deser and Misner (ADM) [24,25], the interpretations of the source terms of a spherically symmetric spacetime in isotropic coordinates requires careful analysis. The results in this section, together with the analysis in Appendix A, extend some of the results in Refs. [24,25] done using the ADM technique on Schwarzschild and Reissner-Nordström spacetimes. We also state the appropriate choice of parameter values for the static spherically symmetric Brans-Dicke electrovac solutions as required by the weak-field approximation. We briefly discuss observational constraints on ω as this is of particular interest due to the fact that scalar fields and the variable cosmological "constant" have become two of the most popular candidates for dark energy [26–30].

In Sec. III, we briefly outline Hadamard's theory of fundamental solutions of curved-space Laplace equations containing first-order terms. In Sec. IV we extend Copson's method to find the first four terms of the fundamental solution describing the potential generated by a static electric point charge placed outside the surface of inversion in a Brans-Dicke Reissner-Nordström background. We then develop a method of identifying the new independent variable using the Brans-Dicke field equations as outlined in Appendix A.

In Sec. V, we solve the linear second-order differential equation to give us a closed-form solution, which can be used to construct the fundamental solution that represents the electric potential generated by a point charge residing outside the surface of inversion in a Brans-Dicke Reissner-Nordstöm spacetime. It is important to note that due to the nature of the background metric in isotropic coordinates, the region interior to the surface of inversion is an exact copy of the exterior. Therefore the closed-form solution obtained has an additional singular point at the inversion point of the perturbing static electric point charge.

In Sec. VI, we introduce a boundary condition that will allow one to determine the relationship between the two constant coefficients and essentially eliminate the singularity that creeps into the spacetime region that lies beyond the inversion surface and which is not covered by the isotropic coordinates. Hence we obtain a process to derive the fundamental solution for a class of curved-space Laplace equations containing first-order terms, thus making it unnecessary to compare with multipole expansion solutions.

Lastly, in Sec. VII we show how our method also yields the fundamental solutions of the three known cases (Schwarzschild, Reissner-Nordström, Brans-Dicke Type I), which are summarized in Table I.

In Appendix A, details of how the scalar field, the metric functions and the electrostatic potential are all essentially a combination of the metric variable $r^2\phi e^{\alpha+\beta}$ are discussed. Appendix A also outlines how the Brans-Dicke Reissner-Nordström background metric can be determined.

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TABLE I. The four cases and their solutions for the electrostatic potential generated by a charged particle at $r = b$ where $k = \frac{C+2}{2\lambda}$.

	Brans-Dicke-Reissner-Nordström	Brans-Dicke	Reissner-Nordström	Schwarzschild
$e^{2\alpha}$	$\frac{e^{2\alpha_0}\lvert\frac{r-B}{r+B}\rvert^{\frac{2}{\lambda}}}{\eta(r)^2}$	$e^{2\alpha_0}\lvert\frac{r-B}{r+B}\rvert^{\frac{2}{\lambda}}$	$\frac{(\frac{r-B}{r+B})^2}{\eta(r)^2}$	$(\frac{r-B}{r+B})^2$
$e^{2\beta}$	$e^{2\beta_0}\eta(r)^2(1+\frac{B}{r})^4\lvert\frac{r-B}{r+B}\rvert^{2(\frac{\lambda-C-1}{\lambda})}$	$e^{2\beta_0}(1+\frac{B}{r})^4\lvert\frac{r-B}{r+B}\rvert^{2(\frac{\lambda-C-1}{\lambda})}$	$\eta(r)^2(1+\frac{B}{r})^4$	$(1+\frac{B}{r})^4$
$\eta(r)$	$p_+^2 - p_-^2\lvert\frac{r-B}{r+B}\rvert^{2k}$	1	$p_+^2 - p_-^2(\frac{r-B}{r+B})^2$	1
ϕ	$\phi_0\lvert\frac{r-B}{r+B}\rvert^{\frac{C}{\lambda}}$	$\phi_0\lvert\frac{r-B}{r+B}\rvert^{\frac{C}{\lambda}}$	ϕ_0	ϕ_0
U_0	$\frac{r}{b}\frac{\eta_0}{\eta(r)}\frac{(r-B)^{k-\frac{1}{2}}}{(r+B)^{k+\frac{1}{2}}}\frac{(b+B)^{k+\frac{1}{2}}}{(b-B)^{k-\frac{1}{2}}}$	$\frac{r}{b}\frac{(r-B)^{k-\frac{1}{2}}}{(r+B)^{k+\frac{1}{2}}}\frac{(b+B)^{k+\frac{1}{2}}}{(b-B)^{k-\frac{1}{2}}}$	$\frac{r}{b}\frac{\eta_0}{\eta(r)}\frac{(r-B)^{\frac{1}{2}}}{(r+B)^{\frac{3}{2}}}\frac{(b+B)^{\frac{3}{2}}}{(b-B)^{\frac{1}{2}}}$	$\frac{r}{b}\frac{(r-B)^{\frac{1}{2}}}{(r+B)^{\frac{3}{2}}}\frac{(b+B)^{\frac{3}{2}}}{(b-B)^{\frac{1}{2}}}$
U_1	$\frac{3B^2(1+\frac{4}{3}(1-k^2))}{2(r^2-B^2)(b^2-B^2)}U_0$	$\frac{3B^2(1+\frac{4}{3}(1-k^2))}{2(r^2-B^2)(b^2-B^2)}U_0$	$\frac{3B^2}{2(r^2-B^2)(b^2-B^2)}U_0$	$\frac{3B^2}{2(r^2-B^2)(b^2-B^2)}U_0$
U_2	$\frac{B^2(1+\frac{4}{3}(1-k^2))(-5+\frac{4}{3}(1-k^2))}{8(r^2-B^2)^2(b^2-B^2)^2}U_0$	$\frac{B^2(1+\frac{4}{3}(1-k^2))(-5+\frac{4}{3}(1-k^2))}{8(r^2-B^2)^2(b^2-B^2)^2}U_0$	$-\frac{5B^4}{8(r^2-B^2)^2(b^2-B^2)^2}U_0$	$-\frac{5B^4}{8(r^2-B^2)^2(b^2-B^2)^2}U_0$
U_3	$\frac{B^6(1+\frac{4}{3}(1-k^2))(-5+\frac{4}{3}(1-k^2))(-7+\frac{4}{3}(1-k^2))}{80(r^2-B^2)^3(b^2-B^2)^3}U_0$	$\frac{B^6(1+\frac{4}{3}(1-k^2))(-5+\frac{4}{3}(1-k^2))(-7+\frac{4}{3}(1-k^2))}{80(r^2-B^2)^3(b^2-B^2)^3}U_0$	$-\frac{7B^6}{80(r^2-B^2)^3(b^2-B^2)^3}U_0$	$-\frac{7B^6}{80(r^2-B^2)^3(b^2-B^2)^3}U_0$
Eq. (52)	$\frac{r\phi_0}{\eta(r)(r+B)^2}(\frac{r-B}{r+B})^{k-1}F(\gamma)$	$\frac{r}{(r+B)^2}(\frac{r-B}{r+B})^{k-1}F(\gamma)$	$\frac{r\phi_0}{\eta(r)(r+B)^2}F(\gamma)$	$\frac{r\phi_0}{(r+B)^2}F(\gamma)$
$V(r,\theta)$	$\frac{\epsilon_0 r}{\eta(r)(r^2-B^2)}[\frac{r-B}{r+B}]^k\frac{bB(b^2-B^2)^{k-1}}{[p_+^2(b+B)^{2k}-p_-^2(b-B)^{2k}]}$ $\times\frac{p_+^2(\sqrt{\gamma+1}+\sqrt{\gamma})^{2k}-p_-^2(\sqrt{\gamma+1}-\sqrt{\gamma})^{2k}}{2\sqrt{\gamma}\sqrt{\gamma+1}}$	$\frac{\epsilon_0 r}{r^2-B^2}[\frac{r-B}{r+B}]^k\frac{bB(b^2-B^2)^{k-1}}{(b+B)^{2k}}$ $\times\frac{(\sqrt{\gamma+1}+\sqrt{\gamma})^{2k}}{2\sqrt{\gamma}\sqrt{\gamma+1}}$	$\frac{\epsilon_0 r}{\eta(r)(r+B)^2}\frac{bB}{[p_+^2(b+B)^2-p_-^2(b-B)^2]}$ $\times\frac{p_+^2(\sqrt{\gamma+1}+\sqrt{\gamma})^2-p_-^2(\sqrt{\gamma+1}-\sqrt{\gamma})^2}{2\sqrt{\gamma}\sqrt{\gamma+1}}$	$\frac{\epsilon_0 r}{(r+B)^2}\frac{bB}{(b+B)^2}$ $\times\frac{(\sqrt{\gamma+1}+\sqrt{\gamma})^2}{2\sqrt{\gamma}\sqrt{\gamma+1}}$

The surface integral inner boundary condition is outlined in Appendix B for the Brans-Dicke Reissner-Nordström background.

II. SCALAR-TENSOR FIELD THEORY

The field equations in the Brans-Dicke theory are

$$R_{ab} - \frac{1}{2}g_{ab}R = \frac{8\pi T_{ab}}{c^4\phi} + \frac{1}{\phi}(\nabla_a\partial_b\phi - g_{ab}\Box\phi) + \frac{\omega}{\phi^2}\left(\partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}g^{cd}\partial_c\phi\partial_d\phi\right), \tag{1}$$

$$\Box\phi = \frac{8\pi T}{(2\omega+3)c^4}, \tag{2}$$

where

$$\Box\phi := \nabla_b(g^{ab}\partial_a\phi) = \frac{1}{\sqrt{-g}}\partial_b(\sqrt{-g}g^{ab}\partial_a\phi) \tag{3}$$

and $\Box$ is the scalar wave operator.

Here the notations have their usual meaning. The contribution of the electromagnetic field, encoded in the Faraday tensor F_{ab}, to the energy-momentum tensor is

$$T_{ab} = F_{ac}F^c_b - \frac{1}{4}g_{ab}F_{cd}F^{cd}, \qquad T^a_a = 0, \tag{4}$$

where F_{ab} satisfies the source-free Maxwell equations

$$\nabla_b F^{ab} = 0, \qquad \nabla_{[c}F_{ab]} = 0. \tag{5}$$

Following the method of Luke and Szamozi [23] while at the same time conforming to the choice of boundary conditions in Brans [31] [see Eqs. (19)–(24)] one can verify that an electrically charged Brans-Dicke field that reduces to the Reisnner-Nordström solution in isotropic coordinates when the long-range field equals the reciprocal of the gravitational constant, i.e. $\phi = (G_0)^{-1}$, can be summarized as follows (see Appendix A for a brief derivation).

Theorem 1.—A static spherically symmetric electrically charged Brans-Dicke Reissner-Nordström (BDRN) solution of Eqs. (1), (2), and (5) in isotropic coordinates (t, r, θ, ϕ) is given by the line element

$$ds^2 = -c^2e^{2\alpha(r)}dt^2 + e^{2\beta(r)}[dr^2 + r^2(d\theta^2 + \sin\theta d\phi^2)], \tag{6}$$

where the static electric potential $V_0(r)$, the Faraday tensor F_{ab} and the corresponding energy-momentum tensor T^a_b are

$$V_0(r) = Q\int_\infty^r \frac{e^{\alpha(r)-\beta(r)}}{r^2}dr, \tag{7}$$

$$F_{ab} = -cV_0'(r)\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{8}$$

$$T^a_b = -\frac{e^{4\beta(r)}Q^2}{2r^4}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}. \tag{9}$$

The reduced long-range scalar field wave equation derived from Eq. (1) is

$$[r^2 \exp(-\alpha(r) + \beta(r))V_0'(r)]' = 0. \tag{10}$$

The metric functions $e^{2\alpha(r)}$ and $e^{2\beta(r)}$ are

$$e^{2\alpha(r)} = \frac{e^{2\alpha_0}|\frac{r-B}{r+B}|^{\frac{2}{\lambda}}}{(p_+^2 - p_-^2|\frac{r-B}{r+B}|^{(\frac{C+2}{\lambda})})^2}, \tag{11}$$

$$e^{2\beta(r)} = e^{2\beta_0}\left(1+\frac{B}{r}\right)^4\left|\frac{r-B}{r+B}\right|^{2(\frac{\lambda-C-1}{\lambda})} \times\left(p_+^2 - p_-^2\left|\frac{r-B}{r+B}\right|^{\frac{C+2}{\lambda}}\right)^2, \tag{12}$$

and the long-range scalar field $\phi(r)$ is

$$\phi = \phi_0\left|\frac{r-B}{r+B}\right|^{\frac{C}{\lambda}}. \tag{13}$$

The functions $V_0(r)$, $e^{2\alpha(r)}$, $e^{2\beta(r)}$ and $\phi(r)$ are defined for all non-negative r except at $r = B$. The boundary values e^{α_0} and e^{β_0} can be rescaled to unity by scaling the t and r coordinates, respectively. The nine parameters Q, B, p_+^2, p_-^2, λ, C, ϕ_0, $e^{2\alpha_0}$ and $e^{2\beta_0}$ in Eqs. (11)–(13) are related via

$$4\lambda^2 = (2\omega+3)C^2 + (C+2)^2, \qquad \omega \geq -\frac{3}{2}, \tag{14}$$

$$B = \frac{1}{2}\sqrt{m_B^2 - q_B^2}, \tag{15}$$

$$m_B := \frac{2M}{c^2\phi_0}\frac{e^{\beta_0}\lambda}{C+2}, \qquad q_B = 2\sqrt{\frac{4\pi}{\phi_0}}\frac{Q}{c^2}\frac{e^{\beta_0}\lambda}{C+2}, \tag{16}$$

$$p_\pm^2 = \frac{m_B \pm \sqrt{m_B^2 - q_B^2}}{2\sqrt{m_B^2 - q_B^2}}, \tag{17}$$

$$p_+^2 - p_-^2 = 1, \tag{18}$$

where M and Q are non-negative real constants and are identified, respectively, as mass measured in conventional units (kg) and charge measured in electrostatic units (e.s.u), which has the dimensions of $[\text{mass}]^{\frac{1}{2}}[\text{length}]^{\frac{3}{2}}[\text{time}]^{-1}$. Here ω is the coupling constant that couples the scalar field to matter, while c is the speed of light in a vacuum.

Taking into account Eqs. (14)–(16), there remain only four essential parameters in the BDRN solution. We adopt the independent parameter set M, Q, C, ϕ_0.

(1) The choice of the physical parameters of mass, M, and charge, Q, in the characterization of the BDRN metric is natural.

(2) As opposite charges neutralize one another, in most astrophysical applications it is reasonable to assume $M \geq \sqrt{4\pi\phi_0}Q \geq 0$, and hence the parameter B in Eq. (15) is non-negative.

(3) ϕ_0 is the value of the long-range scalar field at spatial infinity. It has the dimensions of $[\text{mass}]\times[\text{length}]^{-3}[\text{time}]^2$.

(4) The parameter C is dimensionless and relates to the local strength of the long-range scalar field $\phi(r)$. Equation (14) gives λ^2 as a quadratic expression in C with the discriminant $\Delta = -(2\omega+3)$. Thus when $\omega > -\frac{3}{2}$, C is real and $\sqrt{\frac{2\omega+3}{2\omega+4}} < |\lambda| < \infty$. Constraining the BDRN solutions to conform with the weak-field approximation (see Ref. [32]), we expand the metric functions and the scalar field, Eqs. (11)–(13), to the order of $1/r$, and obtain the following restrictions on the parameters:

$$\alpha_0 = \beta_0 = 0, \tag{19}$$

$$\phi_0 = \frac{1}{G_0}\left(\frac{2\omega+4}{2\omega+3}\right), \tag{20}$$

$$\lambda \cong \sqrt{\frac{2\omega+3}{2\omega+4}}, \tag{21}$$

$$C \cong -\frac{1}{\omega+2}, \tag{22}$$

$$m_B \cong \frac{M}{c^4\phi_0}\sqrt{\frac{2\omega+4}{2\omega+3}}, \tag{23}$$

$$q_B \cong \sqrt{\frac{4\pi}{\phi_0}}\frac{Q}{c^2}\sqrt{\frac{2\omega+4}{2\omega+3}}, \tag{24}$$

where G_0 is defined as the gravitational constant (for the BDRN spacetimes), while G denotes Newton's universal constant of gravity (see Case 2 and Case 3 below).

(5) Observational constraints put even stronger requirements on the values of ω. The latest results obtained from the Cassini-Huygen experiment [33] put the value of ω at over 40 000. The coupling constant ω represents the strength of the coupling between the scalar field and the gravitational field. Therefore its value is of great importance in any discussion regarding (a) the existence and properties of Brans-Dicke black holes and (b) candidates for dark energy.

(6) When an inversion is applied, that is, transforming from r to $r* = \frac{B^2}{r}$, the region $B < r < \infty$ is mapped

one-to-one onto the region $0 < r < B$. Under such a reflection at the sphere of $r_{\mathrm{BDRN}} = B$, the functions $V_0(r)$, $e^{2\alpha(r)}$, $\phi(r)$ and the line element (6) remain invariant while the metric function $e^{2\beta(r)}$ is transformed into $e^{2\beta(r^*)} = \frac{r^4}{B^4} e^{2\beta(r)}$ and the flat 3-metric $d\ell^2 := [dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]$ is mapped conformally onto the flat metric $d(\ell^*)^2 = \frac{r^4}{B^4} d\ell^2$. Therefore the spherical surface at $r_{\mathrm{BDRN}} := B = \frac{1}{2} \times \sqrt{m_B^2 - q_B^2}$ is an inversion surface in the sense that the BDRN solution in isotropic coordinates is invariant under the geometric inversion transformations $r(r*) = B^2$. The two copies of the BDRN spacetime—one exterior to and the other interior to the inversion surface—are identical. At the spherical surface of inversion $r_{\mathrm{BDRN}} = B$, the line element (6) is singular. It is the inaccessible boundary of the two identical copies of the BDRN spacetime in isotropic coordinates. Throughout this article we use the exterior copy where $B < r < \infty$, unless stated otherwise. This will have important consequences (see below) on how to interpret the Copson-Hadamard method [13] in the construction of the fundamental solution to the Laplace equation of a perturbed electrostatic potential in a BDRN background solution.

(7) An investigation by Ref. [34] found that Brans Type I solutions may represent an external gravitational field for nonsingular spherically symmetric matter sources. They concluded, however, that Brans-Dicke black holes cannot exist, as a condition equivalent to Eq. (22) (that is, the weak-field approximation) would require that $-2 > \omega > -(2 + \frac{1}{\sqrt{3}})$, a requirement which clearly violates observational constraints [33].

By choosing various combinations of the four independent parameters M, Q, C and ϕ_0 to vanish, we obtain the following limiting solutions.

Case 1 Brans Type I (BS) metric in isotropic coordinates. By setting the charge parameter Q to zero, it implies that $B = \frac{m_B}{2} = \frac{M}{2c^2\phi_0}$, $p_+^2 = 1$ and $p_-^2 = 0$. We recover the Brans Type I metric [31] of the Brans-Dicke theory,

$$\phi(r) = \phi_0 \left| \frac{r - B}{r + B} \right|^{\frac{C}{\lambda}}, \tag{25}$$

$$e^{2\alpha(r)} = e^{2\alpha_0} \left| \frac{r - B}{r + B} \right|^{\frac{2}{\lambda}}, \tag{26}$$

$$e^{2\beta(r)} = e^{2\beta_0}\left(1 + \frac{B}{r}\right)^4 \left| \frac{r - B}{r + B} \right|^{2\left(\frac{\lambda - C - 1}{\lambda}\right)}, \tag{27}$$

where $B < r < \infty$ and the inversion spherical surface is at $r_{\mathrm{BS}} = B = \frac{M}{2c^2\phi_0}$ and is a curvature singularity.

Case 2 Reissner-Nordström (RN) metric in isotropic coordinates.
By setting the parameters $C = \alpha_0 = \beta_0 = 0$, it implies that $\lambda^2 = 1$, $\phi_0 = (G)^{-1}$, $B = \frac{1}{2} \times \sqrt{m^2 - q^2}$, $p_+^2 = \frac{m + \sqrt{m^2 - q^2}}{2\sqrt{m^2 - q^2}}$ and $p_-^2 = \frac{m - \sqrt{m^2 - q^2}}{2\sqrt{m^2 - q^2}}$, where $m := \frac{GM}{c^2}$ and $q := \frac{\sqrt{4\pi G}Q}{c^2}$ are, respectively, the mass and the electric charge measured in gravitational units. The metric functions reduce to the usual Reissner-Nordström solution in isotropic coordinates,

$$e^{2\alpha(r)} = \frac{\left(r - \frac{\sqrt{m^2 - q^2}}{2}\right)^2 \left(r + \frac{\sqrt{m^2 - q^2}}{2}\right)^2}{(r + \frac{m - q}{2})^2 (r + \frac{m + q}{2})^2}, \tag{28}$$

$$e^{2\beta(r)} = \frac{(r + \frac{m - q}{2})^2 (r + \frac{m + q}{2})^2}{r^4}, \tag{29}$$

where $\frac{1}{2}\sqrt{m^2 - q^2} < r < \infty$, and the inversion spherical surface is at $r_{H+} = \frac{1}{2}\sqrt{m^2 - q^2}$, which is also the outer event horizon of the RN spacetime in isotropic coordinates.
The Reisnner-Nordström metric in isotropic coordinates was first derived in the form given in Eqs. (28) and (29) above using the ADM technique (see Refs. [24,25]).

Case 3 Schwarzschild (S) metric in isotropic coordinates.
By setting the parameters $Q = C = \alpha_0 = \beta_0 = 0$, it implies that $\lambda^2 = 1$, $\phi_0 = (G)^{-1}$, $B = \frac{m}{2} = \frac{GM}{2c^2}$, $p_+^2 = 1$ and $p_-^2 = 0$, where $m = \frac{GM}{c^2}$ is the mass in gravitational units. The metric functions reduce to the well-known Schwarzschild solution in isotropic coordinates,

$$e^{2\alpha(r)} = \left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^2, \qquad e^{2\beta(r)} = \left(1 + \frac{m}{2r}\right)^4, \tag{30}$$

where $\frac{1}{2}m < r < \infty$ and the inversion spherical surface is at $r_H = \frac{1}{2}m$, which is also the event horizon of the Schwarzschild spacetime in isotropic coordinates.

III. ELECTROSTATICS AND THE HADAMARD METHOD

We now consider the electrostatic potential due to a "small" static electric charge $-\epsilon_0(|\epsilon_0| \ll m_B)$ situated outside the spherical surface of inversion B.

We let $V(r, \theta, \phi)$ denote the linearly perturbated electrostatic potential so that the perturbed Faraday tensor F_{ab} takes the form

$$F_{0i} = -F_{i0} = -c\partial_i V(r, \theta, \phi), \qquad F_{ij} = 0; \quad i, j = 1, 2, 3 \ldots. \tag{31}$$

The perturbed Maxwell equations $\nabla_{[a}F_{bc]} = 0$ are automatically satisfied by Eq. (31).

Without loss of generality, the perturbed Maxwell equations due to a single electrostatic charge yield

$$\frac{1}{\sqrt{-g}}\partial_b(\sqrt{-g}F^{ab}) = J^0, \tag{32}$$

which implies

$$\nabla^2 V(r,\theta,\phi) - (\alpha'(r) - \beta'(r))\frac{\partial V(r,\theta,\phi)}{\partial r} \tag{33}$$

$$= ce^{2(\alpha(r)+\beta(r))}J^0, \tag{34}$$

where the current density $J^0 = -\frac{4\pi\epsilon_0}{cr^2}e^{-2\alpha(r)-3\beta}\delta(r - b)\delta(\cos\theta - \cos\theta_0)$. Here $\alpha(r)$ and $\beta(r)$ are given by Eqs. (11) and (2), respectively, and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the three-dimensional Euclidean-space Laplacian with $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$. Note that $\partial_r = \frac{x}{r}\partial_x + \frac{y}{r}\partial_y + \frac{z}{r}\partial_z$. We define

$$\Gamma(r,\theta) = r^2 + b^2 - 2br\cos\theta, \tag{35}$$

which is equal to the square of the "radial" distance from the charged particle at $z = b$.

A brief overview of Hadamard's theory of fundamental solutions [14] is necessary to fully understand Copson's construction [13,21]. We adapt Hadamard's result that includes Eq. (33) as a particular case as follows.

Theorem 2 (Hadamard's Theorem).—Consider a second-order linear partial differential equation of the form

$$\Im(u) = \sum_{i,j=1}^{3}\delta^{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{3} h(r)\frac{x^i}{r}\frac{\partial u}{\partial x^i} = 0, \tag{36}$$

where δ^{ij} is the Kronecker tensor and $h(r)$ is a differentiable function of $r = \delta_{ij}x^i x^j$. The fundamental solution of Eq. (36) is continuous and differentiable everywhere except at the singular point $(r,\theta,\phi) = (b,\theta_0,\phi_0)$ and can be written as

$$u = \frac{U(r,\theta,\phi)}{\Gamma^{\frac{1}{2}}}, \tag{37}$$

where Γ is given by Eq. (35) and the function $U(r,\theta,\phi)$ is real analytic everywhere in the domain of definition of Eq. (36), including the singular point $(r,\theta,\phi) = (b,\theta_0,\phi_0)$. $U(r)$ is expandable as a convergent power series in Γ such that

$$U(r,\theta,\phi) = U_0(r) + U_1(r)\Gamma + U_2(r)\Gamma^2 + \cdots, \tag{38}$$

where U_n is given by the recurrent formula

$$U_n(r) = \frac{U_0}{4(n-\frac{1}{2})s}\int_0^s \frac{s^{n-1}}{U_0}\Im(U_{n-1})ds, \quad n = 1,2,3\cdots, \quad s = \sqrt{r^2 + b^2 - 2rb\cos\theta}, \tag{39}$$

and

$$U_0(r) = \exp\left(-\int_b^r h(r)dr\right). \tag{40}$$

In the case of the BDRN metric, the coefficient $h(r)$ in Eqs. (36) and (40) is given by

$$h(r) = -\alpha'(r) + \beta'(r). \tag{41}$$

IV. EXTENSION OF THE COPSON CONSTRUCTION

Equation (33) for the BDRN metric can be expressed in the form

$$\nabla^2 V + \frac{2B}{r^2 - B^2}\left(2k\left[\frac{\eta^*(r)}{\eta(r)}\right] + \frac{B}{r}\right)\frac{\partial V}{\partial r} = ce^{2(\alpha(r)+\beta(r))}J^0, \tag{42}$$

where

$$k = \frac{C+2}{2\lambda}; \qquad \eta(r) = p_+^2 - p_-^2\left(\frac{r-B}{r+B}\right)^{2k}, \tag{43}$$

$$\eta^*(r) = -p_+^2 - p_-^2\left(\frac{r-B}{r+B}\right)^{2k}. \tag{44}$$

Instead of using the formal expression in Eq. (39), we follow Copson [13] by substituting Eq. (38) into Eq. (42). After some algebra we obtain the first four terms of the recurrent series of the Brans-Dicke Reissner-Nordström metric,

$$U_0(r) = \frac{r}{b}\frac{\eta_0}{\eta(r)}\frac{(r-B)^{k-\frac{1}{2}}}{(r+B)^{k+\frac{1}{2}}}\frac{(b+B)^{k+\frac{1}{2}}}{(b-B)^{k-\frac{1}{2}}}, \tag{45}$$

$$U_1(r) = \frac{3B^2(1+\frac{4}{3}(1-k^2))}{2(r^2-B^2)(b^2-B^2)}U_0, \tag{46}$$

$$U_2(r) = \frac{B^2(-5+\frac{4}{3}(1-k^2))}{4(r^2-B^2)(b^2-B^2)}U_1, \tag{47}$$

$$U_3(r) = \frac{B^2(-7+\frac{4}{3}(1-k^2))}{10(r^2-B^2)(b^2-B^2)}U_2, \tag{48}$$

where $\eta_0 = \eta(b)$. See Table I for the three degenerate cases.

We introduce the method by which the substitution can be determined for any background with a line element of the form given by Eq. (6), which satisfies the Brans-Dicke electrovac field equations (see Appendix A for the governing equations). Like Copson, from the first few terms given above we find that the ratio of the $(n+1)$th term to the nth term of the power series (39) is proportional to

$$\frac{B^2}{b^2 - B^2}\frac{\Gamma}{r^2 - B^2}, \tag{49}$$

where $r^2 - B^2$ is proportional to $\phi r^2 e^{\alpha+\beta}$ [see Eq. (A8) in Appendix A].

Furthermore, the first term of the infinite series, $\frac{U_0}{\Gamma^{1/2}}$, given by Eq. (40) is proportional to

$$\frac{e^{\frac{1}{2}(\alpha-\beta)}}{\Gamma^{1/2}} = \frac{1}{\gamma^{1/2}\sqrt{\phi}re^{\beta}}, \tag{50}$$

where

$$\gamma(r,\theta) = \frac{B^2}{b^2 - B^2}\frac{\Gamma(r,\theta)}{\phi r^2 e^{\alpha(r)+\beta(r)}}. \tag{51}$$

Now we introduce a new dependent variable $F(\gamma)$ such that the perturbed electrostatic potential takes the form

$$V(r,\theta,\phi) = \frac{F(\gamma)}{\sqrt{\phi}re^{\beta(r)}}. \tag{52}$$

For the Brans-Dicke Reissner-Nordström background, Eqs. (52) and (51), become, respectively,

$$V(r,\theta) = \frac{r\phi_0}{\eta(r)(r+B)^2}\left(\frac{r-B}{r+B}\right)^{k-1}F(\gamma), \tag{53}$$

$$\gamma(r,\theta) = \frac{B^2}{b^2 - B^2}\frac{\Gamma(r,\theta)}{r^2 - B^2}. \tag{54}$$

Substituting Eqs. (53) and (54) into Eq. (42) gives us a second-order linear differential equation in $F(\gamma)$,

$$\gamma(\gamma+1)F''(\gamma) + \frac{3}{2}(2\gamma+1)F'(\gamma) + (1-k^2)F(\gamma) = 0. \tag{55}$$

We have allowed the right-hand side of the above equation to vanish as we are only interested in regions away from the point source where the right-hand side of Eq. (55) is zero. We later use our boundary condition to verify that the delta-function source term is satisfied and to also determine the constants of integration of the solution to Eq. (55).

V. FUNDAMENTAL SOLUTIONS

Equation (55) can be solved if we transform the independent variable γ as

$$\gamma = \sinh^2\frac{\zeta}{2}, \tag{56}$$

which implies

$$\gamma + 1 = \cosh^2\frac{\zeta}{2}, \tag{57}$$

and we write the dependent variable $F(\gamma)$ as

$$F(\gamma) = \Phi(\zeta). \tag{58}$$

By using Eqs. (56) and (58), Eq. (55) can be written in terms of the new variables as follows:

$$\Phi''(\zeta) + 2\coth\zeta\,\Phi'(\zeta) + (1-k^2)\Phi(\zeta) = 0, \tag{59}$$

which has the closed-form solution (see Ref. [35])

$$\Phi(\zeta) = \frac{k}{\sinh\zeta}(\hat{W}_1 e^{k\zeta} - \hat{W}_2 e^{-k\zeta}), \tag{60}$$

where $\hat{W}_1$ and $\hat{W}_2$ are integration constants. The solution in terms of γ is therefore

$$F(\gamma) = \frac{k}{2\sqrt{\gamma}\sqrt{\gamma+1}}[\hat{W}_1(\sqrt{\gamma+1}+\sqrt{\gamma})^{2k} - \hat{W}_2(\sqrt{\gamma+1}-\sqrt{\gamma})^{2k}]. \tag{61}$$

Substituting Eq. (61) into Eq. (53) gives the electrostatic potential $V(r,\theta)$ as follows:

$$V(r,\theta) = \frac{k}{2\eta(r)\sqrt{\gamma}\sqrt{\gamma+1}}\frac{r}{(r+B)^2}\left(\frac{r-B}{r+B}\right)^{k-1} \times[\hat{W}_1(\sqrt{\gamma+1}+\sqrt{\gamma})^{2k} - \hat{W}_2(\sqrt{\gamma+1}-\sqrt{\gamma})^{2k}]. \tag{62}$$

Consider the inversion point of the static electric point charge $(0,0,(b*))$, where $(b*) = \frac{B^2}{b}$. Let

$$\gamma* = \frac{B^2}{B^2 - (b*)^2}\frac{\Gamma*(r,\theta)}{B^2 - r^2}, \tag{63}$$

$$\Gamma* = r^2 + (b*)^2 - 2(b*)r\cos\theta. \tag{64}$$

Thus $\Gamma*$ is equal to the square of the "radial" distance from the inversion point at $z = (b*)$. It is straightforward to verify that

$$\gamma + 1 = \gamma*. \tag{65}$$

The electrostatic potential $V(r,\theta)$ in Eq. (62) is therefore singular at the point charge $z = b$ and also at its inversion point $z = (b*)$. One can also verify that as the field point r approaches the inversion surface $r = B$, the potential approaches a finite limit value provided that $C > -2$.

Finally, to determine the fundamental solution for the electrostatic potential, which allows only one free parameter to arise from the presence of the perturbing electrostatic charge, it is necessary to establish the relationship between the two arbitrary constants in Eq. (62).

VI. DETERMINATION OF INTEGRATION CONSTANTS

In 1927, Whittaker, using the method of separation of variables in the usual Schwarzschild coordinates, found the solution expressing the electrostatic potential of a charge in a Schwarzschild background as an infinite series [12]. His result was later confirmed by Cohen and Wald [15] in 1971 and Hanni and Ruffini [16] in 1973. A commonality of these works is the use of a boundary condition stating that a charge should not arise inside the horizon as a result of the presence of the perturbing electric charge situated outside the horizon.

This boundary condition was not implemented by Copson in his determination of integration constants in Refs. [13,21] due to the fact that the region inside the horizon is excised in the isotropic coordinates. Instead, Copson chose values for the integration constants such that the overall solution would be symmetric in interchanging the position of the field point r with the position of the perturbing charge b. As a result, his solution, as he pointed out himself, was in contradiction to Whittaker's solution by the existence of a nonvanishing zero-order term. Linet [17], using the boundary condition at infinity and Gauss' theorem, found that this second charge—which was necessarily excised—gave a result which was in accordance to those given by Refs. [12,15,16].

In Ref. [20], Linet and Teyssandier found a single closed-form solution describing the electrostatic potential generated by a perturbing charge in a Brans-Dicke background. They expressed the fundamental solution as a sum of this solution and Legendre functions before performing a multipole expansion and writing the fundamental solution completely in terms of Legendre functions. By expressing the solution as a multipole expansion they were then able to impose boundary conditions at infinity to get a meaningful solution upon which Gauss' theorem could then be implemented to yield their final closed-form solution.

Here we introduce a method of determining the integration constants of Eq. (62) which does not require one to expand the closed-form solution into an infinite series and which is even more stringent than those set by Refs. [12,15,36]. We impose the condition that any integration over a closed spatial region not containing the perturbing charge must be exactly zero even if that area contains a surface of inversion. Naturally, an integration over an area containing the perturbing charged particle must therefore equal exactly $4\pi\epsilon$, where ϵ is the charge of the particle. From Appendix B we know that for the Brans-Dicke Reissner-Nordström background, the generalized Gauss's theorem can be written as the following:

$$\int_{\Re} J^0 dv = \int_0^{2\pi}\int_{-\pi}^{\pi} \eta(r)^2 (r+B)^2 \left(\frac{r-B}{r+B}\right)^{\frac{\lambda-C-2}{\lambda}} \times \frac{\partial V(r)}{\partial r} \sin\theta d\theta d\phi. \tag{66}$$

Here, $\Re$ is a region of three-dimensional space residing in a hypersurface and $\partial\Re$ is its closed two-dimensional boundary. Again, dv is an element of spatial proper volume in $\Re$. In order to integrate the above we convert Eq. (62) into a function of $\sinh\zeta$ where $\gamma = \sinh^2\frac{\zeta}{2}$. We find that the only term that requires integration is the term containing the integration constants, the integral of which is

$$\int_{-\pi}^{\pi} \frac{\hat{W}_1 e^{k\zeta} - \hat{W}_2 e^{-k\zeta}}{\sinh\zeta} \sin\theta d\theta = \frac{2brB^2[(b+B)^{2k} - (b-B)^{2k}]}{k(b^2-B^2)^{k-1}(r^2-b^2)^{k-1}} \times [\hat{W}_1 (r+B)^{2k} - \hat{W}_2 (r-B)^{2k}] \tag{67}$$

for $B < r < b$ and

$$\int_{-\pi}^{\pi} \frac{\hat{W}_1 e^{k\zeta} - \hat{W}_2 e^{-k\zeta}}{\sinh\zeta} \sin\theta d\theta = \frac{2brB^2[(r+B)^{2k} - (r-B)^{2k}]}{k(b^2-B^2)^{k-1}(r^2-b^2)^{k-1}} \times [\hat{W}_1 (b+B)^{2k} - \hat{W}_2 (b-B)^{2k}] \tag{68}$$

for $B < b < r$.

When we insert Eq. (67) into Eq. (66) it is fairly straightforward to see that for the electrostatic potential to vanish for the region not containing a charge the integration constants must be chosen as the following:

$$\hat{W}_1 = p_+^2 \hat{W}, \tag{69}$$

$$\hat{W}_2 = p_-^2 \hat{W}, \tag{70}$$

where $\hat{W}$ is a constant yet to be determined.

By inserting Eq. (68) into Eq. (66) and under the condition that for this region ($B < b < r$), Eq. (66) must equal $-4\pi\epsilon_0$, we can quickly solve for $\hat{W}$, giving

$$\hat{W} = \frac{bB(b^2-B^2)^{k-1}\sqrt{\phi_0}\epsilon_0}{k[p_+^2(b+B)^{2k} - p_-^2(b-B)^{2k}]}. \tag{71}$$

VII. DEGENERATE CASES

The relationship between the four cases and the process by which one reduces to the other is made obvious in Table I. It is straightforward to convert the equations in the Reissner-Nordström and Schwarzschild spacetimes into their more familiar form when one conducts the transformations given in Sec. II of this paper. When the transformations are made we find that the solutions are in agreement with the closed-form solutions given by Linet [17], Leaute and Linet [19] and Linet and Teyssandier [20], and with the multipole expansions given by Hanni and Ruffini [16] and Cohen and Wald [15].

VIII. CONCLUSION

In this paper we have established an ansatz to solve the perturbed Maxwell equations due to an electrostatic charge in a Brans-type spacetime in isotropic coordinates which is reducible to the Schwarzschild and Reissner-Nordström black hole solutions by extending Copson's method. As Copson's solution is based on Hadamard's theory of fundamental solutions of general Laplace equations it would be interesting to see whether Hadamard's infinite series converges to give Copson's closed-form result.

By finding the coefficients to $U_0, U_1, U_2, \ldots$ through the direct substitution of Hadamard's infinite series into the field equations one is able to compare them with the coefficients given in this paper using Copson's method.

In a separate paper, a formal proof of Hadamard's fundamental solution equation (36) is given. We find that Copson's results in Ref. [13] are in fact exactly equal to those found using Hadamard's method and go on to investigate how the Hadamard method relates to the results obtained by Linet in Ref. [22]. We also find that the discrepancy between Copson and Hadamard and those from the literature—including Whittaker, Hanni and Ruffini, and Cohen and Wald—lies in the domain of definition of fundamental solutions in the presence of a surface of inversion when considering the situation in isotropic coordinates.

Furthermore, we investigate the scope of applicability of Hadamard's theorem, including its application to more general scalar-tensor-vector theories and $f(R)$ theory and in particular to other branches of the Brans-Dicke theory such as the Barker and Schwinger cases (see also Ref. [37]). For detailed discussions on the scope of applicability of the Copson method in higher dimensions see Ref. [22].

In a separate paper, we convert the results given in this paper from isotropic coordinates to the usual Schwarzschild coordinates using Linet's transformation (outlined in Ref. [17]) and plot equipotential surfaces in both coordinate systems. As alluded to in Item 7 of Sec. II, Ref. [34] found that Brans-Dicke black holes cannot exist if the weak-field approximation is to be upheld. In our next paper we postulate that the weak-field approximation need not be satisfied [38–41]. Thus we find it worthwhile to plot the results of this paper in the usual coordinates to gain better insight into the behavior of the scalar field inside the horizon and thereby shed light on the physical possibility of Brans-Dicke black holes.

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APPENDIX A: BRANS-DICKE REISSNER-NORDSTRÖM BACKGROUND

The Brans-Dicke electrovac equations (1), (2), and (5) arising from the static spherically symmetric line element (6) in isotropic coordinates can be simplified when the following substitutions are introduced:

$$\tilde{A}(r) := \alpha(r) + \frac{1}{2}[\ln \phi(r)], \tag{A1}$$

$$\tilde{B}(r) := \beta(r) + \frac{1}{2}[\ln \phi(r)]. \tag{A2}$$

The electrovac equations $[e^{2\beta}G_t^t, e^{2\beta}G_r^r$ and $e^{2\beta}(G_r^r + G_\theta^\theta)]$ from Eq. (1) can then be written as

$$2\tilde{B}''(r) + \tilde{B}'(r)^2 + \frac{4}{r}\tilde{B}'(r) + \frac{4\pi Q^2 e^{-2\tilde{B}(r)}}{c^4} + \frac{2\omega + 3}{4}([\ln \phi(r)]')^2 = 0, \tag{A3}$$

$$\tilde{B}'(r)^2 + 2\tilde{A}'(r)\tilde{B}'(r) + \frac{2}{r}(\tilde{A}'(r) + \tilde{B}'(r)) + \frac{4\pi Q^2 e^{-2\tilde{B}(r)}}{c^4} - \frac{2\omega + 3}{4}([\ln \phi(r)]')^2 = 0, \tag{A4}$$

$$\tilde{A}''(r) + \tilde{B}''(r) + ((\tilde{A}(r) + \tilde{B})')^2 + \frac{3}{r}(\tilde{A}'(r) + \tilde{B}'(r)) = 0. \tag{A5}$$

The above three equations are not linearly independent, but instead are related via the following:

$$-\tilde{A}'(r)e^{2\beta}G_t^t + \left(\frac{d}{dr} + \left(\tilde{A}'(r) + 2\tilde{B}'(r) + \frac{4}{r}\right)\right)e^{2\beta}G_r^r - 2\left(\tilde{B}'(r) + \frac{1}{r}\right)e^{2\beta}(G_r^r + G_\theta^\theta) = 0. \tag{A6}$$

We point out here that the integrations below are carried out formally without taking into account the signature or actual boundary values of $\tilde{A}_b$, $\tilde{B}_b$, $\tilde{A}_b' + \tilde{B}_b'$, ϕ_b and ϕ_b', where the former are the corresponding values of $\tilde{A}(r)$, $\tilde{B}(r)$, $\tilde{A}(r)' + \tilde{B}(r)'$, $\phi(r)$ and $\phi(r)'$ at the boundary point at infinity.

Equation (A5) can be expressed as a Cauchy-Euler equation,

$$(e^{\tilde{A}(r)+\tilde{B}(r)})'' + \frac{3}{r}(e^{\tilde{A}(r)+\tilde{B}(r)})' = 0, \tag{A7}$$

which can be solved to give

$$e^{\tilde{A}(r)+\tilde{B}(r)} = e^{\tilde{A}_b+\tilde{B}_b}\left(1 - \frac{\varepsilon^2 B^2}{r^2}\right), \tag{A8}$$

and

$$\lim_{r\to\infty} r^3(\tilde{A}'(r) + \tilde{B}'(r)) = 2\varepsilon^2 B^2, \tag{A9}$$

$$\varepsilon^2 \in \{-1, +1\}. \tag{A10}$$

The reduced long-range scalar field wave equation Eq. (10) can be written in terms of $\tilde{A}$ and $\tilde{B}$ as

$$\left(\frac{r^2 e^{\tilde{A}+\tilde{B}}\phi'(r)}{\phi}\right)' = 0. \tag{A11}$$

By integrating Eq. (A11) twice from r to infinity we obtain

$$\phi = \phi_0\left(\frac{r-\varepsilon B}{r+\varepsilon B}\right)^{\frac{C}{\varepsilon\lambda}}, \tag{A12}$$

where

$$\lambda^2 = \frac{\varepsilon^2}{4}((2\omega+3)C^2 + (C+2)^2) > 0. \tag{A13}$$

We rewrite the modified field equation (A4) in the following form:

$$\tilde{A}'(r)^2 = (\tilde{A}+\tilde{B})'\left(\tilde{A}' + \tilde{B}' + \frac{2}{r}\right) + \frac{4\pi Q^2 e^{-2\tilde{B}(r)}}{c^4} - \frac{2\omega+3}{4}([\ln\phi(r)]')^2. \tag{A14}$$

Using Eqs. (A8) and (A12), after some algebra we obtain a first-order second-degree separable differential equation,

$$\left(\frac{d}{dr}(e^{-\tilde{A}(r)})\right)^2 = \frac{4\pi Q^2 e^{-2(\tilde{A}_b+\tilde{B}_b)}}{c^4(r^2-\varepsilon^2B^2)^2} \times\left[\left(\frac{c^4B^2e^{2(\tilde{A}_b+\tilde{B}_b)}(C+2)^2}{4\pi Q^2\lambda^2}\right)e^{-2\tilde{A}(r)} + 1\right]. \tag{A15}$$

Since $e^{\tilde{A}_b} = \sqrt{\phi_0}e^{\alpha_b}$ and $e^{\tilde{B}_b} = \sqrt{\phi_0}e^{\beta_b}$ the solution to this equation gives

$$e^{-\alpha(r)} = e^{-\alpha_b}\left(\frac{r-\varepsilon B}{r+\varepsilon B}\right)^{\frac{C}{2\varepsilon\lambda}}\left(p_+^2\left(\frac{r-\varepsilon B}{r+\varepsilon B}\right)^{\frac{C+2}{2\varepsilon\lambda}} - p_-^2\left(\frac{r-\varepsilon B}{r+\varepsilon B}\right)^{-\frac{C}{2\varepsilon\lambda}}\right), \tag{A16}$$

where p_+ and p_- are given in Eq. (17),

$$e^{\beta(r)} = e^{\beta_b}\left(1+\frac{\varepsilon B^2}{r^2}\right)\left(\frac{r-\varepsilon B}{r+\varepsilon B}\right)^{-\frac{C}{2\varepsilon\lambda}}\left(p_+^2\left(\frac{r-\varepsilon B}{r+\varepsilon B}\right)^{\frac{C+2}{2\varepsilon\lambda}} - p_-^2\left(\frac{r-\varepsilon B}{r+\varepsilon B}\right)^{-\frac{C}{2\varepsilon\lambda}}\right). \tag{A17}$$

When $\varepsilon^2 = +1$ the above coincides with the BDRN metric given in Theorem 1 in Sec. II above. The solutions corresponding to the Brans Type II, Type III and IV solutions are given by setting, respectively, $\epsilon^2 = -1$ and taking the limit when $\epsilon \to 0$ (using L'Hopital's rule) in Eqs. (A12), (A16), and (A17).

APPENDIX B: GAUSS' THEOREM

In order to determine the integration constants in Eq. (62) we use Gauss's theorem, a brief overview of which is given here. Let $\mathfrak{R}$ be a region of three-dimensional space residing in a hypersurface ϱ and let $\partial\mathfrak{R}$ be its closed two-dimensional boundary. Gauss' theorem states that for the electric field E^a (and indeed for any given vector field; see Wald Ref. [42])

$$\int_{\mathfrak{R}} \nabla_a E^a dv = \int_{\partial\mathfrak{R}} E^a \cdot n_a dS, \tag{B1}$$

where dv is an element of spatial proper volume in $\mathfrak{R}$, n_a is the outward-facing unit vector orthogonal to the closed two-dimensional boundary $\partial\mathfrak{R}$ and dS is the usual surface element $dS = r^2\sin\theta d\theta d\phi$.

We know that the electric field is related to the Faraday tensor by the following:

$$E^a = F^{ab}n_b. \tag{B2}$$

Using the above and Eq. (31) we find that the electric field is indeed equal to the gradient of the electrostatic potential $V(r,\theta,\phi)$ and therefore the right-hand side of Eq. (B1) can be written as

$$\int_{d\mathfrak{R}} E^a \cdot n_a dS = \int_{d\mathfrak{R}} \nabla V \cdot \hat{n} dS. \tag{B3}$$

From Maxwell's equations the left-hand side of Eq. (B1) can be written as $\int_{\mathfrak{R}} J^0 dv$, where

$$J^0 = -\frac{4\pi\epsilon_0}{cr^2}e^{-2\alpha(r)-3\beta}\delta(r-b)\delta(\cos\theta - \cos\theta_0) \tag{B4}$$

is the charge density.

It follows that for the region $B < b < r$ containing the point charge $-\epsilon_0$ positioned at $r = b$, $\theta = 0$ the left-hand side of Eq. (B1) becomes $-4\pi\epsilon_0$ and for any region not containing the charge, i.e. $B < r < b$, and the left-hand side vanishes.

For the Brans-Dicke Reisner-Nordström spacetime—as n^a is orthogonal to $\partial\mathfrak{R}$—the only term that remains is the r term and Eq. (B1) becomes

$$\int_{\mathfrak{R}} J^0 dv = \eta(r)^2(r+B)^2\left(\frac{r-B}{r+B}\right)^{1-2k} \times \int_0^{2\pi}\int_{-\pi}^{\pi}\frac{\partial V(r,\theta)}{\partial r}\sin\theta d\theta d\phi. \tag{B5}$$

The left-hand side of Eq. (B5) is determined by whether or not the region $\mathfrak{R}$ contains the singular point at $r = b$. In particular, for the purposes of this investigation the theorem determines the choice of integration constants in $V(r,\theta)$, as can be seen in the main section of this article.

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Concluding Remarks

After the publication of the above paper, it was brought to our attention by Professor Robert Wald that although we claimed that we were considering an electromagnetic pertubation of the Einstein-Maxwell equations of the BDRN space-time, this was not strictly true. Professor Wald correctly pointed out that an electromagnetic perturbation however small, would necessarily interact with the background space-time via the energy-momentum tensor thereby producing changes to the background metric. The changes would be of the same order as the initial perturbation. Of course, in the absence of a background charge (the Schwarzschild and Brans-Dicke space-times), the electromagnetic perturbation considered in the above paper is in its most complete form as it does not interact with or affect the background metric via the energy-momentum tensor. In the presence of a background charge (the BDRN and Reisnner-Nordström space-times) the investigations of the above paper can instead be considered to be that of a "model system" whose behaviour is presumably very similar to the actual point charge perturbation solution. A complete investigation of the effect that the electromagnetic pertubation has on the background space, including its contribution to the gravitational field, is conducted by Moncrief in [97] for the Reissner-Nordström background. From the results of the aforementioned paper, it is evidently clear that to consider the perturbation of the BDRN background in its entirety, as opposed to the "model system" studied here, would be extremely complex. Additionally, we do not believe that the results would be significantly different to those found here and feel satisfied to continue to use this "model system" in future investigations.

Declaration of Authorship

In the case of Chapter 3, the nature and extent of my contribution to the work was the following:

Nature of Contribution	Extent of Contribution
Initiation, key ideas, ran programs and generated plots, contributed to discussion and analyses of results, prepared paper	67%

The following co-authors contributed to the work.

Name	Nature of contribution
Anthony W.C Lun	Key ideas, contributed to discussion and analyses of results

The undersigned hereby certify that the above declaration correctly reflects the nature and extent of the candidate's and co-authors' contributions to this work.

Candidate's
Signature: Date:

Main Supervisor's
Signature: Date:

Chapter 3

Analysis of the Equipotential Surfaces in Isotropic and Schwarzschild-type Coordinates

As a natural progression of the research conducting in the first paper, we proceed to analysing the electrostatic potential by studying the equipotential surfaces generated by a point charge in a Brans-Dicke Reissner-Nordström (BDRN) space-time in both isotropic and Schwarzschild-type coordinates.

In the pioneering work of [1], Copson plotted the equipotential surfaces generated by a single "electron" residing in a Schwarzschild space-time. The equipotential surfaces were plotted only for the region exterior to the surface of inversion. From his plots, Copson was able to detect the presence of the inversion point inside the surface of inversion, however he was unable to detect the presence of the additional charge that was inadvertantly placed inside the region not covered by the isotropic coordinates.

The electrostatic potential was amended in [65] to be representative of a single perturbing charge in a Schwarzschild spacetime however the equipotential surfaces generated by this single charge solution was not plotted. Subsequent research into the topic of electrostatic perturbations of various spacetimes such as [95] and [96] also did not include such plots. It was not until 2007, that Harpaz [66] plotted the electric field lines generated by the perturbing charge solution of [65] and proved that they were comparable to the electric field lines generated from the multipole expansion solution of [63].

In fact, one of the few papers to include a pictoral representation of a small electrostatic perturbation was [63] who plotted the electric lines of force for a static charged particle outside the horizon of a Schwarzschild black hole in Schwarzschild coordinates. The lines of force were plotted for three different values of the radial coordinate r and it was found that as the charge "approached" the horizon, the behaviour of the force lines came to mimic the field lines of a Reissner-Nordström black hole. In fact, the authors of [63] were able to show that when the charge was allowed to get sufficiently close to the event horizon of the Schwarzschild black hole, the charge became evenly distributed over the Schwarzschild surface generating, in a smooth manner, the monopole field characteristic of a Reissner-Nordström black hole.

Similar results were obtained earlier by [53] where a multipole expansion solution was found describing the electrostatic potential generated by a perturbing charge in a Schwarzschild spacetime. This solution was found by implementing a boundary condition stating that the electrostatic potential must be well behaved when the charge is situated at the horizon of the Schwarzschild black hole. They found that their multipole expansion solution was regular and to the order of $\frac{e}{r}$, thus concluding, like [63] that a Reissner-Nordström black hole is produced when a charge approaches the horizon.

Black hole uniqueness theories of the early 70s, and in particular those theories devised in [98], [99] and [100], showed that an isolated black hole cannot possess an electromagnetic field unless it is endowed with an electric charge. Furthermore, as the perturbing charges described in the papers above are "test" charges, and are not coupled directly to the metric, the metric describing the Schwarzschild spacetime metric does not convert to that of the Reissner-Nordström spacetime simply by the presence of a perturbing charge at the horizon. When the point charge approaches the horizon it creates instead, a linearly perturbed Schwarzschild spacetime.

In the case of "test" perturbing charges (as opposed to gravitational Regge-Wheeler type perturbations), investigations into the mechanism by which the charge is able to remain static outside the outer "horizon" of a background space are scarce. By graphically representing the equipotential surfaces arising from the exact closed-form electrostatic potential solution of [101] for an electrostatic point charge in a BDRN spacetime we are able to find a mechanism by which the perturbing charge can remain static outside the outer "horizon".

In this paper we extend the work done in [101] by first converting the electrostatic potential generated by a point charge in a BDRN spacetime into a simple form in terms of a single parameter $\Pi(\varsigma)$. We do so by introducing a new variable $\sigma(r,\theta)$ which is the ratio of the distance of a field point from the charge at b to the distance of a field point from the image charge at b^*.

We go on to plot the equipotential surfaces generated by a charged particle at rest in BDRN, Brans-Dicke, Reissner-Nordström and Schwarzschild space-times respectively in isotropic coordinates. We find that the equipotential surfaces arising from the electrostatic potential generated by a point charge in BDRN and Brans-Dicke spacetimes are difficult to distinguish qualitatively. Likewise, the equipotential surfaces of the Reissner-Nordström and Schwarzschild spacetimes are qualitatively similar. Additionally we find that in the presence of a scalar field the equipotential surfaces are able to cross the surface of inversion seamlessly while in the absence of a scalar field they do not. Copson's solution of [1] is also plotted in isotropic coordinates for the entire region $0 < r < \infty$. We find that the region exterior to but in the vicinity of the surface of inversion, i.e the region plotted by Copson in [1], is qualitatively the same as the single charge solution. The difference between Copson's solution and the single charge solution is only apparent when one looks at the entire spacetime from $0 < r < \infty$. This is done by looking at the region $(0 < r < B)$ which is isomorphic to the region outside the surface of inversion $(B < r < \infty)$. The sphere of $r = B$ is essentially a snapshot of the entire spacetime in a bounded region and is a useful feature of the isotropic coordinates.

As a point of interest, and to ascertain whether the boundary condition of [53] is upheld, we allow the perturbing electrostatic charge to approach the surface of inversion $(r = B)$ from above. We find that the electrostatic potential generated by a point charge at the surface of inversion is well behaved and finite in requirement of the boundary condition of [53]. Interestingly, we find that the electrostatic potential is independent of angle θ. We are unable to explain this phenomenon and find it necessary to convert the solutions to Schwarzschild-type coordinates for further explaination.

We do this by introducing a generalized transformation formula by which one is able to convert the isotropic coordinates into Schwarzschild-type coordinates. We convert the BDRN metric into Schwarzschild-type coordinates and find that in addition to the usual two S^2 singularities which are the generalization of the inner (r_-) and outer (r_+) "horizons", there exists a third S^2 singularity at

$$r_0 = \frac{r_+ r_-(r_+^{\frac{1}{k}-1} - r_-^{\frac{1}{k}-1})}{r_+^{\frac{1}{k}} - r_-^{\frac{1}{k}}}. \tag{3.1}$$

We find that this singularity is the singularity one would expect to find at the location of the origin as a point singularity.

The conditions under which r_0 appears as a point singularity is when $k = 1$ and/or when $Q = 0$. Thus this singularity appears as a point singularity in the Brans-Dicke spacetime when $Q = 0$, the Reissner-Nordström spacetime when $k = 1$ and the Schwarzschild spacetime when $Q = 0$ and $k = 1$.

In order to understand this S^2 singularity better and in order to understand the behaviour of the equipotential surfaces of the perturbing electrostatic charge we convert the electrostatic potential of [101] into Schwarzschild-type coordinates.

As before, we are able to write the potential in terms of a single function $\Pi_s(r_s, \theta)$ by introducing a function $\sigma_s(r_s, \theta)$ which is simply the function $\sigma(r, \theta)$ converted into Schwarzschild-type coordinates.

In order to compare with the boundary condition of [53], the perturbing charge is allowed to approach the outer "horizon" r_+. The electrostatic potential remains well behaved even when the perturbing charge is at r_+ and the boundary condition of [53] is upheld. As before, the electrostatic potential generated by a charge at r_+ is found to be independent of the angle θ. We can interpret this as follows: in the region $r_- \leq r_s \leq r_+$ the timelike coordinate becomes spacelike and the radial coordinate becomes timelike, thus surfaces of $r_s = const$ are spacelike surfaces. When the charge approaches the spacelike surface of r_+ the charge is distributed evenly across the entire surface and is thus independent of the angle θ. This is in agreement with [63] who also found that the charge gets distributed evenly across the entire surface when it approaches the horizon of a Schwarzschild black hole.

In the BDRN and Reissner-Nordström spacetimes the radial and time coordinates return to their usual physical interpretation in the region $0 < r_s < r_-$. Thus within this region an image b_s^* to the charge placed at b_s appears. When the charge at b_s approaches r_+, the image charge b_s^* approaches r_- from below. The image charge is also distributed evenly across the inner horizon when $b_s^* = r_-$.

We plot the equipotential surfaces of the electrostatic potential generated by a perturbing charge in a BDRN, Brans-Dicke, Reissner-Nordström and Schwarzschild spacetimes alongside Copson's "symmetrical boundary condition" solution of [1] in Schwarzschild-type coordinates first for a perturbing charge placed in the vicinity of the outer horizon and next for a charge placed far from the outer horizon.

When the perturbing charge is situated in the vicinity of the outer "horizon" of the BDRN and Reissner-Nordström spacetimes one can identify the presence of the image charge within the inner "horizon" r_-. When the perturbing charge is taken far from the outer "horizon", the image charge approaches the singularity. For the Reissner-Nordström this causes the charge distribution within the inner horizon to resemble that of the Brans-Dicke and Schwarzschild spacetimes. The charge distribution within the horizons of a Brans-Dicke and Schwarzschild spacetime is dipole in nature. From the boundary condition implemented in [101], the net charge of the region is zero.

We plot also Copson's solution of [1] and find that the equipotential surfaces inside the horizon is not completely dipole in nature. When the charge is taken far away from the horizon the equipotential surfaces inside the horizon is representative of a single charge located at the singularity and it is clear to see that Copson's solution contains an extra charge there.

Lastly, we allow the perturbing charge of Copson's solution of [1] to approach the horizon r_+ in a Schwarzschild-type coordinates to compare with the boundary condition of [53]. When the perturbing charge is placed at the horizon the electrostatic potential of Copson's solution proves to be well behaved and is also to the order of $\frac{c}{r}$. Thus the boundary condition of [53] is necessary though insufficient in ensuring that the resulting potential is representative of that generated by a single perturbing charge.

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Electrostatic potential of a point charge in a Brans-Dicke Reissner-Nordström field. II. Analysis of the equipotential surfaces in isotropic and Schwarzschild-type coordinates

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In [1] we modeled a single perturbative electrostatic point charge placed outside the surface of inversion in a Brans-Dicke Reissner-Nordström (BDRN) spacetime in isotropic coordinates using a boundary condition, based on Gauss' divergence theorem, regarding the presence or absence of net electric flux. In Part I of this paper, we complete our investigation by plotting the equipotential surfaces of the exact perturbative electrostatic potential. We find that in isotropic coordinates the electrostatic potentials of the BDRN and Brans-Dicke backgrounds are difficult to distinguish qualitatively and the equipotential surfaces are able to cross the surface of inversion seamlessly. The electrostatic potentials of the Reissner-Nordström and Schwarzschild spacetimes are also difficult to distinguish qualitatively. Unlike the BDRN and Brans-Dicke spacetimes, however, the equipotential surfaces do not cross the surface of inversion (see Fig.(1)). In the absence of more information it is difficult to explain this phenomenon and why a model of a static perturbative, electrostatic point charge is physically reasonable. In addition to the above, we also plot Copson's symmetrical boundary condition solution [2] which models two charges, to illustrate the importance of choosing appropriate boundary solutions. In Part II, we introduce a general coordinate transformation equation that converts the isotropic coordinates into the Schwarzschild-type coordinates such that the BDRN metric is transformed into Schwarzschild-type coordinates. We show that in addition to the two S^2 singularities that arise from the generalization of the inner and outer "horizons", there exists a third singular S^2 surface in the BDRN background that replaces the singularity at the origin. Finally, we use the transformation equation to convert the exact perturbative electrostatic potentials into Schwarzschild-type coordinates. We plot the equipotential surfaces in Schwarzschild-type coordinates.

PACS numbers:

I. INTRODUCTION

We recently established an ansatz to solve the perturbed Maxwell equations due to an electrostatic charge in a class of Brans-type spacetimes in isotropic coordinates, which is reducible to the Brans-Dicke Class I, Reissner-Nordström and Schwarzschild black hole solutions [1]. The Brans-Dicke Reissner-Nordström (BDRN) background was perturbed by a small electrostatic charge placed outside the surface of inversion in isotropic coordinates. By extending a method first developed by Copson in [2], we were able to derive a closed-form solution perturbed electrostatic potential and found that the BDRN spacetime was stable under electromagnetic perturbations.

The study of the electromagnetic perturbations of electrovac or vacuum spacetimes can be traced back to 1927, when Whittaker [3] used the separation of variables method to express the electric potential generated by a point charge in a Schwarzschild and quasi-uniform field. This was expanded on by Copson [2] who found a closed-form expression describing the electric potential generated by a point charge in the quasi-uniform and Schwarzschild backgrounds using Hadamard's [4] theory of elementary solutions. Copson went on to plot the equipotential surfaces generated by the point charge in isotropic coordinates for the region outside the surface of inversion.

The electrostatic potential generated by a point charge was reestablished for the Schwarzschild background independently by Cohen and Wald [5] and Hanni and Ruffini [6] who also used the method of separation of variables.

A short while later, Linet [7] found that Copson's original [2] solution did not contain just one perturbing charge outside the surface of inversion $r = B$ but also another implicit image charge inside the surface of $r = B$, the region not covered by the isotropic coordinates. It was for this reason that the second charge did not appear in Copson's original plot of equipotential surfaces. Linet was able to find this additional charge by applying Gauss' law at infinity, thereby correcting Copson's original boundary condition of demanding symmetry between the radial coordinate r and the location of the charge $r = b$. His solution agreed with known multipole expansion solutions given by [5] and [6]. We find that the important difference between Copson's result and the results found by [3] lies in whether or not there is net electric flux over any

* [redacted]

closed oriented surface containing the surface of inversion but excluding the perturbing charge.

Linet and co-authors went on to find closed-form expressions for the electrostatic potential generated by a point charge placed in a Reissner-Nordström [8] and a Brans-Dicke Class I [9] using the Copson-Hadamard method. In 1978, Copson revisited the issue, finding a closed-form expression describing the electrostatic potential generated by a charged particle in a Reissner-Nordström background [10]. However, as Copson used the same boundary condition as before, by the same reasoning this solution also contained an image charge inside the surface of $r = B$. The Copson-Hadamard method to find a closed-form solution describing the electrostatic potential of a perturbing point charge was extended to higher dimensions in [11]. In 2007, [12] plotted the electric field lines of the electrostatic potential solution of [7] for different locations of the point charge to simulate the motion of an accelerating particle. He found that when the charge is brought close to the horizon, the field lines are comparable to that of [6].

Meanwhile in [13], the authors studied the electrostatic perturbation of the Schwarzschild spacetime using hypergeometric functions and investigated the fields of not just a point charges but also of current loops, charged rings and magnetic dipoles. In two following papers they extended their perturbation work to Reissner-Nordström [14] and Kerr black holes [15] and plotted the magnetic field lines generated by the perturbing phenomena.

The perturbed Schwarzschild spacetime was revisited in [16] who used Green's function and multipole expansions to express the electrostatic potential generated by a point charge and in [17] found that the electrostatic potential of a perturbed Schwarzschild spacetime was indeed unique.

In a pivotal paper by Regge and Wheeler [18] the authors developed a perturbation method by which the background metric is perturbed directly. This would lead to several investigations into assorted perturbations of various black hole spacetimes (see for example [19], [20], [21]). Although Regge-Wheeler type perturbations are outside of the scope of this paper as we look at small perturbations by a "test" charge which do not effect the spacetime curvature of the background metric, we point out a few interesting papers that are relevant to our research insofar as they study and plot the electromagnetic perturbation of charged spacetimes.

In 2007, Bini et al [22], [23], found an expression for the electric potential generated by a charged massive particle at rest in a Reissner-Nordström field using the Regge-Wheeler gauge perturbation method and went on to numerically plot the electric force lines in [24]. Paolino et al studied the electric force lines of a double Reissner-Nordström exact solution in [25] while in [26] they plotted the electric force lines of a two body equilibrium model proposed earlier by [27]. Lastly, [28] and [29] found that all linear perturbations of an extreme Reissner-Nordström black hole have long term instability.

We point out that the perturbed Maxwell equations under consideration in [1] did not include the effect the perturbing electromagnetic charge would have on the gravitational field of the BDRN and Reissner-Nordström background space-times. As such we cannot consider it a perturbation in its most complete form but rather a "model system" whereby the perturbing charge does not produce cross-terms in the Maxwell equation via the energy-momentum tensor. A full consideration of the perturbation would be extremely complex (see [30] for a full consideration of the perturbation of the Reissner-Nordström space-time) and the results of a full consideration would not be significantly different to those found in [1]. Thus we continue to use this "model system" when investigating the perturbation of the BDRN and Reissner-Nordström background space-times in this paper. The perturbation of the Brans-Dicke and Schwarzschild space-times considered in [1] on the other hand, are complete as these backgrounds are uncharged and the perturbing charge does not interact with the gravitational field via the energy-momentum tensor.

The structure of this paper is as follows;

In Section II, we give a brief overview of the salient features of the BDRN metric in isotropic coordinates. Throughout the investigations of this paper and [1] we have not implemented the weak field approximation. However, as a point of interest, we include in this section how the weak field limit can be achieved by placing certain restrictions on the parameters of the background metric.

In Section III, we rewrite the closed-form solution describing the electrostatic potential generated by a point charge situated outside the surface of inversion in isotropic coordinates of [1] in a succinct form using a single function $\Pi(\varsigma)$. We go on to plot all the BDRN closed-form perturbed electrostatic potentials found in Paper 1, and the three degenerate solutions (BD, Reissner-Nordström and Schwarzschild) in isotropic coordinates. In [1], we used the generalized Gauss divergence theorem to prove that there exists only one charge at $z = b$. Its image exists at $z = B^2/b$ within the surface of inversion. Both of these charges are clearly visible in all four backgrounds spacetimes. However, the presence of the scalar field in the BDRN and BD backgrounds causes the potential to cross the surface of inversion in a smooth manner. The Reissner-Nordström and Schwarzschild cases, on the other hand, do not display any crossing over at the surface of inversion and the region exterior and interior to the surface of inversion exist as two distinct regions. In addition to the four backgrounds, we also plot Copson's solution [2] and show that in isotropic coordinates, by looking at the region exterior to the surface of inversion alone, it is difficult to detect the existence of a second charge residing within the region excised by the isotropic coordinates. Thus Copson's solution exterior to the surface of inversion in isotropic coordinates is qualitatively the same as Linet's modified solution in [7]. However, in the manner of [7], when one looks at the entire spacetime to infinity, that is, the refraction of the exterior solution inside the sphere of $r = B$, then one can immediately recognize that the solution differs to the single charge solution of [7]. Thus the difference between Linet's [7] solution and Copson's [2] solution only becomes apparent when one either looks at the entire spacetime to infinity in isotropic coordinates, or if one includes the region excised by the isotropic coordinates, that is, the region interior to the surface of $r = B$. This is achieved by using a transformation formula to convert the solutions from isotropic coordinates to Schwarzschild-type coordinates which contains the complete information of the spacetime.

In Section IV, we introduce a method by which to convert the Brans-Dicke Reissner-Nordstöm (BDRN) metric from isotropic coordinates to the usual Schwarzschild-type coordinates. The transformation equation devised (Eq.47) is of a general form and can be applied, by an appropriate choice of constants, to all four background metrics (BDRN, Brans-Dicke, Reissner-Nordstöm and Schwarszschild). We give the BDRN metric in Schwarzschild-type coordinates initially in a notation similar to that used by Campanelli and Lousto [31] before giving in its usual notation in Theorem 1. The BDRN metric and its degenerate spacetime metrics are given in Table I. Upon converting the BDRN metric from isotropic to the usual Schwarzschild-type coordinates, we discover the existence of a S^2 singularity in addition to the usual inner and outer "horizons". The additional S^2 singularity is studied in some detail and we find the conditions under which this S^2 singularity appears as a point singularity at the origin and find that these coincide with the precise conditions underwhich the BDRN reduces to the BD, Reissner-Nordström and Schwarzschild backgrounds and it is thus that the degenerate cases do not possess this unusual singularity.

In Section V, we introduce in Theorem 2, the method by which the transformation equation is used to convert the four closed-form solutions given in [1] into Schwarzschild-type coordinates and present our results in Table.II. As was the case when in isotropic coordinates, the solution can be written in terms of a single parameter $\Pi_s(\varsigma)$. We also give Copson's [2] solution in Schwarzschild-type coordinates which is in agreement with that found earlier by [7]. Furthermore, we allow the perturbing electrostatic charge to approach the horizon to study how the Copson's [2] electric potential solution behaves in this limit. We find that like the ammended single charge solution of [7], the electrostatic potential generated by the perturbing charges of Copson's solution is also well behaved when the external perturbing charge is brought to the horizon. Thus it is clear that the boundary condition of [5] though necessary, is insufficient in ensuring that the electrostatic potential is representative of a single charge perturbation.

In Section VI, we plot our closed-form solutions in Schwarzschild-type coordinates. We find that in doing so we shed some light on the effect the additional S^2 singularity of the BDRN background has on the perturbed electrostatic potential. In particular, we find that the electrostatic potential exhibits no irregular behavior at the inner and outer "horizons" and shows singular behavior only at the location of the additional S^2 singularity. As was the case in the isotropic coordinates, the presence of the scalar field allows the electrostatic potential to cross over smoothly from the exterior to the interior of the BDRN and BD "horizons". One must tread carefully when discussing the region in between the two "horizons" of the BDRN and Reissner-Nordström spacetimes ($r_- < r_s < r_+$) and the region interior to the "horizon" ($r_s < r_+$) in the Brans-Dicke and Schwarzschild spacetimes as it is here that the physical interpretation of the r_s coordinate as being spacelike breaks down, however, we include this region in all our plots to show that, if only mathematically, we are able to plot the electrostatic potential for the entire spacetime and give the charge distribution inside the outer "horizon".

Lastly, we plot Copson's solution in Schwarzschild coordinates (first given by [7]). We find that in Schwarzschild coordinates Copson's solution does differ slightly from our perturbed Schwarzschild solution. However, the difference is subtle which only emphasizes the importance of using appropriate boundary conditions. We find the boundary condition introduced in [1] proves to be necessary and sufficient in ensuring that the solution is representative of a single perturbing charge. We note that the presence of the image charge Copson placed inadvertantly inside the surface $r = B$ does not cause Copson's solution to transition to a Reissner-Nordström solution.

We point out that in order to differentiate the scalar field from the coordinate ϕ, we denote the scalar field here using $\varphi(r)$ instead of $\phi(r)$ which was used throughout [1].

Part I

II. ISOTROPIC COORDINATES

In Paper 1, we considered a solution to the field equations in the Brans-Dicke scalar tensor theory

$$R_{ab} - \frac{1}{2}g_{ab}R = \frac{8\pi T_{ab}}{c^4\varphi} + \frac{1}{\varphi}(\nabla_a\partial_b\varphi - g_{ab}\Box\varphi) + \frac{\omega}{\varphi^2}(\partial_a\varphi\partial_b\varphi - \frac{1}{2}g_{ab}g^{cd}\partial_c\varphi\partial_d\varphi), \quad (1)$$

$$\Box\varphi = \frac{8\pi T}{(2\omega+3)c^4}, \quad (2)$$

where

$$\Box\varphi := \nabla_b(g^{ab}\partial_a\varphi) = \frac{1}{\sqrt{-g}}\partial_b(\sqrt{-g}g^{ab}\partial_a\varphi) \quad (3)$$

and $\Box$ is the scalar wave operator. Here the notations have their usual meaning. We further assume the energy-momentum tensor arising from the contribution of the electromagnetic field encoded in the Faraday tensor F_{ab} is

$$T_{ab} = F_{ac}F^c_b - \frac{1}{4}g_{ab}F_{cd}F^{cd}\ , \quad T^a_a = 0, \tag{4}$$

where F_{ab} satisfies the source-free Maxwell equations

$$\nabla_b F^{ab} = 0\ ,\ \nabla_{[c}F_{ab]} = 0. \tag{5}$$

The metric describing the BDRN background in isotropic coordinates was given in [1] as

$$ds^2 = -c^2e^{2\alpha(r)}dt^2 + e^{2\beta(r)}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{6}$$

where the metric functions $e^{2\alpha(r)}$ and $e^{2\beta(r)}$ are

$$e^{2\alpha(r)} = \frac{e^{2\alpha_0}\left|\frac{r-B}{r+B}\right|^{\frac{2}{\lambda}}}{\left(p_+^2 - p_-^2\left|\frac{r-B}{r+B}\right|^{\frac{C+2}{\lambda}}\right)^2}, \tag{7}$$

$$e^{2\beta(r)} = e^{2\beta_0}\left(1+\frac{B}{r}\right)^4\left|\frac{r-B}{r+B}\right|^{2(\frac{\lambda-C-1}{\lambda})} \times\left(p_+^2 - p_-^2\left|\frac{r-B}{r+B}\right|^{\frac{C+2}{\lambda}}\right)^2, \tag{8}$$

the long range scalar field $\varphi(r)$ is

$$\varphi = \varphi_0\left|\frac{r-B}{r+B}\right|^{\frac{C}{\lambda}}, \tag{9}$$

the static electric potential $V_0(r)$, the Faraday tensor F_{ab} and the corresponding energy-momentum tensor T^a_b are:

$$V_0(r) = Q\int_\infty^r \frac{e^{\alpha(r)-\beta(r)}}{r^2}dr, \tag{10}$$

$$F_{ab} = -cV_0'(r)\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{11}$$

$$T^a_b = -\frac{e^{4\beta(r)}Q^2}{2r^4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{12}$$

In order to ensure that the above metric is real for all values of r we use the following convention for the $\frac{r-B}{r+B}$ terms that appear in the metric coefficients $e^{2\alpha}$ and $e^{2\beta}$:

$$\left(\frac{r-B}{r+B}\right)^{2\alpha} = \left(\left(\frac{r-B}{r+B}\right)^2\right)^\alpha. \tag{13}$$

For the sake of simplicity we write the above as $|\frac{r-B}{r+B}|^{2\alpha}$ but point out that taking the modulus alone is insufficient in ensuring the BDRN metric is real.

The functions $V_0(r)$, $e^{2\alpha(r)}$, $e^{2\beta(r)}$ and $\varphi(r)$ are defined for all non-negative r except at $r = B$. The nine parameters $Q, B, p_+^2, p_-^2, \lambda, C, \varphi_0, e^{2\alpha_0}$ and $e^{2\beta_0}$ in equations (7) to (12) are related via the following

$$4\lambda^2 = (2\omega+3)C^2 + (C+2)^2 > 0, \tag{14}$$

$$B = \frac{\sqrt{m_B^2 - q_B^2}}{2}, \tag{15}$$

$$m_B := \frac{m}{c^2\varphi_0}\left(\frac{1}{e^{\beta_0}}\frac{2\lambda}{C+2}\right), \tag{16}$$

$$q_B := \sqrt{\frac{4\pi}{\varphi_0}}\frac{q}{c^2}\left(\frac{1}{e^{\beta_0}}\frac{2\lambda}{C+2}\right), \tag{17}$$

$$\varphi_0 = \frac{1}{G_{eff}}\left(\frac{2\omega+4}{2\omega+3}\right), \tag{18}$$

$$p_\pm^2 = \frac{m_B \pm \sqrt{m_B^2 - q_B^2}}{2\sqrt{m_B^2 - q_B^2}}, \tag{19}$$

$$p_+^2 - p_-^2 = 1, \tag{20}$$

where m and q are non-negative real constants and are identified, respectively, as mass measured in conventional units (kg) and charge measured in electrostatic units (e.s.u), which has the dimensions of $[mass]^{\frac{1}{2}}[length]^{\frac{3}{2}}[time]^{-1}$. Here ω is the coupling constant that couples the scalar field to matter, while c is the speed of light in a vacuum. In scalar tensor theories, the gravitational constant G_0 that appears in the Einstein-Hilbert term of the standard theory is replaced by G_{eff} which is related to the strength of the scalar field as given above.

Taking into account equations (14) to (20), there are six essential parameters in the BDRN solution. We adopt the independent parameter set $\alpha_0, \beta_0, m, q, C, \varphi_0$. From Eq.(14), for the BDRN metric to be well defined, we further require

$$\omega \geq -\frac{3}{2}. \tag{21}$$

A. Weak Field Approximation

Although there is no a priori reason why the weak field approximation must be upheld for the phenomena discussed here (see [32]), we give below the restrictions on the six parameters of the BDRN solution $\alpha_0, \beta_0, m_0, q_0, C, \phi_0$ such that the BDRN spacetimes are in agreement with weak field approximations should one wish to implement it:

$$\alpha_0 = \beta_0 = 0, \tag{22}$$

$$\varphi_0 = \frac{1}{G_0}\left(\frac{2\omega+4}{2\omega+3}\right), \tag{23}$$

$$\frac{C}{\lambda} \cong \frac{-2}{\sqrt{(2\omega+4)(2\omega+3)(1-\frac{q_B^2}{m_B^2}}}, \tag{24}$$

$$m_B \cong \frac{m}{c^2\varphi_0}\sqrt{\frac{2\omega+3}{2\omega+4}}, \tag{25}$$

$$q_B \cong \sqrt{\frac{4\pi}{\varphi_0}}\frac{q_0}{c^2}. \tag{26}$$

Here G_0 is the Newtonian gravitational constant. When the weak field approximation is implemented there remain only 4 parameters in the BDRN solution: $m_0, q_0, \omega, \varphi_0$. Note that the coupling constant ω replaces C (and thus λ). Here m_0 and q_0 are the Newtonian mass and charge measured in [kg] and e.s.u, respectively.

We also point out an error in [1] where there is a factor of $\sqrt{\frac{2\omega+4}{2\omega+3}}$ that should not appear in Eq.(24) of [1]. With this ammendment Eq.(24) of [1] coincides with (26) above.

III. ELECTROSTATIC POTENTIAL AND EQUIPOTENTIAL SURFACES IN ISOTROPIC COORDINATES

In [1], we considered the electrostatic potential $V(r,\theta)$ due to a "small" perturbative static electric charge $-\epsilon_0$ ($|\epsilon_0| \ll q_B < m_B$) situated outside the spherical surface of inversion B along the azimuthal axis at $z = b$ in the BDRN background.

We found that the potential $V(r,\theta)$ can be written as the following

$$V(r,\theta) = \frac{\epsilon_0 r}{\eta(r)(r^2-B^2)}\left[\frac{r-B}{r+B}\right]^k \frac{bB}{\eta(b)(b^2-B^2)}\left[\frac{b-B}{b+B}\right]^k \times\frac{p_+^2(\sqrt{\gamma+1}+\sqrt{\gamma})^{2k} - p_-^2(\sqrt{\gamma+1}-\sqrt{\gamma})^{2k}}{2\sqrt{\gamma}\sqrt{\gamma+1}} \tag{27}$$

where

$$\eta(\varsigma) = p_+^2 - p_-^2\left|\frac{\varsigma-B}{\varsigma+B}\right|^{2k}, \tag{28}$$

is a dimensionless function and

$$\gamma(r,\theta) = \frac{B^2}{b^2-B^2}\frac{\Gamma(r,\theta)}{r^2-B^2}, \tag{29}$$

where

$$\Gamma(r,\theta) = r^2 + b^2 - 2brcos\theta. \tag{30}$$

Thus $\gamma(r,\theta)$ is proportional to the square of the "radial" distance from the charged particle at $z = b$. Using the same notation as [1]

$$k = \frac{C+2}{2\lambda}. \tag{31}$$

The electrostatic potential satisfies a boundary condition based on Gauss' theorem, that any integration over a simply connected annular domain bounded by the surface of inversion $r = B$ and does not contain the perturbing charge must be exactly zero Additionally, an integration over a region containing the perturbing charge must be exactly $4\pi\epsilon_0$ where ϵ_0 is the magnitude of the perturbing charge.

We begin by first showing that the closed-form electric potential $V(r,\theta)$ in isotropic coordinates due to a point electrostatic charge at $(r,\theta) = (b,0)$, given in Eq.(27), can be expressed succinctly in terms of the function

$$\Pi : \varsigma \mapsto \frac{\varsigma}{\eta(\varsigma)(\varsigma^2-B^2)}\left(\frac{\varsigma-B}{\varsigma+B}\right)^k, \tag{32}$$

such that

$$V(r,\theta) = \frac{\epsilon_0}{2}\frac{\Pi(b)\Pi(r)}{\Pi(\sigma(r,\theta))}, \tag{33}$$

where the composite function

$$\Pi(\sigma(r,\theta)) = (\Pi\circ\sigma)(r,\theta), \tag{34}$$

and

$$\sigma(r,\theta) = b\left(\frac{(r^2+\frac{B^4}{b^2}-\frac{2B^2r}{b}\cos\theta)}{(r^2+b^2-2br\cos\theta)}\right)^{\frac{1}{2}}. \tag{35}$$

$$= \frac{b\Gamma^*(r,\theta)}{\Gamma(r,\theta)} \tag{36}$$

where

$$\Gamma^*(r,\theta) = r^2 + \frac{B^4}{b^2} - 2\frac{B^2}{b}rcos\theta, \tag{37}$$

is the square of the distance from the inversion point at $z = b^* = \frac{B^2}{b}$.

From Eq.(35), it is straightforward to show that

$$\sigma^2(r,\theta) - B^2 = \frac{(b^2-B^2)(r^2-B^2)}{(r^2+b^2-2br\cos\theta)} \tag{38}$$

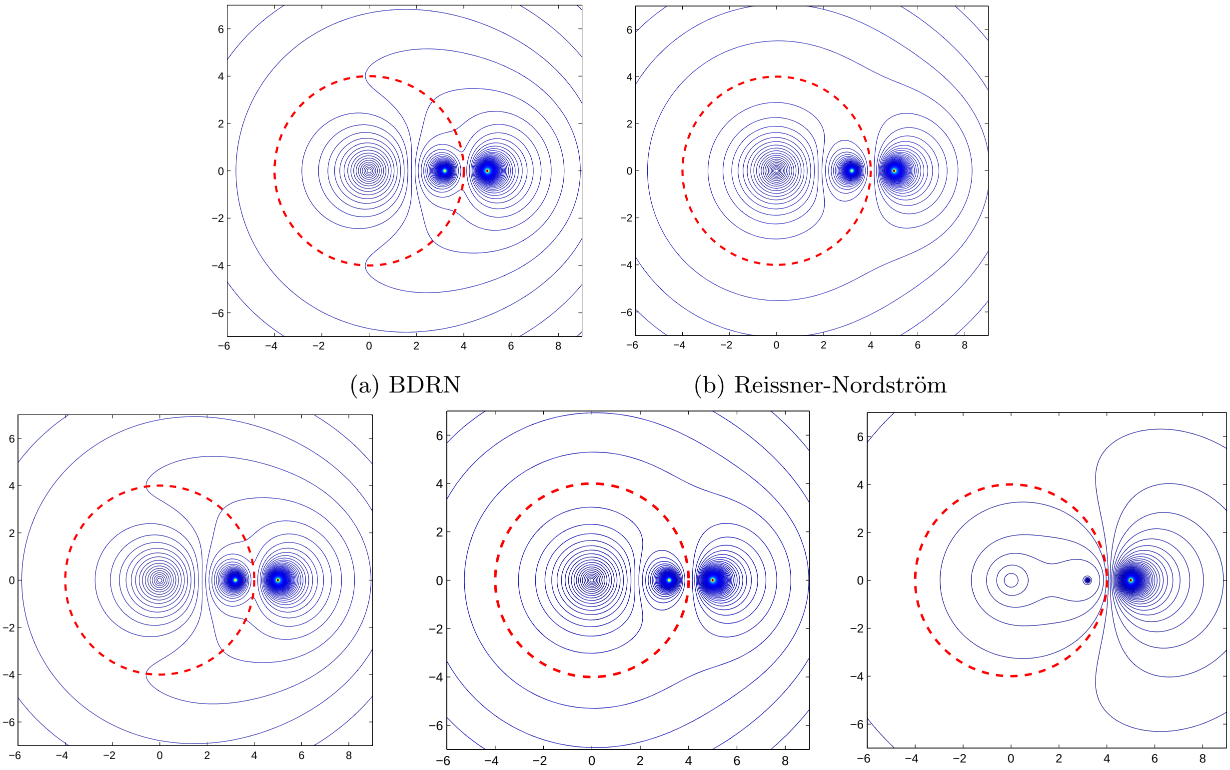
$$= \frac{(b^2-B^2)(r^2-B^2)}{\Gamma(r,\theta)} = \frac{B^2}{\gamma(r,\theta)}. \tag{39}$$

Rearranging terms we have

$$\gamma(r,\theta) = \frac{B^2}{\sigma^2(r,\theta)-B^2} \tag{40}$$

$$\gamma(r,\theta)+1 = \frac{\sigma^2(r,\theta)}{\sigma^2(r,\theta)-B^2}. \tag{41}$$

Substituting Eqs.(40) and (41) into Eq.(32), and after some algebra, we obtain



(a) BDRN (b) Reissner-Nordström

(c) BD (d) Schwarzschild (e) Copson Solution

FIG. 1: A plot of the electrostatic potential generated by a point charge of magnitude 1 situated at $b = 5$ outside the surface of inversion in isotropic coordinates. For all backgrounds $B = 4$ and is denoted by a dashed red line. For the BDRN and Reissner-Nordström backgrounds $m_B = 9$ and $q = \sqrt{17}$ while for the BD and Schwarzschild backgrounds $m_B = 8$. For the BDRN and Brans-Dicke cases, $k = 0.95$ and for the Reissner-Nordström and Schwarzschild backgrounds $k = 1$. The plot shows the presence of two charges, one at $r = b$ and the other at its counterpart within the surface of inversion at $r = \frac{B^2}{b}$. Copson's solution is also given for a point charge at $b = 5$ in a Schwarzschild background where $m_B = 8$. We point out that the equipotential surfaces around the origin of the isotropic coordinates is of decreasing order where at the origin (the counterpart to spatial infinity) the equipotential "surface" is zero. This is in contrast to the equipotential surfaces around the perturbing charge at b which are of increasing order, rising to a peak at b.

$$\Pi(\sigma(r,\theta)) = \frac{\sqrt{\gamma(\gamma+1)}}{B(p_+^2(\sqrt{\gamma+1}+\sqrt{\gamma})^{\frac{C+2}{\lambda}} - p_-^2(\sqrt{\gamma+1}-\sqrt{\gamma})^{\frac{C+2}{\lambda}})}. \tag{42}$$

Hence we verify that Eq.(33) and Eq.(27) are indeed equivalent.

If we allow the perturbing charge to approach the surface of inversion, that is $b \to B$, the electrostatic potential becomes the following

$$\begin{aligned} V(r,\theta) &= \frac{\epsilon_0}{2}\Pi(r) \\ &= \frac{\epsilon_0}{2}\frac{r}{\eta(r)(r^2-B^2)}\left(\frac{r-B}{r+B}\right)^k. \end{aligned} \tag{43}$$

From this we are able to ascertain that the electrostatic potential is well behaved at the surface of inversion as required by the boundary conditions in [5].

Interestingly, we find that the electrostatic potential of the perturbing charge when at the surface of inversion is independent of the angle θ. The reason for this will become clear when we convert the potential into Schwarzschild-type coordinates.

A. Copson's Solution in Isotropic coordinates

In [2], Copson found a closed-form solution for the electrostatic potential generated by a point charge or "electron" in a Schwarzschild background space. The integration constants were chosen such that the potential would be symmetrical in the radial coordinate r and in the position of the point charge at $r = b$. Using the notation of this paper where the surface of inversion of the spacetime is located at $r = B$ (and not scaled to unity as in [2]) we rewrite Copson's solution as the following

$$V^c(r,\theta) = \frac{ebr}{(b+B)^2(r+B)^2}\left[\frac{\sigma^2(r,\theta)+B^2}{\sigma(r,\theta)}\right], \quad (44)$$

where $\sigma(r,\theta)$ is given by Eq.(35). It is straightforward to verify that $V^c(r,\theta)$ is symmetrical in r and b. If we choose the magnitude of the point charge at $r = b$ to be $\epsilon_0 = 2e$ then we are able to compare Copson's solution with the single charge solution given in Eq.34 by setting $p_-^2 = 0, p_+^2 = 1, k = 1, C = 0$ and $\lambda = 1$ (see also [7])

$$V(r,\theta) = \frac{\epsilon_0 br}{2(b+B)^2(r+B)^2}\left[\frac{(\sigma(r,\theta)+B)^2}{\sigma(r,\theta)}\right]. \quad (45)$$

As pointed out in [7], Copson's solution Eq.(44) differs from the single charge solution in that there is an additional potential due to a charge of the following magnitude

$$V^*(r,\theta) = -\frac{\epsilon_0 bBr}{(b+B)^2(r+B)^2}. \quad (46)$$

As Copson only placed a single charge outside the surface of $r = B$, this additional charge must reside in the interior of the spherical surface of $r = B$ and hence in the region not covered by the isotropic coordinates.

B. Equipotential Surface Plots in Isotropic Coordinates

We turn to plotting the solution Eq. (33) in all four backgrounds (BDRN, Brans-Dicke, Reissner-Nordström and Schwarzschild). In order to allow some comparison between the four backgrounds, we fix the surface of inversion at the same value for $B = 4$ for all four backgrounds. We choose $m_B = 9$ and $q = \sqrt{17}$ for the BDRN and Reissner-Nordström backgrounds such that $B = 4$. Likewise, we allow $m_B = 8$ for the Brans-Dicke and Schwarzschild backgrounds, such that $B = 4$. For the BDRN and Brans-Dicke backgrounds we set $k = 0.95$ and as always $k = 1$ for the Reissner-Nordström and Schwarzschild backgrounds. The point charge of magnitude $\epsilon_0 = 1$ is placed at $b = 5$ and the corresponding inversion point $b^* = \frac{B^2}{b} = \frac{16}{5}$.

From the contour diagrams of Fig.(1), it is immediately obvious that there exists two charges, one at the location of the charge at $b = 5$ and the other at its inversion point inside the surface of inversion at $b^* = \frac{B^2}{b} = \frac{16}{5}$. Of great interest, however, is the fact that there are no discernible differences between the BDRN and Brans-Dicke backgrounds, and similarly between the Reissner-Nordström and Schwarzschild backgrounds. It is clear then that the presence or absence of charge in the background metric itself does not affect the perturbative electrostatic potential exterior to the surface of inversion (compare Fig.(1a) with Fig.(1c) and Fig.(1b) with Fig.(1d)). In Section VI, where we plot the electrostatic potential in Schwarzschild-type coordinates, we find that the differences between the charged and uncharged spacetimes manifests only in the region excised by the isotropic coordinates. The presence of the scalar field however, does affect the electrostatic potential and the BDRN and Brans-Dicke backgrounds are clearly distinguishable from their scalar-field-free counterparts, i.e the Reissner-Nordström and Schwarzschild backgrounds. Most notably, the equipotential surfaces smoothly cross the surface of inversion in the presence of a scalar field almost as if the surface of inversion does not exist, while in the absence of a scalar field, the electrostatic potential does not cross the surface of inversion.

Although the interior of Copson's two-charge solution of Eq.(44) (see Fig.(1e)) differs quite significantly from the interior of the single charge solution of Eq.(45), the exterior solution on the other hand, exhibits no qualitative difference, except for a slightly weaker electrostatic potential displayed by Copson's solution in comparison to the single charge solution. From the electrostatic potential exterior to but near the surface of inversion one would be unable to discern that this is a two charge solution. It is only when one takes a look at the entire spacetime stretching from the surface of inversion to infinity that the difference becomes clear. This can be seen from the region $(0 < r < B)$ which is isomorphic to the region from the surface of inversion to infinity $(B < r < \infty)$. The sphere of radius $r = B$ is essentially a refraction of the unbounded region lying exterior to the surface of $r = B$ and thus provides a snapshot of the entire spacetime in a bounded region.

In [2], Copson chose the values for the integration constant such that the solution would be symmetric in the radial coordinate r and the location of the perturbing point charge b in isotropic coordinates. When Copson compared his result term by term to the known multipole expansion solution of [33] he found that his series solution differed in the appearance of an "additional" term. It was not until several years later that [7] applied Gauss' law at infinity and found that this "additional" term was in fact an implicit image charge that appeared as a direct result of Copson's "symmetrical" boundary condition. Thus it is straightforward to see how by looking at the snapshot of the entire spacetime, ie. the region inside the sphere of radius $r = B$, one is able to recognize that Copson's solution Fig.(1e) differs from the single charge solution of Fig.(1d).

We find that the boundary condition introduced in [1] is necessary and sufficient to achieve a stable, linearized, perturbative electrostatic potential that models a single charge perturbation of a BDRN background. An inappropriate choice of boundary condition can result in the appearance of an additional charge, as was the case in Copson's solution, however the actual physical manifestation of this deviation may not be obvious at all. Thus while plots may be useful to gain an immediate physical understanding of a background they cannot be relied on in the absence of robust boundary conditions. See also [5] for a proper choice of boundary condition when using a series solution.

Part II

IV. COORDINATE TRANSFORMATION

To transform the BDRN metric (Eqs.(6), (7), (8)) into Schwarzschild-type coordinates, where the r coordinate will be subscripted with an s, we introduce the following transformation equation

$$r_s = \left(r + \frac{B^2}{r}\right) + m_B. \tag{47}$$

From Eq.(47), it is straightforward to show that

$$\left(\frac{r-B}{r+B}\right)^2 = \frac{sgn(r_s - r_+)}{sgn(r_s - r_-)}, \tag{48}$$

where we define

$$r_\pm = m_B \pm \sqrt{m_B^2 - q_B^2} \tag{49}$$

and

$$sgn(\varsigma - r_\pm)^\alpha = signum(\varsigma - r_\pm)|\varsigma - r_\pm|^\alpha \tag{50}$$

where

$$signum(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases} \tag{51}$$

For convenience we let

$$A(r_s) = \frac{sgn(r_s - r_+)}{sgn(r_s - r_-)}. \tag{52}$$

From Eq.(28) and using Eq.(47) we are able to rewrite $\eta(\varsigma)$ in terms of r_s as

$$\eta_s(\varsigma) \;=\; p_+^2 - p_-^2 \left[\frac{sgn(\varsigma - r_+)}{sgn(\varsigma - r_-)}\right]^k. \tag{53}$$

Using the transformation equation Eq.(47) we are able to write the BDRN metric in Schwarzschild-type coordinates in a form that generalizes the Brans-Dicke solution in Schwarzschild-type coordinates given by Campanelli and Lousto [31] as

$$\begin{aligned} ds^2 = & -\frac{c^2 A(r_s)^{m_0+1}}{\eta_s(r_s)^2} dt^2 + \eta_s(r_s)^2 A(r_s)^{n_0-1} dr_s^2 \\ & + \eta_s(r_s)^2 A(r_s)^{n_0} sgn(r_s - r_-)^2 d\Omega^2, \end{aligned} \tag{54}$$

where

$$d\Omega^2 = (d\theta^2 + \sin^2\theta d\varphi^2), \tag{55}$$

is the metric of the unit 2-sphere and

$$m_0 = \frac{1}{\lambda} - 1, \tag{56}$$

and

$$n_0 = \frac{\lambda - C - 1}{\lambda}. \tag{57}$$

The scalar field is given by

$$\varphi(r_s) = \varphi_0 A(r_s)^{-\frac{C}{2\lambda}}. \tag{58}$$

Here the parameters φ_0, B, m_B, and q_B all maintain their values as given in Eqs. (14) to (20).

As in Eq.(13) for the isotropic coordinates, it is necessary to implement certain conventions to ensure that the BDRN metric (and all its denegerate metrics) are real for all values of r_s. In isotropic coordinates the factor $\left(\frac{r-B}{r+B}\right)$ appears everywhere as $\left(\frac{r-B}{r+B}\right)^2$ and thus the isotropic coordinate is real and regular for all values of r as long as one upholds the convention Eq.(13). In Schwarzschild-type coordinates, from Eq. (48) the corresponding $\frac{r_s-r_+}{r_s-r_-}$ factor that appears in the metric coefficients of the BDRN spacetime does not have the advantage of appearing raised to the power of 2. One finds that for the region lying between the two "horizons" as $r_- < r_s < r_+$ it follows that $r_s - r_+ < 0$ and thus $\left(\frac{r_s-r_+}{r_s-r_-}\right)^{\frac{\alpha}{\beta}}$ is complex for all $\alpha \in \mathbb{R}$ and $\beta \neq 1$. Similarly for the region inside the inner "horizon" where both $r_s - r_+ < 0$ and $r_s - r_- < 0$, again $\left(\frac{r_s-r_+}{r_s-r_-}\right)^{\frac{\alpha}{\beta}}$ is complex for all $\alpha \in \mathbb{R}$ and $\beta \neq 1$. It is therefore necessary to implement the sgn convention used in Eq.(48) above to ensure that the metric is real for all values of r_s.

The exterior region of the Schwarzschild-type coordinate, where $r_+ < r_s < \infty$, is doubly covered by the isotropic coodinate such that in
Region 1: by the inverse transformation

$$\begin{aligned} r &= \frac{1}{2}\left(r_s - m_B + \sqrt{(r_s - r_-)(r_s - r_+)}\right), \\ B &= \frac{\sqrt{m_B^2 - q_B^2}}{2} < r < \infty; \end{aligned} \tag{59}$$

Region 2: by the inverse transformation:

	Brans-Dicke-Reissner-Norström	Brans-Dicke	Reissner-Norström	Schwarzschild
C and λ	$C \in \mathbb{R}$ and $\lambda^2 > 0$	$C \in \mathbb{R}$ and $\lambda^2 > 0$	$C = 0, \lambda = 1$	$C = 0, \lambda = 1$
Q	$Q \neq 0$	$Q = 0$	$Q \neq 0$	$Q = 0$
$\Sigma(r_s)$	$\frac{sgn(r_s - r_-)^{\frac{C+2}{\lambda}}}{\rho_s^2(r_s)} \left(\frac{sgn(r_s - r_+)}{sgn(r_s - r_-)}\right)^{\frac{1}{\lambda}}$	$(1 - \frac{2m_B}{r_s})^{\frac{1}{\lambda}}$	$(1 - \frac{2m_B}{r_s} + \frac{q_B^2}{r_s^2})$	$(1 - \frac{2m_B}{r_s})$
$\Upsilon(r_s)$	$\frac{\rho_s^2(r_s)}{sgn(r_s - r_+)^{\frac{C+2}{\lambda}}} \left(\frac{sgn(r_s - r_+)}{sgn(r_s - r_-)}\right)^{\frac{1}{\lambda}}$	$\frac{1}{(1 - \frac{2m_B}{r_s})^{\frac{C+1}{\lambda}}}$	$\frac{1}{(1 - \frac{2m_B}{r_s} + \frac{q_B^2}{r_s^2})}$	$\frac{1}{(1 - \frac{2m_B}{r_s})}$
$\Lambda^2(r_s)$	$\frac{\rho_s^2(r_s)}{sgn(r_s - r_+)^{\frac{C+2}{\lambda} - 2}} \left(\frac{sgn(r_s - r_-)}{sgn(r_s - r_+)}\right)^{\frac{\lambda-1}{\lambda}}$	$(r_s)^2(1 - \frac{2m_B}{r_s})^{\frac{\lambda - C - 1}{\lambda}}$	r_s^2	r_s^2
$\rho(r_s)$	$p_+^2 sgn(r_s - r_-)^{\frac{C+2}{2\lambda}} - p_-^2 sgn(r_s - r_+)^{\frac{C+2}{2\lambda}})$	$r_s^{\frac{C+2}{2\lambda}}$	$r_+\lvert r_s - r_-\rvert - r_-\lvert r_s - r_+\rvert)$	r_s
$\varphi(r_s)$	$\varphi_0 \left(\frac{sgn(r_s - r_+)}{sgn(r_s - r_-)}\right)^{\frac{C}{2\lambda}}$	$\varphi_0 \left(\frac{sgn(r_s - r_+)}{sgn(r_s - r_-)}\right)^{\frac{C}{2\lambda}}$	φ_0	φ_0

TABLE I: The metric coefficients and scalar field of the BDRN, Brans-Dicke, Reissner-Nordström and Schwarzschild spacetimes in Schwarzschild-type coordinates given by Eq.(62).

$$
\begin{aligned}
r &= \frac{1}{2}\left(r_s - m_B - \sqrt{(r_s - r_-)(r_s - r_+)}\right), \quad (60) \\
0 &< r < \frac{\sqrt{m_B^2 - q_B^2}}{2} = B.
\end{aligned}
$$

Points in Region 1 and Region 2 of the isotropic coodinate r are connected via the inversion map

$$
\iota : (r, \theta, \varphi) \leftrightarrow \left(\frac{B^2}{r}, \theta, \varphi\right), \quad (61)
$$

where $r = B$ is the surface of inversion.

For the region where the Schwarzschild-type coordinate is $r_- < r_s < r_+$, i.e. the spherical shell region between the inner "horizon" and the outer "horizon", $(r_s - r_-)(r_s - r_+) < 0$, the inverse transformations (59) and (60) are not defined and the isotropic coordinate r is not a real coordinate.

For the region where the Schwarzschild-type coordinate is $0 < r_s < r_-$, i.e. the spherical region inside the inner "horizon", $(r_s - r_-)(r_s - r_+) > 0$, so that now
Region 1: the inverse transformation (59) gives

$$
-B = -\frac{\sqrt{m_B^2 - q_B^2}}{2} < r < -\frac{m_B - q_B}{2};
$$

and in
Region 2: the inverse transformation in (60) gives

$$
-\frac{m_B + q_B}{2} < r < -\frac{\sqrt{m_B^2 - q_B^2}}{2} = -B.
$$

Hence when $0 < r_s < r_-$, the isotropic coordinate $-\frac{m_B + q_B}{2} < r < -\frac{m_B - q_B}{2} < 0$. Since r is interpreted as some form of radius of sphere , we can safely discard this region.

We now describe the metric (54) in notation more familiar to readers as follows.

Theorem 1 The Brans-Dicke-Reissner-Norström spacetime in Schwarzschild-type coordinates $(t, r_s, \theta, \varphi)$ is given by the metric

$$
ds^2 = -c^2 \Sigma(r_s)\, dt^2 + \Upsilon(r_s)\, dr_s^2 + \Lambda^2(r_s)\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \quad (62)
$$

The metric coefficients are

$$
\begin{aligned}
\Sigma(r_s) &= \frac{sgn(r_s - r_-)^{\frac{C+1}{\lambda}} sgn(r_s - r_+)^{\frac{1}{\lambda}}}{\rho_s^2((r_s)}, \quad (63) \\
\Upsilon(r_s) &= \frac{\rho_s^2(r_s)}{sgn(r_s - r_+)^{\frac{C+1}{\lambda}} sgn(r_s - r_-)^{\frac{1}{\lambda}}}, \quad (64) \\
\Lambda^2(r_s) &= \frac{\rho_s^2(r_s)\, sgn(r_s - r_-)^{\frac{\lambda-1}{\lambda}}}{sgn(r_s - r_+)^{\frac{C+1-\lambda}{\lambda}}}, \quad (65)
\end{aligned}
$$

where

$$
\rho_s(r_s) = p_+^2 sgn(r_s - r_-)^{\frac{C+2}{2\lambda}} - p_-^2 sgn(r_s - r_+)^{\frac{C+2}{2\lambda}}. \quad (66)
$$

The scalar field is

$$
\varphi(r_s) = \varphi_0 \left(\frac{sgn(r_s - r_+)}{sgn(r_s - r_-)}\right)^{\frac{C}{2\lambda}}. \quad (67)
$$

The parameter λ is defined in terms of C and satisfies the quadratic equations

$$
\begin{aligned}
4\lambda^2 &= (2\omega + 3)C^2 + (C + 2)^2 \quad (68) \\
&= \frac{1}{2}(\omega + 2)C^2 + C + 1. \quad (69)
\end{aligned}
$$

When the coupling constant ω is choosen such that

$$
-\frac{3}{2} \leq \omega < \infty, \quad (70)
$$

then the parameter $C \in \mathbb{R}$ and $\lambda^2 \geq \frac{2\omega+3}{2(\omega+2)}$.

For the BDRN solutions that conform to the weak field approximations and with the coupling constant confined to the range given in Eq.(70), we can, without loss of generality, choose the following

$$
\lambda \geq 0, \quad C \leq 0, \quad \frac{C}{\lambda} \leq 0, \quad (71)
$$

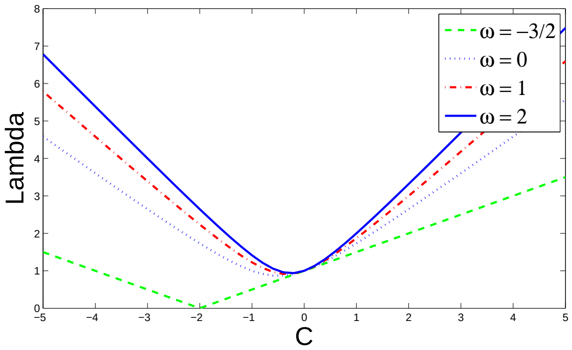


FIG. 2: Plot of λ vs. C for all real values of λ and C

so that

$$\lambda = \sqrt{\frac{1}{2}\left(\omega+2\right)C^2 + C + 2}, \quad (72)$$

and the scalar field satisfies the boundary conditions

$$\lim_{r_s\to\infty} \varphi\left(r_s\right) = \varphi_0, \quad \lim_{r_s\to r_+} \varphi\left(r_s\right) = 0. \quad (73)$$

in the neighbourhood of $C \cong -\frac{1}{\omega+2}$ and $\lambda \cong \sqrt{\frac{2\omega+3}{2\omega+4}}$. The relationship between C and λ is given in Fig.(2) for different values of ω.

Analysis of the BDRN spacetime

From Eqs.(62) to (65) one is able to identify three singular points where the metric (62) is not defined. The three singularities arise when

1. The metric coefficient $\Sigma(r_s) = 0$ and is located at $r_+ = m_B + \sqrt{m_B^2 - q_B^2}$.
2. The metric coefficient $\Sigma(r_s) = 0$ and is located at $r_- = m_B + \sqrt{m_B^2 - q_B^2}$.
3. The metric coefficient $\Sigma(r_s) \to \infty$ due to $\rho(r_s) = 0$ and is located at $r_0 = \frac{r_+ r_-(r_+^{\frac{1}{k}-1} - r_-^{\frac{1}{k}-1})}{r_+^{\frac{1}{k}} - r_-^{\frac{1}{k}}}$.

From Eq.(94) it follows that the third singularity which we shall subscript with a 0 is located at

$$r_0 = \frac{r_+ r_-(r_+^{\frac{1}{k}-1} - r_-^{\frac{1}{k}-1})}{r_+^{\frac{1}{k}} - r_-^{\frac{1}{k}}}. \quad (74)$$

The reason for subscripting with a zero is due to the fact that this is the singularity that one would normally expect to see located as a point singularity at $r_s = 0$ (see Eqs.(76) and (78) below). For the BDRN background we find that it has shifted to the location given by Eq.(74) and is no longer a point singularity, but a "horizon".

The conditions under which the r_0 singularity exists at the origin ($r_s = 0$) is given by

$$[r_+][r_-][r_-^{\frac{1}{k}-1} - r_+^{\frac{1}{k}-1}] = 0. \quad (75)$$

which has two solutions, the first being

$$[r_+][r_-] = q^2 = 0. \quad (76)$$

When $r_+ \neq r_-$ then $q = 0$ is a condition underwhich the r_0 singularity will exist at $r_s = 0$.

The second solution is

$$\left(\frac{r_-}{r_+}\right)^{\frac{1}{k}-1} = 1. \quad (77)$$

For this to be satisfied, as $r_+ \neq r_-$, then

$$k \equiv 1. \quad (78)$$

Thus we find that there are three situations where the spherical singularity degenerates to a point singularity at the origin:

1. In the uncharged case ($q = 0$) where $k \neq 0$, which is equivalent to the Brans-Dicke background.
2. In the charged case ($q \neq 0$) where $k = 1$, which is equivalent to the Reissner-Nordström background.
3. In the uncharged case ($q = 0$) and $k = 1$, which is equivalent to the Schwarzschild background.

Since $r_- < r_+$ then $\frac{1}{r_+} < \frac{1}{r_-}$. Hence

$$\frac{r_+^{\frac{1}{k}}}{r_+} - \frac{r_-^{\frac{1}{k}}}{r_-} < \frac{r_+^{\frac{1}{k}} - r_-^{\frac{1}{k}}}{r_+}, \quad (79)$$

which implies

$$\frac{r_+^{\frac{1}{k}-1} - r_-^{\frac{1}{k}-1}}{r_+^{\frac{1}{k}} - r_-^{\frac{1}{k}}} < \frac{1}{r_+} < \frac{1}{r_-}. \quad (80)$$

Multiplying the above inequality throughout by $[r_+][r_-] > 0$ gives us

$$r_0 := \frac{(r_+^{\frac{1}{k}-1} - r_-^{\frac{1}{k}-1})r_- r_+}{r_+^{\frac{1}{k}} - r_-^{\frac{1}{k}}} < \frac{r_- r_+}{r_+} < \frac{r_- r_+}{r_-}, \quad (81)$$

which is equivalent to

$$r_0 < r_- < r_+, \tag{82}$$

that is, the singularity at r_0 is inside the inner "horizon" and the outer "horizons".

In addition, we find that if $k > 1$ then

$$\frac{1}{k} - 1 < 0, \tag{83}$$

and as $r_+ > r_-$ therefore

$$r_+^{\frac{1}{k}-1} < r_-^{\frac{1}{k}-1}. \tag{84}$$

From the numerator of Eq.(74) this implies that when $k > 1$, then $r_0 < 0$. Similarly, when $k < -1$ then from the denominator of Eq.(74) we find that $r_0 < 0$. This means that when $k > 1$ or $k < -1$ the additional spherical singularity will have a negative radius. The condition that the singularity will have a positive radius is when $-1 < k < 1$. We point out that when $-1 < k < 1$, then from Eq.(14), $\omega \geq -\frac{3}{2}$.

Degeneracy of the BDRN metric in Schwarzschild-type coordinates

The degenerate cases of the BDRN spacetime described by Eq.(54) can be quickly recovered in Schwarzschild-type coordinates and are:

Case 1: Brans Type I (BD) metric in Schwarzschild-type coordinates

The Brans-Dicke metric is recovered when $q = 0$ implying that $B = \frac{m_B}{2}$ and thus that $r_+ = 2m_B$ and $r_- = 0$. See Table I.

Case 2: Reissner-Nordström (RN) metric in Schwarzschild-type coordinates

Setting the parameters as $C = \alpha_0 = \beta_0 = 0$ implies that $\lambda = 1, \varphi_0 = (G_0)^{-1}, B = \frac{1}{2}\sqrt{m^2 - q^2}$ where $m := \frac{G_0 M}{c^2}$ and $q := \frac{\sqrt{4\pi G_0}Q}{c^2}$ are respectively the mass and the electric charge measured in gravitational units. The inner and outer "horizons" exist at

$$r_\pm = m \pm \sqrt{m^2 - q^2}, \tag{85}$$

respectively. The full metric for the Reissner-Nordström metric in Schwarzschild-type coordinates is given in Table I.

Case 3: Schwarzschild (S) metric in Schwarzschild coordinates

Setting the parameters as $Q = C = \alpha_0 = \beta_0 = 0$ implies that $\lambda = 1, \varphi_0 = (G_0)^{-1}, B = \frac{m}{2} = \frac{GM}{2c^2}, r_+ = 2m_B$ and $r_- = 0$ where $m = \frac{G_0 M}{c^2}$ is the mass in gravitational units. The metric functions reduce to the well known Schwarzschild solution in the usual coordinates

$$r_+ = 2m, \tag{86}$$

as expected. The full metric for the Schwarzschild metric in Schwarzschild coordinates is given in Table I.

The additional S^2 singularity and the "horizons" that arise in the BDRN background in Schwarzschild-type coordinates must be examined further to determine whether or not they are true physical singularities or anomolies of the coordinate system. To do so it is neccessary to look at the scalar invariant quantities arising from the curvature tensor in a manner similar to that done by Campanelli and Lousto in [31] and Bhadra and Sarkar [34] for the Brans-Dicke background in Schwarzschild-type coordinates. We will present our results in a forthcoming paper [35].

V. ELECTROSTATIC POTENTIAL IN SCHWARZSCHILD-TYPE COORDINATES

To transform the potential $V(r, \theta)$ from isotropic coordinates (t, r, θ, ϕ) into the Schwarzschild-type coodinates (t, r_s, θ, ϕ), we generalize the transformation equation (9) from above. The equation $r_s = (r + \frac{B^2}{r}) + m_B$ can be interpreted as follows: Let (r, θ) be a point exterior to the surface of inversion $r = B$. Then $(\frac{B^2}{r}, \theta)$ is the inversion point of (r, θ) in the interior of the sphere $r = B$. Thus the corresponding point (r_s, θ) in the Schwarzschild-type coordinate is the summation of the distance to the point (r, θ), to the distance of its inversion point $(\frac{B^2}{r}, \theta)$ and the distance to the point (m_B, θ) in the isotropic coordinates. We are then able to define

$$b_s = \left(b + \frac{B^2}{b}\right) + m_B, \tag{87}$$

so that the position $(b_s, 0)$ of the electrostatic charge in Schwarzschild-type coordinate is the summation of the distance to the charge at $(b, 0)$, to the distance to its inversion point $(\frac{B^2}{b}, 0)$ and the distance to the point $(m_B, 0)$ in the isotropic coordinates. Now we define

$$\sigma_s = \left(\sigma + \frac{B^2}{\sigma}\right) + m_B. \tag{88}$$

Using $\sigma = \sigma(r, \theta)$ from Eq.(35), which is substituted into equation (88), we get

$$\sigma_s - m_B = \frac{b^2(r^2 + \frac{B^4}{b^2} - \frac{2B^2 r}{b}\cos\theta) + B^2\Gamma(r, \theta)}{b(\Gamma(r, \theta)(r^2 + \frac{B^4}{b^2} - \frac{2B^2 r}{b}\cos\theta))^{\frac{1}{2}}}, \tag{89}$$

where, following the notation of [1], $\Gamma(r, \theta) = r^2 + b^2 - 2br\cos\theta$ is the square of the "radial" distance from the charged particle at b.

By regrouping terms, the numerator N of Eq.(89) reduces to

$$N = br\left(\left(r + \frac{B^2}{r}\right)\left(b + \frac{B^2}{b}\right) - 4B^2\cos\theta\right); \tag{90}$$

Similarly, by expanding and then regrouping terms, the denominator D reduces to

	Brans-Dicke-Reissner-Nordström	Brans-Dicke	Reissner-Nordström	Schwarzschild
r_+	$m_B + \sqrt{m_B^2 - q_B^2}$	$2m_B$	$m_B + \sqrt{m_B^2 - q_B^2}$	$2m$
r_-	$m_B - \sqrt{m_B^2 - q_B^2}$	0	$m_B - \sqrt{m_B^2 - q_B^2}$	0
$\Pi_s(\varsigma)$	$\frac{[sgn(\varsigma-r_-)sgn(\varsigma-r_+)]^{\frac{1}{2}(k-1)}}{p_+^2 sgn(\varsigma-r_-)^k - p_-^2 sgn(\varsigma-r_+)^k}$	$\frac{sgn(\varsigma-r_+)^{\frac{1}{2}(k-1)}}{\varsigma^{\frac{1}{2}(k+1)}}$	$\frac{1}{p_+^2 sgn(\varsigma-r_-) - p_-^2 sgn(\varsigma-r_+)}$	$\frac{1}{\varsigma}$

TABLE II: The Brans-Dicke Reissner-Nordström, Brans-Dicke, Reissner-Nordström and Schwarzschild backgrounds and their solutions for the electrostatic potential generated by a charged particle at $r_s = b_s > r_+$ in Schwarzschild-type coordinates. Referring to Eqs. (93), the potential is simply $V(r_s, \theta) = \frac{\epsilon \Pi_s(b_s)\Pi_s(r_s)}{2\Pi_s(\sigma_s)}$. Here, as always, $k = \frac{C+2}{2\lambda}$ and σ is given by Eq.(92).

$$D = br\left(\left(r + \frac{B^2}{r}\right)^2 + \left(b + \frac{B^2}{b}\right)^2 - 2\left(r + \frac{B^2}{r}\right)\left(b + \frac{B^2}{b}\right)\cos\theta - 4B^2\sin^2\theta\right)^{\frac{1}{2}}. \tag{91}$$

Using Eqs.(47) and (87), we obtain an explicit expression, in Schwarzschild-type coordinates, for $\sigma(r_s, \theta)$

$$\sigma_s(r_s, \theta) = \frac{((r_s - m_B)(b_s - m_B) - 4B^2\cos\theta)}{((r_s - m_B)^2 + (b_s - m_B)^2 - 4B^2 - 2(r_s - m_B)(b_s - m_B)\cos\theta + 4B^2\cos^2\theta)^{\frac{1}{2}}} + m_B. \tag{92}$$

We can now transform $\Pi(\varsigma)$ from isotropic coordinates into Schwarzschild-type coordinates. Using the definition Eq.(32), Eq.(47), and Eq.(48) we have

$$\begin{aligned}\Pi_s(\varsigma) &= \frac{[sgn(\varsigma_s - r_-)sgn(\varsigma_s - r_+)]^{\frac{1}{2}(k-1)}}{\rho(\varsigma_s)} \\ : &= \Pi_s(b_s).\end{aligned} \tag{93}$$

and following Eq.(94) we have defined the following

$$\rho_s(\varsigma_s) = p_+^2 sgn(\varsigma_s - r_-)^k - p_-^2 sgn(\varsigma_s - r_+)^k. \tag{94}$$

Thus the electrostatic potential Eq.27) can be written in Schwarzschild-type coordinates simply as

$$V(r_s, \theta) = \frac{\epsilon_0}{2}\frac{\Pi_s(b_s)\Pi_s(r_s)}{\Pi_s(\sigma_s(r_s, \theta))} \tag{95}$$

where r_s, b_s and σ_s are given by Eqs.(47), (87) and (88) respectively.

Theorem 2 *Consider the BDRN metric (62) in Schwarzschild-type coordinates $(t, r_s, \theta, \varphi)$. The perturbed electrostatic potential $V_s(r_s, \theta)$ due to a point charge ϵ_0 at $(r_s, \theta) = (b_s, 0)$ exterior to r_+ reduces to a single second order linear partial differential equation:*

$$\begin{aligned}L[V_s(r_s, \theta)] &\equiv \left(\begin{array}{c}[sgn(r_s - r_+)sgn(r_s - r_-)]\frac{\partial}{\partial r_s}\left(\frac{\partial}{\partial r_s}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) \\ + \left(2(r_s - m_B) - \frac{(C+2)\sqrt{m_B^2 - q_B^2}}{\lambda}\Omega_s(r_s)\right)\frac{\partial}{\partial r_S}\end{array}\right)V_s(r_s, \theta) \\ &= \frac{\epsilon_0[sgn(r_s - r_+)sgn(r_s - r_-)]^{\frac{C+2}{2\lambda}}\delta(r_s - b_s)\delta(\theta)}{\rho_s^2(r_s)\sin\theta},\end{aligned} \tag{96}$$

where the functions

$$\begin{aligned}\Omega_s(r_s) &= \frac{p_+^2 sgn(r_s - r_-)^{\frac{C+2}{2\lambda}} + p_-^2 sgn(r_s - r_+)^{\frac{C+2}{2\lambda}}}{\rho_s(r_s)} \\ \rho_s(r_s) &= p_+^2 sgn(r_s - r_-)^{\frac{C+2}{2\lambda}} - p_-^2 sgn(r_s - r_+)^{\frac{C+2}{2\lambda}}.\end{aligned} \tag{97}$$

The linear operator $L[\cdot]$ is elliptic when $r_s\epsilon(0, (r_s)_-) \cup ((r_s)_+, \infty)$ and hyperbolic when $(r_s)_- < r_s < (r_s)_+$. The electrostatic potential can be expressed explicitly in the form

$$V_s(r_s, \theta) = \frac{\epsilon_0\rho_s(\sigma_s)}{2\rho_s(b_s)\rho_s(r_s)}\left(\frac{[sgn(\sigma_s(r_s, \theta) - r_+)sgn(\sigma_s(r_s, \theta) - r_-)]}{[sgn(r_s - r_+)sgn(r_s - r_-)][sgn(b_s - r_+)sgn(b_s - r_-)]}\right)^{\frac{1}{2}\left(1 - \frac{C+2}{2\lambda}\right)}, \tag{98}$$

where the function $\rho_s(\sigma_s) = \rho_s \circ \sigma_s$ *and*

$$\sigma_s(r_s, \theta) = m_B + \frac{(r_s - m_B)(b_s - m_B) - 4B^2 \cos\theta}{\sqrt{(r_s - b_s)^2 - 4B^2 + 2(r_s - m_B)(b_s - m_B)(1 - \cos\theta) + 4B^2 \cos^2\theta}}. \tag{99}$$

If we place the perturbing charge closer and closer to the outer "horizon", that is $b_s \to r_+$, then the electrostatic potential becomes

$$\begin{aligned} V(r,\theta) &= \frac{\epsilon_0}{2}\Pi_s(r_s) \\ &= \frac{\epsilon_0}{2\rho_s(r_s)}\left(\frac{1}{sgn(r_s - r_-)sgn(r_s - r_+)}\right)^{\frac{1}{2}\left(1-\frac{C+2}{2\lambda}\right)} \end{aligned} \tag{100}$$

As before, we find that the perturbative electrostatic potential is well behaved even when the perturbing charge is brought to the outer "horizon". Moreover, the electrostatic potential generated by a perturbing charge located at r_+ is independent of the angle θ. In Schwarzschild-type coordinates it is straightforward to understand the reason for this; as the radial coordinate becomes timelike and vice versa for the region between the inner and outer "horizons" of a BDRN spacetime, the $r = const$ surfaces inside $r_- < r_s < r_+$ are spacelike surfaces. As such, when the charge is brought to r_+ it is immediately distributed evenly across the whole surface of r_+ and is therefore independent of the angle θ. This is in agreement with [6] who found that the charge is evenly distributed across the whole horizon of a Schwarzschild black hole when a perturbing charge is brought to the horizon.

When the perturbing charge approaches the outer "horizon" $b_s = r_+$, its image point b_s^* (which lies within the inner "horizon" $0 \leq b_s^* \leq r_-$) approaches the inner "horizon" from below (see Fig.(4c)). When the image point arrives at the inner "horizon" $b_s^* = r_-$, the charge is evenly distributed across the entire surface of r_-.

A. Copson's Solution in Schwarzschild-type Coordinates

Copson's solution was first converted into Schwarzschild-type coordinates in [7]. If we choose the magnitude of the point charge ϵ_0 to be related to the magnitude of the point charge e given in [7] as $\epsilon_0 = 2e$ then we are able to express Copson's solution in terms of the notation used in this paper as the following

$$V_s^c(r,\theta) = \frac{((r_s - m_B)(b_s - m_B) - m^2\cos\theta)}{2br((r_s - m_B)^2 + (b_s - m_B)^2 - m^2 - 2(r_s - m_B)(b_s - m_B)\cos\theta + m^2\cos^2\theta)^{\frac{1}{2}}}. \tag{101}$$

We point out that this differs from the single charge solution given in [7] and this paper in the presence of an additional charge of the following magnitude

$$V^*(r,\theta) = -\frac{\epsilon_0 m_B}{2br}. \tag{102}$$

B. Limits of Copson's Schwarzschild Solution

The integration constants of the multipole expansion solution of [5] describing a charge situated in a Schwarzschild spacetime for the region $r_s < b$ was found using the boundary condition that the electrostatic potential must be well behaved at the horizon $r_s = 2m$.

As the perturbing point charge at $r_s = b_s$ is allowed to slowly approach the horizon $b_s \to r_+$ in the single charge solution of the Schwarzschild spacetime [7] and [1] it is straightforward to verify that the electrostatic potential approaches the following

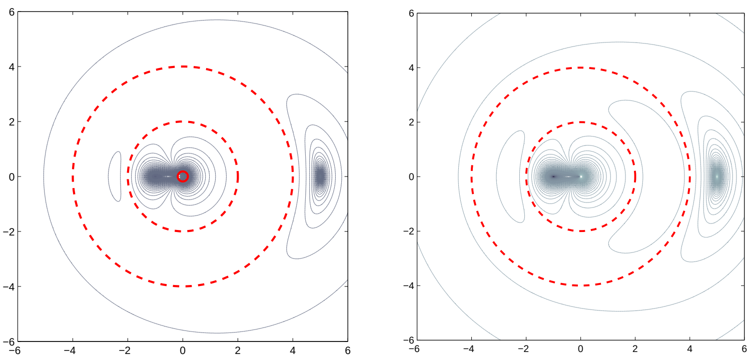
$$V(r,\theta) = \frac{\epsilon_0}{2r}. \tag{103}$$

This is in agreement with the result found by [5] and the electrostatic potential proves to be well behaved at the horizon. We point out that the electrostatic potential Eq.(103) is not representative of a Reissner-Nordström black hole as has already been proved by the theorems of [36], [37] and [38].

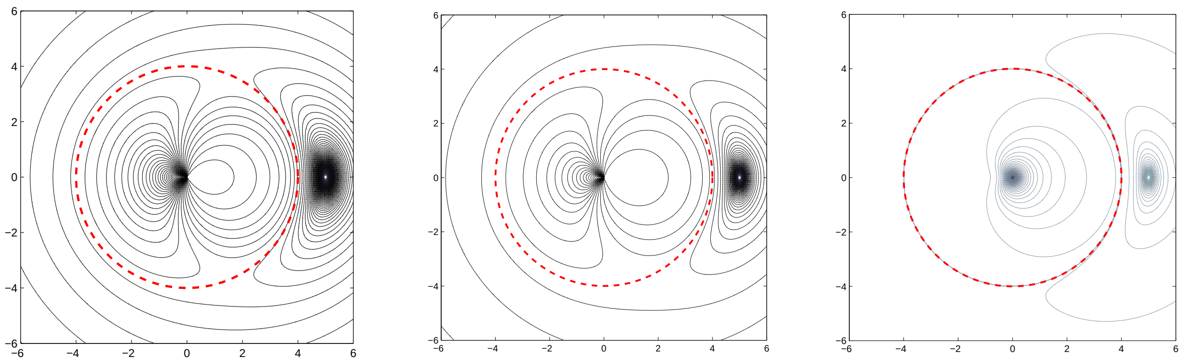
Similarly Copson's expression for the electostatic potential Eq.(101) approaches the following when the perturbing charge is allowed to slowly approach the horizon:

$$V(r,\theta) = \frac{\epsilon_0}{4r}. \tag{104}$$

It is evident that Copson's solution is also well behaved at the horizon thus a boundary condition stating that the electrostatic potential must be well behaved at the horizon is insufficient in ensuring the solution is representative of a single charge (see also [5]).



(a) BDRN (b) Reissner-Nordström



(c) BD (d) Schwarzschild (e) Copson Solution

FIG. 3: A plot of the electrostatic potential generated by a point charge of magnitude 1 situated at $b = 5$ in Brans-Dicke Reissner-Nordström, Brans-Dicke, Reissner-Nordström and Schwarzschild backgrounds. In all four backgrounds, the outer "horizon" is located at $r_+ = 4$ and is denoted by a dashed red line. For the BDRN and Reissner-Nordström backgrounds $m_B = 3$ and $q_B = \sqrt{8}$. Thus for these two backgrounds, the outer "horizon" is located at $r_+ = 4$ and the inner "horizon" is at $r_- = 2$ (also denoted by a dashed red line). The additional S^2 singlarity in the BDRN background is located at $r_0 = 0.1833$ and is denoted by a solid red line. For the Brans-Dicke and Schwarzschild backgrounds $m_B = 2$ and $q_B = 0$ with the horizon located at $r_+ = 4$. For the BDRN and Brans-Dicke cases, $k = 0.95$, while for the Reissner-Nordström and Schwarzschild backgrounds $k = 1$. The Copson two charge solution is also given with $m_B = 2$ and $q_B = 0$ as in the single charge solution of the Schwarzschild background.

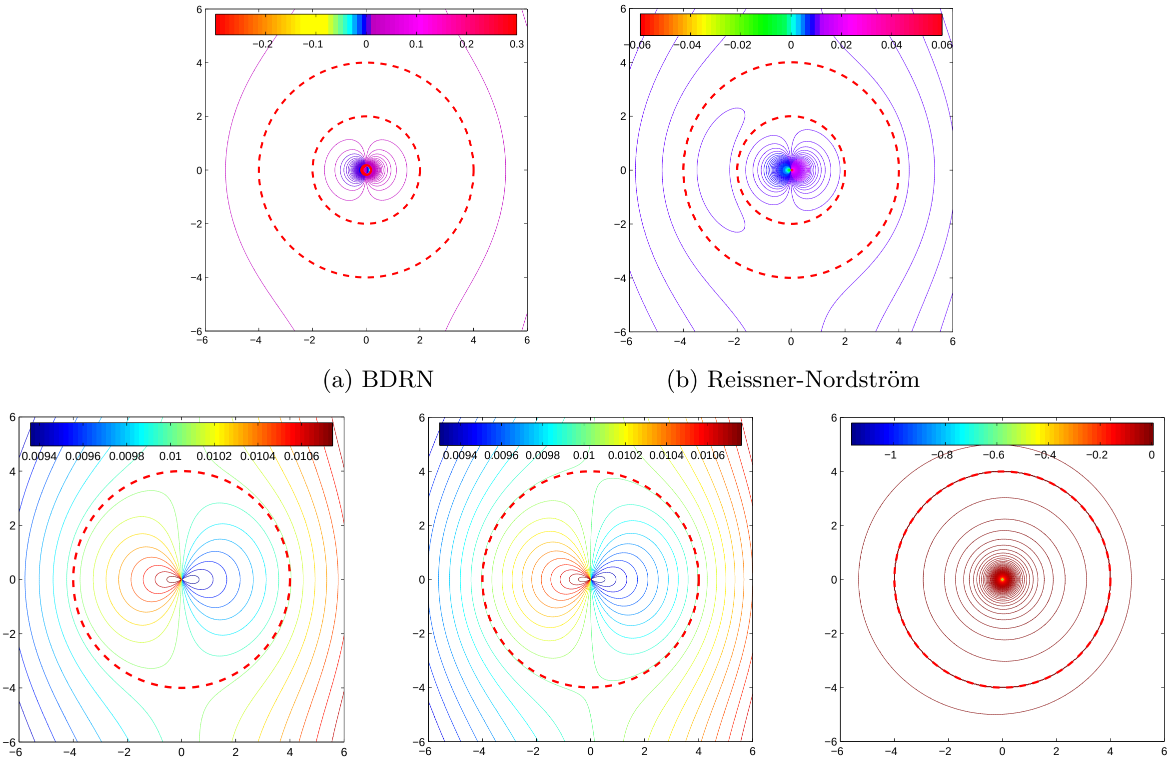
VI. EQUIPOTENTIAL SURFACES IN SCHWARZSCHILD-TYPE COORDINATES

Using the results given in Table II, we are able to plot the equipotential lines generated by a point charge located at b_s in the usual Schwarzshild type coordinates.

Very little can be said about the meaning of the equipotential lines that appear in the region $r_- < r_s < r_+$ as it is here that the radial coordinate becomes timelike and the time coordinate becomes spacelike for the BDRN and Reissner-Nordström spacetimes. For the Brans-Dicke and Schwarzschild spacetimes the radial coordinate is timelike, and vice versa for the entire region within the "horizon" $0 < r_s < r_+$. It is for these reasons that the physical interpretation of any phenomena occuring within the outer "horizon" is not straightforward and why this region is usual excised from any plots and discussion regarding the matter. We have chosen to retain the complete information of the entire spacetime to prove that, if only mathematically, the electrostatic potential can be plotted for all values of r_s.

As in the isotropic case, for the sake of continuity we fix the location of the outer "horizon" at $r_+ = 4$. By setting $m_B = 3$ and $q_B = \sqrt{8}$ in the BDRN and Reissner-Nordström backgrounds we ensure that $r_+ = 4$ and $r_- = 2$. For the BDRN and Brans-Dicke backgrounds, in order for the additional singularity to possess a positive radius the auxillary parameter k must satisfy $-1 < k < 1$ and thus we choose $k = 0.95$. The additional S^2 singularity of the BDRN background is thus located at $r_0 = 0.1833$. For the Brans-Dicke and Schwarzschild background we set $m_B = 2$ and $q_B = 0$, and it follows that $r_+ = 4$. In Fig.(3) we place the point charge of magnitude $\epsilon = 1$ just outside the outer "horizon" at $b = 5$ in all four spacetimes. In Fig.(4) we place the point charge of magnitude $\epsilon_0 = 1$ far from the "horizon" at $b = 50$.

One of the main efficacies of transforming the solu-



(a) BDRN (b) Reissner-Nordström

(c) BD (d) Schwarzschild (e) Copson Solution

FIG. 4: The equipotential lines describing the charge distribution inside the "horizon" of r_+ in the Brans-Dicke Reissner-Nordström, Brans-Dicke, Reissner-Nordström and Schwarzschild backgrounds generated by a perturbing charge of magnitude $\epsilon_0 = 1$ which is situated at $b = 50$,far from the singularities. In all four backgrounds, the outer "horizon" is located at $r_+ = 4$ and is denoted by a dashed red line. For the BDRN and Reissner-Nordström backgrounds $m_B = 3$ and $q_B = \sqrt{8}$. Thus for these two backgrounds, the outer "horizon" is located at $r_+ = 4$ and the inner "horizon" is at $r_- = 2$ (also denoted by a dashed red line). The additional S^2 singlarity of the BDRN background is located at $r_0 = 0.1833$ and is denoted by a solid red line. For the Brans-Dicke and Schwarzschild backgrounds $m_B = 2$ and $q_B = 0$ with the "horizon" located at $r_+ = 4$. For the BDRN and Brans-Dicke cases, $k = 0.95$, while for the Reissner-Nordström and Schwarzschild backgrounds $k = 1$. The Copson two charge solution is also given with $m_B = 2$ and $q_B = 0$ similar in the single charge solution of the Schwarzschild background.

tions to Schwarzschild-type coordinates is that now all four backgrounds are distinctly different from one another. As before, in the presence of a scalar field, it appears that the electrostatic potential is able to cross the outer "horizon" into the interior seamlessly. This is in contrast to the Reissner-Nordström and Schwarzschild spacetimes where the inner and outer regions are separate and disconnected.

Another great advantage of the Schwarzschild-type coordinates is that we are able to see how the perturbing charge is held in place at b without the need for struts or strings. This equilibrium is achieved in all four spacetimes between the singularity and the perturbing charge. We first outline the behaviour of the equipotential surfaces in all four backgrounds before giving the interpretation of our results.

In the BDRN and Reissner-Nordström an image charge appears inside the inner "horizon", see Figs.(3a) and (3b) alongside a dipole-like singularity. The location of the image charge is proportional to the inverse of the distance between the perturbing charge at b and the outer "horizon". Thus we can see that when the perturbing charge is taken sufficiently far away from the outer "horizon" r_+, see Figs.(4a) and (4b), the image charge approaches the singularity. From Figs.(4a) and (4b) one can see the polarity of the dipole-like configuration of the region inside the inner "horizon". As the perturbing charge at $b = 5$ has positive sign, the area closest to the charge has positive polarity. Naturally the other side of the dipole is of negative sign.

In the Brans-Dicke and Schwarzschild backgrounds, although the region inside r_+ is where the physical meaning of the radial and time coordinates are not upheld, one can still detect the presence of a "dipole-like" singularity, see Figs.(3c) and (3d). In the uncharged spacetimes, however, an induced charge does not appear inside the "horizon" as it does in the charged spacetimes.

This phenomena can be interpreted as follows: the presence of the perturbing charge at b causes the singularity to display dipole-like behaviour. As a result of this dipole behaviour the charge configuration of the r_+ "horizon" is altered such that a charge of like sign to the perturbing charge is induced in the region of the "horizon" close to the perturbing charge. A charge of opposite sign is induced in the opposite side of the surface of r_+. The total charge of the "horizon" is maintained at zero and the net ingoing electric flux is equal to the net outgoing electric flux.

Lastly we look at the Copson two charge solution Fig.(3e). As would be expected, the region immediately exterior to the horizon is indistinguishable from the single charge solution Fig.(3d). While the electrostatic potential at the singularity of the single charge solution Fig.(3d) displays dipole behavior, the singularity of the two charge solution Fig.(3e) appears to be a dipole solution in addition to a charged singularity at $r_s = 0$ generating its own electrostatic potential. When the point charge is placed far from the event horizon the effect of the charge on the singularity is diminished and one can see that there is indeed a charge at the location of the singularity in Copson's solution, see Fig.(4e). The Schwarzschild spacetime perturbed by a charge placed inside the horizon does not however, come to resemble the Reissner-Nordström spacetime. It can only be assumed that the perturbing charge is too small to change the nature of the spacetime in any significant way.

VII. CONCLUSION

In this paper and in [1], although we have concentrated only on the BDRN spacetime that reduces to the Brans Class I and Reissner-Nordström backgrounds it is worthwhile to study the charged counterparts to the entire class of solutions of [39]. In an extension of the work done in this paper and of [1] we continue our study of an entire class of charged Brans-Dicke (CBD) spacetimes that reduce to the four known classes of the Brans-Dicke solution when the charge vanishes. We go on to investigate the stability of those charged background spacetimes to electrostatic perturbations and discuss the possibility of charged Brans-Dicke black holes.

In this paper, we have introduced a transformation equation by which one is able to convert a Brans Class I metric from isotropic coordinates to the usual Schwarzschild-type coordinates. We discovered the presence of a S^2 singularity in the BDRN background when in the usual Schwarzschild-type coordinates in addition to the usual inner and outer "horizons". Further investigations into these three singularities is neccessary to determine their true nature. A useful method to understand the behavior of a background is to look to the curvature invariants of that spacetime [40]. In a follow-up paper we use the invariants of the generalized CBD background to discuss the nature of the CBD spacetimes in a manner similar to that conducted by [31] and [34].

The study into Brans-Dicke wormholes has been carried out quite thoroughly [41], [42], [43], [44], [45], [46]. As the BDRN spacetime is stable under electrostatic perturbations we find it worthwhile to investigate the possibility of BDRN and indeed CBD wormholes by studying the CBD metric in a way analogous to that first done by [47]. We devise a method similar to the method devised in this paper to convert the CBD metric in isotropic coordinates to Morris-Thorne canonical form and discuss the possibility of traversable wormholes in CBD spacetimes in a separate paper.

In this paper we plotted the equipotential surfaces in a manner similar to that done in [2]. Another interesting physical phenomenon that would be worthwhile to investigate is the electric lines of force as an extension of work done by [6] and [24]. By plotting the force lines for decreasing values of the radial coordinate one is able to track the evolution of a point charge "falling" into the CBD black hole.

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Declaration of Authorship

In the case of Chapter 4, the nature and extent of my contribution to the work was the following:

Nature of Contribution	Extent of Contribution
Initiation, key ideas, devised theories and methodology, contributed to discussion and analyses of results, prepared paper	67%

The following co-authors contributed to the work.

Name	Nature of contribution
Anthony W.C Lun	Key ideas, contributed to discussion and analyses of results

The undersigned hereby certify that the above declaration correctly reflects the nature and extent of the candidate's and co-authors' contributions to this work.

Candidate's
Signature: Date:

Main Supervisor's
Signature: Date:

Chapter 4

On the Stability of Static Spherically Symmetric Electrovac Brans-Dicke Spacetimes

The electrostatic perturbation of the BDRN spacetime proves to be both interesting and fruitful in explaining not only the stability of the BDRN spacetime but also in understanding the nature of the background space and in finding a mechanism by which the charge distribution within the outer "horizon" of the background can hold an electric charge static. For these reasons, and to address the paucity within the literature regarding charged Brans-Dicke spacetimes (see Chapter 1), we find it pertinent to extend our study to more generalized static spherically symmetric Brans-Dicke electrovac spacetimes. As we have discussed at length the history of BD theory and static spherically symmetric electrovac BD theory in Chapter 1, to avoid repetition, we delve immediately into the most pertinent studies and discuss the key findings of our investigation.

In order to make a spacetime metric physically meaningful, it is imperative that the parameters of the spacetime metric be endowed with some physical interpretation. In the case of Luke and Szamosi [42], although the charged BD spacetime was derived by a direct integration of the field equations, the parameters of the metric were left undetermined. A result of this was that the authors were unable to obtain the degeneracy of their solution to give the BD solutions. Even when they were able to "turn off" the charge, they were unable to obtain the BD solutions. The only degeneracy they were able to obtain was the Reissner-Nordström metric when the scalar field was made constant. Although they were able to obtain four classes of solutions representing the charged Brans-Dicke spacetime, their choice of the constants of integration inhibited them from obtaining the full set of nine solutions that we are able to derive in our investigations.

Luke and Szamosi were reluctant to implement the weak field approximation as they found no justification in doing so and as a result were unable to determine their constants of integration or give them any physical interpretation. We agree with their reasoning for not implementing the weak field approximation. Indeed there is no a priori reason to implement it in the context of the phenomena discussed here. However, we find that we are able to gain great insight from the weak field approximation which enables us to infer the physical interpretation of the parameters of the background metric. Thus we use the weak field approximation, and the form of the parameters under the weak field approximation, as tools to determine the physical interpretation of the parameters in the general spacetime.

The CBD electrovac field equations arising from the static spherically symmetric electrovac CBD spacetime are integrated directly to give for the first time a generalized charged Brans-Dicke solution that reduces to the four known BD spacetimes. There are a total of nine classes of solutions based upon the sign of the parameter ab and κ. The sign of ab determines the class of solution where $ab > 0$, $ab < 0$ and $ab = 0$ correspond to Class I, II and III/IV respectively. The sign of κ is denoted by a superscript to the roman numeral such that $\kappa > 0$, $\kappa < 0$ and $\kappa = 0$ correspond to the superscripts $+, -$ and 0 respectively. The nine classes of solutions are thus CBDI$^+$, CBDI$^-$, CBDI^0, CBDII$^+$, CBDII$^-$, CBDII^0, CBDIII/IV$^+$, CBDIII/IV$^-$, and CBDIII/IV^0.

From the physical interpretations of the sign of ab we are able to infer the following: CBD Class I solutions are representative of spatial singularities where the mass is larger than the electromagnetic charge $m > q$. This is physically meaningful for cosmological phenomena where the net charge of the universe must be zero. CBD Class II solutions are representative of spacetimes singularities where the mass is smaller than the electromagnetic charge $m < q$. This may be interesting in the quantum field gauge where such strongly charged particles are conceivable. CBD Class III/IV solutions are representative of spatial singularities exhibiting extreme behaviour where the charge is exactly equal to the charge $m = q$. The analogue to this is the extreme Reissner-Nordström black hole. The ramifications of this interpretation on the classes of solution are apparent when we discuss the degeneracy of the CBD spacetime metric.

The curvature invariants of the general CBD spacetime is investigated in the same manner as Campanelli and Lousto [79] and Bhadra and Sarkar [90]. We find that the only nonsingular spacetime that may give rise to black holes or wormholes is the CBDII$^-$ Class of solutions when $\omega < -2$. The analysis into the nature of the horizon in order to determine whether the spacetime is representative of a black hole or a wormhole is very delicate. Indeed, in the BD theory, the findings of Campanelli and Lousto in [79] regarding the existence of BD black holes was not refuted until almost two decades later by [80] who found they were indicative of wormholes instead. For these reasons, we leave the analysis of physical nature of the CBDII$^-$ spacetime with $\omega < -2$ to a later paper.

We go on to electrostatically perturb the generalized CBD spacetime. Using the ansatz developed in Chapter 2 we are able to find a general solution for the electrostatic potential generated by a point charge in a generalized CBD spacetime. The ansatz developed in Chapter 2 proves to be robust and allows us to find the form of the substitution term directly from the background generalized CBD metric. We also implement the boundary condition introduced in Chapter 2 and find that it is necessary and sufficient in ensuring that the electrostatic potential represents a single perturbing charge. We are thus able to find an exact closed-form solution describing the electrostatic potential generated by an electrostatic point charge in a general CBD spacetime. The solution can be divided into nine cases, corresponding to the electrostatic perturbation of the nine classes of CBD spacetimes, by making an appropriate choice on the parameter $NX^{\delta}(r)$ where X is I, II or III/IV and represents the class of solutions and $\delta \in \{+, -, 0\}$ represents the sign of κ.

Lastly, along with a conclusion of the findings, we give the future investigations that may be of astrophysical interest and importance.

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On the Stability of Static Spherically Symmetric Electrovac Brans-Dicke Spacetimes

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In Part I of this paper, we generalize the Brans-Dicke scalar-tensor theory to investigate static spherically symmetric electrovac Brans-Dicke spacetimes in isotropic coordinates. We derive a general charged Brans-Dicke (CBD) solution which can be divided into three classes of solutions: CBDI, CBDII and CBDIII/IV solutions depending on the sign of the parameter ab. These can be further divided into three branches of solutions each, depending on the sign of the auxillary parameter κ to give a total of nine branches of solutions. Three of the nine solutions reduce to the three classes of solutions of the Brans-Dicke spacetime. It is well known that Class III of the BD solutions is simply the Class IV solution under an inverse transform thus we combine them into a single branch denoted by III/IV. As there is no a priori reason to implement the weak field approximation we give the CBD metric and its parameters in their most general form. We find, however, that the weak field approximation is a useful tool in understanding the physical interpretation of the parameters and give, in addition to the general form, the parameters as restricted by the weak field approximation. We are thus able to interpret the CBD III/IV solutions as being that of extreme CBD spacetimes as an analogue to extreme Reissner-Nordström black holes. In the absence of charge, a subclass of the CBD III/IV solutions reduce to the BD III/IV solutions and we find that the BD Class III/IV background contains a massless, scalar field generating singularity. As such, this subclass of CBDIII/IV solutions and consequently, the BDIII/IV solutions, degenerate to Minkowski space in the absence of the scalar field. The curvature invariants are scrutinized to gain a better understanding of the background metrics and we find that the CBDII$^-$ background may allow black holes or wormholes when the coupling constant ω is restricted such that $\omega < -2$. We find that the CBDIII/IV backgrounds represent naked singularities. In Part II of this paper, we perturb the CBD spacetime with a small electrostatic point charge using a method devised in [1] and find the electrostatic potential generated by the perturbing charge placed outside the surface of inversion. We use a boundary condition based on Gauss' theory introduced in [1] to ensure that the solution represents a single charge perturbation. We find that all nine classes of solutions are stable under static electromagnetic perturbations.

PACS numbers:

I. INTRODUCTION

General relativity (GR) is one of the most well-known physical theories due to both its beautiful nature and its ability to withstand several observational tests including but not limited to Mercury's precession, gravitational lensing and gravitational redshift. However GR is not without its limitations; namely its inability to reconcile with quantum mechanics and its inconsistency with the current acceleration of the universe. It seems neccessary then to modify GR somewhat to overcome these problems. One such modified theory is the scalar-tensor theory, originally conceived by Jordan [2] who coupled a scalar field to the Ricci scalar in the general Lagrangian. This scalar-tensor theory was adopted and modified by Brans and Dicke in 1961 [3] who decoupled the matter part of the Lagrangian from the scalar field and ensured that, unlike Jordan's earlier model, the Weak Equivalence Principle (WEP) would be upheld.

Scalar-tensor theories have gained particular momentum as they are able to explain phenomena that have been proposed and supported by more recent cosmological observations including but not limited to the expansion of the universe [4], [5], [6], [7], [8]. Another advantage of scalar-tensor theories is its diverse range of applicability, spanning from cosmology to quantum field theory. Cosmologically speaking, a prominent scalar field is quintessence [9], [10], [11], [12], which is a candidate for the missing energy component that must be added to the baryonic and matter density in order for the universe to reach critical density [13]. In quantum field theory, dilatons and the size of compactified internal space in string theory and Kaluza-Klein (KK) theory can be identified as scalar fields. Scalar fields also appear in brane theory (see [14] for more details). Perhaps most intriguingly, scalar-tensor theories have yet again entered the spotlight with the recent discovery of the Higgs boson [15] and the fact that the Higgs field can be expressed as a scalar field. Mass arises in the Higgs mechanism through the coupling of the gravitational scalar field ϕ to the matter scalar field Φ in the matter Lagrangian. Of course, cou-

[redacted]

pling the gravitational scalar field to matter violates the basic premise of the Brans-Dicke (BD) model discussed earlier, and it then follows that the WEP is violated. For a more comprehensive history of scalar-tensor theory see [14], [16], [17].

Since the formulation of scalar-tensor theory in 1961, the parameter range of the the coupling constant ω has garnered great interest while at the same time been the source of some controversy and disagreement. The Shapiro time-delay measurements of the Viking Project in the 1970s placed the constraint at $\omega \gtrsim 1000$ while twenty years later that was revised from VLBI experiments to $\omega \gtrsim 3.6 \times 10^3$. The current accepted value for ω lies at over 4×10^4 due to the Cassini-Huygen experiment of 2003 [18].

In theoretical circles, the discussion regarding the parameter range ω continues in spite of the observational findings and are largely unresolved even to this day. It is the intention of this paper to shed some light on the issue. While recent observations place the value of ω at over 40000, this does not neccessarily eliminate other possibilities for ω. In fact it is simple to see that if the corresponding force-range of the scalar force is smaller than the size of the solar system, and consequently no longer affects the perihelion advance of Mercury, then the observational constraints are irrelevant [14].

But what can be said about other restrictions on ω? It would appear from the Lagrangian that negative values for ω should immediately be eliminated from all discussions as they appear to violate the weak energy condition. However, even this is premature for two reasons:

- Due to the presence of the nonminimal coupling term in the Lagrangian the dominating energy density may still be positive. This is the case where the scalar field arises from the size of compactified internal space in KK theory which has a negative ω whilst still maintaining an overall positive energy due to the mixing interaction with the spinless component of the metric field (the role played by the nonminimal coupling term). This is also the case with dilatons in string theory which possess a negative value for ω. However the overall positivity of the physical mode is assured as it is easy to restrict other parameters in order for the "diagnolized" scalar field to be a nonghost field. One last example is in brane theory where in order to exist in a negative-tension brane (which is required to solve the hierarchy problem), ω must be negative but larger than $-\frac{3}{2}$ which again does not violate the positivity condition as the "diagnolized" scalar field can still be expressed as a nonghost field. Even the extreme circumstance where $\omega = 0$ need not be eliminated from consideration [14].

- Negative energy is no longer the impasse it was once perceived to be. There are several instances where negative energy has been theorized to exist such as squeezed light states [19], the Casimir effect [20] and moving boundaries [21]. While for the above theories researchers have only been able to measure the indirect effects of negative energy, it may be possible to detect them directly using atomic spins [22]. Another famous example of negative energy is the Hawking prediction that black holes evaporate by emitting radiation [23] which would manifest as an absorption of negative energy. Lastly, in [24], the authors found that in order for a wormhole to be traversable, it is in fact *necessary* for the wormhole throat to possess negative energy.

Another cause for debate regarding the parameter range for ω has been the implementation of the weak field approximation. In [3], Brans and Dicke applied the weak field approximation which restricted the range of ω to $\omega > -\frac{3}{2}$. It is well known that the BD theory has given rise to numerous studies into the possibility of the existence of BD black holes [25], [26], [27] and wormholes (see for example [28], [29], [30], [31], [32], [33]). The neccessity of the implementation of the weak field approximation and its subsequent effect on the permissible range of ω became the subject of some debate starting from the late 1990s (see [30], [34], [30]). In [35], the authors implemented a variation of the weak field approximation and together with observational data ruled out the possibility of BD black holes in all four cases of the BD spacetime. However as pointed out in [36] there is no reason why the weak field approximation must be upheld in the presence of post-Newtonian compact objects with strong gravitational fields.

The independency of the four classes of solutions in the BD theory came under question in 2002, with the claim that two new solutions of the BD theory were discovered by [31]. This idea was challenged by Bhadra et. al [35] who pointed out that the solutions in [31] were simply limiting cases of the Brans Class I solution. Previously, Bhadra et. al stated in [37] that Brans' class III and IV given in [3] were not independent. In [35] it was stated that in fact of the four classes of solutions only Class I and Class IV solutions were independent. We show that this is not strictly true, and that while the Class III and IV solutions are the same under an inverse transformation and thus are denoted by III/IV throughout this paper, Classes I and II are indeed independent. Thus there are three independent classes of solution: I, II and III/IV.

In regards to charged BD theories, it was in 1972 that the authors of [38] found an approximate solution describing a static spherically symmetric metric due to a point charged mass in BD theory. This was followed by an exact solution by [39] and independently by [40] who were able to describe a gravitational field with a charged mass point in Brans-Dicke theory that degenerated to the Brans-Dicke Class I and Reissner-Nordström solutions. Later this work was extended by [41], [42], [43], [44], [45]. In a previuos paper [1], we were able to integrate the Einstein field equations arising from the gravitational field generated by a charged mass point in scalar-tensor theory. However, in [1], as is the case with the studies cited above, the solutions found only reduced to one of the four classes of BD solutions (Class I) and the Reissner-Nordström solution. In this paper we formally integrate the Einstein field equations arising from a scalar-tensor field with a gravitational field generated by a charged mass point and are able to give a solution describing a static spherically symmetric charged BD background that reduces to all four of the BD solutions.

In Part I of this paper we give a detailed overview of the static spherically symmetric Brans-Dicke electrovac (or "Charged Brans-Dicke") spacetime and the integration constants that arise from integrating the field equations. We find that the Charged Brans-Dicke (CBD) spacetime gives rise to three independent solutions, CBD I, II and III/IV, based on the sign of the constants of integration ab. The general electrovac Brans-Dicke solution has six constants of integration, two of which can be determined by scaling the coordinates r and t. The solutions are also in terms of two auxillary constants, $N(r)$ and κ. We place no restrictions on the integration constants other than boundary conditions at infinity and find that the three independent solutions can further be broken down into three separate classes each depending on the sign of the auxillary constant κ. Of the resulting nine cases, denoted by superscripting Classes I, II and III/IV with $+, -$ or 0, only three reduce to a Brans-Dicke equivalent (with $Q = 0$) to give the three independent Brans-Dicke classes (I, II and III/IV). The remaining six cases only exist when the point contains a charge but as we discover, not necessarily a mass.

In Part II, we electrostatically perturb the CBD spacetime in its most general form using the Copson-Hadamard method developed in [1]. We apply the boundary condition, also introduced in [1] to determine our constants of integration such that the resultant solution on the electrostatic potential generated by a perturbing point charge is representative of a single charge perturbation as required. We find that all nine classes of the CBD spacetime are stable under electrostatic perturbations.

The structure of this paper is as follows: In Section II we explain how the CBD spacetime metric is derived directly from the BD field equations. One of the primary benefits of using the choice of integration constants of this paper and integrating the field equations directly is that we are able to gain immediate insight into the physical meaning behind the constants of integration. Also by placing no restrictions on the integration constants we are able to give the solutions in their most general form without the neccesity of making any assumptions regarding the physical nature of the individual solutions.

In Section III, we investigate the weak field approximation and show how implementating it sheds light on the physical interpretation of the background and in particular the meaning of the constants of integration. We emphasize that there is no a priori reason why the weak field approximation must be implemented on such post-Newtonian phenomena [36]. The weak field approximation is simply a useful tool that allows one to better understand the spacetime and the physical meaning of the integration constants.

In Section IV, we return to the most general form of the CBD metric and briefly relate the constants of integration given in this paper to that of Brans and Dicke [46], a more thorough comparison between the two is give in Appendix B. We are thus able to classify the generalized CBD solution into three solutions based on the sign of the parameters ab and κ.

In Section V, we formally give the representation of the nine classes of solutions in Table I and II. We show how the nine classes of the CBD spacetime degenerate to known spacetimes and discuss the effect the electromagnetic charge of the background space has on the perihelion shift of Mercury and observational data.

In Section VI, as an extension of work done previously in [47] we introduce a transformation equation by which one is able to convert the CBD metric of Section II from isotropic coordinates to the more familiar Schwarzschild-type coordinates. In doing so, one is able to gain an immediate appreciation for how the CBDI$^+$ solution reduces to the known Brans-Dicke, Reissner-Nordström and Schwarzschild solutions.

In Section V, we study the curvature invariants of the CBD spacetime in isotropic coordinates and find that of the nine backgrounds only the CBDII$^-$ spacetime with $\omega < -2$ possesses a nonsingular horizon and may give rise to wormholes or black holes. The CBDIII/IV$^+$ background is found to represent a naked singularity.

In Part II, we perturb the CBD background by placing a point charge outside the surface of inversion. Following a method developed in a previous paper [1], we are able to derive and then solve the governing Maxwell equations to give a closed-form solution for the electrostatic potential generated by a point charge in the CBD spacetime in its most general form. We make use of the same boundary condition outlined in [1] to determine the constant coefficients of the closed-form solution and in doing so also eliminate the singularity that creeps into the spacetime region that lies beyond the inversion surface and which is not covered by the isotropic coordinates. We find that the CBD spacetime is linearly stable under electrostatic perturbations.

In Appendix A, we give a detailed overview of how the CBD background metric is determined starting from the transformation from the Jordan to the Einstein frame. In Appendix B, we relate our notation to that given in [3] and used in [1] and [47].

We point out that the perturbed Maxwell equations considered in [1] did not include the effect the perturbing electromagnetic charge would have on the gravitational field of the BDRN and Reissner-Nordström background space-times. As such we cannot consider it a perturbation in its most complete form but rather a "model system" whereby the perturbing charge does not produce cross-terms in the Maxwell equation via the energy-momentum tensor. A full consideration of the perturbation would be extremely complex (see [48] for a full consideration of the perturbation of the Reissner-Nordström space-time) and the results of a full consideration would not be significantly different to those found in [1]. Thus we used this "model system" for the BDRN and Reissner-Nordström space-times in [47] and will use it in this paper. The perturbation of the Brans-Dicke and Schwarzschild space-times considered in [1] on the other hand, are complete as these backgrounds are uncharged and the perturbing charge does not interact with the gravitational field via the energy-momentum tensor.

Part I

II. STATIC SPHERICALLY SYMMETRIC BRANS-DICKE ELECTROVAC SPACETIMES

Theorem: Consider the static spherically symmetric spacetime metric

$$ds^2 = -c^2 A(r)\,dt^2 + B(r)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]. \tag{1}$$

in isotropic coordinates $(t, r, \theta.\phi)$. The electrovac Brans-Dicke field equations in the Jordan frame are:

$$\begin{aligned}
B(r)\,G^t_t &= \frac{B''(r)}{B} - \frac{3B'(r)^2}{4B^2} + \frac{2}{r}\frac{B'(r)}{B} && (2)\\
&= -\frac{Q^2}{r^4\varphi(r)\,B(r)} - \frac{\varphi''(r)}{\varphi} - \left(\frac{2}{r} + \frac{B'(r)}{2B}\right)\frac{\varphi'(r)}{\varphi}\\
&\quad -\frac{\omega}{2}\frac{\varphi'(r)^2}{\varphi^2}, && (3)
\end{aligned}$$

$$\begin{aligned}
B(r)\,G^r_r &= \frac{B'(r)^2}{4B^2} + \frac{A'(r)\,B'(r)}{2AB} + \frac{1}{r}\left(\frac{A'(r)}{A} + \frac{B'(r)}{B}\right)\\
&= -\frac{Q^2}{r^4\varphi(r)\,B(r)} - \left(\frac{2}{r} + \frac{A'(r)}{2A} + \frac{B'(r)}{B}\right)\frac{\varphi'(r)}{\varphi}\\
&\quad +\frac{\omega}{2}\frac{\varphi'(r)^2}{\varphi^2}, && (4)
\end{aligned}$$

$$\begin{aligned}
B(r)\,G^\theta_\theta &= \frac{A''(r)}{2A} + \frac{B''(r)}{2B} - \frac{A'(r)^2}{4A^2} - \frac{B'(r)^2}{2B^2} && (5)\\
&\quad +\frac{1}{2r}\left(\frac{A'(r)}{A} + \frac{B'(r)}{B}\right) && (6)\\
&= \frac{Q^2}{r^4\varphi(r)\,B(r)} - \frac{\varphi''(r)}{\varphi} - \left(\frac{1}{r} + \frac{A'(r)}{2A}\right)\frac{\varphi'(r)}{\varphi}\\
&\quad -\frac{\omega}{2}\frac{\varphi'(r)^2}{\varphi^2}; && (7)
\end{aligned}$$

the scalar wave equation is

$$\frac{d\left(r^2\sqrt{A(r)\,B(r)}\varphi'(r)\right)}{dr} = 0, \tag{8}$$

while the electrostatic force equation, which is obtained from the Maxwell equations on the vector potential $-cV(r)\,dt$, is

$$\left(r^2\sqrt{\frac{B(r)}{A(r)}}V'(r)\right)' = 0. \tag{9}$$

The solutions to equations (2) to (9), called the Static Spherically Symmetric Charged Brans-Dicke (CBD) solutions, are given formally by

$$\begin{aligned}
\varphi(r) &= \varphi_0 \exp\left(\frac{\varphi_1}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r+1}\right)\right), && (10)\\
a, b &\in \mathbb{R}, \quad \text{if} \quad ab = 0 \Rightarrow \begin{cases} a = 0, & b \neq 0\\ b = 0, & a \neq 0\end{cases}\\
\varphi_0 &> 0, \quad \varphi_1 \in \mathbb{R}; && (11)\\
A(r) &= \frac{\varphi_0 A_0}{\varphi(r)\,N^2(r)}, \qquad A_0 > 0; && (12)\\
&= \frac{A_0}{\exp\left(\frac{\varphi_1}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r+1}\right)\right)N^2(r)}, && (13)\\
B(r) &= \frac{\left(ar^2-b\right)^2 N^2(r)}{\varphi_0 A_0 r^4 \varphi(r)} && (14)\\
&= \frac{a^2\left(1-\frac{b}{a}\frac{1}{r^2}\right)^2 N^2(r)}{\varphi_0^2 A_0 \exp\left(\frac{\varphi_1}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r+1}\right)\right)}; && (15)\\
V'(r) &= \frac{c^2 Q\varphi_1}{\sqrt{4\pi}}\frac{A_0\varphi_0}{a\left(r^2-\frac{b}{a}\right)N^2(r)}, \quad Q \in \mathbb{R}; && (16)
\end{aligned}$$

and

$$Q^2 A_0\varphi_0 + \kappa > 0. \tag{17}$$

See Appendix A for details on the derivation of the metric coefficients and in particular, refer to Eq. (144) for an explanation on the condition Eq. (17).

Here c is the speed of light in vacuum and the function $N(r)$ is expressed as

$$\begin{aligned}
N(r) &:= p_+^2 \exp\left(-\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r+1}\right)\right)\\
&\quad -p_-^2 \exp\left(\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r+1}\right)\right), && (18)
\end{aligned}$$

The nine parameters $\left\{a, b, A_0, \varphi_1, \varphi_2, m_B, q_B, p_-^2, p_+^2\right\}$ that appear in the Eqs.(10)-(18) are are related via the following

$$\frac{4ab}{\varphi_1^2} = (m_B^2 - q_B^2)\left(\frac{(2\omega+3)c^2\varphi_0)}{2m_B}\right)^2, \quad (19)$$

$$\frac{\varphi_1}{a} = -\frac{2m_B}{c^2\varphi_0}\left(\frac{1}{2\omega+3}\right),) \quad (20)$$

$$\frac{b}{a} = \frac{m_B^2 - q_B^2}{4}, \quad (21)$$

$$m = \frac{m}{c^2\varphi_0}\frac{1}{e^{\beta_0}}\left(\frac{2\sqrt{ab}}{\varphi_1\sqrt{\kappa}}\right), \quad (22)$$

$$q = \sqrt{\frac{4\pi}{\varphi_0}}\frac{q}{c^2}e^{\beta_0}\left(\frac{2\sqrt{ab}}{\varphi_1\sqrt{\kappa}}\right), \quad (23)$$

$$\varphi_0 = \frac{1}{G_{eff}}\left(\frac{2\omega+4}{2\omega+3}\right), \quad (24)$$

$$p_\pm^2 = \frac{1}{2}\left(\sqrt{1+\frac{Q^2A_0\varphi_0}{\kappa}} \pm 1\right), \quad (25)$$

and

$$\kappa := \frac{4ab}{\varphi_0^2} - \frac{2\omega+3}{4}, \qquad Q^2 := \frac{4\pi q^2}{c^4\varphi_1^2} \quad (26)$$

Comments on Theorem 1:

1. The six real parameters

$$\{a, b, \varphi_1, Q\} \in \mathbb{R}, \quad \{\varphi_0, A_0\} \in \mathbb{R}^+, \quad (27)$$

are constants of integrations, where a, b, A_0 arise from boundary conditions on the tensor gravitational field, Q arises from the static electric field (more precisely, the vector electromagnetic potential), while φ_1 and φ_2 arise from the boundary condition on the scalar field.

2. The parameters m and q are non-negative real constants and are identified, respectively, as mass measured in convetional units (kg) and charge measured in electrosatic units (e.s.u), which has the dimensions of $[\text{mass}]^{\frac{1}{2}}[\text{length}]^{\frac{3}{2}}[\text{time}]^{-1}$.

3. The dimensionless real parameter

$$\omega \in \mathbb{R}. \quad (28)$$

is the coupling constant, whose inverse measures the strength of scalar field.

4. The three auxillary parameters

$$\{\kappa, p_+^2, p_-^2\} \quad (29)$$

defined in Eqs.(25) and (26) are given in terms of the constants of integration (27) and the coupling constant (28).

5. The positive constants φ_0 and a have the dimension of $[\text{length}]^{-3}[\text{time}]^2[\text{mass}]^1$, i.e., it has the dimension of the reciprocal of the gravitational constant; the constant b has the dimension of $[\text{length}]^{-1}[\text{time}]^2[\text{mass}]^1$, the constant φ_2 has the dimension of $[\text{length}]^{-2}[\text{time}]^2[\text{mass}]^1$, the constant $Q = \frac{\sqrt{4\pi}q_0}{c^2\varphi_1}$ has dimension $[\text{length}]^{\frac{3}{2}}[\text{time}]^{-1}[\text{mass}]^{-\frac{1}{2}} = [\text{charge}]^1[\text{mass}]^{-1}$, and the constant A_0 is dimensionless.

6. In scalar-tensor theories, the gravitational constant G_0 that appears in the Einstein-Hilbert term of the standard theory is replaced by G_{eff} which is related to the strength of the scalar field as given above.

7. Under the scaling transformation of the coordinates t and r :

$$t \to t' = \tau_0 t, \quad r \to r' = \rho_0 r, \quad (30)$$

where τ_0 and ρ_0 are arbitrary constants, the integration constants transform according to

$$A_0 \to A_0' = \tau_0^2 A_0, \quad Q \to Q' = \tau_0^{-1}Q,$$

$$a \to a' = \frac{\tau_0}{\rho_0}a, \quad b \to b' = \tau_0\rho_0 b, \quad (31)$$

$$\varphi_0 \to \varphi_0' = \varphi_0, \quad \varphi_1 \to \varphi_1' = \tau_0\varphi_1, \quad (32)$$

while the products

$$ab \to a'b' = \tau_0^2 ab, \quad (33)$$

$$\frac{b}{a} \to \frac{b'}{a'} = \rho_0^2\frac{b}{a}. \quad (34)$$

There are two degrees of freedom to fix two of the four real parameters: $\{A_0, a.b, \varphi_2\}$; however the scalar field and the vector field parameters φ_1 and the charge q_0 are not affected by coordinate scaling (30).

8. Under the inversion transformation

$$r \to r_* = \left|\frac{b}{a}\right|\frac{1}{r}, \quad (35)$$

the regions

$$\begin{cases} \left(0 < r < \sqrt{\left|\frac{b}{a}\right|}\right) \leftrightarrow \left(\sqrt{\left|\frac{b}{a}\right|} < r_* < \infty\right) \\ \left(\sqrt{\left|\frac{b}{a}\right|} < r < \infty\right) \leftrightarrow \left(0 < r_* < \sqrt{\left|\frac{b}{a}\right|}\right) \end{cases} \quad (36)$$

are mapped one-to-one and onto from points outside the sphere of radius $\sqrt{\left|\frac{b}{a}\right|}$ to points inside, and vice versa. Under such a coordinate transformation the flat 3-metric

$$dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{37}$$

is mapped one-to-one and onto the conformally flat 3-metric

$$\left(\frac{b}{a}\frac{1}{r_*^2}\right)^2\left(dr_*^2 + r_*^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right). \tag{38}$$

III. WEAK FIELD APPROXIMATION

In the theorem outlined above we have given the CBD solution formally in its most general form without placing any restrictions on any of the six real constants of integration: $\{a, b, \varphi_1, Q\} \in \mathbb{R}$ and without loss of generality $\{\varphi_0, A_0\} \in \mathbb{R}^+$, except boundary conditions at spatial infinity for the vanishing of the electrostatic force. Here we implement the weak field approximation to see what restrictions are placed upon these parameters when the formal CBD solution is required to agree to the first order with the weak field approximation. We point out that there is no a priori reason why the weak field approximation needs to be applied to phenomena such as that considered here [36], however we find the weak field approximation a useful tool in interpretating the physical meaning of the parameters.

The derivation of the weak field approximation is given in detail in [3] and we give only the results here. The linearized metric coefficients obtained from the weak field approximation are

$$h_{ij} = \frac{4}{\varphi_1 c^4}\int \frac{T_{ij}}{r}d^3x - \frac{4}{\varphi_1 c^4}\left(\frac{\omega+1}{2\omega+3}\right)\eta_{ij}\int \frac{T}{r}d^3x. \tag{39}$$

For a static electrically charged point mass with mass M and charge Q, series expansion to the first order of $\frac{1}{r}$ give

$$\begin{aligned}
\varphi(r) &= \varphi_0\left(1 + \frac{2m_0}{\varphi_0 c^2}\frac{1}{2\omega+3}\frac{1}{r}\right), && (40)\\
g_{00} &= -1 + \frac{2m_0}{\varphi_0 c^2}\left(1 + \frac{1}{2\omega+3}\right)\frac{1}{r}, && (41)\\
g_{11} &= 1 + \frac{2m_0}{\varphi_0 c^2}\left(1 - \frac{1}{2\omega+3}\right)\frac{1}{r}. && (42)
\end{aligned}$$

We find that the charge Q only affects the metric at order $O\left(\frac{1}{r^2}\right)$ and higher orders of $\frac{1}{r}$, and thus the weak field approximation remains unchanged to that given in [3]. A series expansion of Eqs.(10), (12) to (14) in powers of $\left(\frac{\varphi_1}{ar}\right)$ give

$$\varphi(r) = \varphi_0\left(1 - \frac{\varphi_1}{ar} + \frac{1}{2}\left(\frac{\varphi_1}{ar}\right)^2 + O\left(\frac{1}{r^3}\right)\right) \tag{43}$$

$$A(r) = A_0\begin{pmatrix} 1 - \left(2\sqrt{\kappa + Q^2\varphi_0 A_0} - 1\right)\frac{\varphi_1}{ar} \\ +\frac{1}{2}\left(4\kappa - 4\sqrt{\kappa + Q^2\varphi_0 A_0} + 6Q^2\varphi_0 A_0 + 1\right)\left(\frac{\varphi_2}{ar}\right)^2 \\ +O\left(\left(\frac{\varphi_1}{ar}\right)^3\right)\end{pmatrix} \tag{44}$$

$$B(r) = \frac{a^2}{\varphi_0^2 A_0}\begin{pmatrix} 1 + \left(2\sqrt{\kappa + Q^2\varphi_0 A_0} + 1\right)\frac{\varphi_1}{ar} \\ +\frac{1}{2}\left(1 + 4\kappa - \frac{4ab}{\varphi_2^2} + 4\sqrt{\kappa + Q^2\varphi_0 A_0} + 2Q^2\varphi_0 A_0\right)\left(\frac{\varphi_1}{ar}\right)^2 \\ +O\left(\left(\frac{\varphi_1}{ar}\right)^3\right)\end{pmatrix} \tag{45}$$

Comparing CBD expansion equations (43) to (45) with weak field approximation equations (40) to (42), we immediately obtain

$$\begin{aligned}
\frac{\varphi_1}{a} &= -\frac{2m_0}{\varphi_0 c^2}\left(\frac{1}{2\omega+3}\right), && (46)\\
A_0 &= 1, && (47)\\
\frac{a^2}{\varphi_0^2 A_0} &= 1, && (48)
\end{aligned}$$

and

$$\sqrt{k + Q^2 A_0\varphi_0} = -\frac{2\omega+3}{2}. \tag{49}$$

Substituting Eqs.(46), (48) and (26) into Eq.(49), after some algebra, we find

$$\frac{b}{a} = \frac{1}{4}\left(\frac{m_0^2}{\varphi_0^2 c^4}\left(\frac{2\omega+4}{2\omega+3}\right) - \frac{4\pi q_0^2}{c^4\varphi_0}\right). \tag{50}$$

Thus in the weak field approximation, the mass m_B and charge q_B become respectively,

$$m_B \cong \frac{m_0}{c^2\varphi_0}\sqrt{\frac{2\omega+4}{2\omega+3}}, \quad (51)$$

$$q_B \cong \sqrt{\frac{4\pi}{\varphi_0}}\frac{q_0}{c^2}, \quad (52)$$

In the weak field approximation G_{eff} approaches G_0 and thus the parameter φ_0 in the weak field limit becomes

$$\varphi_0 = \frac{1}{G_0}\left(\frac{2\omega+4}{2\omega+3}\right). \quad (53)$$

Substiting Eq.(53) into Eq.(46) we have

$$\frac{\varphi_1}{a} = -\frac{2m_B}{\sqrt{(2\omega+3)(2\omega+4)}}. \quad (54)$$

In summary, the seven parameters $\{a, b, \varphi_0, \varphi_1, A_0, m_0, q_0\}$ as restricted by the weak field approximation are the following

$$A_0 = 1, \qquad a^2 = \varphi_0^2, \quad (55)$$

$$\frac{4ab}{\varphi_1^2} = \frac{(2\omega+3)(2\omega+4)}{4}\left(1-\frac{q_B^2}{m_B^2}\right), \quad (56)$$

$$\frac{b}{a} = \frac{m_B^2 - q_B^2}{4}, \quad (57)$$

$$\frac{\varphi_1}{a} = -\frac{2m_0}{\sqrt{(2\omega+3)(2\omega+4)}}, \quad (58)$$

$$\varphi_0 = \frac{1}{G_0}\left(\frac{2\omega+4}{2\omega+3}\right), \quad (59)$$

$$m_B \cong \frac{m_0}{c^2\varphi_0}\sqrt{\frac{2\omega+4}{2\omega+3}}, \quad (60)$$

$$(61)$$

$$q_B \cong \sqrt{\frac{4\pi}{\varphi_0}}\frac{q_0}{c^2}. \quad (62)$$

Thus there are remain only four parameters $\{\omega, \varphi_0, m_0, q_0\}$. Here m_0 and q_0 are the Newtonian mass and charge measured in (kg) and (e.s.u) respectively.

IV. CLASSIFICATION OF CBD SPACETIMES

The form of the CBD spacetimes are determined by (1) the exponents $\frac{2\sqrt{ab}}{\varphi_2}$ and $\frac{1}{2}\left(1 \pm 2\sqrt{\kappa}\right)$ that arise in the product of the scalar field $\varphi(r)$ and the function $N^2(r)$, viz. $\varphi(r) N^2(r)$ given by Eqs. (10) and (18), and (2) the auxillary constants $p_\pm^2$ that are constrained by the requirement $Q^2\varphi_1 A_0 + \kappa > 0$ given by Eq.(17). This is because even though all the integration constants and the coupling constants are real, ab and κ can take negrative values, and hence$\sqrt{ab}$ and $\sqrt{\kappa}$ are purely imaginary. Under such circumstances, it is necessary to examine their effects on the formal CBD solution. From the complex exponents $\frac{1}{2}\left(1 \pm 2\sqrt{\kappa}\right)$, we define the complex parameter

$$\begin{aligned}\frac{1}{C} \;:\; &= -\frac{1}{2}\left(1+2\sqrt{\kappa}\right)\\ &= -\frac{1}{2}\left(1+2\sqrt{\frac{4ab}{\varphi_2^2}-\frac{2\omega+3}{4}}\right),\end{aligned} \quad (63)$$

which we call the Brans-Dicke parameter (see [3], [46] and for further discussion see Appendix B). Eqs.(161) and (161) of Appendix B gives

$$\begin{aligned}\frac{1}{C}+1 &= \frac{1}{2}\left(1-2\sqrt{\kappa}\right)\\ &= \frac{1}{2}\left(1-2\sqrt{\frac{4ab}{\varphi_2^2}-\frac{2\omega+3}{4}}\right).\end{aligned} \quad (64)$$

The product of Eqs. (161) and (64) gives

$$\frac{1}{C}\left(\frac{1}{C}+1\right) = \frac{4ab}{\varphi_2^2} - \frac{\omega+2}{2}. \quad (65)$$

Re-arranging terms in Eq. (65), we obtain identities for $\frac{4ab}{\varphi_2^2}$ and κ as quadratic expressions of the Brans-Dicke parameter $\frac{1}{C}$:

$$\frac{4ab}{\varphi_2^2} \equiv \frac{1}{C^2}+\frac{1}{C}+\frac{\omega+2}{2} \quad (66)$$

$$\kappa \equiv \left(\frac{1}{C}+\frac{1}{2}\right)^2. \quad (67)$$

The discriminant of the quadratics expression (66) is

$$\Delta_1 = -2\omega - 3. \quad (68)$$

Specifically, the classification and the representations of the formal CBD solution, Eqs. (10) to (17), are based upon three conditions:

Condition 1: the sign of the product of integration constants $\frac{4ab}{\varphi_2^2} \equiv \frac{1}{C^2}+\frac{1}{C}+\frac{\omega+2}{2} \gtreqless 0$ (note that the exponent $\frac{\varphi_2}{2\sqrt{ab}} = \left(\frac{4ab}{\varphi_2^2}\right)^{-\frac{1}{2}}$), The range of the coupling constant is dependent on the sign of $\frac{4ab}{\varphi_2^2}$ and is the following

(1) When $\frac{4ab}{\varphi_2^2} > 0$; the graph of the quadratic inequality does not cut the horizontal axis and hence $\Delta_1 < 0$ implying that $\omega > -\frac{3}{2}$,

(2) When $\frac{4ab}{\varphi_2^2} < 0$; the graph of the quadratic inequality cuts the horizntal axis twice and hence $\Delta_1 > 0$ implying that $\omega < -\frac{3}{2}$,

(3) When $\frac{4ab}{\varphi_2^2} = 0$; the graph of the quadratic inequality has either a pair of complex conjugate roots, exactly one real root, or two real roots and hence $\Delta_1 \gtreqless 0$ implying that $\omega \in \mathbb{R}$;

Condition 2: the sign of the auxillary parameter $\kappa := \frac{4ab}{\varphi_2^2} - \frac{2\omega+3}{4} = \left(\frac{1}{C} + \frac{1}{2}\right)^2 \gtreqless 0$

Condition 3: $Q^2\varphi_1 A_0 + \kappa > 0$ (equivalently $\kappa := \frac{4ab}{\varphi_2^2} - \frac{2\omega+3}{4} = \left(\frac{1}{C} + \frac{1}{2}\right)^2 > -Q^2\varphi_1 A_0$, where $\varphi_1 A_0 > 0$).

The nine Classes of solutions are given in Table (I).

V. REPRESENTATION OF THE CBD SPACETIMES

The CBD solutions can be classified according to the signs of the parameters ab and κ.

The roman numerals I,II and III/IV are representative of the three class of solutions that can be catergorized by the sign of ab and are respectively when ab is $> 0, < 0$ and $= 0$. The Class III/IV solution is made up of two subclasses, Class III where $a = 0$, $b \neq 0$ and Class IV where $a \neq b = o$. It is easy to show that Class III is equivalent to the Class IV solutions under an inverse transformation and that the two solutions are not independent. Thus without loss of generality we are able to classify them together as Class III/IV.

The superscript to the roman numerals are representative of the sign of κ and are $+, -, 0$ for when κ is $> 0, < 0$ and $= 0$ respectively. There are thus a total of nine classes of solutions for the CBD spacetime, viz. CBDI$^+$, CBDI$^-$, CBDI^0, CBDII$^+$, CBDII$^-$, CBDII^0, CBDIII/IV$^+$, CBDIII/IV$^-$, CBDIII/IV^0.

A. Degeneracy of the CBD solutions

The Degeneracy of the CBD solutions are given quickly below:

- CBDI$^+$: When $q_B = 0$ and $\varphi_1 \neq 0$, the BD Class I solution is recovered. When $q_B \neq 0$ and $\varphi_1 = 0$, the Reissner-Nodrström solution is recovered. When $q_B = 0$ and $\varphi_1 = 0$, the Schwarzschild solution is recovered.
- CBDI$^-$ and CBDI^0: From Condition 3 and Eq.(26), $q_B \neq 0$ and $\varphi_1 \neq 0$ respectively and thus there are no degenerate solutions.
- CBDII$^+$: When $q_B = 0$, the BD Class II solution is recovered. From Eq.(26) $\varphi_1 \neq 0$ and thus there is no further degeneracy.
- CBDII$^-$: When $\varphi_1 = 0$, from Eq.(19) a Reissner-Nordström-type spacetime is recovered where $m_B < q_B$. From Condition 3, $q_B \neq 0$ and thus there is no further degeneracy.
- CBDII^0: From Condition 3 and Eq.(26), $q_B \neq 0$ and $\varphi_1 \neq 0$ respectively and thus there are no degenerate solutions.
- CBDIII/IV$^+$: When $q_B = 0$, the BD Class III/IV solution is recovered. When $q_B \neq 0$ but $\varphi_1 = 0$, a massless Reissner-Nodrström solution is recovered where $m_B = 0$. When $q_B = 0$ and scalar field $\varphi_1 = 0$, the solution reduces to Minkowski space.
- CBDIII/IV$^-$ and CBDIII/IV^0: When $\varphi_1 = 0$, an extreme Reissner-Nodrström solution is recovered where $m_B = q_B$. From Condition 3, $q_B \neq 0$ and thus there is no further degeneracy.

B. Perihelion Shift of Mercury

By implementing the weak field approximation we are able to determine the parameterized post-Newtonian (PPN) parameters β and γ for the CBD metric. By expanding Eq.(12) to the second order and Eq. (14) to the first order we obtain the following

$$\beta = \frac{(\omega+2)^2 - \frac{1}{2}Q^2\varphi_0 A_0}{(\omega+2)^2}, \quad (69)$$
$$\gamma = \frac{\omega+1}{\omega+2},$$

respectively. The relativistic rate of shift in the perhelion of a planetary orbit is thus

$$\frac{(\omega+2)^2(3\omega+4) + \frac{2\omega+3}{4}\frac{Q^2 A_0}{G_{eff}}}{3(\omega+2)^3} \times \text{value of general relativity} \quad (70)$$

The rate of perihelion shift ϖ in seconds of arc per century is

$$\varpi = 42.6\text{''}\left(\frac{1}{3}(2+2\gamma-\beta) + 3\times 10^{-4}\frac{J_2}{10^{-7}}\right), \quad (71)$$

where the solar quadrupole moment J_2 is estimated to be of the order $\sim 10^{-7}$. If we accept the value for γ a priori from the Cassini experiments on the time delay of light as $\gamma - 1 = 2.3 \times 10^{-5}$, then one is able to determine

	CBDI			CBDII			CBDIII/CBDIV		
ab	>0			<0			$=0$ (CBDIII: $a=0$, $b>0$; CBDIV: $b=0$, $a>0$)		
ω	$>-\frac{3}{2}$			$<-\frac{3}{2}$			$>-\frac{3}{2}$	$<-\frac{3}{2}$	$=-\frac{3}{2}$
Eq.(19)	$m>q$			$m<q$			$m=q$		
$\varphi(r)$	$\varphi_0 \left\lvert \frac{1-\sqrt{\frac{b}{a}}\frac{1}{r}}{1+\sqrt{\frac{b}{a}}\frac{1}{r}} \right\rvert^{\frac{\varphi_1}{2\sqrt{ab}}}$			$\varphi_0 \exp\left(\frac{\varphi_1}{2i\sqrt{-ab}} \ln\left(\frac{1+\frac{i}{r}\sqrt{\frac{-b}{a}}}{1-\frac{i}{r}\sqrt{\frac{-b}{a}}}\right)\right)$ $= \varphi_0 \exp\left(\frac{\varphi_1}{\sqrt{-ab}} \arctan\sqrt{-\frac{b}{a}\frac{1}{r}}\right)$			$\varphi_0 \exp\left(\frac{\varphi_1 r}{-b}\right)$ (III) (35) maps (III) 1-1 onto (IV) $\varphi_0 \exp\left(-\frac{\varphi_1}{ar}\right)$ (IV) $r\in(0,\infty)$		
	CBDI$^+$	CBDI$^-$	CBDI^0	CBDII$^+$	CBDII$^-$	CBDII^0	CBDIII$^+$/CBDIV$^+$	CBDIII$^-$/CBDIV$^-$	CBDIII^0/CBDIV^0
κ	>0	<0	$=0$	>0	<0	$=0$	>0	<0	$=0$
φ_1	$\in\mathbb{R}$	$\neq 0$	$\neq 0$	$\neq 0$	$\in\mathbb{R}$	$\neq 0$	$\in\mathbb{R}$	$\in\mathbb{R}$	$\in\mathbb{R}$
A_0	$\in\mathbb{R}^+$	$>\frac{4(-\kappa)}{Q^2\varphi_0}$	$A_0 Q^2 \varphi_0 > 0$	$\in\mathbb{R}^+$	$>\frac{4(-\kappa)}{Q^2\varphi_0}$	$A_0 Q^2 \varphi_0 > 0$	$\in\mathbb{R}^+$	$>\frac{4(-\kappa)}{Q^2\varphi_0}$	$A_0 Q^2 \varphi_0 > 0$
Q	$\in\mathbb{R}$	$\neq 0$	$\neq 0$	$\in\mathbb{R}$	$\neq 0$	$\neq 0$	$\in\mathbb{R}$	$\neq 0$	$\neq 0$
$A(r)$	$\frac{A_0}{\left\lvert \frac{1-\sqrt{\frac{b}{a}}\frac{1}{r}}{1+\sqrt{\frac{b}{a}}\frac{1}{r}} \right\rvert^{\frac{\varphi_1}{2\sqrt{ab}}} \left(NI^\delta(r)\right)^2}$ $\delta\in\{+,-,0\}$			$\frac{A_0}{\exp\left(\frac{\varphi_1}{\sqrt{-ab}}\left(\arctan\sqrt{\frac{-b}{a}\frac{1}{r}}\right)\right)\left(NII^\delta(r)\right)^2}$ $\delta\in\{+,-,0\}$			$\frac{A_0}{\exp\left(\frac{\varphi_1 r}{-b}\right)\left(NIII^\delta(r)\right)^2}$, $\delta\in\{+,-,0\}$, $r\in(0,\infty)$ (III) $\updownarrow$ (35) maps (III) 1-1 onto (IV) $\frac{A_0}{\exp\left(-\frac{\varphi_1}{ar}\right)\left(NIV^\delta(r)\right)^2}$, $\delta\in\{+,-,0\}$, $r\in(0,\infty)$ (IV)		
$B(r)$	$\frac{a^2\left(1-\frac{b}{a}\frac{1}{r^2}\right)^2\left(NI^\delta(r)\right)^2}{\varphi_0^2 A_0 \left\lvert \frac{1-\sqrt{\frac{b}{a}}\frac{1}{r}}{1+\sqrt{\frac{b}{a}}\frac{1}{r}} \right\rvert^{\frac{\varphi_1}{2\sqrt{ab}}}}$ $\delta\in\{+,-,0\}$			$\frac{a^2\left(1+\frac{b}{a}\frac{1}{r^2}\right)^2\left(NII^\delta(r)\right)^2}{\varphi_0^2 A_0 \exp\left(\frac{\varphi_1}{\sqrt{-ab}}\left(\arctan\sqrt{\frac{-b}{a}\frac{1}{r}}\right)\right)}$ $\delta\in\{+,-,0\}$			$\frac{b^2\left(NIII^\delta(r)\right)^2}{\varphi_0^2 A_0 r^4 \exp\left(\frac{\varphi_1 r}{-b}\right)}$, $\delta\in\{+,-,0\}$, $r\in(0,\infty)$ (III) $\updownarrow$ (35) maps (III) 1-1 onto (IV) $\frac{a^2\left(NIV^\delta(r)\right)^2}{\varphi_0^2 A_0 \exp\left(-\frac{\varphi_1}{ar}\right)}$, $\delta\in\{+,-,0\}$, $r\in(0,\infty)$ (IV)		
$V'(r)$	$\frac{c^2}{\sqrt{4\pi}}\frac{Q\varphi_1 A_0\varphi_0}{a\left(r^2-\frac{b}{a}\right)\left(NI^\delta(r)\right)^2}$ $\delta\in\{+,-,0\}$			$\frac{c^2}{\sqrt{4\pi}}\frac{Q\varphi_1 A_0\varphi_0}{a\left(r^2+\frac{b}{a}\right)\left(NII^\delta(r)\right)^2}$ $\delta\in\{+,-,0\}$			$\frac{c^2}{\sqrt{4\pi}}\frac{Q\varphi_1 A_0\varphi_0}{(-b)\left(NIII^\delta(r)\right)^2}$, $\delta\in\{+,-,0\}$, $r\in(0,\infty)$ (III) $\updownarrow$ (35) maps (III) 1-1 onto (IV)) $\frac{c^2}{\sqrt{4\pi}}\frac{Q\varphi_1 A_0\varphi_0}{ar^2\left(NIV^\delta(r)\right)^2}$, $\delta\in\{+,-,0\}$, $r\in(0,\infty)$ (IV)		

TABLE I: Static Spherically Symmetric Charged Brans-Dicke (CBD) Solutions in Isotropic Coordinates: $ds^2 = -c^2 A(r)\,dt^2 + B(r)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$. The representations of the formal solutions depends on the form of the scalar field $\varphi(r)$ and the function $N(r)$ with the function $N(r)$ given in Table II. The three branches of solutions are dependent on the sign of the parameter ab, with $ab>0$ corresponding to the CBD Class I, $ab<0$ corresponding to the CBD Class II and $ab=0$ corresponding to the CBD Class III/IV solutions. The CBD Class III solution is when $a=0$ and its inverse transformation is the CBD Class IV solution when $b=0$. The three branches of solutions are further divided into three categories each depending on the sign of the auxillary parameter κ where $\kappa := \frac{4ba}{\varphi_1^2} - \frac{2\omega+3}{4}$. The CBD Class I solutions are provided for the region $\left(\sqrt{\left\lvert\frac{b}{a}\right\rvert} < r < \infty\right)$, while the other 2 classes of solutions are for the region $(0<r<\infty)$. The CBD Class I solutions for the region $\left(0 < r < \sqrt{\left\lvert\frac{b}{a}\right\rvert}\right)$ can be quickly recovered with the use of Eq.(35). The metric coefficients $A(r)$ and $B(r)$, and the electrostatic field $V'(r)$ are given in terms of $N(r)$ for the three branches of solutions and are further divided into three categories each when one substitutes the appropriate value for $N(r)$ for when $\kappa \gtreqless 0$ from Table II. The branches of solutions for which the parameters φ_1 and Q are not allowed to vanish are given explicitly and thus it is clear why only 3 branches of the total 9 solutions degenerate to the 3 known BD solutions (refer to Eq. (17)).

	CBDI	CBDII	CBDIII/CBDIV
$N^+(r)$	$NI^+(r) = p_+^2 \left\lvert\frac{1-\sqrt{\frac{b}{a}\frac{1}{r}}}{1+\sqrt{\frac{b}{a}\frac{1}{r}}}\right\rvert^{-\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}} - p_-^2 \left\lvert\frac{1-\sqrt{\frac{b}{a}\frac{1}{r}}}{1+\sqrt{\frac{b}{a}\frac{1}{r}}}\right\rvert^{\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}}$ where $p_\pm^2 = \frac{1}{2}\left(\sqrt{1+\frac{Q^2\varphi_0 A_0}{\kappa}} \pm 1\right)$ $p_+^2 + p_-^2 = 1$	$NII^+(r) = p_+^2\left(\frac{1-\frac{i}{r}\sqrt{\frac{-b}{a}}}{1+\frac{i}{r}\sqrt{\frac{-b}{a}}}\right)^{-i\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}} - p_-^2\left(\frac{1-\frac{i}{r}\sqrt{\frac{-b}{a}}}{1+\frac{i}{r}\sqrt{\frac{-b}{a}}}\right)^{i\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}}$ $= p_+^2 \exp\left(-\frac{\varphi_1\sqrt{\kappa}}{\sqrt{-ab}}\tan^{-1}\sqrt{\frac{-b}{a}\frac{1}{r}}\right)$ $-p_-^2 \exp\left(\frac{\varphi_1\sqrt{\kappa}}{\sqrt{-ab}}\left(\tan^{-1}\sqrt{\frac{-b}{a}\frac{1}{r}}\right)\right)$ where $p_\pm^2 = \frac{1}{2}\left(\sqrt{1+\frac{Q^2\varphi_0 A_0}{\kappa}} \pm 1\right)$ $p_+^2 - p_-^2 = 1$	$NIII^+(r) = p_+^2 e^{\sqrt{-\frac{2\omega+3}{4}\frac{\varphi_1}{-b}}r} - p_-^2 e^{-\sqrt{-\frac{2\omega+3}{4}\frac{\varphi_1}{-b}}r}$, (III) (35) maps (III) 1-1 onto (IV) $NIV^+(r) = p_+^2 e^{-\sqrt{-\frac{2\omega+3}{4}\frac{\varphi_1}{ar}}} - p_-^2 e^{\sqrt{-\frac{2\omega+3}{4}\frac{\varphi_1}{ar}}}$ (IV) where $p_\pm^2 = \frac{1}{2}\left(\sqrt{1+\frac{4Q^2\varphi_0 A_0}{-(2\omega+3)}} \pm 1\right)$ $p_+^2 + p_-^2 = 1$
$N^-(r)$	$NI^-(r) = p_+^2 \left\lvert\frac{1-\sqrt{\frac{b}{a}\frac{1}{r}}}{1+\sqrt{\frac{b}{a}\frac{1}{r}}}\right\rvert^{-i\frac{\varphi_1\sqrt{-\kappa}}{2\sqrt{ab}}} - \overline{p_+^2} \left\lvert\frac{1-\sqrt{\frac{b}{a}\frac{1}{r}}}{1+\sqrt{\frac{b}{a}\frac{1}{r}}}\right\rvert^{i\frac{\varphi_1\sqrt{-\kappa}}{2\sqrt{ab}}}$ $= \cos\left(\frac{\varphi_1\sqrt{-\kappa}}{2\sqrt{ab}}\ln\left\lvert\frac{1-\sqrt{\frac{b}{a}\frac{1}{r}}}{1+\sqrt{\frac{b}{a}\frac{1}{r}}}\right\rvert\right)$ $+\sqrt{\frac{Q^2\varphi_0 A_0}{(-\kappa)} - 1}\sin\left(\frac{\varphi_1\sqrt{-\kappa}}{2\sqrt{ab}}\ln\left\lvert\frac{1-\sqrt{\frac{b}{a}\frac{1}{r}}}{1+\sqrt{\frac{b}{a}\frac{1}{r}}}\right\rvert\right)$ where $p_\pm^2 = \frac{1}{2}\left(i\sqrt{\frac{Q^2\varphi_0 A_0}{(-\kappa)} - 1} \mp 1\right)$; $p_-^2 = -\overline{p_+^2}, \quad p_+^2 - p_-^2 = 1$	$NII^-(r) = p_+^2\left(\frac{1-\frac{i}{r}\sqrt{\frac{-b}{a}}}{1+\frac{i}{r}\sqrt{\frac{-b}{a}}}\right)^{-\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}} - \overline{p_+^2}\left(\frac{1-\frac{i}{r}\sqrt{\frac{-b}{a}}}{1+\frac{i}{r}\sqrt{\frac{-b}{a}}}\right)^{\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}}$ $= \cos\left(\frac{\varphi_1\sqrt{-\kappa}}{\sqrt{-ab}}\arctan\sqrt{\frac{-b}{a}\frac{1}{r}}\right) +$ $\sqrt{\frac{Q^2\varphi_0 A_0}{(-\kappa)} - 1}\sin\left(\frac{\varphi_1\sqrt{-\kappa}}{\sqrt{-ab}}\arctan\sqrt{\frac{-b}{a}\frac{1}{r}}\right)$ where $p_\pm^2 = \frac{1}{2}\left(i\sqrt{\frac{Q^2\varphi_0 A_0}{(-\kappa)} - 1} \mp 1\right)$; $p_-^2 = -\overline{p_+^2}, \quad p_+^2 - p_-^2 = 1$	$NIII^-(r) = p_+^2 e^{i\sqrt{\frac{2\omega+3}{4}\frac{\varphi_1}{-b}}r} - \overline{p_+^2} e^{-i\sqrt{\frac{2\omega+3}{4}\frac{\varphi_1}{-b}}r}$ $= \cos\sqrt{\frac{2\omega+3}{4}\frac{\varphi_1}{-b}}r +$ $\sqrt{\frac{4Q^2\varphi_0 A_0}{(2\omega+3)} - 1}\sin\sqrt{\frac{2\omega+3}{4}\frac{\varphi_1}{-b}}r$ (III) (35) maps (III) 1-1 onto (IV) $NIV^-(r) = p_+^2 e^{-i\sqrt{\frac{2\omega+3}{4}\frac{\varphi_1}{ar}}} - \overline{p_+^2} e^{i\sqrt{\frac{2\omega+3}{4}\frac{\varphi_1}{ar}}}$ $= \cos\sqrt{\frac{2\omega+3}{4}\frac{\varphi_1}{ar}} + \sqrt{\frac{4Q^2\varphi_0 A_0}{(2\omega+3)} - 1}\sin\sqrt{\frac{2\omega+3}{4}\frac{\varphi_1}{ar}}$ (IV) where $\frac{1}{2}\left(i\sqrt{\frac{4Q^2\varphi_0 A_0}{(2\omega+3)} - 1} \mp 1\right)$ $p_-^2 = \overline{p_+^2}, \quad p_+^2 - p_-^2 = 1$
$N^0(r)$	$NI^0(r) = 1 - \frac{Q\varphi_1\sqrt{\varphi_0 A_0}}{2\sqrt{ab}}\ln\left\lvert\frac{1-\sqrt{\frac{b}{a}\frac{1}{r}}}{1+\sqrt{\frac{b}{a}\frac{1}{r}}}\right\rvert$	$NII^0(r) = 1 - \frac{Q\varphi_1\sqrt{\varphi_0 A_0}}{\sqrt{-ab}}\arctan\sqrt{\frac{-b}{a}\frac{1}{r}}$	$NIII^0(r) = 1 + \frac{Q\varphi_1\sqrt{\varphi_0 A_0}}{-b}r$ (III) $\updownarrow$ (35) maps (III) 1-1 onto (ii) $NIV^0(r) = 1 - \frac{Q\varphi_1\sqrt{\varphi_0 A_0}}{ar}$ (IV)

TABLE II: The values for the parameter $N(r)$ for the nine classes of solutions. As before the roman numerals denote the sign of the parameter ab such that I, II and III/IV represent respectively when ab is $> 0, < 0$ and $= 0$. The superscript to the roman numerals denote the sign of the parameter κ such that $+, -, 0$ represent respectively when κ is $> 0, < 0$ and $= 0$.

a bound on β . The bound on β is thus estimated to be $\beta-1=(-4.1\pm 7.8)\times 10^{-5}$ (see [14]).

Any realistic approximation for the value of the parameter Q, of Eq.(26), such that the overall charge of the universe remains neutral would require Q to be very small. As such the Q^2 term of Eq.(70) is negligible and the relativistic rate of shift in the perihelion as calculated from the CBD background reduces to the known rate for the BD background.

VI. TRANSFORMATION FROM ISOTROPIC TO SCHWARZSCHILD-TYPE COORDINATES

In [25] the authors expressed the Brans-Dicke metric in Schwarzschild-type coordinates as opposed to the usual isotropic form given in [46]. The Schwarzschild form lends itself readily to physical interpretation and degenerates quickly to the Schwarzschild and Reissner-Nordström solutions. The transformation equation that will convert all nine CBD solutions from isotropic into Schwarzschild-type coordinates (denoted by a subscripted S) is found to be

$$r_s=\frac{ar^2-b}{r}\left[p_+^2\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r+1}\right)^{-1}-p_-^2\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r+1}\right)\right] \tag{72}$$

after some algebra we find

$$\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r+1}\right)^2=\frac{\sqrt{\frac{a}{b}}r_s-4ap_+^2}{\sqrt{\frac{a}{b}}r_s-4ap_-^2}. \tag{73}$$

In the Schwarzschild-type coordinates the expressions for $p_\pm^2$ remain unchanged, however $N(r)$ and $\varphi(r)$ are now expressed in terms of the Schwarzschild radial coordinate r_s as $N(r_s)$ and $\varphi(r_s)$ and are given in Table III.

The CBD metric in Schwarzschild-type coordinates is thus

$$\begin{aligned} ds^2 &= -c^2A\left(r_s\right)dt^2+B\left(r_s\right)dr_s^2 && (74)\\ &+B\left(r_s\right)\frac{b}{a}\left(\sqrt{\frac{a}{b}}r_s-4ap_+^2\right)\left(\sqrt{\frac{a}{b}}r_s-4ap_-^2\right)\left[\left(d\theta^2+\sin^2\theta d\phi^2\right)\right], && (75)\end{aligned}$$

where the scalar field and metric coefficients are

$$\begin{aligned}\varphi\left(r_s\right) &= \varphi_0\exp\left(\frac{\varphi_1}{4\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r_s-4ap_+^2}{\sqrt{\frac{a}{b}}r_s-4ap_-^2}\right)\right), && (76)\\ a,b &\in \mathbb{R},\quad \text{if}\quad ab=0\Rightarrow\begin{cases}a=0, & b\neq 0\\ b=0, & a\neq 0\end{cases}\\ \varphi_0 &> 0,\quad \varphi_1\in\mathbb{R}; && (77)\end{aligned}$$

$$\begin{aligned}A\left(r_s\right) &= \frac{A_0\varphi_0}{\varphi\left(r_s\right)N^2\left(r_s\right)},\qquad A_0>0;\\ &= \frac{A_0}{\exp\left(\frac{\varphi_1}{4\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r_s-4ap_+^2}{\sqrt{\frac{a}{b}}r_s-4ap_-^2}\right)\right)N^2\left(r_s\right)}, && (78)\end{aligned}$$

$$\begin{aligned}B\left(r_s\right) &= \frac{N^2\left(r_s\right)}{A_0\varphi_0\varphi\left(r_s\right)}\\ &= \frac{N^2\left(r_s\right)}{\varphi_0^2A_0\exp\left(\frac{\varphi_1}{4\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r_s-4ap_+^2}{\sqrt{\frac{a}{b}}r_s-4ap_-^2}\right)\right)}. && (79)\end{aligned}$$

The auxillary parameter $N(r_s)$ is

$$\begin{aligned}N\left(r\right) &:= p_+^2\exp\left(-\frac{\varphi_1\sqrt{\kappa}}{4\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r_s-4ap_+^2}{\sqrt{\frac{a}{b}}r_s-4ap_-^2}\right)\right)\\ &-p_-^2\exp\left(\frac{\sqrt{\kappa}}{4\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r_s-4ap_+^2}{\sqrt{\frac{a}{b}}r_s-4ap_-^2}\right)\right), && (80)\end{aligned}$$

while the parameters $p_\pm^2$ and κ remain unchanged.

VII. INVARIANTS

It was demonstrated in [25] that the Brans Class I solution of [3] can exhibit black hole behaviour for a certain limited range of the solution parameters. However in [35] the authors implemented an interior to exterior matching criteria (a variant of the weak field approximation) which restricted the coupling constant ω to a very narrow, negative range, $-(2+1/\sqrt{3})<\omega<-2$, which was incompatible with observational data. As such they ruled out the possibility of Brans Dicke black holes.

Earlier, [28] had shown independently of the work by [25], that the BD spacetimes give rise to wormholes when $\omega<-2$, or naked singularities when $\omega>-3/2$.

Recently [49] revisited the solutions of [25] and was able to confirm the findings of [28]. By studying the

	CBDI	CBDII	CBDIII/CBDIV
ab	>0	<0	$=0 \left(\begin{array}{l} \text{CBDIII:} \quad a=0, b>0 \\ \text{CBDIV:} \quad b=0, a>0 \end{array}\right)$
$\varphi(r)$	$\varphi_0 \left\lvert \frac{4ap_+^2 - \sqrt{\frac{a}{b}} r_s}{4ap_-^2 + \sqrt{\frac{a}{b}} r_s} \right\rvert^{\frac{\varphi_1}{4\sqrt{ab}}}$	$\varphi_0 \exp\left(\frac{\varphi_1}{4i\sqrt{-ab}} \ln \left(\frac{i\sqrt{\frac{a}{-b}} r_s - 4ap_+^2}{i\sqrt{\frac{a}{-b}} r_s - 4ap_-^2} \right) \right)$	$\varphi_0 \exp\left(-\varphi_1 r_s\right),$ (III) (35) maps (III) 1-1 onto (IV) $\varphi_0 \exp\left(-\frac{\varphi_1}{r_s}\right)$ (IV)
$N^+(r)$	$p_+^2 \left\lvert \frac{4ap_+^2 - \sqrt{\frac{a}{b}} r_s}{4ap_-^2 - \sqrt{\frac{a}{b}} r_s} \right\rvert^{-\frac{\varphi_1\sqrt{\kappa}}{4\sqrt{ab}}} - p_- \left\lvert \frac{4ap_+^2 - \sqrt{\frac{a}{b}} r_s}{4ap_-^2 - \sqrt{\frac{a}{b}} r_s} \right\rvert^{\frac{\varphi_1\sqrt{\kappa}}{4\sqrt{ab}}}$ where $p_\pm^2 = \frac{1}{2}\left(\sqrt{1 + \frac{Q^2\varphi_0 A_0}{\kappa}} \pm 1\right),$ $p_+^2 - p_-^2 = 1$	$p_+^2 \left(\frac{i\sqrt{\frac{a}{-b}} r_s - 4ap_+^2}{i\sqrt{\frac{a}{-b}} r_s - 4ap_-^2} \right)^{i\frac{\varphi_1\sqrt{\kappa}}{4\sqrt{-ab}}} + p_- \left(\frac{i\sqrt{\frac{a}{-b}} r_s - 4ap_+^2}{i\sqrt{\frac{a}{-b}} r_s - 4ap_-^2} \right)^{-i\frac{\varphi_1\sqrt{\kappa}}{4\sqrt{-ab}}}$ where $p_\pm^2 = \frac{1}{2}\left(\sqrt{1 + \frac{Q^2\varphi_0 A_0}{\kappa}} \pm 1\right)$ $p_+^2 - p_-^2 = 1$	$p_+^2 e^{\sqrt{-\frac{2\omega+3}{4}\frac{\varphi_1}{r_s}}} - p_-^2 e^{-\sqrt{-\frac{2\omega+3}{4}\frac{\varphi_1}{r_s}}}$ where $p_\pm^2 = \frac{1}{2}\left(\sqrt{1 - \frac{4Q^2\varphi_0 A_0}{2\omega+3}} \pm 1\right)$ $p_+^2 - p_-^2 = 1$
$N^-(r)$	$p_+^2 \left\lvert \frac{4ap_+^2 - \sqrt{\frac{a}{b}} r_s}{4ap_-^2 - \sqrt{\frac{a}{b}} r_s} \right\rvert^{-i\frac{\varphi_1\sqrt{-\kappa}}{4\sqrt{ab}}} - \overline{p_+^2} \left\lvert \frac{4ap_+^2 - \sqrt{\frac{a}{b}} r_s}{4ap_-^2 - \sqrt{\frac{a}{b}} r_s} \right\rvert^{i\frac{\varphi_1\sqrt{-\kappa}}{4\sqrt{ab}}}$ $= \cos\left(\frac{\varphi_1\sqrt{-\kappa}}{4\sqrt{ab}} \ln \left\lvert \frac{4ap_+^2 - \sqrt{\frac{a}{b}} r_s}{4ap_-^2 - \sqrt{\frac{a}{b}} r_s} \right\rvert \right)$ $+\sqrt{\frac{Q^2\varphi_0 A_0}{(-\kappa)} - 1} \sin\left(\frac{\varphi_1\sqrt{-\kappa}}{4\sqrt{ab}} \ln \left\lvert \frac{4ap_+^2 - \sqrt{\frac{a}{b}} r_s}{4ap_-^2 - \sqrt{\frac{a}{b}} r_s} \right\rvert \right)$ where $p_\pm^2 = \frac{1}{2}\left(i\sqrt{\frac{Q^2\varphi_0 A_0}{(-\kappa)} - 1} \mp 1\right)$ $p_-^2 = -\overline{p_+^2}, \quad p_+^2 - p_-^2 = 1$	$p_+^2 \left(\frac{i\sqrt{\frac{a}{b}} r_s - 4ap_+^2}{i\sqrt{\frac{a}{b}} r_s - 4ap_-^2} \right)^{-\frac{\varphi_1\sqrt{-\kappa}}{4\sqrt{-ab}}} - \overline{p_+^2} \left(\frac{i\sqrt{\frac{a}{b}} r_s - 4ap_+^2}{i\sqrt{\frac{a}{b}} r_s - 4ap_-^2} \right)^{\frac{\varphi_1\sqrt{-\kappa}}{4\sqrt{-ab}}}$ $= \cos\left(\frac{\varphi_1\sqrt{-\kappa}}{4\sqrt{ab}} \ln \left\lvert \frac{4ap_+^2 - \sqrt{\frac{a}{b}} r_s}{4ap_-^2 - \sqrt{\frac{a}{b}} r_s} \right\rvert \right)$ $+\sqrt{\frac{Q^2\varphi_0 A_0}{(-\kappa)} - 1} \sin\left(\frac{\varphi_1\sqrt{-\kappa}}{4\sqrt{ab}} \ln \left\lvert \frac{4ap_+^2 - \sqrt{\frac{a}{b}} r_s}{4ap_-^2 - \sqrt{\frac{a}{b}} r_s} \right\rvert \right)$ where $p_\pm^2 = \frac{1}{2}\left(i\sqrt{\frac{Q^2\varphi_0 A_0}{(-\kappa)} - 1} \mp 1\right)$ $p_-^2 = -\overline{p_+^2}, \quad p_+^2 - p_-^2 = 1$	$p_+^2 e^{i\sqrt{\frac{2\omega+3}{4}\frac{\varphi_1}{r_s}}} - \overline{p_+^2} e^{-i\sqrt{\frac{2\omega+3}{4}\frac{\varphi_1}{r_s}}}$ $= \cos\sqrt{\frac{2\omega+3}{4}\frac{\varphi_1}{r_s}} +$ $\sqrt{\frac{4Q^2\varphi_0 A_0}{(2\omega+3)} - 1} \sin\sqrt{\frac{2\omega+3}{4}\frac{\varphi_1}{r_s}}$ where $\frac{1}{2}\left(1 \mp i\sqrt{\frac{4Q^2\varphi_0 A_0}{(2\omega+3)} - 1}\right)$ $p_-^2 = -\overline{p_+^2}, \quad p_+^2 - p_-^2 = 1$
$N^0(r)$	$1 - \frac{Q\varphi_1\sqrt{\varphi_0 A_0}}{4\sqrt{ab}} \ln \left\lvert \frac{4ap_+^2 - \sqrt{\frac{a}{b}} r_s}{4ap_- + \sqrt{\frac{a}{b}} r_s} \right\rvert$	$1 - \frac{Q\varphi_1\sqrt{\varphi_0 A_0}}{4i\sqrt{-ab}} \ln \frac{i\sqrt{-\frac{a}{b}} r_s - 4ap_+^2}{i\sqrt{-\frac{a}{b}} r_s + 4ap_-}$	$1 - \frac{Q\varphi_1\sqrt{\varphi_0 A_0}}{r_s}$
$A(r)$	$\frac{A_0}{\varphi_0 \left\lvert \frac{4ap_+^2 - \sqrt{\frac{a}{b}} r_s}{4ap_- + \sqrt{\frac{a}{b}} r_s} \right\rvert^{\frac{\varphi_1}{4\sqrt{ab}}} \left(NI^\delta(r)\right)^2}, \quad \delta \in \{+,-,0\}$	$\frac{A_0}{\varphi_0 \exp\left(\frac{\varphi_1}{4i\sqrt{-ab}} \ln\left(\frac{i\sqrt{\frac{a}{-b}} r_s - 4ap_+^2}{i\sqrt{\frac{a}{-b}} r_s + 4ap_-} \right)\right) \left(NII^\delta(r_s)\right)^2}, \quad \delta \in \{+,-,0\}$	$\frac{A_0}{\exp(-\varphi_1 r_s)\left(NIII^\delta(r_s)\right)^2}, \quad \delta \in \{+,-,0\},$ (III) $\updownarrow$ (35) maps (III) 1-1 onto (IV) $\frac{A_0}{\exp\left(-\frac{\varphi_1}{r_s}\right)\left(NIV^\delta(r_s)\right)^2}, \quad \delta \in \{+,-,0\},$ (IV)
$B(r)$	$\frac{\left(NI^\delta(r_s)\right)^2}{A_0\varphi_0 \left\lvert \frac{4ap_+^2 - \sqrt{\frac{a}{b}} r_s}{4ap_- + \sqrt{\frac{a}{b}} r_s} \right\rvert^{\frac{\varphi_1}{4\sqrt{ab}}}}, \quad \delta \in \{+,-,0\}$	$\frac{\left(NII^\delta(r_s)\right)^2}{A_0\varphi_0 \exp\left(\frac{\varphi_1}{4i\sqrt{-ab}} \ln\left(\frac{i\sqrt{\frac{a}{-b}} r_s - 4ap_+^2}{i\sqrt{\frac{a}{-b}} r_s + 4ap_-} \right)\right)}, \quad \delta \in \{+,-,0\},$	$\frac{\left(NIII^\delta(r_s)\right)^2}{\varphi_0 A_0 \exp(-\varphi_1 r_s)}, \quad \delta \in \{+,-,0\},$ (III) $\updownarrow$ (35) maps (III) 1-1 onto (IV) $\frac{\left(NIV^\delta(r_s)\right)^2}{\varphi_0 A_0 \exp\left(-\frac{\varphi_1}{r_s}\right)}, \quad \delta \in \{+,-,0\},$ (IV)

TABLE III: Static Spherically Symmetric Charged Brans-Dicke (CBD) Solutions in Schwarzschild-type coordinates Eq.(75): The representations of the formal solutions depends on the form of the scalar field $\varphi(r_S)$ and the function $N(r_S)$. The three branches of solutions are dependent on the sign of the parameter ab, with $ab<0$ corresponding to the CBD Class I solutions, $ab<0$ corresponding to the CBD Class II solutions and $ab=0$ corresponding to the CBD Class III/IV solutions. The CBD Class III solution is when $a=0$ and its inverse transformation is the CBD Class IV solution when $b=0$. The three branches of solutions are further divided into three categories each depending on the sign of the auxillary parameter κ where $\kappa>0, \kappa<0, \kappa=0$ is denoted by the superscript of $+,-,0$ respectively to $N\chi(r)$ where χ is the roman numeral I, II or III/IV depending on the class of solution.The CBD solutions are valid over the entire region of $(0<r<\infty)$.

behaviour of the horizon of the nonsingular solutions of [25], they found that the behaviour of the horizon was indicative of a wormhole and not of a black hole.

It is pertinent then to study the behaviour of the electrovac solutions for possible black hole and wormhole nature and we do so by looking at the curvature invariants of the metric in the same way as [25] and [35]. By studying the curvature invariants of a metric one is able to distinguish true spacetime singularities from coordinate system pathologies.

For a spherically symmetric background, such as the CBD background under discussion in this paper, the complete set of invariants include only eight of the usual sixteen invariants (see [50] and [51]). This is due to the fact that the complex scalars vanish for spherically symmetric backgrounds. The remaining eight invariants are defined as the following

$$\begin{aligned} R &:= R^a_a, && (81)\\ R1 &:= \frac{1}{4}S^a_b S^b_a, && (82)\\ R2 &:= -\frac{1}{8}S^a_b S^b_c S^c_a, && (83)\\ R3 &:= \frac{1}{16}S^a_b S^b_c S^c_d S^d_a, && (84)\\ W &:= \frac{1}{8}C_{abcd}C^{abcd}, && (85)\\ W1 &:= \frac{1}{8}S^{ab}S^{cd}C_{acdb}, && (86)\\ W2 &:= \frac{1}{16}S^{bc}S_{ef}C_{abcd}C^{aefd}, && (87)\\ W3 &:= \frac{1}{32}S^{cd}S^{ef}C^{aghb}C_{acdb}C_{gefh}, && (88) \end{aligned}$$

where R is the Ricci scalar, R_{ab} is the Ricci tensor, S_{ab} is the trace-free Ricci tensor and C_{abcd} is the Weyl tensor.

For the sake of brevity we concentrate only on the Ricci and Weyl scalars as these are two the fundamental invariants with the remaining 6 invariants being contractions of these.

The Ricci scalar R for the CBD metric is

$$R = \frac{A_0\varphi_0\varphi(r)r^4}{2(ar^2-b)^4N(r)^2}\left[16ab - (3+4\kappa)\,\varphi_1^2\right] \qquad (89)$$

The Weyl scalar W for the CBD metric is

$$W = \frac{2}{3}\frac{A_0\varphi_0\varphi(r)r^6}{\left(ar^2-b\right)^8 N(r)}\left[\left(3N^*(r)-N(r)\right)r\varphi_1^2\kappa - \left(2ar^2 + \left(\left(ar^2-b\right)^2 - 4b^2\right)N(r)\right)N^*(r)\varphi_1\sqrt{\kappa} + 4abrN^2(r)\right] \quad (90)$$

It is easy to see that in general, as pointed out by [35] for the BD background, the curvature invariants diverge when $r \to \sqrt{\frac{b}{a}}$ and the solution exhibits a naked singularity. However, there are cases when $\kappa \neq 0$ that the invariants do not diverge. These cases are discussed below:

A. CBD I and II

Using Eq.(18) for $\kappa \neq 0$, from the denominator of the Weyl scalar one can see that the Weyl scalar will not diverge when

$$\varphi_1\sqrt{\frac{\kappa}{ab}} \geq 2. \qquad (91)$$

The Ricci scalar will also be non-singular under the more stringent condition of

$$\varphi_1\sqrt{\frac{\kappa}{ab}} \geq 4. \qquad (92)$$

The only time that Eqs.(91) and (92) can be satisfied is when the sign of κ and ab are the same and

$$\frac{\kappa}{ab} > 0. \qquad (93)$$

This corresponds to the CBDI$^+$ and CBDII$^-$ backgrounds only. Furthermore, using Eq.(93) we can rewrite Eqs.(91) and (92) as the following

$$0 < \frac{2}{\sqrt{\frac{\kappa}{ab}}} \leq \varphi_1, \tag{94}$$

and

$$0 < \frac{4}{\sqrt{\frac{\kappa}{ab}}} \leq \varphi_1. \tag{95}$$

Thus we find that the only solution available is when the weak field approximation is violated and φ_1 is allowed to be positive. From Eq.(58), φ_1 can be positive when $\omega < -2$.

We find that this range on ω is only permitted in the CBDII$^-$ spacetime and we find that nonsingular solutions are available in the CBDII$^-$ spacetime when $\omega < -2$. The behaviour of the horizon in the CBD$^-$ spacetime with the coupling parameter restricted to $\omega < -2$ must be studied further to determine whether the spacetime is representative of a black hole or a wormhole. We leave this to a later paper.

B. CBD III/IV

As the CBDIII and CBDIV solutions are the inverse transform of one another we give here the curvature invariants for the CBDIV solution only but point out that the CBDIII invariants can be quickly recovered using Eq.(72). The Ricci and Weyl scalars for the CBDIV background when $\kappa \neq 0$ are respectively

$$R = \frac{A_0\varphi_0\varphi(r)\omega\varphi_1^2}{a^4r^4N(r)^2}, \tag{96}$$

$$W = \frac{2}{3}\frac{\left(\varphi_0 A_0\varphi(r)\right)^2 r^6}{N(r)^8a^4r^2}\left(\frac{\varphi_1^2}{a^2r^4}\left(\frac{2\omega+3}{4}\right)\left(rN^2(r)-3\left(N^*(r)\right)^2\right)-\frac{3\varphi_1}{ar^2}\sqrt{-\frac{2\omega+3}{4}}N^*(r)N(r)\right)^2. \tag{97}$$

For the CBDIV$^+$ and CBDIV$^+$ backgrounds the invariants diverge only at the location of the singularity at $r = 0$. It is clear to see that there exists no other singularity and no horizon in the CBDIII/IV$^+$ and CBDIII/IV$^-$ backgrounds and that $r = 0$ is a naked singularity.

Part II

Perturbing the CBD spacetime with a point charge

In a previous paper [1] we considered the electrostatic field of an electric point charge placed outside the surface of inversion in the CBDI$^+$ or "Brans-Dicke Reisnner-Norström" spacetime. We went on to introduce a method based on the Copson-Hadamard method [52] by which we were able to find a closed-form solution to the Maxwell equation for the electrostatic potential generated by the point charge.

Here we extend our method to the general form of the CBD metric and perturb the CBD spacetime with an electrostatic point charge. As before we consider the electrostatic potential due to a "small" static electric charge $-\epsilon_0$ situated outside the spherical surface of inversion $\sqrt{\frac{b}{a}}$.

We let $V(r,\theta,\phi)$ denote the linearly perturbated electrostatic potential so that the perturbed Faraday tensor F_{ab} takes the form

$$F_{0i} = -F_{i0} = -c\partial_i V(r,\theta,\phi) \tag{98}$$
$$F_{ij} = 0\,;\ i,j = 1,2,3\cdots.$$

The perturbed Maxwell equations $\nabla_{[a}F_{bc]} = 0$ is automatically satisfied by Eq.(98).

As before, the perturbed Maxwell equations due to a single electrostatic charge is

$$\frac{1}{\sqrt{-g}}\partial_b(\sqrt{-g}F^{ab}) = J^0 \tag{99}$$

which implies

$$\nabla^2 V(r,\theta,\phi) - \left(\ln\sqrt{\frac{A(r)}{B(r)}}\right)'\frac{\partial V(r,\theta,\phi)}{\partial r} = cA(r)B(r)J^0 \tag{100}$$

where the current density $J^0 = -\frac{4\pi\epsilon_0}{cr^2}\frac{1}{A(r)B(r)^{\frac{3}{2}}}\delta(r-b_0)\delta(\cos\theta-\cos\theta_0)$. Here $A(r)$ and $B(r)$ are given by Eqs (12) and (15) respectively, $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is the 3-dimensional Euclidean space Laplacian with $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$. Note that $\partial_r = \frac{x}{r}\partial_x + \frac{y}{r}\partial_y + \frac{z}{r}\partial_z$. As in [1], we define

$$\Gamma(r,\theta) = r^2 + b_0^2 - 2b_0 r\cos\theta \tag{101}$$

which is equal to the square of the "radial" distance from the charged particle at $z = b_0$.

Eq.(100) for the CBD spacetime in its general form is

$$\nabla^2 V + 2\left(\frac{N'(r)}{N(r)} + \frac{ar}{ar^2 - b} - \frac{1}{r}\right)\frac{\partial V}{\partial r} = cA(r)B(r)J^0 \tag{102}$$

In the CBD spacetime Eqs.(51) and (50) of [1] give us the requisite form of the substitution that will convert Eq.(102) into a solvable ordinary differential equation, and are

$$V = \frac{\sqrt{A_0\varphi_0}r}{N(r)(ar^2 - b)}F(\gamma), \tag{103}$$

$$\gamma = \frac{ab}{ab_0^2 - b}\frac{r^2 + b_0^2 - 2b_0 r\cos\theta}{ar^2 - b}. \tag{104}$$

Substituting Eqs.(103) and (104) into (102) gives us a second order linear differential equation on $F(\gamma)$.

$$\gamma(\gamma+1)F''(\gamma) + \frac{3}{2}(2\gamma+1)F'(\gamma) + \left(1 - \frac{\varphi_1^2\kappa}{4ab}\right)F(\gamma) = 0.$$

This is the general form of Eq.(55) of [1] and can be solved using the same method given therein, by transforming the independent variable γ as

$$\gamma = \sinh^2\frac{\zeta}{2}, \tag{105}$$

such that the dependent variable is now $F(\gamma) = \Phi(\zeta)$.

Using Eq. (105), one can rewrite Eq. (105) in terms of the new variables as follows

$$\Phi''(\zeta) + 2\coth\zeta : \Phi'(\zeta) + \left(1 - \frac{\varphi_1^2\kappa}{4ab}\right)\Phi(\zeta) = 0 \tag{106}$$

which has the closed-form solution (see [53])

$$\Phi(\zeta) = \frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}\frac{\hat{W}_1 e^{\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}\zeta} - \hat{W}_2 e^{-\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}\zeta}}{\sinh\zeta} \tag{107}$$

where $\hat{W}_1$ and $\hat{W}_2$ are integration constants. The solution in terms of γ is therefore

$$\begin{aligned} F(\gamma) \;=\; & \frac{\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}}{2\sqrt{\gamma}\sqrt{\gamma+1}}[\hat{W}_1(\sqrt{\gamma+1} + \sqrt{\gamma})^{\frac{\varphi_1\sqrt{\kappa}}{\sqrt{ab}}} \\ & - \hat{W}_2(\sqrt{\gamma+1} - \sqrt{\gamma})^{\frac{\varphi_1\sqrt{\kappa}}{\sqrt{ab}}}]. \end{aligned} \tag{108}$$

Substituting Eq.(108) into Eq.(103) gives the electrostatic potential $V(r,\theta)$ as follows

$$\begin{aligned} V(r,\theta) \;=\; & \frac{\sqrt{A_0\varphi_0}\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}}{2N(r)(ar^2 - b)\sqrt{\gamma}\sqrt{\gamma+1}} \\ & \times[\hat{W}_1(\sqrt{\gamma+1} + \sqrt{\gamma})^{\frac{\varphi_1\sqrt{\kappa}}{\sqrt{ab}}} \\ & - \hat{W}_2(\sqrt{\gamma+1} - \sqrt{\gamma})^{\frac{\varphi_1\sqrt{\kappa}}{\sqrt{ab}}}]. \end{aligned} \tag{109}$$

Consider the inversion point of the static electric point charge $(0, 0, (b_0*))$, where $(b_0*) = \frac{b}{ab_0}$. Let

$$\gamma* \;=\; \frac{\frac{b}{a}}{\frac{b}{a} - (b_0*)^2}\frac{\Gamma*(r,\theta)}{\frac{b}{a} - r^2} \tag{110}$$

$$\Gamma* \;=\; r^2 + (b_0*)^2 - 2(b_0*)r\cos\theta. \tag{111}$$

Thus $\Gamma*$ is equal to the square of the "radial" distance from the inversion point at $z = (b_0*)$. It is straightforward to verify that

$$\gamma + 1 = \gamma*. \tag{112}$$

The electrostatic potential $V(r,\theta)$ in Eq.(109) is therefore singular at the point charge $z = b_0$ and also at its inversion point $z = (b_0*)$.

The integration constants can be determined using the boundary condition introduced in [1]; any integration over a closed spatial region not containing the perturbing charge must be exactly zero even if that region contains a surface of inversion, and conversely, an integretaion over an area containing the perturbing charged particle must equal exactly $4\pi\epsilon_0$ where ϵ_0 is the charge of the particle.

$$\int_{\Re} J^0 dv = \int_0^{2\pi}\int_{-\pi}^{\pi}\frac{N(r)^2(ar^2 - b)}{A_0\varphi_0}\frac{\partial V(r)}{\partial r}\sin\theta d\theta d\phi. \tag{113}$$

Here, $\Re$ is a region of 3-dimensional space residing in a hypersurface and $\partial\Re$ is its closed 2-dimensional boundary. Again, dv is an element of spatial proper volume in $\Re$. In order to integrate the above we convert Eq. (109) into a function of $\sinh\zeta$ where $\gamma = \sinh^2\frac{\zeta}{2}$. We find that the only term that requires integration is the term containing the integration constants, the integral of which is

$$\int_{-\pi}^{\pi} \frac{\hat{W}_1 \exp\left(\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}\zeta\right) - \hat{W}_2 \exp\left(\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}\zeta\right)}{\sinh\zeta} \sin\theta d\theta$$

$$= \frac{(ar^2-b)(ab_0^2-b)}{2abb_0 r}\left[\left(\frac{\sqrt{\frac{a}{b}}b_o-1}{\sqrt{\frac{a}{b}}b_o-1}\right)^{-\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}} - \left(\frac{\sqrt{\frac{a}{b}}b_o-1}{\sqrt{\frac{a}{b}}b_o-1}\right)^{\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}}\right]\left[\hat{W}_1\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r-1}\right)^{-\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}} - \hat{W}_2\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r-1}\right)^{\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}}\right]$$

for $\sqrt{\frac{b}{a}} < r < b_0$, where $k = \frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}$ and

$$\int_{-\pi}^{\pi} \frac{\hat{W}_1 \exp\left(\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}\zeta\right) - \hat{W}_2 \exp\left(\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}\zeta\right)}{\sinh\zeta} \sin\theta d\theta$$

$$= \frac{(ar^2-b)(ab_0^2-b)}{2abb_0 r}\left[\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r-1}\right)^{-\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}} - \left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r-1}\right)^{\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}}\right]\left[\hat{W}_1\left(\frac{\sqrt{\frac{a}{b}}b_o-1}{\sqrt{\frac{a}{b}}b_o-1}\right)^{-\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}} - \hat{W}_2\left(\frac{\sqrt{\frac{a}{b}}b_o-1}{\sqrt{\frac{a}{b}}b_o-1}\right)^{\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}}\right]$$

(116)

for $\sqrt{\frac{b}{a}} < b_0 < r$.

When we return Eq.(114) into Eq.(109) it is fairly straightforward to see that for the electrostatic potential to vanish for the region not containing a charge the integration constants must be chosen as the following

$$\hat{W}_1 = p_+^2 \hat{W}, \qquad (117)$$

$$\hat{W}_2 = -p_-^2 \hat{W}, \qquad (118)$$

where $\hat{W}$ is a constant yet to be determined.

By returning Eq.(115) into Eq.(109) and under the condition that for this region $\left(\sqrt{\frac{b}{a}} < b_0 < r\right)$ Eq.(113) must equal $-4\pi\epsilon_0$ we can quickly solve for $\hat{W}$ giving

$$\hat{W} = \frac{2abb_0\sqrt{\varphi_0 A_0}\epsilon_0}{\varphi_1\sqrt{k}\,(ab_0^2-b)\,N_b}, \qquad (119)$$

where

$$N_b = p_+^2\left(\frac{\sqrt{\frac{a}{b}}b_o-1}{\sqrt{\frac{a}{b}}b_o-1}\right)^{-\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}} + p_-^2\left(\frac{\sqrt{\frac{a}{b}}b_o-1}{\sqrt{\frac{a}{b}}b_o-1}\right)^{\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}} \qquad (120)$$

The electrostatic potential of a point charge ϵ_0 residing at b_0 outside the surface of inversion in a CBD spacetime is therefore

$$V(r,\theta) = \frac{\sqrt{A_0\varphi_0}r\epsilon_0}{N(r)(ar^2-b)}\frac{b_0}{N_b(ab_0^2-b)}\frac{\sqrt{ab}}{2\sqrt{\gamma}\sqrt{\gamma+1}}\left(p_+^2(\sqrt{\gamma+1}+\sqrt{\gamma})^{\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}} + p_-^2(\sqrt{\gamma+1}-\sqrt{\gamma})^{\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}}\right),$$

where $N(r)$ and $p_\pm^2$ are given in Table II.

We are able to show, in a manner similar to that demonstrated in [47], that the closed-form electric potential can be expressed succinctly in terms of the function

$$\Pi : \varsigma \mapsto \frac{\varsigma}{(a\varsigma^2-b)N\chi^\delta(r)}, \quad \delta \in \{+,-,0\}, \qquad (121)$$

where χ is either I, II, or III/IV corresponding to the three branches of the CBD solutions and the values for $N\chi^\delta(r)$ for each branch of solutions are given in Table

IV, such that

$$V(r,\theta) = \frac{A_0\varphi_0\epsilon_0}{2}\frac{\Pi(b)\Pi(r)}{\Pi(\sigma(r,\theta))}, \tag{122}$$

and the composite function is

$$\Pi(\sigma(r,\theta)) = (\Pi\circ\sigma)(r,\theta). \tag{123}$$

We define the new variable $\sigma(r,\theta)$ as the following

$$\sigma(r,\theta) = \left(\frac{b_0^2 r^2 + \left(\frac{b}{a}\right)^2 - 2\frac{b}{a}b_0 r\cos\theta)}{(r^2+b^2-2br\cos\theta)}\right)^{\frac{1}{2}}. \tag{124}$$

From Eq.(124), it is straightforward to show that

$$\begin{aligned}\sigma^2(r,\theta) - B^2 &= \frac{(ab_0^2-b)(ar^2-b)}{a^2(r^2+b^2-2br\cos\theta)}, \\ &= \frac{(ab_0^2-b)(ar^2-b)}{a^2\Gamma(r,\theta)} = \frac{\frac{b}{a}}{\gamma(r,\theta)}, \end{aligned}\tag{125}$$

where $\Gamma(r,\theta)$ is given in Eq.(101). Rearranging terms we have

$$\gamma(r,\theta) = \frac{\frac{b}{a}}{\sigma^2(r,\theta)-\frac{b}{a}} \tag{126}$$

$$\gamma(r,\theta)+1 = \frac{\sigma^2(r,\theta)}{\sigma^2(r,\theta)-\frac{b}{a}}. \tag{127}$$

Substituting Eqs. (126) and (127) into Eq. (123), and after some algebra, we obtain Eq.(122).

VIII. CONCLUSION

In this paper we were able to describe the static spherically symmetric charged Brans-Dicke spacetime in its most general form, placing no restrictions on the constants of integration except boundary conditions at infinity. In doing so we found that the CBD (and indeed the BD) metric gives rise to three independent classes of solutions, I, II and III/IV. In the absence of a charge ($Q=0$) the CBD I$^+$, II$^+$, III$^+$/IV$^+$ classes reduce to the BD Classes I, II, and III/IV.

We found that the CBD Class I solutions represent the case where $m_B > q_B$, the CBD Class II solutions represent the unusual case where $q_B > m_B$ and the Class III/IV solution represents an extreme CBD spacetime as an analogy to extreme Reissner-Nordström black holes where $q=m$. Interestingly, as an extension of our findings on the CBD Class III/IV$^+$ solution we can infer that the BD Class III/IV solution contains no mass and instead describes a spacetime with a massless, scalar field generating singularity. By looking at the curvature invariants of the nine classes of solutions we find that the CBDII$^-$ background may admit black hole or wormhole nature when $\omega < -2$. As there is no range of ω that must be dismissed on grounds of being unphysical there is no reason why the possibility of the existence of CBD or BD black holes should be dismissed. Further investigations are warranted and are left to a following paper.

All nine classes of solutions of the CBD spacetime prove to be stable under small electrostatic perturbations. This includes the CBDIII/IV or “extreme CBD” spacetimes. Although the extreme Reissner-Nordström black hole counterpart has been proven to be unstable under linear perturbations by [54] and particularly electromagnetic perturbations by [55] we point out that these studies were on electrodynamic perturbations and the decay occured only at late times. It would be worthwhile to investigate the matter further by perturbing the CBD spacetime with an electrodynamic charge by adding a time dependent contribution to the charge density of Eq.(100).

IX. ACKNOWLEDGEMENTS

We would like to thank Professor Bob Wald for pointing out key points regarding the electromagnetic perturbation of a charged space-time, namely that the perturbation considered in this paper is not complete but rather a “model system”. The curvature invariants in this paper were calculated using the package GRTensorII.

X. APPENDIX A

The CBD field equations arising from the static, spherically symmetric metric Eq.(1) in isotropic coordinates can be simplified when the following substitutions are introduced

$$A(r) = \frac{\mathcal{A}(r)}{\varphi(r)} \Leftrightarrow \mathcal{A}(r) = \varphi(r)A(r), \tag{128}$$

$$B(r) = \frac{\mathcal{B}(r)}{\varphi(r)} \Leftrightarrow \mathcal{B}(r) = \varphi(r)B(r); \tag{129}$$

Lemma 1: The reduced field Eqs.(2), (4), (5), the scalar field wave equation (8), in the Einstein frame are thus simplified to

$$\begin{aligned}B(r)G_t^t &= \frac{\mathcal{B}''(r)}{\mathcal{B}} - \frac{3\mathcal{B}'(r)^2}{4\mathcal{B}^2} + \frac{2\mathcal{B}'(r)}{r\mathcal{B}} \\ &= -\frac{4\pi q_0^2}{c^4 r^4\mathcal{B}(r)} - \frac{2\omega+3}{4}\left((\ln\varphi(r))'\right)^2 \end{aligned}\tag{130}$$

$$\begin{aligned}B(r)G_r^r &= \frac{1}{4}\left(\frac{\mathcal{B}'(r)}{\mathcal{B}}\right)^2 + \frac{\mathcal{A}'(r)\mathcal{B}'(r)}{2\mathcal{A}\mathcal{B}} \\ &\quad + \frac{1}{r}\left(\frac{\mathcal{A}'(r)}{\mathcal{A}} + \frac{\mathcal{B}'(r)}{\mathcal{B}}\right) \\ &= -\frac{4\pi q_0^2}{c^4 r^4\mathcal{B}(r)} + \frac{2\omega+3}{4}\left((\ln\varphi(r))'\right)^2 \end{aligned}\tag{131}$$

$$\begin{aligned} B(r)\,G^{\theta}_{\theta} &= \frac{\mathcal{A}''(r)}{2\mathcal{A}} + \frac{\mathcal{B}''(r)}{2\mathcal{B}} - \frac{\mathcal{A}'(r)^2}{4\mathcal{A}^2} \\ &\quad -\frac{\mathcal{B}'(r)^2}{2\mathcal{B}^2} + \frac{1}{2r}\left(\frac{\mathcal{A}'(r)}{\mathcal{A}} + \frac{\mathcal{B}'(r)}{\mathcal{B}}\right) \\ &= \frac{4\pi q_0^2}{c^4 r^4 \mathcal{B}(r)} - \frac{2\omega+3}{4}\left((\ln\varphi(r))'\right)^2, \end{aligned} \tag{132}$$

and the scalar field satisfies

$$\left(r^2\left(\mathcal{A}(r)\,\mathcal{B}(r)\right)^{\frac{1}{2}}(\ln\varphi(r))'\right)' = 0. \tag{133}$$

The reduced field equations (130), (131) and (132) are linearly dependent, and satisfy the condition

$$\left(\begin{array}{c} \left(\frac{d}{dr} + \frac{\mathcal{A}'(r)}{2\mathcal{A}} + \frac{2}{r}\right)\text{Eq.}(131) \\ -\frac{\mathcal{A}'(r)}{2\mathcal{A}}\text{Eq.}(130) - 2\left(\frac{\mathcal{B}'(r)}{2\mathcal{B}} + \frac{1}{r}\right)\text{Eq.}(132) \end{array}\right) = 0. \tag{134}$$

Using Lemma 1 and introducing appropriate boundary conditions at spatial infinity $r \to \infty$, the static spherically symmetric electrovac Brans-Dicke spacetimes metric in isotropic coordinates is characterised by the nontrivial Maxwell tensor component $F_{01} = -\frac{cV_0' e^{\mathcal{A}(r)-\mathcal{B}(r)}}{r^2}$ and the following system of three boundary-value ordinary differential equations on: (I) $(\mathcal{A}(r)\,\mathcal{B}(r))^{\frac{1}{2}}$, (II) $\ln\varphi(r)$, and (III) $(\mathcal{A}(r))^{-\frac{1}{2}}$.

The sum of the modified reduced field equations (131) and (132) yields a second order equation on $(\mathcal{A}(r)\,\mathcal{B}(r))^{\frac{1}{2}}$:

$$\left(r^3\left((\mathcal{A}(r)\,\mathcal{B}(r))^{\frac{1}{2}}\right)'\right)' = 0, \tag{135}$$

This can be integrated twice to give

$$\begin{aligned} r^2\left(\mathcal{A}(r)\,\mathcal{B}(r)\right)^{\frac{1}{2}} &= \left(ar^2 - b\right); \\ \Leftrightarrow \quad \left(\mathcal{A}(r)\,\mathcal{B}(r)\right)^{\frac{1}{2}} &= \left(a - \frac{b}{r^2}\right), \end{aligned} \tag{136}$$

where a and b are real arbitrary constants of integration. However, because when $a = b = 0$, the metric becomes singular we exclude the case when both the real constants a and b vanish, ie.

$$ab = 0 \Rightarrow \left\{\begin{array}{c} a = 0, \quad b \neq 0 \\ \text{or} \\ b = 0, \quad a \neq 0 \end{array}\right. . \tag{137}$$

The modified scalar field wave equation (133) can be expressed as a total derviative

$$\left(r^2\left(\mathcal{A}(r)\,\mathcal{B}(r)\right)^{\frac{1}{2}}(\ln\varphi(r))'\right)' = 0. \tag{138}$$

The formal solution of Eq. (138) can be expressed in terms of Eq. (136) in the following form

$$(\ln\varphi(r))' = \frac{\varphi_1}{b\left(\frac{a}{b}r^2 - 1\right)}, \tag{139}$$

where the constant of integration φ_1 is an arbitrary real number. Integrating Eq. (139), the formal solution of the scalar field is expressed as

$$\varphi(r) = \varphi_0 \exp\left(\frac{\varphi_1}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r - 1}{\sqrt{\frac{a}{b}}r + 1}\right)\right), \qquad \varphi_0 > 0, \tag{140}$$

where, without loss of generality, the constant of integration φ_0 is real and positive. Since the product ab, determined by boundary conditions, can be $\gtreqless 0$, i.e.,

$$ab \gtreqless 0, \tag{141}$$

the terms $\sqrt{ab}$ and $\sqrt{\frac{a}{b}}$ can be either real or imaginary as the function $\frac{\varphi_1}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r+1}\right)$ will always remain real, and hence the scalar field $\varphi(r)$ likewise will always remain real. In this sense, the solution (140) is said to be formal.

Using Eq. (136) the modified reduced field equation (131) can be rewritten in the form

$$\left(\frac{\mathcal{A}'(r)}{2\mathcal{A}}\right)^2 = \frac{\varphi_1^2}{b^2\left(\frac{ar^2}{b} - 1\right)^2}\left(\frac{4\pi q_0^2}{c^4\varphi_1^2}\mathcal{A}(r) + \left(\frac{4ab}{\varphi_1^2} - \frac{2\omega+3}{4}\right)\right). \tag{142}$$

After some manipulation we are able to obtain a first order second degree differential equation on $\left(\frac{\kappa}{Q^2\mathcal{A}(r)}\right)^{\frac{1}{2}}$

$$\frac{\frac{d}{dr}\left(-\left(\frac{\kappa}{Q^2\mathcal{A}(r)}\right)^{\frac{1}{2}}\right)}{\sqrt{1 + \frac{\kappa}{Q^2\mathcal{A}(r)}}} = \frac{d}{dr}\left(\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r - 1}{\sqrt{\frac{a}{b}}r + 1}\right)\right) \tag{143}$$

where

$$\kappa + Q^2\mathcal{A}_0 > 0. \tag{144}$$

where we have defined the auxillary constants as follows

$$\kappa := \frac{4ab}{\varphi_1^2} - \frac{2\omega+3}{4}, \tag{145}$$

$$Q^2 := \frac{4\pi q_0^2}{c^4\varphi_1^2}. \tag{146}$$

Equation (143) can be solved to give, after some algebra, the following

$$\mathcal{A}(r) = \mathcal{A}_0\left(\begin{array}{c} p_+^2 \exp\left(-\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r+1}\right)\right) \\ +p_-^2 \exp\left(\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r+1}\right)\right) \end{array}\right)^2 \tag{147}$$

$$A_0 := \frac{\mathcal{A}_0}{\varphi_0} > 0, \tag{148}$$

where the positive constant $\mathcal{A}_0 > 0$ is determined by the boundary condition on $\mathcal{A}(r)$ and the auxillary constants are

$$p_{\pm}^2 = \frac{1}{2}\left(\sqrt{1 + \frac{A_0\varphi_0 Q^2}{\kappa}} \pm 1\right). \tag{149}$$

If we define

$$N(r) \ : \ = p_+^2 \exp\left(-\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r - 1}{\sqrt{\frac{a}{b}}r + 1}\right)\right) \tag{150}$$

$$-p_-^2 \exp\left(\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r - 1}{\sqrt{\frac{a}{b}}r + 1}\right)\right), \tag{151}$$

then we are able to give $A(r)$ succinctly in the following form

$$A(r) := \frac{\mathcal{A}(r)}{\varphi(r)} = \frac{A_0}{\exp\left(\frac{\varphi_1}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r+1}\right)\right)N^2(r)}. \tag{152}$$

The limit of $N(r)$ when $\kappa \to 0$ is found to be

$$N(r) = 1 - \sqrt{A_0\varphi_0}\left(\frac{Q\,\varphi_1}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r - 1}{\sqrt{\frac{a}{b}}r + 1}\right)\right), \quad \kappa = 0. \tag{153}$$

Substituting Eq.(152) into Eq.(136), we obtain the formal expressions for the metric coefficient $B(r)$

$$B(r) := \frac{\mathcal{B}(r)}{\varphi(r)} = \frac{a^2}{\varphi_0^2 A_0}\frac{\left(1 - \frac{b}{a}\frac{1}{r^2}\right)^2 N^2(r)}{\exp\left(\frac{\varphi_1}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r+1}\right)\right)}. \tag{154}$$

From the above and Eq. (9) the formal expression for the electrostatic force field is thus

$$V'(r) = \frac{c^2 Q\varphi_1}{\sqrt{4\pi}}\frac{A_0\varphi_0}{a\left(r^2 - \frac{b}{a}\right)N^2(r)}. \tag{155}$$

Appendix B

The constants of integration are related to the BD parameters given in [3] so that one may compare the CBD solutions and their degenerate solutions to the known BD solutions.

The metric coefficients $A(r)$ and $B(r)$ in equations (12) and (14) have exponents

$$\frac{1}{\lambda} := -\frac{\varphi_1}{4\sqrt{ab}}\left(1 + 2\sqrt{\kappa}\right), \tag{156}$$

which appear in the products $\varphi(r)N^2(r)$ and $\frac{\varphi(r)}{N^2(r)}$. From the scalar field we can establish the following correspondence

$$\frac{C}{\lambda} := \frac{\varphi_1}{2\sqrt{ab}}. \tag{157}$$

so that

$$\frac{C^2}{\lambda^2} = \frac{\varphi_1^2}{4ab}, \tag{158}$$

By Eq. (156), the parameter

$$\begin{aligned} C &= \frac{\varphi_1}{2\sqrt{ab}}\lambda = \frac{-\frac{\varphi_1}{2\sqrt{ab}}}{\frac{\varphi_1}{4\sqrt{ab}}\left(1 + 2\sqrt{\kappa}\right)} \\ &= -\frac{1}{\frac{1}{2}\left(1 + 2\sqrt{\frac{4b}{a}\left(\frac{a}{\varphi_1}\right)^2 - \frac{2\omega+3}{4}}\right)} \end{aligned} \tag{159}$$

Using Eqs.(156) and (159), we are able to prove the identity

$$\begin{aligned} \lambda^2 &\equiv \frac{\omega+2}{2}C^2 + C + 1 \\ &= \frac{2\omega+3}{4}C^2 + \frac{(C+2)^2}{4}, \end{aligned} \tag{160}$$

which was first introduced in [3] and was extended to other branches of solutions in [46]. Consider the reciprocal of Eq.(157):

$$\frac{4ab}{\varphi_1^2} = \frac{\lambda^2}{C^2} = \frac{1}{C^2} + \frac{1}{C} + \frac{\omega+2}{2}, \tag{161}$$

which is a quadratic equation in $\frac{1}{C}$. Since

$$ab \in \mathbb{R}, \quad \varphi_1^2 \geq 0, \tag{162}$$

we have

$$\frac{\lambda^2}{C^2} = \frac{\omega+2}{2} + \frac{1}{C} + \frac{1}{C^2} \gtreqless 0, \tag{163}$$

depending of the sign of the product $ab \gtreqless 0$.

Additionally, the auxillary parameter $k = \frac{C+2}{2\lambda}$ as used in [1] and [47] is related to the present notation employed in this paper via the following

$$k = \frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}. \tag{164}$$

The correspondence between the Brans-Dicke constants [46] and the integration constants used here (CBD) is given in Table IV:

Using Eq. (160), we have the auxillary constant

$$\begin{aligned} \frac{\varphi_1^2\kappa}{4ab} &= 1 - \frac{2\omega+3}{4}\frac{\varphi_1^2}{4ab} \\ &= \frac{(C+2)^2}{\lambda^2}. \end{aligned} \tag{165}$$

Note that the CBD constants of integration are real: $\{a, b, \varphi_1, Q\} \in \mathbb{R}$, $\{\varphi_0, A_0\} \in \mathbb{R}^+$; while the Brans-Dicke type parameters $\left\{e^{2\alpha_0}, e^{2\beta_0}, B^2, Q\right\} \in \mathbb{R}$, $\{\phi_0\} \in$

CBD	Brans-Dicke	Brans-Dicke	CBD
a	$\sqrt{\phi_0}e^{\alpha_0+\beta_0}$	$e^{2\alpha_0}$	A_0
b	$\sqrt{\phi_0}e^{\alpha_0+\beta_0}B^2$	$e^{2\beta_0}$	$\frac{a^2}{\varphi_0 A_0}$
A_0	$e^{2\alpha_0}$	B^2	$\frac{b}{a}$
φ_0	ϕ_0	ϕ_0	φ_0
φ_1	$\frac{2C}{\lambda}\sqrt{\phi_0}Be^{\alpha_0+\beta_0}$	C	$-\frac{\frac{\varphi_1}{2\sqrt{ab}}}{\frac{1}{2}\left(\frac{\varphi_1\pm2\sqrt{\kappa}}{2\sqrt{ab}}\right)}$

TABLE IV: Corrspondence between CBD integration constants and Brans-Dicke constants

$\mathbb{R}^+$, $\{C\} \in \mathbb{C}$, although $\frac{C^2}{\lambda^2} = \frac{\varphi_1^2}{4ab} \in \mathbb{R}$ with $\lambda^2 = \frac{\omega+2}{2}C^2 + C + 1$. By comparing the Brans-Dicke parameters with the CBD constants of integration we are able to shed some light on the physical interpretation of the former. We are able to extrapolate that the parameter

$$\frac{\varphi_1}{2\sqrt{ab}} = \frac{C}{\lambda} \tag{166}$$

is the ratio of strength of the scalar field

$$\frac{\varphi_1}{2a} = \frac{2BC}{\lambda} \tag{167}$$

to the strength of the gravitational field

$$\sqrt{\frac{b}{a}} = B. \tag{168}$$

The auxillary parameter $N(r)$ is related to the parameter $\eta(r)$ of [1] and [47] via the following

$$N(r) := \eta(r)\left(\frac{\sqrt{\frac{a}{b}}r - 1}{\sqrt{\frac{a}{b}}r + 1}\right)^{-\frac{\varphi_1\sqrt{\kappa}}{2\sqrt{ab}}}, \tag{169}$$

where $\eta(r)$ can be expressed in the more generalized notation used in this paper as

$$\eta(r) = p_+^2 - p_-^2\left(\frac{\sqrt{\frac{a}{b}}r - 1}{\sqrt{\frac{a}{b}}r + 1}\right)^{\frac{\varphi_1\sqrt{\kappa}}{\sqrt{ab}}}. \tag{170}$$

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Chapter 5

Conclusion

The aim of this body of research was to study the stability of static spherically symmetric electrovac BD spacetimes. We have been successful in giving, for the first time, a form of the static, spherically symmetric electrovac BD spacetimes that is reducible to all three independent classes of the BD spacetime as given in [8]. We have been able to prove that a class of the CBD metric, the BDRN metric, is also reducible to the Reissner-Nordström spacetime in the absence of a scalar field, and to the Schwarzschild spacetime in the absence of both charge and scalar fields. We have paid particular attention to the BDRN metric as it seems physically relevant in the context of its degeneracy.

To study the stability of the BDRN spacetime, and later the CBD spacetime, we electrostatically perturbed it with a point charge. In order to find an exact, closed-form solution we generalized a method introduced nearly a century ago by Copson by which we were able to convert the partial differential equation on the electrostatic potential into a solvable ordinary differential equation. Additionally, we introduced a boundary condition that proved to be sufficient in ensuring that the resultant solution was representative of a single perturbing charge. In this way we were able to find an exact, closed-form solution for the electrostatic potential generated by a single perturbing charge in a BDRN, and later CBD, spacetime. We found that the BDRN, and the CBD spacetimes in general, are stable under electromagnetic perturbations.

Through the introduction of a transformation formula we were able to express the BDRN, and later CBD, spacetime in Schwarzschild-type coordinates. In doing so we discovered the presence of an additional S^2 singularity which is the analogue of the singularity at the origin for the Reissner-Nordström spacetime. This singularity is in addition to the generalized inner and outer "horizons". We find that, as expected, this singularity does not manifest as an S^2 singularity in the degenerate spacetimes due to the range of the parameters and is thus unique to the BDRN spacetime.

Through the graphical representation of the equipotential surfaces generated by the perturbing point charge in the BDRN spacetime we were able to discover a mechanism by which equilibrium can be achieved. In the uncharged BD and Schwarzschild spacetimes the singularity exhibits dipole behaviour as a direct consequence of the presence of the perturbing point charge. The interpretation of this dipole behavior is as follows: the dipole nature of the singularity creates an induced charge on the outer "horizon" while maintaining an overall zero charge and zero net electric flux.

In the charged spacetimes (BDRN and Reissner-Nordström) we find that in addition to the dipole-like singularity there appears an image charge located inside the inner "horizon", r_-. We can interpret the image charge thusly: an image charge within the inner "horizon" of the BDRN and Reissner-Nordström backgrounds arises from the interaction between the perturbing charge and the background charge.

We were able to show through the curvature invariants that only the CBDII$^-$ spacetime permits black holes or wormholes when $\omega < -2$, and it is worthwhile, and indeed, pertinent to investigate whether it is representative of wormholes or black holes or both.

In regards to wormhole investigations, the CBD metric lends itself quickly to Morris-Thorne canonical form [14] from which the redshift function $\Phi(R)$ and the shape function $b(R)$ can be determined, where R is the new radial coordinate and is related to r through a transformation equation. From the shape and redshift functions one can find the minimum allowable radius R_{min} which represents the wormhole throat. If the wormhole R_{min} is real and the redshift function is nonsingular for all $R_{min} < R$ and finite everywhere then it can be inferred that a two way traversable wormhole is possible. Of course there are several other factors that must be considered also. First, the Weak Energy Condition (WEC), must be violated for wormholes to exist [14]. The energy density of the wormhole material, $\rho(R)$ is given by

$$\rho(R) = (8\pi R^{-2})(\frac{db}{dR}. \tag{5.1}$$

It is therefore a trivial matter to determine the condition upon the metric parameters for which the energy density is negative. Although a wormhole may be theoretically traversable this does not mean that it is suitable for interstellar (or time) travel. To determine whether a wormhole is suitable for a human traveller several factors must be taken into consideration. The tidal forces upon and the acceleration experienced by the human traveller must be acceptable levels (around 1 Earth gravity) and finally for travel through a wormhole to be practical the journey time should be of a reasonable length (the acceptable range is around 1 year [89]). It would be interesting and meaningful to study two way traversable CBD wormholes in this context. These conditions are all fairly easy to verify in the manner already done by several researchers for the BD spacetime, for example, [83], [90], and [91] (See Section 1.5 for a more comprehensive literature review).

Of course if the possibility of CBD wormholes is established, the discussion can be taken further and the amount of exotic matter required to produce the wormhole could be investigated. As exotic matter seems troublesome, it can be understood that the less one requires to build a wormhole the more physically viable a wormhole becomes [14]. In order to limit the amount of exotic material required three scenarios that are proposed: 1. use exotic material throughout the wormhole, but require the density of it to fall of drastically the further one moves away from the throat. 2. Use exotic material but cut it off completely at a given radius. Or 3. Use exotic material for a tiny central region around the throat. The energy density profiles for each of these three scenarios are different and it would be meaningful to investigate whether any of them could give rise to realistic CBD wormholes.

Of personal interest, and a direct extension of the research conducted in this thesis, would be to study the stability of the wormhole to perturbations in the manner of Boisseau and Linet [93]. A spacecraft traversing the wormhole can be considered a perturbation of the wormhole and thus it would be pertinent to first determine the stability of the spacetime to something smaller, such as an electrostatic charge!

There are other directions the research contained in this thesis could pursue. One such avenue is to allow the perturbing charge to have time dependency such that the perturbation would be electrodynamic. It would be straightforward to add time dependency to the perturbing charge by adding a δ function on time to the right hand side of Eq.(34) of Paper 1, Chapter 2. In this way, one could see how the electric potential of the charge varies with time and how the equipotential surfaces behave inside the horizon with the passage of time. We can investigate and extend to the CBD spacetime, the results of [102] and [103] who found that linear electromagnetic perturbations of extreme Reissner-Nordström black holes are unstable at late times.

Another natural step forward would be to extend the results of this research to Bergmann-Wagoner-Nordtvedt (BWN) theory where the coupling parameter ω is allowed to be a function of the scalar field φ. Closed-form solutions for the Barker and Schwinger subcases of the static, spherically symmetric, electrovac BWN spacetime were found by Singh and Singh [51] following the manner of Luke and Szamosi who found a subclass of the electrovac BD spacetime [42]. However, their resulting metric, like Luke and Szamosi's, was incomplete in that they were unable to determine any physical interpretation for the constants of integration. It is entirely possible that, like Luke and Szamosi, who were only able to find four solutions of the CBD spacetime, Singh and Singh may have unknowingly excluded several classes of solutions as well. It seems pertinent to replicate the study of the BWN theory following the procedure of this research where the field equations are integrated directly without placing any restrictions or assumptions on the parameters except for boundary conditions at infinity. In this way, the full class of solutions can be obtained. Of course there is no reason to restrict our investigation to the Barker and Schwinger subcases and a following investigation would involve the generalized static spherically symmetric electrovac BWN spacetimes.

The CBD spacetime discussed throughout this body of research has been without the presence of a cosmological constant. It is altogether possible to investigate charged Brans-Dicke spacetimes with the addition of a cosmological constant in the Lagrangian as given by Eq.(1.2) of Chapter 1. The BD field equations and the scalar field equation are both supplemented by a term arising from Λ. A consequence of the addition of the cosmological constant in the Lagrangian Eq.(1.2) is that the universe is now described as being static [4]. Clearly this is not in agreement with obversational data. If one moves from the Jordan frame to the Einstein frame however this inconsistency is removed and the universe appears to be expanding as desired. Thus BD theory with the presence of the cosmogical constant may still be physically interesting. It is intriguing to see if the field equations with this extra term can be integrated in the way we were able to demonstrate for the electrovac BD field equations in this thesis. Furthermore, if we are able to find a solution for the field equations it would be fascinating to see how the presence of the cosmological constant will affect the form of the partial differential equation on the electrostatic potential of a perturbing charge and to see whether the Copson-Hadarmard method developed in Chapter 2 may still be applicable.

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